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# An example of instability in high-dimensional Hamiltonian systems

Abed Bounemoura \*

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## Abstract

In this article, we use a mechanism introduced by Herman, Marco and Sauzin to show that if a perturbation of a quasi-convex integrable Hamiltonian system is not too small with respect to the number of degrees of freedom, then the classical exponential stability estimates do not hold. Indeed, we construct an unstable solution whose drifting time is polynomial with respect to the inverse of the size of the perturbation. A different example was already given by Bourgain and Kaloshin, with a linear time of drift but with a perturbation which is larger than ours. As a consequence, we obtain a better upper bound on the threshold of validity of exponential stability estimates.

## 1 Introduction

1. Consider a near-integrable Hamiltonian system of the form

$$\begin{cases} H(\theta, I) = h(I) + f(\theta, I) \\ |f| < \varepsilon \end{cases}$$

with angle-action coordinates  $(\theta, I) \in \mathbb{T}^n \times \mathbb{R}^n$ , and where  $f$  is a small perturbation, of size  $\varepsilon$ , in some suitable topology defined by a norm  $|\cdot|$ .

If the system is analytic and  $h$  satisfies a generic condition, it is a remarkable result due to Nekhoroshev ([Nek77], [Nek79]) that the action variables are stable for an exponentially long interval of time with respect to the inverse of the size of the perturbation: one has

$$|I(t) - I_0| \leq c_1 \varepsilon^b, \quad |t| \leq c_2 \exp(c_3 \varepsilon^{-a}),$$

for some positive constants  $c_1, c_2, c_3, a, b$  and provided that the size of the perturbation  $\varepsilon$  is smaller than a threshold  $\varepsilon_0$ . Of course, all these constants

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strongly depend on the number of degrees of freedom  $n$ , and when the latter goes to infinity, the threshold  $\varepsilon_0$  and the exponent of stability  $a$  go to zero.

More precisely, in the case where  $h$  is quasi-convex and the system is analytic or even Gevrey, then we know that the exponent  $a$  is of order  $n^{-1}$  and this is essentially optimal (see [LN92], [Pös93], [MS02], [LM05] [Zha09] and [BM10] for more information on the optimality of the stability exponent).

**2.** This fact was used by Bourgain and Kaloshin in [BK05] to show that if the size of the perturbation is

$$\varepsilon_n \sim e^{-n},$$

then there is no exponential stability: they constructed unstable solutions for which the time of drift is linear with respect to the inverse of the size of the perturbation, that is

$$|I(\tau_n) - I_0| \sim 1, \quad \tau_n \sim \varepsilon_n^{-1}.$$

Their motivation was the implementation of stability estimates in the context of Hamiltonian partial differential equations, which requires to understand the relative dependence between the size of the perturbation and the number of degrees of freedom. Their result indicates that for infinite dimensional Hamiltonian systems, Nekhoroshev's mechanism does not survive and that "fast diffusion" should prevail. Of course, in their example, one cannot simply take  $n = \infty$  as the size of the perturbation  $\varepsilon_n \sim e^{-n}$  goes to zero and the time of instability  $\tau_n \sim e^n$  goes to infinity exponentially fast with respect to  $n$ . A more precise interpretation concerns the threshold of validity  $\varepsilon_0$  in Nekhoroshev's theorem: it has to satisfy

$$\varepsilon_0 \ll e^{-n},$$

and so it deteriorates faster than exponentially with respect to  $n$ .

**3.** As was noticed by the authors in [BK05], their examples share some similarities with the mechanism introduced by Herman, Marco and Sauzin in [MS02] (see also [LM05]). In this present article, we use the approach of [MS02] to show, using simpler arguments, that if the size of the perturbation is

$$\varepsilon_n \sim e^{-n \ln(n \ln n)},$$

then it is still too large to have exponential stability: we will show that one can find an unstable solution where the time of drift is polynomial, more precisely

$$|I(\tau_n) - I_0| \sim 1, \quad \tau_n \sim \varepsilon_n^{-n}.$$

We refer to Theorem 2.1 below for a statement. Hence one can infer that the threshold  $\varepsilon_0$  in Nekhoroshev's theorem further satisfies

$$\varepsilon_0 \ll e^{-n \ln(n \ln n)} \ll e^{-n},$$

and this gives another evidence that the finite dimensional mechanism of stability cannot extend so easily to infinite dimensional systems. Let us point out that our time of drift is worst than the one obtained in [BK05], but this stems from the fact that the size of our perturbation is smaller than theirs and so it is natural for the time of instability to be larger. Moreover, our exponent  $n$  in the time of drift can be a bit misleading since in any cases, that is even for a linear time of drift,  $\tau_n$  goes to infinity exponentially fast with  $n$ .

## 2 Main result

1. Now let us state our result more precisely. Let  $n \geq 3$  be the number of degrees of freedom, and given  $R > 0$ , let  $B = B_R$  be the open ball of  $\mathbb{R}^n$  around the origin, of radius  $R > 0$  with respect to the supremum norm  $|\cdot|$ , and  $\overline{B}$  its closure.

The phase space is  $\mathbb{T}^n \times B$ , and we consider a Hamiltonian system of the form

$$H(\theta, I) = h(I) + f(\theta, I), \quad (\theta, I) \in \mathbb{T}^n \times B.$$

Our quasi-convex integrable Hamiltonian  $h$  is the simplest one, namely

$$h(I) = \frac{1}{2}(I_1^2 + \dots + I_{n-1}^2) + I_n, \quad I = (I_1, \dots, I_n) \in B.$$

Let us recall that, given  $\alpha \geq 1$  and  $L > 0$ , a function  $f \in C^\infty(\mathbb{T}^n \times \overline{B})$  is  $(\alpha, L)$ -Gevrey if, using the standard multi-index notation,

$$|f|_{\alpha, L} = \sum_{l \in \mathbb{N}^{2n}} L^{|l|} (\ell!)^{-\alpha} |\partial^l f|_{C^0(\mathbb{T}^n \times \overline{B})} < \infty.$$

The space of such functions, with the above norm, is a Banach space that we denote by  $G^{\alpha, L}(\mathbb{T}^n \times \overline{B})$ . One can see that analytic functions correspond exactly to  $\alpha = 1$ .

2. Now we can state our theorem.

**Theorem 2.1.** *Let  $n \geq 3$ ,  $R > 1$ ,  $\alpha > 1$  and  $L > 0$ . Then there exist positive constants  $c, \gamma, C$  and  $n_0 \in \mathbb{N}^*$  depending only on  $R, \alpha$  and  $L$  such that for any  $n \geq n_0$ , the following holds: there exists a function  $f_n \in G^{\alpha, L}(\mathbb{T}^n \times \overline{B})$  with  $\varepsilon_n = |f_n|_{\alpha, L}$  satisfying*

$$e^{-2(n-2) \ln(4n \ln 2n)} \leq \varepsilon_n \leq c e^{-2(n-2) \ln(n \ln 2n)},$$

*such that the Hamiltonian system  $H_n = h + f_n$  has an orbit  $(\theta(t), I(t))$  for which the estimates*

$$|I(\tau_n) - I_0| \geq 1, \quad \tau_n \leq C \left( \frac{c}{\varepsilon_n} \right)^{n\gamma},$$

*hold true.*

As we have already explained, this statement gives an upper bound on the threshold of applicability of Nekhoroshev's estimates, which is an important issue when trying to use abstract stability results for "realistic" problems, for instance for the so-called planetary problem (see [Nie96]).

So let us consider the set of Gevrey quasi-convex integrable Hamiltonians  $\mathcal{H} = \mathcal{H}(n, R, \alpha, L, M, m)$  defined as follows:  $h \in \mathcal{H}$  if  $h \in G^{\alpha, L}(\overline{B})$  and satisfies both

$$\forall I \in B, \quad |\partial^k h(I)| \leq M, \quad 1 \leq |k_1| + \dots + |k_n| \leq 3,$$

and

$$\forall I \in B, \forall v \in \mathbb{R}^n, \quad \nabla h(I) \cdot v = 0 \implies \nabla^2 h(I) v \cdot v \geq m|v|^2.$$

From Nekhoroshev's theorem (see [MS02] for a statement in Gevrey classes), we know that there exists a positive constant  $\varepsilon_0(\mathcal{H}) = \varepsilon_0(n, R, \alpha, L, M, m)$  such that the following holds: for any  $h \in \mathcal{H}$ , there exist positive constants  $c_1, c_2, c_3, a$  and  $b$  such that if

$$f \in G^{\alpha, L}(\mathbb{T}^n \times \overline{B}), \quad |f|_{\alpha, L} < \varepsilon_0(\mathcal{H}),$$

then any solution  $(\theta(t), I(t))$  of the system  $H = h + f$ , with  $I(0) \in B_{R/2}$ , satisfies

$$|I(t) - I_0| \leq c_1 \varepsilon^b, \quad |t| \leq c_2 \exp(c_3 \varepsilon^{-a}).$$

Then we can state the following corollary of our Theorem 2.1.

**Corollary 2.2.** *With the previous notations, one has the upper bound*

$$\varepsilon_0(\mathcal{H}) < e^{-2(n-2) \ln(4n \ln 2n)}.$$

This improves the upper bound  $\varepsilon_0(\mathcal{H}) < e^{-n}$  obtained in [BK05]. For concrete Hamiltonians like the planetary problem, the actual distance to the integrable system is slightly less than  $10^{-3}$ , hence the above corollary yields the impossibility to apply Nekhoroshev's estimates for  $n > 3$ .

**3.** Theorem 2.1 is easily obtained from the example in [MS02]: in fact, the construction is completely similar but one has to pay attention to the dependence with respect to  $n$  of the various constants involved. Such a result can easily be obtained in the analytic category, using the more elaborated techniques developed in [LM05], but we will restrict to the Gevrey case in order to describe entirely this very simple mechanism of instability.

As the reader will see, we will use only rough estimates leading to the factor  $n$  in the time of instability: this can be easily improved but we do not know if it is possible in our case (that is with a perturbation of size  $e^{-n \ln(n \ln n)}$ ) to obtain a linear time of drift.

Finally, as in [MS02], we have restricted the perturbation to a compact subset of  $\mathbb{A}^n = \mathbb{T}^n \times \mathbb{R}^n$  just in order to evaluate Gevrey norms. In fact,

the Hamiltonian vector field generated by  $H_n = h + f_n$  is complete and the unstable solution  $(\theta(t), I(t))$  satisfies

$$\lim_{t \rightarrow \pm\infty} I_1(t) = \pm\infty,$$

which means that it is bi-asymptotic to infinity.

**4.** In this text, we will have to deal with time-one maps associated to Hamiltonian flows. So given a function  $H$ , we will denote by  $\Phi_t^H$  the time- $t$  map of its Hamiltonian flow and by  $\Phi^H = \Phi_1^H$  the time-one map. We shall use the same notation for time-dependent functions  $H$ , that is  $\Phi^H$  will be the time-one map of the Hamiltonian isotopy (the flow between  $t = 0$  and  $t = 1$ ) generated by  $H$ .

### 3 Proof of Theorem 2.1

The proof of Theorem 2.1 is contained in section 3.2, but first in section 3.1 we recall the mechanism of instability presented in the paper [MS02] (see also [MS04]).

This mechanism has two main features. The first one is that it deals with perturbation of integrable maps rather than perturbation of integrable flows, and then the latter is recovered by a suspension process. This point of view, which is only of technical matter, was already used for example in [Dou88] and offers more flexibility in the construction. The second feature, which is the most important one, is that instead of trying to detect instability in a map close to integrable by means of the usual splitting estimates, we will start with a map having already “unstable” orbits and try to embed it in a near-integrable map. This will be realized through a “coupling lemma”, which is really the heart of the mechanism.

As we will see, the construction offers an easy and very efficient way of computing the drifting time of unstable solutions, therefore avoiding all the technicalities that are usually required for such a task.

#### 3.1 The mechanism

**1.** Given a potential function  $U : \mathbb{T} \rightarrow \mathbb{R}$ , we consider the following family of maps  $\psi_q : \mathbb{A} \rightarrow \mathbb{A}$  defined by

$$\psi_q(\theta, I) = (\theta + qI, I - q^{-1}U'(\theta + qI)), \quad (\theta, I) \in \mathbb{A},$$

for  $q \in \mathbb{N}^*$ . If we require  $U'(0) = -1$ , for example if we choose

$$U(\theta) = -(2\pi)^{-1} \sin 2\pi\theta, \quad U'(\theta) = -\cos(2\pi\theta),$$

then it is easy to see that  $\psi_q(0, 0) = (0, q^{-1})$  and by induction

$$\psi_q^k(0, 0) = (0, kq^{-1}) \tag{1}$$

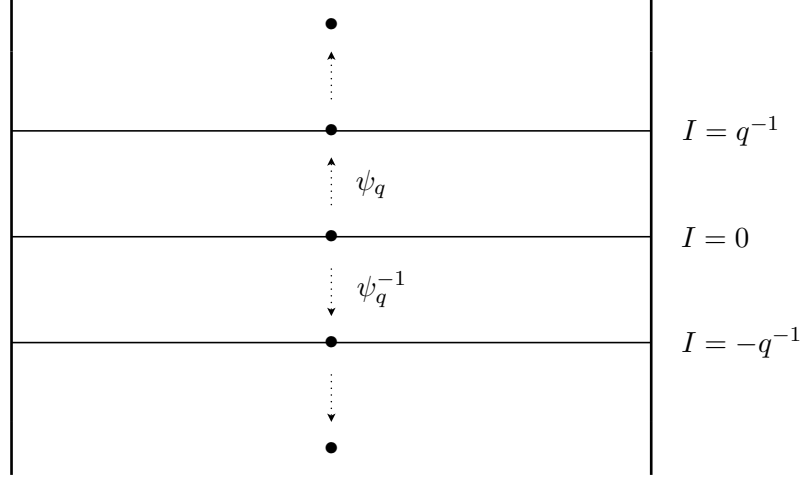


Figure 1: Drifting point for the map  $\psi_q$

for any  $k \in \mathbb{Z}$  (see figure 1). After  $q$  iterations, the point  $(0,0)$  drifts from the circle  $I = 0$  to the circle  $I = 1$  and it is bi-asymptotic to infinity, in the sense that the sequence  $(\psi_q^k(0,0))_{k \in \mathbb{Z}}$  is not contained in any semi-infinite annulus of  $\mathbb{A}$ . Clearly this map is exact-symplectic, but it has obviously no invariant circles and so it cannot be “close to integrable”. However, we will use the fact that it can be written as a composition of time-one maps,

$$\psi_q = \Phi^{q^{-1}U} \circ \left( \Phi^{\frac{1}{2}I^2} \circ \dots \circ \Phi^{\frac{1}{2}I^2} \right) = \Phi^{q^{-1}U} \circ \left( \Phi^{\frac{1}{2}I^2} \right)^q, \quad (2)$$

to embed  $\psi_q$  in the  $q^{\text{th}}$ -iterate of a near-integrable map of  $\mathbb{A}^n$ , for  $n \geq 2$ . To do so, we will use the following “coupling lemma”, which is easy but very clever.

**Lemma 3.1** (Herman-Marco-Sauzin). *Let  $m, m' \geq 1$ ,  $F : \mathbb{A}^m \rightarrow \mathbb{A}^m$  and  $G : \mathbb{A}^{m'} \rightarrow \mathbb{A}^{m'}$  two maps, and  $f : \mathbb{A}^m \rightarrow \mathbb{R}$  and  $g : \mathbb{A}^{m'} \rightarrow \mathbb{R}$  two Hamiltonian functions generating complete vector fields. Suppose there is a point  $a \in \mathbb{A}^{m'}$  which is  $q$ -periodic for  $G$  and such that the following “synchronisation” conditions hold:*

$$g(a) = 1, \quad dg(a) = 0, \quad g(G^k(a)) = 0, \quad dg(G^k(a)) = 0, \quad (\text{S})$$

for  $1 \leq k \leq q - 1$ . Then the mapping

$$\Psi = \Phi^{f \otimes g} \circ (F \times G) : \mathbb{A}^{m+m'} \longrightarrow \mathbb{A}^{m+m'}$$

is well-defined and for all  $x \in \mathbb{A}^m$ ,

$$\Psi^q(x, a) = (\Phi^f \circ F^q(x), a).$$

The product of functions acting on separate variables was denoted by  $\otimes$ , *i.e.*

$$f \otimes g(x, x') = f(x)g(x') \quad x \in \mathbb{A}^m, x' \in \mathbb{A}^{m'}.$$

Let us give the elementary proof of this lemma since it is a crucial ingredient.

*Proof.* First note that since the Hamiltonian vector fields  $X_f$  and  $X_g$  are complete, an easy calculation shows that for all  $x \in \mathbb{A}^m, x' \in \mathbb{A}^{m'}$  and  $t \in \mathbb{R}$ , one has

$$\Phi_t^{f \otimes g}(x, x') = \left( \Phi_t^{g(x')f}(x), \Phi_t^{f(x)g}(x') \right) \quad (3)$$

and therefore  $X_{f \otimes g}$  is also complete. Using the above formula and condition (S), the points  $(F^k(x), G^k(a))$ , for  $1 \leq k \leq q-1$ , are fixed by  $\Phi^{f \otimes g}$  and hence

$$\Psi^{q-1}(x, a) = (F^{q-1}(x), G^{q-1}(a)).$$

Since  $a$  is  $q$ -periodic for  $G$  this gives

$$\Psi^q(x, a) = \Phi^{f \otimes g}(F^q(x), a),$$

and we end up with

$$\Psi^q(x, a) = (\Phi^f(F^q(x)), a)$$

using once again (S) and (3).  $\square$

Therefore, if we set  $m = 1$ ,  $F = \Phi^{\frac{1}{2}I_1^2}$  and  $f = q^{-1}U$  in the coupling lemma, the  $q^{\text{th}}$ -iterate  $\Psi^q$  will leave the submanifold  $\mathbb{A} \times \{a\}$  invariant, and its restriction to this annulus will coincide with our “unstable map”  $\psi_q$ . Hence, after  $q^2$  iterations of  $\Psi$ , the  $I_1$ -component of the point  $((0, 0), a) \in \mathbb{A}^2$  will move from 0 to 1.

**2.** The difficult part is then to find what kind of dynamics we can put on the second factor to apply this coupling lemma. In order to have a continuous system with  $n$  degrees of freedom at the end, we may already choose  $m' = n - 2$  so the coupling lemma will give us a discrete system with  $n - 1$  degrees of freedom.

First, a natural attempt would be to try

$$G = G_n = \Phi^{\frac{1}{2}I_2^2 + \dots + \frac{1}{2}I_{n-1}^2}.$$

Indeed, in this case

$$F \times G_n = \Phi^{\frac{1}{2}I_2^2 + \dots + \frac{1}{2}I_n^2} = \Phi^h$$

and the unstable map  $\Psi$  given by the coupling lemma appears as a perturbation of the form  $\Psi = \Phi^u \circ \Phi^h$ , with  $u = f \otimes g$ . However, this cannot work.



Indeed, for  $j \in \{2, \dots, n-1\}$ , one can choose a  $p_j$ -periodic point  $a^{(j)} \in \mathbb{A}$  for the map  $\Phi^{\frac{1}{2}I_j^2}$ , and then setting

$$a_n = (a^{(2)}, \dots, a^{(n-1)}) \in \mathbb{A}^{n-2}, \quad q_n = p_2 \cdots p_{n-1},$$

the point  $a_n$  is  $q_n$ -periodic for  $G_n$  provided that the numbers  $p_j$  are mutually prime. One can easily see that the latter condition will force the product  $q_n$  to converge to infinity when  $n$  goes to infinity. So necessarily the point  $a_n$  gets arbitrarily close to its first iterate  $G_n(a_n)$  when  $n$  (and therefore  $q_n$ ) is large: this is because  $q_n$ -periodic points for  $G_n$  are equi-distributed on  $q_n$ -periodic tori. As a consequence, a function  $g_n$  with the property

$$g_n(a_n) = 1, \quad g_n(G_n(a_n)) = 0,$$

will necessarily have very large derivatives at  $a_n$  if  $q_n$  is large. Then as the size of the perturbation is essentially given by

$$|f \otimes g_n| = |q_n^{-1}U \otimes g_n| = |q_n^{-1}||g_n|,$$

one can check that it is impossible to make this quantity converge to zero when the number of degrees of freedom  $n$  goes to infinity.

**3.** As in [MS02], the idea to overcome this problem is the following one. We introduce a new sequence of “large” parameters  $N_n \in \mathbb{N}^*$  and in the second factor we consider a family of suitably rescaled penduli on  $\mathbb{A}$  given by

$$P_n(\theta_2, I_2) = \frac{1}{2}I_2^2 + N_n^{-2}V(\theta_2),$$

where  $V(\theta) = -\cos 2\pi\theta$ . The other factors remain unchanged, so

$$G_n = \Phi^{\frac{1}{2}(I_2^2 + I_3^2 + \dots + I_{n-1}^2) + N_n^{-2}V(\theta_2)}.$$

In this case, the map  $\Psi$  given by the coupling lemma is also a perturbation of  $\Phi^h$  but of the form  $\Psi = \Phi^u \circ \Psi^{h+v}$ , with  $v = N_n^{-2}V$ . But now for the map  $G_n$ , due to the presence of the pendulum factor, it is now possible to find a periodic orbit with an irregular distribution: more precisely, a  $q_n$ -periodic point  $a_n$  such that its distance to the rest of its orbit is of order  $N_n^{-1}$ , no matter how large  $q_n$  is.

**4.** Let us denote by  $(p_j)_{j \geq 0}$  the ordered sequence of prime numbers and let us choose  $N_n$  as the product of the  $n-2$  prime numbers  $\{p_{n+3}, \dots, p_{2n}\}$ , that is

$$N_n = p_{n+3}p_{n+4} \cdots p_{2n} \in \mathbb{N}^*.$$

Our goal is to prove the following proposition.

**Proposition 3.2.** *Let  $n \geq 3$ ,  $\alpha > 1$  and  $L_1 > 0$ . Then there exist a function  $g_n \in G^{\alpha, L_1}(\mathbb{A}^{n-2})$ , a point  $a_n \in \mathbb{A}^{n-2}$  and positive constants  $c_1$  and  $c_2$  depending only on  $\alpha$  and  $L_1$  such that if*

$$M_n = 2 \left[ c_1 N_n e^{c_2(n-2)P_{2n}^{\frac{1}{\alpha}}} \right], \quad q_n = N_n M_n,$$

*then  $a_n$  is  $q_n$ -periodic for  $G_n$  and  $(g_n, G_n, a_n, q_n)$  satisfy the synchronization conditions (S):*

$$g_n(a_n) = 1, \quad dg_n(a_n) = 0, \quad g_n(G_n^k(a_n)) = 0, \quad dg_n(G_n^k(a_n)) = 0,$$

*for  $1 \leq k \leq q_n - 1$ . Moreover, the estimate*

$$q_n^{-1} |g_n|_{\alpha, L_1} \leq N_n^{-2}, \tag{4}$$

*holds true.*

The rest of this section is devoted to the proof of the above proposition. Note that together with the coupling lemma (Lemma 3.1), this proposition easily gives a result of instability (analogous to Proposition 2.1 in [MS02]) for a discrete system which is a perturbation of the map  $\Phi^h$ .

**5.** We first consider the simple pendulum

$$P(\theta, I) = \frac{1}{2}I^2 + V(\theta), \quad (\theta, I) \in \mathbb{A}.$$

With our convention, the stable equilibrium is at  $(0, 0)$  and the unstable one is at  $(0, 1/2)$ . Given any  $M \in \mathbb{N}^*$ , there is a unique point  $b^M = (0, I_M)$  which is  $M$ -periodic for  $\Phi^P$  (this is just the intersection between the vertical line  $\{0\} \times \mathbb{R}$  and the closed orbit for the pendulum of period  $M$ ). One can check that  $I_M \in ]2, 3[$  and as  $M$  goes to infinity,  $(0, I_M)$  tends to the point  $(0, 2)$  which belongs to the upper separatrix. Since  $P_n(\theta, I) = \frac{1}{2}I^2 + N_n^{-2}V(\theta)$ , then one can see that

$$\Phi^{P_n} = (S_n)^{-1} \circ \Phi^{N_n^{-1}P} \circ S_n,$$

where  $S_n(\theta, I) = (\theta, N_n I)$  is the rescaling by  $N_n$  in the action components. Therefore the point  $b_n^M = (0, N_n^{-1}I_M)$  is  $q_n$ -periodic for  $\Phi^{P_n}$ , for  $q_n = N_n M$ . Let  $(\Phi_t^P)_{t \in \mathbb{R}}$  be the flow of the pendulum, and

$$\Phi_t^P(0, I_M) = (\theta_M(t), I_M(t)).$$

The function  $\theta_M(t)$  is analytic. The crucial observation is the following simple property of the pendulum (see Lemma 2.2 in [MS02] for a proof).

**Lemma 3.3.** *Let  $\sigma = -\frac{1}{2} + \frac{2}{\pi} \arctan e^\pi < \frac{1}{2}$ . For any  $M \in \mathbb{N}^*$ ,*

$$\theta_M(t) \notin [-\sigma, \sigma],$$

*for  $t \in [1/2, M - 1/2]$ .*

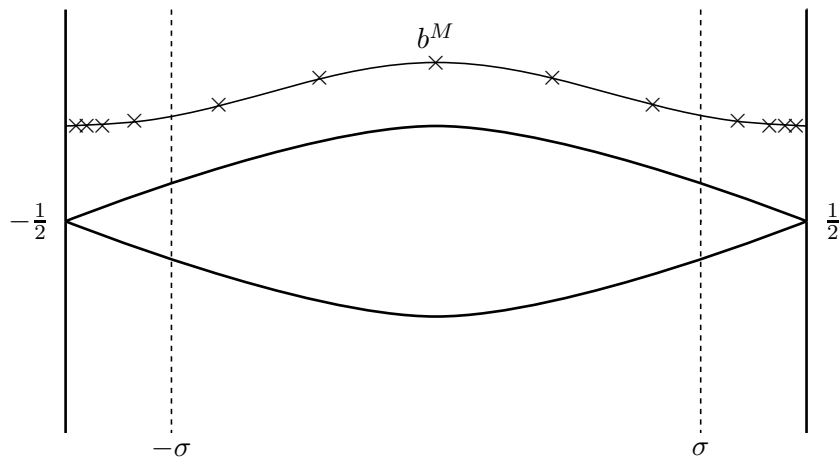


Figure 2: The point  $b^M$  and its iterates

Hence no matter how large  $M$  is, most of the points of the orbit of  $b^M \in \mathbb{A}$  will be outside the set  $\{-\sigma \leq \theta \leq \sigma\} \times \mathbb{R}$  (see figure 2). The construction of a function that vanishes, as well as its first derivative, at these points, will be easily arranged by means of a function, depending only on the angle variables, with support in  $\{-\sigma \leq \theta \leq \sigma\}$ .

As for the other points, it is convenient to introduce the function

$$\tau_M : [-\sigma, \sigma] \longrightarrow ] -1/2, 1/2[$$

which is the analytic inverse of  $\theta_M$ . One can give an explicit formula for this map:

$$\tau_M(\theta) = \int_0^\theta \frac{d\varphi}{\sqrt{I_M^2 - 4 \sin^2 \pi \varphi}}.$$

In particular, it is analytic and therefore it belongs to  $G^{\alpha, L_1}([-\sigma, \sigma])$  for  $\alpha \geq 1$  and  $L_1 > 0$ , and one can obtain the following estimate (see Lemma 2.3 in [MS02] for a proof).

**Lemma 3.4.** *For  $\alpha > 1$  and  $L_1 > 0$ ,*

$$\Lambda = \sup_{M \in \mathbb{N}^*} |\tau_M|_{\alpha, L_1} < +\infty.$$

Note that  $\Lambda$  depends only on  $\alpha$  and  $L_1$ . Under the action of  $\tau_M$ , the points of the orbit of  $b^M$  whose projection onto  $\mathbb{T}$  belongs to  $\{-\sigma \leq \theta \leq \sigma\}$  get equi-distributed, and we can use the following elementary lemma.

**Lemma 3.5.** *For  $p \in \mathbb{N}^*$ , the analytic function  $\eta_p : \mathbb{T} \rightarrow \mathbb{R}$  defined by*

$$\eta_p(\theta) = \left( \frac{1}{p} \sum_{l=0}^{p-1} \cos 2\pi l\theta \right)^2$$

satisfies

$$\eta_p(0) = 1, \quad \eta'_p(0) = 0, \quad \eta_p(k/p) = \eta'_p(k/p) = 0,$$

for  $1 \leq k \leq p-1$ , and

$$|\eta_p|_{\alpha, L_1} \leq e^{2\alpha L_1 (2\pi p)^{\frac{1}{\alpha}}}.$$

The proof is trivial (see [MS02], Lemma 2.4).

**6.** We can now pass to the proof of Proposition 3.2.

*Proof of Proposition 3.2.* For  $\alpha > 1$  and  $L_1 > 0$ , consider the bump function  $\varphi_{\alpha, L_1} \in G^{\alpha, L_1}(\mathbb{T})$  given by Lemma A.1 (see Appendix A).

We choose our function  $g_n \in G^{\alpha, L_1}(\mathbb{A}^{n-2})$ , depending only on the angle variables, of the form

$$g_n = g_n^{(2)} \otimes \cdots \otimes g_n^{(n-1)},$$

where

$$g_n^{(2)}(\theta_2) = \eta_{p_{n+3}}(\tau_{M_n}(\theta_2)) \varphi_{\alpha, (4\sigma)^{-\frac{1}{\alpha} L_1}}((4\sigma)^{-1} \theta_2),$$

and

$$g_n^{(i)}(\theta_i) = \eta_{p_{n+1+i}}(\theta_i), \quad 3 \leq i \leq n-1.$$

Let us write

$$c_1 = \left| \varphi_{\alpha, (4\sigma)^{-\frac{1}{\alpha} L_1}} \Big|_{\alpha, (4\sigma)^{-\frac{1}{\alpha} L_1}} \right|.$$

Now we choose our point  $a_n = (a_n^{(2)}, \dots, a_n^{(n-1)}) \in \mathbb{A}^{n-2}$ . We set

$$a_n^{(2)} = b_n^{M_n} = (0, N_n^{-1} I_{M_n}),$$

and

$$a_n^{(i)} = (0, p_{n+1+i}^{-1}), \quad 3 \leq i \leq n-1.$$

Let us prove that  $a_n$  is  $q_n$ -periodic for  $G_n$ . We can write

$$G_n = \Phi^{\frac{1}{2} I_2^2 + N_n^{-2} V(\theta_2)} \times \Phi^{\frac{1}{2} (I_3^2 + I_4^2 + \cdots + I_{n-1}^2)} = \Phi^{P_n} \times \widehat{G}.$$

Since  $p_{n+4}, \dots, p_{2n}$  are mutually prime, the point  $(a_n^{(3)}, \dots, a_n^{(n-1)}) \in \mathbb{A}^{n-3}$  is periodic for  $\widehat{G}$ , with period

$$N'_n = p_{n+4} \cdots p_{2n}.$$

By construction, the point  $a_n^{(2)} = b_n^{M_n} \in \mathbb{A}$  is periodic for  $\Phi^{P_n}$ , with period  $q_n = N_n M_n$ , where

$$N_n = p_{n+3} p_{n+4} \cdots p_{2n}.$$

This means that  $a_n$  is periodic for the product map  $G_n$ , and the exact period is given by the least common multiple of  $q_n$  and  $N'_n$ . Since  $N'_n$  divides  $q_n$ , the period of  $a_n$  is  $q_n$ .

Now let us show that the synchronization conditions (S) hold true, that is

$$g_n(a_n) = 1, \quad dg_n(a_n) = 0, \quad g_n(G_n^k(a_n)) = 0, \quad dg_n(G_n^k(a_n)) = 0,$$

for  $1 \leq k \leq q_n - 1$ . Since  $\varphi_{\alpha, L_1}(0) = 1$ , then

$$g_n(a_n) = g_n^{(2)}(0) \cdots g_n^{(n-1)}(0) = 1$$

and as  $\varphi'_{\alpha, L_1}(0) = 0$ , then

$$dg_n(a_n) = 0.$$

To prove the other conditions, let us write  $G_n^k(a_n) = (\theta_k, I_k) \in \mathbb{A}^{n-2}$ , for  $1 \leq k \leq q_n - 1$ .

If  $\theta_k^{(2)}$  does not belong to  $] -\sigma, \sigma[$ , then  $g_n^{(2)}$  and its first derivative vanish at  $\theta_k^{(2)}$  because it is the case for  $\varphi_{\alpha, (4\sigma)^{-\frac{1}{\alpha}}L}$ , so

$$g_n(\theta_k) = dg_n(\theta_k) = 0.$$

Otherwise, if  $-\sigma < \theta_k^{(2)} < \sigma$ , one can easily check that

$$-\frac{N_n - 1}{2} \leq k \leq \frac{N_n - 1}{2}$$

and therefore

$$\tau_{M_n}(\theta_k^{(2)}) = \frac{k}{N_n},$$

while

$$\theta_k^{(i)} = \frac{k}{p_{n+i+1}}, \quad 3 \leq i \leq n - 1.$$

If  $N'_n = p_{n+4} \cdots p_{2n}$  divides  $k$ , that is  $k = k'N'_n$  for some  $k' \in \mathbb{Z}$ , then

$$\tau_{M_n}(\theta_k^{(2)}) = \frac{k}{N_n} = \frac{k'}{p_{n+3}}$$

and therefore, by Lemma 3.5,  $\eta_{p_{n+3}}$  vanishes with its differential at  $\theta_k^{(2)}$ , and so does  $g_n^{(2)}$ . Otherwise,  $N'_n$  does not divide  $k$  and then, for  $3 \leq i \leq n - 1$ , at least one of the functions  $\eta_{p_{n+i+1}}$  vanishes with its differential at  $\theta_k^{(2)}$ , and so does  $g_n^{(i)}$ . Hence in any case

$$g_n(\theta_k) = dg_n(\theta_k) = 0, \quad 1 \leq k \leq q_n - 1,$$

and the synchronization conditions (S) are satisfied.

Now it remains to estimate the norm of the function  $g_n$ . First, using Lemma A.2, one finds

$$|g_n|_{\alpha, L_1} \leq \left| \varphi_{\alpha, (4\sigma)^{-\frac{1}{\alpha}}L_1} \right|_{\alpha, (4\sigma)^{-\frac{1}{\alpha}}L_1} \left| \eta_{p_{n+3}} \circ \tau_{M_n} \right|_{\alpha, L_1} \left| \eta_{p_{n+4}} \right|_{\alpha, L_1} \left| \eta_{p_{2n}} \right|_{\alpha, L_1},$$

which by definition of  $c_1$  gives

$$|g_n|_{\alpha, L_1} \leq c_1 |\eta_{p_{n+3}} \circ \tau_{M_n}|_{\alpha, L_1} |\eta_{p_{n+4}}|_{\alpha, L_1} |\eta_{p_{2n}}|_{\alpha, L_1}.$$

Then, by definition of  $\Lambda$  (Lemma 3.4) and using Lemma A.3 (with  $\Lambda_1 = \Lambda^{\frac{1}{\alpha}}$ ),

$$|\eta_{p_{n+3}} \circ \tau_{M_n}|_{\alpha, L_1} \leq |\eta_{p_{n+3}}|_{\alpha, \Lambda^{\frac{1}{\alpha}}},$$

so using Lemma 3.5 and setting  $c_2 = 2\alpha \sup \left\{ \Lambda^{\frac{1}{\alpha}}, L_1 \right\} (2\pi)^{\frac{1}{\alpha}}$ , this gives

$$\begin{aligned} |g_n|_{\alpha, L_1} &\leq c_1 e^{2\alpha(\Lambda^{\frac{1}{\alpha}} + (n-3)L_1)(2\pi p_{2n})^{\frac{1}{\alpha}}} \\ &\leq c_1 e^{2\alpha \sup \left\{ \Lambda^{\frac{1}{\alpha}}, L_1 \right\} (n-2)(2\pi p_{2n})^{\frac{1}{\alpha}}} \\ &\leq c_1 e^{c_2(n-2)p_{2n}^{\frac{1}{\alpha}}}. \end{aligned}$$

Finally, by definition of  $M_n$  we obtain

$$|g_n|_{\alpha, L_1} \leq M_n N_n^{-1},$$

and as  $q_n = N_n M_n$ , we end up with

$$q_n^{-1} |g_n|_{\alpha, L_1} \leq N_n^{-2}.$$

This concludes the proof.  $\square$

## 3.2 Proof of Theorem 2.1

**7.** In the previous section, we were concerned with a perturbation of the integrable diffeomorphism  $\Phi^h$ , which can be written as  $\Phi^u \circ \Phi^{h+v}$ . So now we will briefly describe a suspension argument to go from this discrete case to a continuous case (we refer once again to [MS02] for the details).

Here we will make use of bump functions, however the process is still valid, though more difficult, in the analytic category, (see for example [Dou88] or [KP94]). The basic idea is to find a time-dependent Hamiltonian function on  $\mathbb{A}^n$  such that the time-one map of its isotopy is  $\Phi^u \circ \Phi^{h+v}$ , or, equivalently, an autonomous Hamiltonian function on  $\mathbb{A}^{n+1}$  such that its first return map to some  $2n$ -dimensional Poincaré section coincides with our map  $\Phi^u \circ \Phi^{h+v}$ .

Given  $\alpha > 1$  and  $L > 1$ , let us define the function

$$\phi_{\alpha, L} = \left( \int_{\mathbb{T}} \varphi_{\alpha, L} \right)^{-1} \varphi_{\alpha, L},$$

where  $\varphi_{\alpha, L}$  is the bump function given by Lemma A.1. If  $\phi_0(t) = \phi_{\alpha, L}(t - \frac{1}{4})$  and  $\phi_1(t) = \phi_{\alpha, L}(t - \frac{3}{4})$ , the time-dependent Hamiltonian

$$H^*(\theta, I, t) = (h(I) + v(\theta)) \otimes \phi_0(t) + u(\theta) \otimes \phi_1(t)$$

clearly satisfies

$$\Phi^{H^*} = \Phi^u \circ \Phi^{h+v}.$$

But as  $u$  and  $v$  go to zero,  $H^*$  converges to  $h \otimes \phi_0$  rather than  $h$ . However, using classical generating functions, it is not difficult to modify the Hamiltonian in order to prove the following proposition (see Lemma 2.5 in [MS02]).

**Lemma 3.6.** *Let  $n \geq 1$ ,  $R > 1$ ,  $\alpha > 1$ ,  $L_1 > 0$  and  $L > 0$  satisfying*

$$L_1^\alpha = L^\alpha(1 + (L^\alpha + R + 1/2)|\phi_{\alpha,L}|_{\alpha,L}). \quad (5)$$

*If  $u, v \in G^{\alpha,L_1}(\mathbb{T}^{n-1})$ , there exists  $f \in G^{\alpha,L}(\mathbb{T}^n \times \overline{B})$ , independent of the variable  $I_n$ , such that if*

$$H(\theta, I) = \frac{1}{2}(I_1^2 + \cdots + I_{n-1}^2) + I_n + f(\theta, I), \quad (\theta, I) \in \mathbb{A}^n,$$

*for any energy  $e \in \mathbb{R}$ , the Poincaré map induced by the Hamiltonian flow of  $H$  on the section  $\{\theta_n = 0\} \cap H^{-1}(e)$  coincides with the diffeomorphism*

$$\Phi^u \circ \Phi^{h+v}.$$

*Moreover, one has*

$$\sup\{|u|_{C^0}, |v|_{C^0}\} \leq |f|_{\alpha,L} \leq c_3 \sup\{|u|_{\alpha,L_1}, |v|_{\alpha,L_1}\}, \quad (6)$$

*where  $c_3 = 2|\phi_{\alpha,L}|_{\alpha,L}$  depends only on  $\alpha$  and  $L$ .*

**8.** Now we can finally prove our theorem.

*Proof of Theorem 2.1.* Let  $R > 1$ ,  $\alpha > 1$  and  $L > 0$ , and choose  $L_1$  satisfying the relation (5). The constants  $c_1$  and  $c_2$  of Proposition 3.2 depend only on  $\alpha$  and  $L_1$ , hence they depend only on  $R$ ,  $\alpha$  and  $L$ .

We can define  $u_n, v_n \in G^{\alpha,L_1}(\mathbb{T}^{n-1})$  by

$$u_n = q_n^{-1}U \otimes g_n, \quad v_n = N_n^{-2}V,$$

where  $U(\theta_1) = -(2\pi)^{-1} \sin 2\pi\theta_1$ ,  $V(\theta_2) = -\cos 2\pi\theta_2$  (so  $v_n$  is formally defined on  $\mathbb{T}$  but we identify it with a function on  $\mathbb{T}^{n-1}$ ) and  $g_n$  is the function given by Proposition 3.2. Let us apply Lemma 3.6: there exists  $f_n \in G^{\alpha,L}(\mathbb{T}^n \times \overline{B})$ , independent of the variable  $I_n$ , such that if

$$H_n(\theta, I) = \frac{1}{2}(I_1^2 + \cdots + I_{n-1}^2) + I_n + f_n(\theta, I), \quad (\theta, I) \in \mathbb{A}^n,$$

for any energy  $e \in \mathbb{R}$ , the Poincaré map induced by the Hamiltonian flow of  $H$  on the section  $\{\theta_n = 0\} \cap H^{-1}(e)$  coincides with the diffeomorphism

$$\Phi^{u_n} \circ \Phi^{\frac{1}{2}(I_1^2 + \cdots + I_{n-1}^2) + v_n} = \Phi^{u_n} \circ \Phi^{h+v_n}.$$

Let us show that our system  $H_n$  has a drifting orbit. First consider its Poincaré section defined by

$$\Psi_n = \Phi^{u_n} \circ \Phi^{\frac{1}{2}(I_1^2 + \dots + I_{n-1}^2) + v_n} = \Phi^{f_n \otimes g_n} \circ (F \times G_n),$$

with

$$f_n = q_n^{-1}U, \quad F = \Phi^{\frac{1}{2}I_1^2}, \quad G_n = \Phi^{\frac{1}{2}(I_2^2 + I_3^2 + \dots + I_{n-1}^2) + N_n^{-2}V(\theta_2)}.$$

By Proposition 3.2, we can apply the coupling lemma (Lemma 3.1), so

$$\Psi_n^{q_n}((0, 0), a_n) = (\Phi_n^f \circ F^{q_n}(0, 0), a_n).$$

Then, using (2), observe that

$$\Phi_n^{f_n} \circ F^{q_n} = \Phi^{q_n^{-1}U} \circ \left( \Phi^{\frac{1}{2}I_1^2} \right)^{q_n} = \psi_{q_n},$$

so

$$\begin{aligned} \Psi_n^{q_n}((0, 0), a_n) &= ((\Phi_n^{f_n} \circ F^{q_n})^{q_n}(0, 0), a_n) \\ &= (\psi_{q_n}^{q_n}(0, 0), a_n) \\ &= ((0, 1), a_n), \end{aligned}$$

where the last equality follows from (1). Hence, after  $q_n^2$  iterations, the  $I_1$ -component of the point  $x_n = ((0, 0), a_n) \in \mathbb{A}^{n-1}$  drifts from 0 to 1. Then, for the continuous system, the initial condition  $(x_n, t = 0, I_n = 0)$  in  $\mathbb{A}^n$  gives rise to a solution  $(x(t), t, I_n(t)) = (x(t), \theta_n(t), I_n(t))$  of the Hamiltonian vector field generated by  $H_n$  such that

$$x(k) = \Psi_n^k(x_n), \quad k \in \mathbb{Z}.$$

So after a time  $\tau_n = q_n^2$ , the point  $(x_n, (0, 0))$  drifts from 0 to 1 in the  $I_1$ -direction, and this gives our drifting orbit.

Now let  $\varepsilon_n = |f_n|_{\alpha, L_1}$  be the size of our perturbation. Using the estimate (4) and (6) one finds

$$N_n^{-2} \leq \varepsilon_n \leq c_3 N_n^{-2}. \quad (7)$$

By the prime number theorem,  $p_n$  is equivalent to  $n \ln n$ , so there exists  $n_0 \in \mathbb{N}^*$  such that for  $n \geq n_0$ , one can ensure that

$$p_{2n}/4 \leq p_{n+i} \leq p_{2n}, \quad 3 \leq i \leq n,$$

which gives

$$(p_{2n}/4)^{n-2} \leq N_n \leq p_{2n}^{n-2}, \quad N_n^{\frac{1}{n-2}} \leq p_{2n} \leq 4N_n^{\frac{1}{n-2}}. \quad (8)$$



We can also assume by the prime number theorem that for  $n \geq n_0$ , one has

$$2n \ln 2n \leq p_{2n} \leq 2(2n \ln 2n) = 4n \ln 2n. \quad (9)$$

From the above estimates (8) and (9) one easily obtains

$$e^{(n-2) \ln(2^{-1}n \ln 2n)} \leq N_n \leq e^{(n-2) \ln(4n \ln 2n)}, \quad (10)$$

and, together with (7), one finds

$$e^{-2(n-2) \ln(4n \ln 2n)} \leq \varepsilon_n \leq c_3 e^{-2(n-2) \ln(2^{-1}n \ln 2n)}. \quad (11)$$

Finally it remains to estimate the time  $\tau_n$ . First recall that

$$M_n = 2 \left[ c_1 N_n e^{c_2(n-2)p_{2n}^{\frac{1}{\alpha}}} \right],$$

and with (8)

$$q_n = N_n M_n \leq 3c_1 N_n^2 e^{4c_2(n-2)N_n^{\frac{1}{\alpha(n-2)}}}.$$

Hence

$$q_n^2 \leq 9c_1^2 N_n^4 e^{8c_2(n-2)N_n^{\frac{1}{\alpha(n-2)}}}.$$

Then using (10) we have

$$N_n^{\frac{1}{\alpha(n-2)}} \leq (4n \ln 2n)^{\frac{1}{\alpha}}$$

and from (7) we know that

$$N_n^4 \leq \left( \frac{c_3}{\varepsilon_n} \right)^2,$$

so we obtain

$$q_n^2 \leq 9c_1^2 \left( \frac{c_3}{\varepsilon_n} \right)^2 e^{8c_2(n-2)(4n \ln 2n)^{\frac{1}{\alpha}}}.$$

Now taking  $n_0$  larger if necessary, as  $\alpha > 1$ , one can ensure that for  $n \geq n_0$ ,

$$(4n)^{\frac{1}{\alpha}} \leq n, \quad (\ln 2n)^{\frac{1}{\alpha}} \leq \ln(2^{-1}n \ln 2n),$$

so

$$8c_2(n-2)(4n \ln 2n)^{\frac{1}{\alpha}} \leq 8c_2(n-2)n \ln(2^{-1}n \ln 2n).$$

Therefore

$$\begin{aligned} q_n^2 &\leq 9c_1^2 \left( \frac{c_3}{\varepsilon_n} \right)^2 e^{8c_2 n(n-2) \ln(2^{-1}n \ln 2n)} \\ &\leq 9c_1^2 \left( \frac{c_3}{\varepsilon_n} \right)^2 \left( e^{2(n-2) \ln(2^{-1}n \ln 2n)} \right)^{4c_2 n}. \end{aligned}$$

Finally by (11) we obtain

$$\begin{aligned} q_n^2 &\leq 9c_1^2 \left(\frac{c_3}{\varepsilon_n}\right)^2 \left(\frac{c_3}{\varepsilon_n}\right)^{4c_2n} \\ &\leq C \left(\frac{c}{\varepsilon_n}\right)^{n\gamma} \end{aligned}$$

with  $C = 9c_1^2$ ,  $c = c_3$  and  $\gamma = 2 + 4c_2$ . This ends the proof.  $\square$

## A Gevrey functions

In this very short appendix, we recall some facts about Gevrey functions that we used in the text. We refer to [MS02], Appendix A, for more details.

The most important property of  $\alpha$ -Gevrey functions is the existence, for  $\alpha > 1$ , of bump functions.

**Lemma A.1.** *Let  $\alpha > 1$  and  $L > 0$ . There exists a non-negative 1-periodic function  $\varphi_{\alpha,L} \in G^{\alpha,L}([-\frac{1}{2}, \frac{1}{2}])$  whose support is included in  $[-\frac{1}{4}, \frac{1}{4}]$  and such that  $\varphi_{\alpha,L}(0) = 1$  and  $\varphi'_{\alpha,L}(0) = 0$ .*

The following estimate on the product of Gevrey functions follows easily from the Leibniz formula.

**Lemma A.2.** *Let  $L > 0$ , and  $f, g \in G^{\alpha,L}(\mathbb{T}^n \times \overline{B})$ . Then*

$$|fg|_{\alpha,L} \leq |f|_{\alpha,L} |g|_{\alpha,L}.$$

Finally, estimates on the composition of Gevrey functions are much more difficult (see Proposition A.1 in [MS02]), but here we shall only need the following statement.

**Lemma A.3.** *Let  $\alpha \geq 1$ ,  $\Lambda_1 > 0$ ,  $L_1 > 0$ , and  $I, J$  be compact intervals of  $\mathbb{R}$ . Let  $f \in G^{\alpha,\Lambda_1}(I)$ ,  $g \in G^{\alpha,L_1}(J)$  and assume  $g(J) \subseteq I$ . If*

$$|g|_{\alpha,L_1} \leq \Lambda_1^\alpha,$$

*then  $f \circ g \in G^{\alpha,L_1}(J)$  and*

$$|f \circ g|_{\alpha,L_1} \leq |f|_{\alpha,\Lambda_1}.$$

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