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IBRAHIM ASSEM, FLÁVIO U. COELHO, AND SONIA TREPODE

Abstract. We prove that, if A is a strongly simply connected algebra of polynomial growth then A is torsionless-finite. In particular, its representation dimension is at most three.

Among the most useful algebraic invariants are the homological dimensions, which are meant to measure how much an algebra or a module deviates from a situation considered to be ideal. Introduced by Auslander in the early seventies [9], the representation dimension of an Artin algebra was long left aside from the mainstream of the theory, until a marked renewal of interest about ten years ago. It measures the least global dimension of all endomorphism rings of those finitely generated modules which are both generators and cogenerators of the module category. Part of the reason for this new interest comes from the fact that Igusa and Todorov have shown that, if the representation dimension of an Artin algebra is at most three, then its finitistic dimension is finite [19]. Also, Iyama has proved that, for any Artin algebra A, the representation dimension rep.dim.A is finite [20]. Since then, there have been several attempts to understand this invariant and to calculate it for classes of algebras, see for instance [2, 16, 17, 22] or the survey [26]. It was shown by Auslander that an Artin algebra is representation-finite if and only if its representation dimension is at most two [9]. Since Auslander's expectation was that this dimension would measure how far an algebra is from being representation-finite, it is natural to ask whether tame algebras have representation dimension at most three. The answer to this question is known to be positive for some classes of tame algebras, such as special biserial algebras [18] and domestic self-injective algebras socle equivalent to a weakly symmetric algebra of euclidean type [13].

The objective of our paper is to prove that the representation dimension of a strongly simply connected algebra of polynomial growth (over an algebraically

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closed field) is at most three. These algebras form a nice class which has been extensively studied by Skowroński, de la Peña and others, see, for instance, the survey [7]. In particular, it is shown in [28] that strongly simply connected algebras of polynomial growth are multicoil algebras. We recall also that, as pointed out in [26], an algebra which is torsionless-finite (that is, such that any indecomposable projective module has only finitely many isomorphism classes of indecomposable submodules) has its representation dimension at most three. Our main result is the following theorem.

Theorem A. *Let A be a strongly simply connected algebra of polynomial growth. Then A is torsionless-finite. In particular, rep.dim.$A \leq 3$.*

In the course of the proof, we found that the following result, of independent interest, was necessary.

Theorem B. *Let C be a tame concealed algebra of type distinct of $\widetilde{\mathbb{A}}$, and $(H_\lambda)_\lambda$ be an infinite family of pairwise non-isomorphic simple homogeneous C-modules. Suppose M is a C-module such that $H_\lambda \subset M$ for all λ. Then $C[M]$ is wild.*

The paper is organized as follows. After a short preliminary section, in which we fix the notations and recall facts about multicoil algebras, we prove our Theorem(B) in section 2 and Theorem(A) in section 3.

1. Multicoil algebras

1.1. **Notation.** In this paper, k denotes a fixed algebraically closed field. By an algebra A is meant a basic, connected, associative finite dimensional k-algebra with an identity. Thus, there exists a connected bound quiver (Q_A, I) and an isomorphism $A \simeq kQ_A/I$. Equivalently, A may be viewed as a k-category, of which the object class A_0 is the set of points of Q_A and the set of morphisms from x to y is the quotient of the k-vector space $kQ_A(x,y)$ of linear combinations of paths in Q_A from x to y by $I(x,y) = I \cap kQ_A(x,y)$, see [15]. A full subcategory C of A is *convex* (in A) if, for any path $x_0 \to x_1 \to \cdots \to x_t$ in A with $x_0, x_t \in C_0$, we have $x_i \in C_0$ for all i. The algebra A is *triangular* if Q_A is acyclic.

By A-module is meant a finitely generated right A-module. We denote by mod$A$ the category of A-modules and by ind$A$ a full subcategory consisting of a complete set of representatives of the isomorphism classes of indecomposable A-modules. We recall that, if $A \simeq kQ/I$, then an A-module M is identified to a corresponding representation $(M(x)_{x \in Q_0}, M(\alpha)_{\alpha \in Q_1})$ of the bound quiver (Q, I), see [3]. For a point $x \in Q_0$, we denote by P_x (or I_x, or S_x) the indecomposable projective (or injective, or simple, respectively) A-module corresponding to x. The *support* of an A-module M is the full subcategory Supp$M$ of A with objects those $x \in A_0$ such that $M(x) \neq 0$. For a full subcategory $\mathcal{C}$ of mod$A$, we denote by add$\mathcal{C}$ the additive full subcategory with objects the direct sums of direct summands of objects in $\mathcal{C}$. If $\mathcal{C}$ contains a single module M, we write add$\mathcal{C}$ =

add$M$. For two full subcategories $\mathcal{C}, \mathcal{C}'$ of ind$A$, the notation $\mathrm{Hom}_A(\mathcal{C}, \mathcal{C}') = 0$ means that $\mathrm{Hom}_A(M, M') = 0$ for all $M \in \mathcal{C}, M \in \mathcal{C}'$. We then denote by $\mathcal{C}' \vee \mathcal{C}$ the full subcategory of ind$A$ having as objects those of $\mathcal{C}'_0 \cup \mathcal{C}_0$.

A *path* in ind$A$ from M to N (sometimes denoted as $M \rightsquigarrow N$) is a sequence of non-zero morphisms

$$(*) \qquad M = M_0 \longrightarrow M_1 \longrightarrow \cdots \longrightarrow M_t = N$$

with all M_i in ind$A$. A path $(*)$ is a *cycle* if $M = N$ and at least one of the morphisms is not an isomorphism. An indecomposable module is *directed* if it lies on no cycle.

We use freely properties of the Auslander-Reiten translations $\tau_A = \mathrm{DTr}$ and $\tau_A^{-1} = \mathrm{TrD}$ and the Auslander-Reiten quiver $\Gamma(\mathrm{mod}A)$ of A for which we refer to [3, 25]. We identify points in $\Gamma(\mathrm{mod}A)$ with the corresponding indecomposable A-modules, and components with the corresponding full subcategories of ind$A$. A component Γ is *standard* if the category Γ is equivalent to the mesh category $k(\Gamma)$. For tubes, tubular extensions and coextensions, we refer to [25] and for tame algebras, we refer to [27].

1.2. **One-point extensions.** The *one-point extension* of an algebra A by an A-module M is the matrix algebra

$$A[M] \;=\; \begin{pmatrix} A & 0 \\ M & k \end{pmatrix}$$

with the usual addition and multiplication of matrices. The quiver of $A[M]$ contains Q_A as a full convex subquiver and there is an additional (extension) point which is a source. The $A[M]$-modules are identified with triples (V, X, φ), where V is a k-vector space, X an A-module and $\varphi\colon V \longrightarrow \mathrm{Hom}_A(M, X)$ is a k-linear map. An $A[M]$-linear map $(V, X, \varphi) \longrightarrow (V', X', \varphi')$ is a pair (f, g) where $f\colon V \longrightarrow V'$ is k-linear and $g\colon X \longrightarrow X'$ is A-linear such that $\varphi' f = \mathrm{Hom}_A(M, g)\varphi$. The dual notion is that of *one-point coextension.*

A *vector space category* [24, 25] $\mathcal{K}$ is a k-category together with a faithful k-linear functor $|\cdot|\colon \mathcal{K} \longrightarrow \mathrm{mod}k$. The *subspace category* $\mathcal{U}(\mathcal{K})$ of $\mathcal{K}$ has as objects the triples (V, X, φ) where V is a k-vector space, X an object in $\mathcal{K}$ and $\varphi\colon V \longrightarrow |X|$ is a k-linear monomorphism. A morphism $(V, X, \varphi) \longrightarrow (V', X', \varphi')$ is a pair (f, g) where $f\colon V \longrightarrow V'$ is k-linear and $g\colon X \longrightarrow X'$ is a morphism in $\mathcal{K}$ such that $\varphi' f = |g|\varphi$.

If A is an algebra and M an A-module, one considers the vector space category $\mathrm{Hom}_A(M,\ \mathrm{mod}A)$ whose objects are of the form $\mathrm{Hom}_A(M, X)$ with X an A-module, and morphisms are of the form

$$\mathrm{Hom}_A(M, f)\colon\ \mathrm{Hom}_A(M, X) \longrightarrow\ \mathrm{Hom}_A(M', X'),$$

where $f\colon X \longrightarrow X'$ is A-linear. Then $|\mathrm{Hom}_A(M, X)|$ is just the underlying k-vector space of $\mathrm{Hom}_A(M, X)$. It is shown in [24] that $\mathcal{U}(\mathrm{Hom}_A(M, \mathrm{mod}A))$ is equivalent to the full subcategory of $\mathrm{mod}A[M]$ consisting of the triples (V, X, φ)

without non-zero direct summands of the form $(k, 0, 0)$ or $(0, Y, 0)$ where $\mathrm{Hom}_A(M, Y) = 0$. We need essentially the following lemma.

LEMMA. *Let A be a tame algebra, and M be an A-module. If L is a submodule of M such that $A[L]$ is wild, then $A[M]$ is wild.*

Proof. Since A is tame, while $A[L]$ is wild, then the vector space category $\mathcal{U}(\mathrm{Hom}_A(L,\mathrm{mod}A))$ is wild. Now the inclusion $L \hookrightarrow M$ induces an epifunctor

$$\mathcal{U}(\mathrm{Hom}_A(M, \mathrm{mod}A)) \longrightarrow \mathcal{U}(\mathrm{Hom}_A(L, \mathrm{mod}A)).$$

Since the latter is wild, then so is $\mathcal{U}(\mathrm{Hom}_A(M, \mathrm{mod}A))$. Therefore, $A]M]$ is wild. □

1.3. **Coils.** Let A be an algebra, and Γ a standard component of $\Gamma(\mathrm{mod}A)$. Given a module $X \in \Gamma$, called the *pivot*, the *support* $\mathrm{Supp}(X, -)|_\Gamma$ of the functor $\mathrm{Hom}_A(X, -)|_\Gamma$ is defined as follows. Let $\mathcal{H}_X$ be the full subcategory of $\mathrm{ind}A$ consisting of the $Y \in \Gamma$ such that $\mathrm{Hom}_A(X, Y) \neq 0$, and $\mathcal{I}_X$ be the ideal of $\mathcal{H}_X$ consisting of the morphisms $f\colon Y \longrightarrow Y'$ (with Y, Y' in $\mathcal{H}_X$) such that $\mathrm{Hom}_A(X, f) \neq 0$. Then $\mathrm{Supp}(X, -)|_\Gamma = \mathcal{H}_X/\mathcal{I}_X$. We define three admissible operations:

(ad1) Assume $\mathrm{Supp}(X, -)|_\Gamma$ consists of an infinite sectional path starting at X

$$X = X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots$$

Let $t \geq 1$, $D = \mathbb{T}_t(k)$ be the full $t \times t$ lower triangular matrix algebra, and Y be the unique indecomposable projective-injective D-module. The *modified algebra* of A is $A' = (A \times D)[X \oplus Y]$. If $t = 0$, it is simply $A' = A[X]$.

(ad2) Assume $\mathrm{Supp}(X, -)|_\Gamma$ consists of two sectional paths starting at X, the first infinite and the second finite with at least one arrow

$$Y_t \longleftarrow \cdots \longleftarrow Y_2 \longleftarrow Y_1 \longleftarrow X = X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots$$

with $t \geq 1$. The *modified algebra* is $A' = A[X]$.

(ad3) Assume $\mathrm{Supp}(X, -)|_\Gamma$ consists of two parallel sectional paths, the first infinite and starting at X, and the second finite with at least one arrow

$$\begin{array}{ccccccccc} Y_1 & \longrightarrow & Y_2 & \longrightarrow & \cdots & \longrightarrow & Y_t & & \\ \uparrow & & \uparrow & & & & \uparrow & & \\ X = X_0 & \longrightarrow & X_1 & \longrightarrow & \cdots & \longrightarrow & X_{t-1} & \longrightarrow X_t & \longrightarrow \cdots \end{array}$$

where $t \geq 2$. In particular, X_{t-1} is injective. The *modified algebra* is $A' = A[X]$.

In each case, t is the *parameter* of the operation, and the component Γ' of $\Gamma(\mathrm{mod}A')$ containing X is the *modified component*. We also consider the duals of the above operations, denoted by (ad1*), (ad2*) and (ad3*), respectively. These six operations are the *admissible operations.*

An Auslander-Reiten component Γ is a *coil* if there exists a sequence $\Gamma_0, \Gamma_1, \cdots, \Gamma_m = \Gamma$ where Γ_0 is a stable tube and, for each i, Γ_{i+1} is obtained from Γ_i by an admissible operation.

Let A be an algebra A. A family $\mathcal{R} = (\mathcal{R}_\lambda)_{\lambda\in\Lambda}$ of components of $\Gamma(\text{mod}A)$ is *weakly separating* if the indecomposable A-modules not in $\mathcal{R}$ split into two classes $\mathcal{P}$ and $\mathcal{Q}$ such that

(1) The $\mathcal{R}_\lambda$ are standard and pairwise orthogonal.
(2) $\text{Hom}_A(\mathcal{Q}, \mathcal{P}) = \text{Hom}_A(\mathcal{Q}, \mathcal{R}) = \text{Hom}_A(\mathcal{R}, \mathcal{P}) = 0$.
(3) Any morphism from $\mathcal{P}$ to $\mathcal{Q}$ factors through add$\mathcal{R}$.

If $\mathcal{R}$ is a weakly separating family in $\Gamma(\text{mod}A)$ consisting of stable tubes, then an algebra B is a coil enlargement of A using modules from $\mathcal{R}$ if there exists a finite sequence of algebras $A = A_0, A_1, \cdots, A_m = B$ such that, for each i, A_{i+1} is obtained from A_i by an admissible operation with pivot either on a stable tube of $\mathcal{R}$ or on a coil of $\Gamma(\text{mod}A_i)$ obtained from a stable tube of $\mathcal{R}$ by means of the admissible operations done so far. The *coil type* $c_B = (c_B^-, c_B^+)$ of B is a pair of functors $\Lambda \longrightarrow \mathbb{N}$ defined by induction on i, for each $\lambda \in \Lambda$, as follows.

(a) $c_A = (c_0^-, c_0^+)$ is such that $c_0^-(\lambda) = c_0^+(\lambda)$ is the rank of the stable tube $\mathcal{R}_\lambda$.
(b) If $c_{A_{i-1}} = (c_{i-1}^-, c_{i-1}^+)$ is known and t_i is the parameter of the operation from A_{i-1} to A_i, then $c_{A_i} = (c_i^-, c_i^+)$ is defined by:

$$c_i^-(\lambda) = \begin{cases} c_{i-1}^-(\lambda) + t_i + 1 & \text{if the operations is (ad1*), (ad2*), (ad3*)} \\ & \text{with pivot in the coil of } \Gamma(\text{mod}A_{i-1}) \\ & \text{arising from } \mathcal{R}_\lambda. \\ c_{i-1}^-(\lambda) & \text{otherwise} \end{cases}$$

and

$$c_i^+(\lambda) = \begin{cases} c_{i-1}^+(\lambda) + t_i + 1 & \text{if the operations is (ad1), (ad2), (ad3)} \\ & \text{with pivot in the coil of } \Gamma(\text{mod}A_{i-1}) \\ & \text{arising from } \mathcal{R}_\lambda. \\ c_{i-1}^+(\lambda) & \text{otherwise} \end{cases}$$

If all but at most finitely many values of each of c_B^-, c_B^+ equal 1, we replace each sequence by a finite sequence containing at least two terms, and including all those exceeding 1.

THEOREM. [8] *Let A be an algebra having a weakly separating family of stable tubes $\mathcal{R}$, and B be a coil enlargement of A using modules from $\mathcal{R}$. Then*

(a) *There is a unique maximal branch coextension B^- (or extension B^+) of A which is a full convex subcategory of B, and having c_B^- as coextension type (or c_B^+ as extension type, respectively).*

(b) *ind* $B = \mathcal{P}' \vee \mathcal{R}' \vee \mathcal{Q}'$, *where* $\mathcal{R}'$ *is a weakly separating family of ind*B *obtained from* $\mathcal{R}$ *by the sequence of admissible operations, and separating* $\mathcal{P}'$ *from* $\mathcal{Q}'$. *Moreover,* $\mathcal{P}'$ *consists of* B^-*-modules, while* $\mathcal{Q}'$ *consists of* B^+*-modules.*

1.4. **Tame coil enlargements.** Let B be a coil enlargement of a tame concealed algebra. Its *coil type* $c_B = (c_B^-, c_B^+)$ is tame if each of the sequences c_B^-, c_B^+ is one of the following: (p, q) with $1 \leq p \leq q$, $(2, 2, r)$ with $r \geq 2$, (2,3,3), (2,3,4), (2,3,5), (2,4,4), (2,3,6) or(2,2,2,2). We have the following result.

COROLLARY. [8](4.3) *Let* B *a coil enlargement of a tame concealed algebra. The following conditions are equivalent:*

(a) B *is tame.*
(b) B *is of polynomial growth.*
(c) c_B *is tame.*
(d) B^- *and* B^+ *are tame.*

Moreover, B *is domestic if and only if* B^- *and* B^+ *are tilted of euclidean type.*

1.5. **Multicoil algebras.** An Auslander-Reiten component Γ is a *multicoil* if it contains a full translation subquiver Γ' which is a disjoint union of coils such that no point in $\Gamma \setminus \Gamma'$ belongs to a cyclical path.

An algebra A is a *multicoil algebra* if, for any cycle $M_0 \to M_1 \to \cdots \to M_t = M_0$ in ind$A$, all the M_i lie in one standard coil of a multicoil of $\Gamma(\text{mod}A)$. The first part of the following theorem is [5] (4.6), the second part is [28](4.1).

THEOREM. [5, 28] *Let* A *be a multicoil algebra, then* A *is of polynomial growth. If* A *is strongly simply connected, then the converse also holds.*

If A is a multicoil algebra, then it is triangular (hence of finite global dimension) [6](3.5). Also, any full convex subcategory of A is a multicoil algebra [6](5.6).

1.6. **Supports.** Supports of indecomposable modules over multicoil algebras are characterised in the following lemma.

LEMMA. Let M be an indecomposable module over a multicoil algebra A, and let $B = \text{Supp}M$. Then:

(a) If M is directed, then B is tame and tilted.
(b) If M is not directed, then B is a tame coil enlargement of a tame concealed algebra.

Proof. Since B is a full subcategory of A, which is tame, then B is also tame. Then (a) follows from [3](IX.2.8, p. 366) and (b) follows from [6](5.9). □

Note that, if L is an A-submodule of M, and $x \in A_0$ is such that $L(x) \neq 0$, then $M(x) \neq 0$. Hence, L is also a B-submodule of M.

2. One-point extension of tame concealed algebras

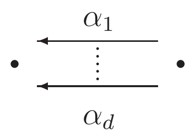
2.1. In this section, we prove that, if C is a tame concealed algebra and M is a C-module containing an infinity of pairwise non-isomorphic simple homogeneous submodules, then $C[M]$ is wild. We start with reduction lemmata.

Lemma. *Let C be a tame concealed algebra, let M be a preinjective C-module, and H a simple homogeneous submodule of M. Then:*

(a) *M is sincere.*
(b) *If $C[M]$ is not wild, then $dim_k(topM) \leq 2$. Then, either M is indecomposable, or $M \simeq M_1 \oplus M_2$, with M_1, M_2 indecomposables with simple top.*

Proof. (a) This follows from the sincerity of H.

(b) If $\dim_k(\mathrm{top}M) = d > 2$, then $C[M]$ contains a wild full subcategory of the form



a contradiction. The second statement follows from Nakayama's lemma, and (1.2). □

2.2. Let $A = kQ/I$ be a bound quiver algebra. It is well-known (see, for instance, [3](III.2.2, p.77)) that, if M is an A-module, then $(\mathrm{top}M)(x) = M(x)$ is x is a source, and $(\mathrm{top}M)(y) = \sum\{\mathrm{Coker}(M(\alpha))\colon \alpha\colon x \longrightarrow y\}$, if y is not a source.

If M satisfies the hypothesis of (2.1)(b), and x is a source, we then have $\dim_k M(x) \leq 2$.

Corollary. *Let C be a tame concealed algebra, let M be a preinjective C-module, and H a simple homogeneous submodule of M. If $C[M]$ is not wild, then:*

(a) *C has at most two sources.*
(b) *If C has just one source s and $dim_k H(s) = d \geq 2$, then $d = 2$ and $topH = topM$.*
(c) *If C has two sources s_1, s_2 then $dim_k H(s_1) = dim_k H(s_2) = 1$ and $topH = topM$.*

Proof. (a) If C has at least three sources s_1, s_2, s_3, the remark above implies that $\mathrm{top}M$ has $S_{s_1} \oplus S_{s_2} \oplus S_{s_3}$ as a summand, and $\dim_k(\mathrm{top}M) \geq 3$, a contradiction.

(b) Since $H(s) \subset M(s)$, which is at most two-dimensionsl, then $d = 2$ and $\mathrm{top}H = \mathrm{top}M = S^2$.

(c) Clearly, $S_{s_1} \oplus S_{s_2}$ is a summand of both $\mathrm{top}H$ and $\mathrm{top}M$. Since $\dim_k(\mathrm{top}M) \leq 2$, then $\mathrm{top}M = S_{s_1} \oplus S_{s_2}$. We claim that $\mathrm{top}H = S_{s_1} \oplus S_{s_2}$. If this is not the case, then there exists $a \notin \{s_1, s_2\}$ such that S_a is a summand of $\mathrm{top}H$. Hence $(\mathrm{top}H)(a) \neq 0$. By the remark above, there exists an arrow $\alpha\colon b \longrightarrow a$ in C

such that $H(\alpha)$ is not surjective. On the other hand, since S_a is not a summand of top$M$, then $M(\alpha)$ is surjective. Now there is a path from one of the s_i (with $i \in \{1, 2\}$) to a passing through α, which we may assume, without loss of generality, to be of minimal length:

$$(*) \qquad s_i = b_t \longrightarrow b_{t-1} \longrightarrow \cdots \longrightarrow b_1 = b \xrightarrow{\alpha} a$$

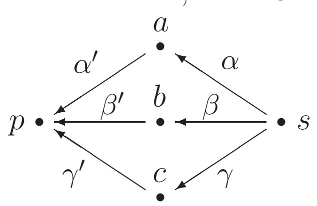
We claim that $\dim_k M(b) \geq 2$. Indeed, if this in not the case, then the inclusion $j\colon H \hookrightarrow M$ yields a commutative square

$$\begin{array}{ccc} H(b) & \xrightarrow{\quad H(\alpha) \quad} & H(a) \\ {\scriptstyle j_b}\downarrow & & \downarrow{\scriptstyle j_a} \\ M(b) & \xrightarrow{\quad M(\alpha) \quad} & M(a) \end{array}$$

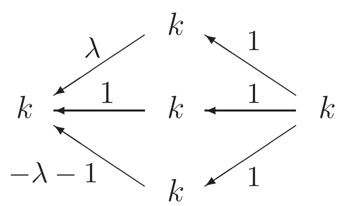
and $M(b) \simeq H(b) \simeq k$ gives that j_b is an isomorphism. Also, since H is sincere and $M(\alpha)$ is surjective, we have $M(a) \simeq H(a) \simeq k$. So $H(\alpha)$ is an isomorphism, a contradiction to its non-surjectivity. Since $\dim_k M(s_i) = 1$, there exists an arrow $\beta\colon c \longrightarrow d$ on the path $(*)$ above such that $1 = \dim_k M(c) < \dim_k M(d)$. Therefore Coker$M(\beta) \neq 0$, hence S_d is a summand of top$M$, a contradiction to top$M = S_{s_1} \oplus S_{s_2}$ and $d \notin \{s_1, s_2\}$. □

2.3. Lemma. *Let C be a tame concealed algebra of type $\widetilde{\mathbb{D}}_4$ and $(H_\lambda)_\lambda$ be an infinite family of pairwise non-isomorphic simple homogeneous modules. If M is a module such that $H_\lambda \subset M$ for all λ, then $C[M]$ is wild.*

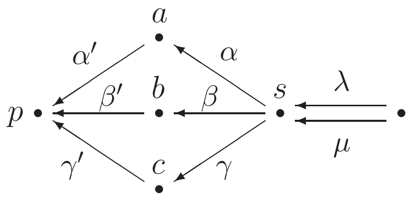
Proof. Assume first C to be non-schurian, then C is given by the quiver



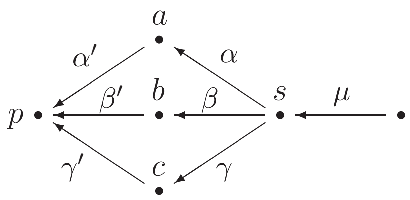
bound by $\alpha\alpha' + \beta\beta' + \gamma\gamma' = 0$. For $\lambda \in k$, H_λ is given by



Since $H_\lambda \subset M$ for any λ, then M is preinjective and we may further assume that $\dim_k(\text{top}M) \leq 2$. Also, soc$H_\lambda = S_p$ is a summand of soc$M$ while S_s is a summand of top$M$. A straightforward examination of the preinjective component of C shows that M is indecomposable and is one of the four modules $I_p, \tau_C^2 I_a, \tau_C^2 I_b$ or $\tau_C^2 I_c$. In the first case, $C[I_p]$ has quiver

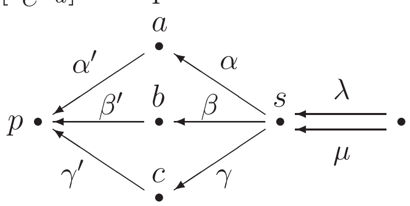


bound by $\lambda\alpha = \mu\alpha$, $\lambda\beta = \mu\beta$, $\lambda\gamma = \mu\gamma$, $\alpha\alpha' + \beta\beta' + \gamma\gamma' = 0$. Changing the presentation (replacing λ by $\lambda - \mu$) gives the same quiver bound by $\lambda\alpha = 0, \lambda\beta = 0, \lambda\gamma = 0, \alpha\alpha' + \beta\beta' + \gamma\gamma' = 0$. This is a split extension [1] of the algebra given by the quiver

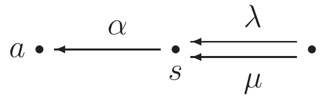


bound by $\alpha\alpha' + \beta\beta' + \gamma\gamma' = 0$, which is evidently wild.

In the second case, $C[\tau_C^2 I_a]$ has quiver

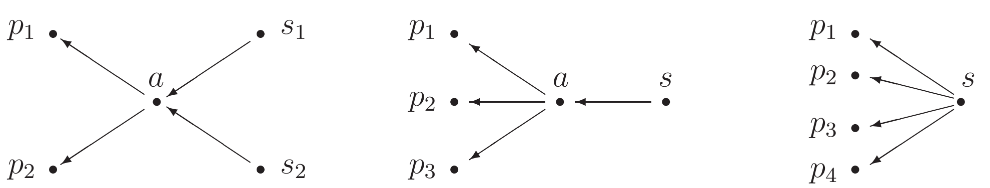


bound by $\lambda\beta = \mu\beta$, $\lambda\gamma = \mu\gamma$, $\alpha\alpha' + \beta\beta' + \gamma\gamma' = 0$. It contains as full convex subcategory the wild hereditary algebra with quiver

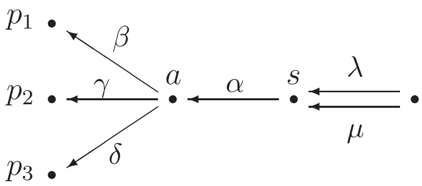


The remaining cases follow by symmetry.

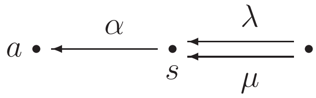
We may thus assume that C is schurian. If $C[M]$ is not wild, then C is hereditary and has one of the quivers



Recall that socH_λ is a summand of soc$M$ and $\dim_k(\text{top}M) \leq 2$. It is easily seen that in the first and third cases, no preinjective C-module satisfies these conditions. In the second case, however, we have the possibility $M = \tau_C^2 I_s$, but then $C[M]$ is given by the quiver

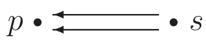


bound by $\lambda\alpha\beta = \mu\alpha\beta$, $\lambda\alpha\gamma = \mu\alpha\gamma, \lambda\alpha\delta = \mu\alpha\delta$. It contains the wild hereditary full subcategory

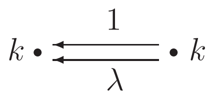


This establishes the lemma. □

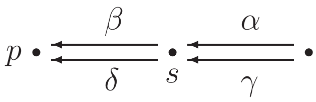
Example. Before proving our next result, we observe that the above lemma does not hold true for tame concealed algebras of type $\widetilde{\mathbb{A}}$. Indeed, let C to be the Kronecker algebra, that is, the hereditary algebra given by the quiver



Observe that the indecomposable injective C-module I_p at the point p contains all the indecomposable regular modules H_λ given by

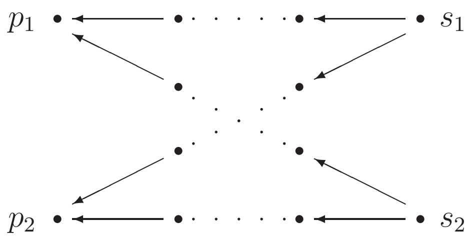


Now, the one point extension $C[I_p]$ is the radical square zero algebra given by the quiver



bounded by $\alpha\delta = 0$, $\gamma\beta = 0$ and $\alpha\beta = \gamma\delta$, which is clearly tame.

Remark. Notice, however, that if C is tame concealed of type $\widetilde{\mathbb{A}}$ and schurian and M satisfies the conditions of (2.3), then $C[M]$ is wild. Indeed, if this is the case then, because of (2.2), C has at most 2 sources, hence is hereditary with quiver



Here, $\mathrm{soc}H_\lambda = S_{p_1} \oplus S_{p_2}$ is a summand of $\mathrm{soc}M$ while $\mathrm{top}M = \mathrm{top}H_\lambda = S_{s_1} \oplus S_{s_2}$. It is easily seen that no preinjective C-module satisfies these conditions.

2.4. Lemma. *Let C be a tame concealed algebra of type $\widetilde{\mathbb{D}}_n$, with $n \geq 5$, or $\widetilde{\mathbb{E}}$ and H be a simple homogeneous C-module. If M is preinjective such that $H \subset M$, then $C[M]$ is wild.*

Proof. We claim that $C[M]$ is wild. Indeed, the tubular type of C is of the form (a, b, c) with at least one of a, b, c larger than or equal to 3. Taking then a one-point extension by a simple homogeneous module yields a wild algebra. This establishes our claim. Applying now (1.2), we get that $C[M]$ is wild, as required. □

2.5. We are now able to prove the main result of this section.

Theorem. *Let C be a tame concealed algebra of type distinct of $\widetilde{\mathbb{A}}$, and $(H_\lambda)_\lambda$ be an infinite family of pairwise non-isomorphic modules. Suppose M is a C-module such that $H_\lambda \subset M$ for all λ. Then $C[M]$ is wild.*

Proof. This follows immediately from (2.3) and (2.4). □

3. Torsionless finiteness and representation dimension.

3.1. Let A be an Artin algebra. An A-module M is called a *generator* (of $\mathrm{mod}A$) if $A_A \in \mathrm{add}M$ and a *cogenerator* if $(\mathrm{D}A)_A = \mathrm{Hom}_k({}_AA, k) \in \mathrm{add}M$. Let A be a non-semisimple algebra. The *representation dimension* rep.dim.A of A is the infimum of the global dimensions of the algebras $\mathrm{End}_A M$, where M is a generator and a cogenerator of $\mathrm{mod}A$, see [9].

An Artin algebra A is called *torsionless-finite* if every indecomposable projective A-module has only finitely many isomorphism classes of indecomposable submodules. One defines *cotorsionless-finite* dually. An Artin algebra is torsionless-finite if and only if it is cotorsionless-finite [10]. We need essentially the following result (see [26]).

Theorem. *If A is torsionless-finite, then rep.dim.$A \leq 3$.*

3.2. Lemma. *Let A be a (possibly wild) branch enlargement of a tame concealed algebra. Then A is torsionless-finite. In particular, rep.dim.$A \leq 3$.*

Proof. Using the description of mod$A$ in [21], we see that, if P_A is an indecomposable projective A-module, then either P_A is postprojective (in which case it clearly has only finitely many isomorphism classes of indecomposable modules) or it lies in an inserted tube Γ. But in this latter case, P has only finitely many indecomposable submodules lying in Γ and, using [11], there are only finitely many postprojective modules X such that $\dim_k X \leq \dim_k P$. Since the other tubes in the same family as Γ are orthogonal to Γ, the proof is complete. □

3.3. Corollary. *Let A be a tame quasi-tilted algebra, then A is torsionless-finite. In particular, rep.dim.$A \leq 3$.*

Proof. This follows from (3.2) and [29] □

Remark. This corollary is a particular case of the main result of [22]. Further, the same proof as in (3.2) and [4] give that if A is iterated tilted of euclidean type, then A is torsionless-finite, and so rep.dim.$A \leq 3$. This is a particular case of the main result of [16].

3.4. Lemma. *Let A be a strongly simply connected tame coil enlargement of a tame concealed algebra, then A is torsionless-finite. In particular, rep.dim.≤ 3.*

Proof. By (1.3), A contains as full convex subcategories a unique maximal branch coextension A^- and a unique maximal branch extension A^+ of a tame concealed algebra, and mod$A$ contains a weakly separating family of coils $\mathcal{T}'$ such that ind$A = \mathcal{P}' \vee \mathcal{T}' \vee \mathcal{Q}'$ where $\mathcal{P}' \subset$ indA^- and $\mathcal{Q}' \subset$ indA^+. Let P_A be an indecomposable projective module. We have three cases:

(a) Assume $P \in \mathcal{P}'$, then P is an indecomposable projective A^--module. Because of (3.2), and the fact that A^- is a branch enlargement of a tame concealed algebra, P has only finitely many isomorphism classes of indecomposable submodules in modA^-. Now, the indecomposable submodules of P in mod$A$ and modA^- coincide.

(b) Assume $P \in \mathcal{Q}'$. For the same reason, P has only finitely many isomorphism classes of indecomposable submodules in modA^+, hence in mod$A$.

(c) Assume $P \in \mathcal{T}'$, and let $M=$ rad$P$. There exists a sequence of full convex subcategories of A

$$A^- = A_0 \subset A_1 \subset \cdots \subset A_t = A$$

which are iterated one-point extensions of A^-, and an index i such that $A_{i+1} = A_i[M]$, that is, P is the unique indecomposable projective in modA_{i+1} which is not in modA_i. Also, M is the pivot of an admissible operation and then is indecomposable except perhaps in the case (ad1). Then, $M = M' \oplus M''$, where M'' is a directed indecomposable, while M' is an indecomposable lying in a coil. Since M'' has only finitely many isomorphism classes of indecomposable submodules and A_{i+1} is of the form $(A'_i \times D)[M' \oplus M'']$, where D is a triangular matrix algebra, and A'_i is the full subcategory of A_i with objects $(A'_i)_0 = (A_i)_0 \setminus D_0$, then it suffices to consider the submodules of M'. We may thus for simplicity assume that $M = M'$.

Let M^- be the largest A^--submodule of M. Assume that M^- has infinitely many isomorphim classes of indecomposable A^--submodules. Because of (3.2), M^- has to be a preinjective A^--module. By (2.5), $A^-[M^-]$ is wild. Since A^- is a full convex subcategory of A_i, then $A_i[M^-]$ is wild. By (1.2), and the tameness of A_i, we get that $A_{i+1} = A_i[M]$ is wild. This is absurd, because A_{i+1} is a full convex subcategory of A which is tame. This shows that M^- has only finitely many non-isomorphic indecomposable submodules. Since a submodule of M is either a submodule of M^-, or lies in the coil containing M, we are done. □

3.5. Corollary. *Let A be a multicoil algebra, and P be an indecomposable projective A-module lying in a coil Γ of $\Gamma(modA)$, then P has only finitely many isomorphism classes of indecomposable submodules.*

Proof. By [6](5.9), we can assume that Γ is obtained from a stable tube over a tame concealed algebra C by a sequence of admissible operations and the support algebra B of Γ is obtained from C by the corresponding sequence of one-point extensions and coextensions. Since the A-submodules and the B-submodules of P coincide, the result follows from (3.4). □

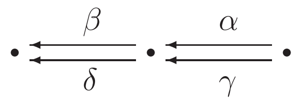
3.6. We are now able to state and prove the main result of this paper.

Theorem. *Let A be a strongly simply connected algebra of polynomial growth, then A is torsionless-finite. In particular, rep.dim.$A \leq 3$.*

Proof. We may clearly assume that A is representation-infinite. Let P be an indecomposable projective A-module. By (1.6), if P is not directed, then the support algebra B of P is a tame coil enlargement of a tame concealed algebra. By (3.4), P has only finitely many isomorphism classes of indecomposable B-submodules, hence A-submodules. On the other hand, if P is directed, then, by (1.6) again, B is tame and tilted. Since, clearly, B has a sincere directed indecomposable module (namely P), then, by [23], B is domestic in at most two one-parameters. Moreover, by [14], B is a full convex subcategory of A.

Let Γ denote the component of $\Gamma(\text{mod}B)$ containing P, and Σ denote the full subquiver of Γ consisting of the indecomposable modules $X \in \Gamma$ such that there is a path $X \rightsquigarrow P$, and every such path is sectional. By [3](IX.2.6, p. 364), Σ is a complete slice in $\Gamma(\text{mod}B)$. Note also that in $\Gamma(\text{mod}B)$, there are finitely many directed components preceding Γ (actually, by [23], at most two). Since indecomposable modules lying in a directed component are uniquely determined by their composition factors [11], then P has infinitely many isomorphism classes of indecomposable submodules if and only if there exists an infinite family $(\mathcal{T}_\lambda)_\lambda$ of homogeneous tubes over a tame concealed algebra C, and an infinite family of homogeneous modules $H_\lambda \in \mathcal{T}_\lambda$ such that $H_\lambda \subset \mathrm{rad}P$, for each λ. Clearly, then, all the H_λ are contained in the largest C-submodule R of $\mathrm{rad}P$. Moreover, by the structure of homogeneous tubes, we can assume the H_λ to be simple homogeneous. By (2.5), if this is the case, then $C[R]$ is wild, hence so is B. This shows that P has only finitely many isomorphism classes of indecomposable B-submodules, hence A-submodules. □

3.7. Remark. As is seen from the example in Section 2, it is not true in general that multicoil algebras are torsionless-finite. However, notice that the algebra



bounded by $\alpha\delta = 0$, $\gamma\beta = 0$ and $\alpha\beta = \gamma\delta$ is tilted and so, by [2], has representation dimension at most 3.

References

1. Assem, I., Coelho, F.U. and Trepode, S., *The bound quiver of a split extension*, J. Algebra and its Appl. **7**, no. 4 (2008), 405-423.
2. Assem, I., Platzeck, M. I. and Trepode, S., *On the representation dimension of tilted and laura algebras*, J. Algebra **296** (2006), 426-439.
3. Assem, I., Simson, D. and Skowroński, A., *Elements of representation theory of associative algebras.* London Math. Soc. Student texts 65, Cambridge University Press (2006).
4. Assem, I. and Skowroński, A., *Algebras with cycle-finite derived categories*, Math. Ann. **280** (1988), 441-463.
5. Assem, I. and Skowroński, A., *Indecomposable modules over multicoil algebras*, Math. Scand. **71** (1992), 31-61.
6. Assem, I. and Skowroński, A., *Multicoil algebras*, Can. Math. Soc. Conf. Proc. Vol **14** (1993), 29-68.
7. Assem, I. and Skowroński, A., *Coils and multicoil algebras*, Can. Math. Soc. Conf. Proc. vol **19** (1996), 1-24.
8. Assem, I. and Skowroński, A. and Tomé, B., *Coil enlargements of algebras*, Tsukuba J. Math., vol. **19**, 2 (1995), 153-179.
9. Auslander, M., *Representation dimension of artin algebras*, Math. Notes, Queen Mary College, London (1971).
10. Auslander, M. and Bridger, M., *Stable module theory*, Memoirs Amer. Math. Soc. no. **94**, Proveidence, R.I. (1969).
11. Auslander, M. and Reiten, I., *Modules determined by their composition factors*, Illinois J. Math. Vol. **29**, no.2 (1985), 280-301.
12. Auslander, M., Reiten, I. and Smalø, S., *Representation Theory of Artin Algebras*, Cambridge Studies in Advanced Mathematics Vol. **36**, Cambridge University Press (1995).
13. Bocian, R., Holm, T. and Skowroński, A., *The representaion dimension of domestic weakly symmetric algebras*, CEJM **2**, no.1 (2004), 67-75.
14. Bongartz, K., *Algebras and quadratic forms*, J. London Math. Soc. (2) **28** (1983), 461-469.
15. Bongartz, K. and Gabriel, P., *Covering spaces in representation theory*, Invent. Math. **65**, 3 (1981/82), 331-378.
16. Coelho, F. U., Happel, D. and Unger, L., *Auslander generators for iterated tilted algebras*, Proc. Amer. Math. Soc. **138**, 5 (2010), 1587-1593.

17. Coelho, F. U. and Platzeck, M. I., *On the representation dimension of some classes of algebras*, J. Algebra **275** (2004), 615-628.
18. Erdmann, K., Holm, T., Iyama, O. and Schröer, J., *Radical embeddings and representation dimensions*, Adv. Math. **185**, no.1 (2005), 159-177.
19. Igusa, K. and Todorov, G. *On the finitistic dimension conjecture for Artin algebras*, Fields Inst. Comm., Vol. **45**, Amer. Math. Soc. Providende, R.I. (2005), 201-204.
20. Iyama, O., *Finiteness of representation dimensions*, Proc. Amer. Math. Soc. **131** (2003), 1011-1014.
21. Lenzing, H., Skowroński, A., *Quasi-tilted algebras of canonical type*, Colloq. Math. **71** (1996), 161-181.
22. Oppermann, S., *Representation dimension of quasi-tilted algebras*, preprint (2008).
23. de la Peña, J. A., *Tame algebras with sincere directing modules*, J. Algebra **161** (1993), 171-185.
24. Ringel, C. M., *Tame algebras*, Proc. ICRA II (Ottawa,1979), Lecture Notes in Math. vol. **831**, Springer, Berlin (1980), 137-287.
25. Ringel, C. M., *Tame algebras and integral quadratic forms*, Lecture Notes in Math., vol. **1099**, Springer-Verlag, Berlin-Heidelberg (1984).
26. Ringel, C. M., *The representation dimension of Artin algebras*, survey, http://www.math.uni-bielefeld.de/birep/2008/
27. Skowroński, A., *Algebras of polynomial growth*, Topics in Algebra, Banach Center Publ. Vol. **26**, Part I, PWN-Polish Scientific Publishers, Warsaw (1990), 535-568.
28. Skowroński, A., *Simply connected algebras of polynomial growth*, Compositio Math. **109** (1997), 99-133.
29. Skowroński, A., *Tame quasi-tilted algebras*, J. Algebra **203** (1998) 470-490.

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