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Embedding mapping-class groups of orientable surfaces with one boundary component

Lluís Bacardit*

Abstract

We denote by $S_{g,b,p}$ an orientable surface of genus g with b boundary components and p punctures. We construct homomorphisms from the mapping-class groups of $S_{g,1,p}$ to the mapping-class groups of $S_{g',1,(b-1)}$, where $b \geq 1$. Our main result is that these homomorphisms are injective. A particular case of these homomorphisms is a well-known embedding of the braid group on p strands into the mapping-class group of $S_{(p-2)/2,1,1}$ if p is even, or into the mapping-class group of $S_{(p-1)/2,1,0}$ if p is odd. We give a short proof of a theorem of Birman and Hilden [4] for surfaces with one boundary component.

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1 Main results

We fix non-negative integers g, p and a positive integer b . We denote by $S_{g,b,p}$ an orientable surface of genus g with b boundary components and p punctures.

Our main theorem is the following.

1.1 Theorem. *Suppose $(g, p) \neq (0, 2)$. Let $\kappa : S_{g',b,0} \rightarrow S_{g,1,0}$ be a finite index regular cover with p branching points in $S_{g,1,0}$ which lift to q points in $S_{g',b,0}$. Suppose every branching point of $S_{g,1,0}$ lift to the same number of points in $S_{g',b,0}$. Let $\kappa' : S_{g',b,q} \rightarrow S_{g,1,p}$ be the corresponding unbranched cover. Let h be a homeomorphism of $S_{g,1,p}$ which fixes the boundary component pointwise. Suppose h lifts to $S_{g',b,q}$. Let \hat{h} be the lift of h which fixes the b -th boundary component pointwise. Let \hat{f} be the extension of \hat{h} to $S_{g',b,0}$. If the restriction*

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of \hat{f} to $S_{g',1,(b-1)} \subseteq S_{g',b,0}$ is isotopic to the identity relative to the boundary component of $S_{g',1,(b-1)}$, then h is isotopic to the identity relative to the boundary component of $S_{g,1,p}$.

Let $\mathcal{M}_{g,b,p}$ be the mapping-class group of $S_{g,b,p}$ relative to the boundary components. That is, $\mathcal{M}_{g,b,p}$ is the group of homeomorphisms of $S_{g,b,p}$ which fix the boundary components pointwise modulo isotopy relative to the boundary components of $S_{g,b,p}$. Since $b \geq 1$, we are restricting ourselves to orientation-preserving homeomorphisms of $S_{g,b,p}$.

The following result is immediate from Theorem 1.1.

1.2 Corollary. *Suppose $(g, p) \neq (0, 2)$. Let $\kappa : S_{g',b,0} \rightarrow S_{g,1,0}$ be a finite index regular cover with p branching points in $S_{g,1,0}$ which lift to q points in $S_{g',b,0}$. Suppose every branching point of $S_{g,1,0}$ lift to the same number of points in $S_{g',b,0}$. Let $\kappa' : S_{g',b,q} \rightarrow S_{g,1,p}$ be the corresponding unbranched cover. If every homeomorphism of $S_{g,1,p}$ which fixes the boundary component pointwise lifts to a homeomorphism of $S_{g',b,q}$, then $\mathcal{M}_{g,1,p}$ embeds in $\mathcal{M}_{g',1,(b-1)}$.*

Suppose $p \geq 3$. If p is odd, there exists a well-known index 2 regular cover $\kappa : S_{(p-1)/2,1,0} \rightarrow S_{0,1,0}$ with p branching points in $S_{0,1,0}$ which lift to p branching points in $S_{(p-1)/2,1,0}$. Every homeomorphism of $S_{0,1,p}$ which fixes the boundary component pointwise lifts to a homeomorphism of $S_{(p-1)/2,1,p}$ which fixes the boundary component pointwise. Corollary 1.2 says that $\mathcal{M}_{0,1,p}$ embeds in $\mathcal{M}_{(p-1)/2,1,0}$. If p is even, there exists a well-known index 2 regular cover $\kappa : S_{(p-2)/2,2,0} \rightarrow S_{0,1,0}$ with p branching points in $S_{0,1,0}$ which lift to p branching points in $S_{(p-2)/2,2,0}$. Every homeomorphism of $S_{0,1,p}$ which fixes the boundary component pointwise lifts to a homeomorphism of $S_{(p-2)/2,2,p}$ which fixes the second boundary component pointwise. Corollary 1.2 says that $\mathcal{M}_{0,1,p}$ embeds in $\mathcal{M}_{(p-2)/2,1,1}$.

From Theorem 1.1 we can prove the following theorem, which is an analog for surfaces with one boundary component of a theorem of Birman and Hilden [4, Theorem 2].

1.3 Theorem. *Suppose $(g, p) \neq (0, 2)$. Let $\kappa : S_{g',b,0} \rightarrow S_{g,1,0}$ be a finite index regular cover with p branching points in $S_{g,1,0}$. Suppose every branching point of $S_{g,1,0}$ lift to the same number of points in $S_{g',b,0}$. Let \hat{f} be a homeomorphism of $S_{g',b,0}$ which fixes the b -th boundary component pointwise and preserves the fibers of $\kappa : S_{g',b,0} \rightarrow S_{g,1,0}$. Then \hat{f} induces a homeomorphism f of $S_{g,1,0}$ such that $\kappa\hat{f} = f\kappa$. If \hat{f} is isotopic to the identity relative to the b -th boundary component, then f is isotopic to the identity relative to the boundary.*

Proof. It is a general fact that if \hat{f} preserves the fibers of $\kappa : S_{g',b,0} \rightarrow S_{g,1,0}$, then \hat{f} induces a homeomorphism f of $S_{g,1,0}$ such that $\kappa\hat{f} = f\kappa$. In particular, f sends branching points to branching points.

Let $\kappa' : S_{g',b,q} \rightarrow S_{g,1,p}$ be the corresponding unbranched cover. Since f sends branching points to branching points, f restricts to a homeomorphism h of $S_{g,1,p}$. Let \hat{h} be the lift of h which fixes the b -th boundary component of $S_{g',b,q}$ pointwise. Notice \hat{h} extends to a homeomorphism of $S_{g',b,0}$. This extension of \hat{h} coincides with \hat{f} . If \hat{f} is isotopic to the identity relative to the b -th boundary component, then the restriction of \hat{f} to $S_{g',1,(b-1)} \subseteq S_{g',b,0}$ is isotopy to the identity relative to the boundary component of $S_{g',1,(b-1)}$. Then, by Theorem 1.1, h is isotopic to the identity relative to the boundary component of $S_{g,1,p}$. This isotopy extends to an isotopy relative to the boundary component of $S_{g,1,0}$ from f to the identity. \square

2 The algebraic analog

Instead of dealing with $S_{g,b,p}$ and homeomorphisms of $S_{g,b,p}$ which fix the boundary components pointwise, we will deal with the fundamental group of $S_{g,b,p}$, denoted $\pi_1(S_{g,b,p})$. Since $b \geq 1$, we choose the base point of $\pi_1(S_{g,b,p})$ in the b -th boundary component. In this way, a homeomorphism of $S_{g,b,p}$ which fixes the boundary components pointwise induces an automorphism of $\pi_1(S_{g,b,p})$.

2.1 Notation. Let G be a group and let g, h be elements of G .

We denote by \bar{g} the invers of g . We denote by g^h the conjugated of g by h , that is, $g^h = \bar{h}gh$. We denote by $[g]$ the conjugacy class of G , that is, $[g] = \{g^a \mid a \in G\}$. We denote by $[g, h]$ the element $\bar{g}hgh$. Let g_1, g_2, \dots, g_k be elements of G . We denote by $\prod_{i=1}^k g_i$ the product $g_1g_2 \cdots g_k$.

We denote by $\text{Aut}(G)$ the automorphism group of G , and, by $\text{Out}(G)$ the group of extern automorphism of G . Given $\phi \in \text{Aut}(G)$, we denote by g^ϕ the image of g by ϕ .

2.2 Notation. Let $\Sigma_{g,b,p}$ be the rank $2g + (b - 1) + p$ free group with generating set $\{x_i, y_i\}_{1 \leq i \leq g} \cup \{z_l\}_{1 \leq l \leq (b-1)} \cup \{t_k\}_{1 \leq k \leq p}$. We view $\Sigma_{g,b,p}$ as a presentation of $\pi_1(S_{g,b,p}, *)$, the fundamental group of $S_{g,b,p}$ based at a point $*$ in the b -th boundary component. In addition, for every $1 \leq l \leq (b - 1)$, z_l represents a loop around the l -th boundary component; for every $1 \leq k \leq p$, t_k represents a loop around the k -th puncture, and $(\prod_{i=1}^g [x_i, y_i] \prod_{l=1}^{b-1} z_l \prod_{k=1}^p t_k)^{-1}$ represents a loop around the b -th boundary component. Note that, if $p = 0$, there is no puncture in $S_{g,b,p} = S_{g,b,0}$, and $\prod_{k=1}^p t_k = 1$.

Let f be a homeomorphism of $S_{g,b,p}$ which fixes the boundary components pointwise. Then f induces an automorphism f_* of $\Sigma_{g,b,p}$ which fixes the set of conjugacy classes of t_1, t_2, \dots, t_p . Since f fixes the boundary components of $S_{g,b,p}$ pointwise, we see that f_* fixes $(\prod_{i=1}^g [x_i, y_i] \prod_{l=1}^{b-1} z_l \prod_{k=1}^p t_k)^{-1}$ and the conjugacy class of z_l , for all $1 \leq l \leq b - 1$. Two isotopic homeomorphisms of $S_{g,b,p}$ induce the same automorphism of $\Sigma_{g,b,p}$. Recall we consider isotopies relative to the boundary components. Notice the Dehn twist with respect a loop around a boundary component is isotopic to the identity, but it is not isotopic to the

identity relative to the boundary. To capture this fact, we associate to f an automorphism of $\Sigma_{g,b,p} * \langle e_1, e_2, \dots, e_{(b-1)} \mid \rangle$ which maps $\Sigma_{g,b,p}$ to itself and respects the following sets

$$(2.2.1) \quad \begin{aligned} & \text{(i)} \quad \{ \Pi_{i=1}^g [x_i, y_i] \Pi_{l=1}^{b-1} z_l \Pi_{k=1}^p t_k \}, \\ & \text{(ii)} \quad \{ \bar{z}_1^{e_1} \}, \{ \bar{z}_2^{e_2} \}, \dots, \{ \bar{z}_{(b-1)}^{e_{(b-1)}} \}, \\ & \text{(iii)} \quad \{ [\bar{t}_k] \}_{1 \leq k \leq p}. \end{aligned}$$

Recall z_l represents a loop around the l -th boundary component which is based at a point in the b -boundary component. For every $1 \leq l \leq (b-1)$, we view e_l as an arc from the base point in the b -th boundary component to a chosen point in the l -th boundary component. We view $\bar{e}_l z_l e_l = z_l^{e_l}$ as a loop around the l -boundary component and based at the chosen point in the l -boundary component. Since the homeomorphism f fixes the l -boundary component pointwise, the automorphism f_* fixes $\bar{z}_l^{e_l}$. For example, the Dehn twist with respect to the loop represented by $z_l^{e_l}$ gives the following automorphism of $\Sigma_{g,b,p} * \langle e_1, e_2, \dots, e_{(b-1)} \mid \rangle$

$$\begin{cases} e_l & \mapsto z_l e_l, \\ a & \mapsto a, \quad a \in \{x_i, y_i\}_{1 \leq i \leq g} \cup \{t_k\}_{1 \leq k \leq p} \cup \{z_l\}_{1 \leq l \leq b} \cup \{e_{l'}\}_{1 \leq l' \leq b, l' \neq l}. \end{cases}$$

2.3 Definition. We denote by $\mathcal{AM}_{g,b,p}$ the subgroup of $\text{Aut}(\Sigma_{g,b,p} * \langle e_1, e_2, \dots, e_{b-1} \mid \rangle)$ consisting of all the automorphisms of $\Sigma_{g,b,p} * \langle e_1, e_2, \dots, e_{(b-1)} \mid \rangle$ which map $\Sigma_{g,b,p}$ to itself and respect the sets of (2.2.1).

We call $\mathcal{AM}_{g,b,p}$ the *algebraic mapping-class group* of a surface of genus g with b boundary components and p punctures, $S_{g,b,p}$.

The mapping-class group of $S_{g,b,p}$, denoted $\mathcal{M}_{g,b,p}$, is defined as the group of homeomorphisms of $S_{g,b,p}$ modulo isotopy relative to the boundary components. The above discussion shows that there exists a map $\mathcal{M}_{g,b,p} \rightarrow \mathcal{AM}_{g,b,p}$. We sketch the proof that this map is an isomorphism.

Since $S_{g,0,p}$ has empty boundary, we cannot suppose that the basepoint of $\Pi_1(S_{g,0,p})$ is fixed by homeomorphisms of $S_{g,0,p}$. A discussion as above shows that there exists a map $\mathcal{M}_{g,0,p} \rightarrow \text{Out}(\Pi_1(S_{g,0,p}))$. The Dehn-Nielsen-Baer theorem says that this map is an isomorphism onto the subgroup of $\text{Out}(\Pi_1(S_{g,0,p}))$ consisting of classes of automorphisms which respect the set $\{[t_k]\}_{1 \leq k \leq p}$. See, for example, [11, Theorem 3.6], [12, Theorem 2.9.A]. In particular, $\mathcal{M}_{g,0,p} \leq \text{Out}(\Pi_1(S_{g,0,p}))$.

We say that $S_{g,b-1,p+1}$ is obtained from $S_{g,b,p}$ by converting a boundary component into a puncture if $S_{g,b-1,p+1} \subset S_{g,b,p}$ and $S_{g,b,p} - S_{g,b-1,p+1}$ is homeomorphic to a circle. If $b \geq 1$, by converting all the boundary components of $S_{g,b,p}$ into punctures, from the Dehn-Nielsen-Baer theorem we can deduce $\mathcal{M}_{g,b,p} \simeq \mathcal{AM}_{g,b,p}$, [10, Theorem 9.6]. See [10] for a background on algebraic mapping-class groups, with some changes of notation.

For $(g, b) = (0, 1)$ and $p \geq 1$, $\mathcal{AM}_{0,1,p}$ is isomorphic to the p -string braid group. We have $\mathcal{AM}_{0,1,p} = \langle \sigma_1, \sigma_2, \dots, \sigma_{p-1} \rangle$, where for all $1 \leq i \leq (p-1)$, $\sigma_i \in \text{Aut}(\Sigma_{0,1,p})$ is defined by

$$(2.3.1) \quad \sigma_i := \begin{cases} t_i & \mapsto t_{i+1}, \\ t_{i+1} & \mapsto t_i^{t_{i+1}}, \\ t_k & \mapsto t_k, \end{cases} \quad \text{for all } 1 \leq k \leq p, k \neq i, i+1.$$

Let $d \in \mathbb{Z}$, $d \geq 2$.

2.4 Notation. Let N_d be the normal closure of t_1, t_2, \dots, t_p in $\Sigma_{g,1,p}$. We denote by $\Sigma_{g,1,p^{(d)}}$ the group $\Sigma_{g,1,p}/N_d$. For every $1 \leq k \leq p$, we denote by τ_k the image of t_k by the natural homomorphism $\Sigma_{g,1,p} \rightarrow \Sigma_{g,1,p^{(d)}}$.

Notice that if $p = 0$, then $N_d = 1$ and $\Sigma_{g,1,p^{(d)}} = \Sigma_{g,1,p}$.

2.5 Definition. Let $\mathcal{AM}_{g,1,p^{(d)}}$ denote the group of all automorphisms of $\Sigma_{g,1,p^{(d)}}$ that respect the sets

$$\{\Pi_{i=1}^g[x_i, y_i] \Pi_{l=1}^p \tau_l\}, \quad \{\{\bar{t}_k\}\}_{1 \leq k \leq p}.$$

Since the elements of $\mathcal{AM}_{g,1,p}$ respect the set $\{\{\bar{t}_k\}\}_{1 \leq k \leq p}$, the natural homomorphism $\Sigma_{g,1,p} \rightarrow \Sigma_{g,1,p^{(d)}}$ induces a natural homomorphism

$$\psi : \mathcal{AM}_{g,1,p} \rightarrow \mathcal{AM}_{g,1,p^{(d)}}.$$

If $p = 0$, then $\Sigma_{g,1,p} = \Sigma_{g,1,p^{(d)}}$ and ψ is the identity.

2.6 Theorem. *The homomorphism $\psi : \mathcal{AM}_{g,1,p} \rightarrow \mathcal{AM}_{g,1,p^{(d)}}$ is injective.*

We proof Theorem 2.6 in Section 4.

Let $\kappa : S_{g',b,0} \rightarrow S_{g,1,0}$ be an index m regular cover with p branching points in $S_{g,1,0}$ which lift to q points in $S_{g',b,0}$. Suppose every branching point of $S_{g,1,0}$ lift to the same number of points in $S_{g',b,0}$. Notice that $q = 0$ if and only if $p = 0$. Let $\kappa' : S_{g',b,q} \rightarrow S_{g,1,p}$ be the corresponding unbranched cover. We identify $\Sigma_{g',b,q}$ with $\kappa'_*(\Sigma_{g',b,q})$. Hence, $\Sigma_{g',b,q}$ is a normal subgroup of $\Sigma_{g,1,p}$ of index m . We set $G := \Sigma_{g,1,p}/\Sigma_{g',b,q}$ the group of deck transformations.

We put $\varrho = \Pi_{i=1}^g[x_i, y_i] \Pi_{k=1}^p t_k \Sigma_{g',b,q} \in G$. Let c be the order of ϱ in G . Since $\varrho^c = 1$ in G , we see that $(\Pi_{i=1}^g[x_i, y_i] \Pi_{k=1}^p t_k)^c \in \Sigma_{g',b,q}$. Notice that $(\Pi_{i=1}^g[x_i, y_i] \Pi_{k=1}^p t_k)^{-c}$ represents a loop around the b -th boundary component. We take a basis $\{\hat{x}_i, \hat{y}_i\}_{1 \leq i \leq g'} \cup \{\hat{z}_l\}_{1 \leq l \leq (b-1)} \cup \{\hat{t}_k\}_{1 \leq k \leq q}$ of $\Sigma_{g',b,q}$ such that

$$\Pi_{i=1}^{g'}[\hat{x}_i, \hat{y}_i] \Pi_{l=1}^{b-1} \hat{z}_l \Pi_{k=1}^q \hat{t}_k = (\Pi_{i=1}^g[x_i, y_i] \Pi_{k=1}^p t_k)^c.$$

Recall G has cardinality m . The subgroup $\langle \varrho \rangle \leq G$ has index $b = m/c$. For every $1 \leq l \leq b-1$, we take $w_l \in \Sigma_{g,1,p} - \Sigma_{g',b,q}$ such that

$$\hat{z}_l = \bar{w}_l (\Pi_{i=1}^g[x_i, y_i] \Pi_{k=1}^p t_k)^{-c} w_l.$$

We put $\rho_l = w_l \Sigma_{g',b,q} \in G$. Then $G = \langle \varrho \rangle \rho_1 \cup \langle \varrho \rangle \rho_2 \cdots \cup \langle \varrho \rangle \rho_{(b-1)} \cup \langle \varrho \rangle$. That is, the boundary components of $S_{g',b,p}$ are image by deck transformations of the b -th boundary component.

For every $1 \leq k \leq p$, we put $\varrho_k = t_k \Sigma_{g',b,q} \in G$. Let d_k be the order of ϱ_k in G . Since t_k corresponds to a branching point, $t_k \notin \Sigma_{g',b,q}$ and $d_k \geq 2$. Since $\varrho_k^{d_k} = 1$ in G , we see that $t_k^{d_k} \in \Sigma_{g',b,q}$. Notice that $t_k^{d_k}$ represents a loop around a lift of the k -th puncture of $S_{g,1,p}$. The subgroup $\langle \varrho_k \rangle$ has index m/d_k in G . Since all the branching point of $S_{g,1,0}$ lift to the same number of points in $S_{g',b,0}$, $m/d_1 = m/d_k$ for all $2 \leq k \leq p$. Hence, $d_1 = d_k$ for all $2 \leq k \leq p$. Let $d = d_1$. We have $G = \langle \varrho_k \rangle \rho_{1,k} \cup \langle \varrho_k \rangle \rho_{2,k} \cup \cdots \cup \langle \varrho_k \rangle \rho_{m/d,k}$, where $\rho_{j,k} = u_{j,k} \Sigma_{g',b,q} \in G$ for all $1 \leq j \leq m/d$. Notice that $(t_k^d)^{u_{1,k}}, (t_k^d)^{u_{2,k}}, \dots, (t_k^d)^{u_{m/d,k}}$ represent loops around the m/d lifts of the k -th puncture. We choose $u_{1,k}, u_{2,k}, \dots, u_{m/d,k} \in \Sigma_{g,1,p}$ such that $\{(t_k^d)^{u_{1,k}}, (t_k^d)^{u_{2,k}}, \dots, (t_k^d)^{u_{m/d,k}}\} \subseteq \{\hat{t}_1, \hat{t}_2, \dots, \hat{t}_q\}$. Then

$$(2.6.1) \quad \{\hat{t}_1, \hat{t}_2, \dots, \hat{t}_q\} = \bigcup_{k=1}^p \{(t_k^d)^{u_{1,k}}, (t_k^d)^{u_{2,k}}, \dots, (t_k^d)^{u_{m/d,k}}\}.$$

Recall N_d is the normal closure of $t_1^d, t_2^d, \dots, t_p^d$ in $\Sigma_{g,1,p}$.

2.7 Lemma. *With the above notation, N_d is equal to the normal closure of $\hat{t}_1, \hat{t}_2, \dots, \hat{t}_q$ in $\Sigma_{g',b,q}$.*

Proof. By (2.6.1), the normal closure of $\hat{t}_1, \hat{t}_2, \dots, \hat{t}_q$ in $\Sigma_{g',b,q}$ is a subgroup of N_d .

Let $1 \leq k \leq p$ and $w \in \Sigma_{g,1,p}$. By (2.6.1), it is enough to proof $(t_k^d)^w = (t_k^d)^{u_{j,k}v}$ for some $1 \leq j \leq (m/d)$ and $v \in \Sigma_{g',b,q}$. Recall $G = \Sigma_{g,1,p}/\Sigma_{g',b,q}$, $\varrho_k = t_k \Sigma_{g',b,q} \in G$ and $G = \langle \varrho_k \rangle \rho_{1,k} \cup \langle \varrho_k \rangle \rho_{2,k} \cup \cdots \cup \langle \varrho_k \rangle \rho_{m/d,k}$, where $\rho_{j,k} = u_{j,k} \Sigma_{g',b,q} \in G$ for all $1 \leq j \leq (m/d)$. Let $1 \leq j \leq (m/d)$ such that $w \Sigma_{g',b,q} \in \langle \varrho_k \rangle \rho_{j,k}$. Let $1 \leq r \leq d$ such that $w \Sigma_{g',b,q} = \varrho_k^r \rho_{j,k} = t_k^r u_{j,k} \Sigma_{g',b,q}$. Then $w = t_k^r u_{j,k} v$, for some $v \in \Sigma_{g',b,q}$ and $(t_k^d)^w = (t_k^d)^{t_k^r u_{j,k} v} = (t_k^d)^{u_{j,k} v}$. \square

Recall $\Sigma_{g,1,p}/N_d = \Sigma_{g,1,p^{(d)}}$, and for every $1 \leq k \leq p$, we denote by τ_k the image of t_k by the natural homomorphism $\Sigma_{g,1,p} \rightarrow \Sigma_{g,1,p^{(d)}}$.

2.8 Notation. Let $H \leq \Sigma_{g,1,p}$ be a normal subgroup of finite index such that $N_d \leq H$. Notice $H/N_d \leq \Sigma_{g,1,p^{(d)}}$. We set

$$\mathcal{AM}_{g,1,p}(H) = \{\phi \in \mathcal{AM}_{g,1,p} \mid H^\phi = H\},$$

and

$$\mathcal{AM}_{g,1,p^{(d)}}(H/N_d) = \{\tilde{\phi} \in \mathcal{AM}_{g,1,p^{(d)}} \mid (H/N_d)^{\tilde{\phi}} = H/N_d\}.$$

2.9 Proposition. *Suppose $(g, p, d) \neq (0, 2, 2)$. Let $H \leq \Sigma_{g,1,p}$ be a normal subgroup of finite index such that $N_d \leq H$. Let $\phi \in \mathcal{AM}_{g,1,p}(H)$. Then $\psi(\phi) \in \mathcal{AM}_{g,1,p^{(d)}}(H/N_d)$. If $\psi(\phi)|_{H/N_d} = 1$, then $\phi = 1$.*

Proof. Since N_d and H are ϕ -invariant, we see H/N_d is $\psi(\phi)$ -invariant. Since $\psi(\phi) \in \mathcal{AM}_{g,1,p^{(d)}}$, we have $\psi(\phi) \in \mathcal{AM}_{g,1,p}(H/N_d)$

Since H has finite index in $\Sigma_{g,1,p}$, there exists $r \in \mathbb{Z}$, $r \geq 1$, such that

$$(\Pi_{i=1}^g [x_i, y_i] \Pi_{k=1}^p t_k)^r \in H.$$

Fix $1 \leq k \leq p$. Since H is normal in $\Sigma_{g,1,p}$, we see

$$\bar{t}_k (\Pi_{i=1}^g [x_i, y_i] \Pi_{k'=1}^p t_{k'})^r t_k \in H.$$

If $\psi(\phi)|_{H/N_d} = 1$, in $\Sigma_{g,1,p^{(d)}}$,

$$\begin{aligned} & \bar{\tau}_k (\Pi_{i=1}^g [x_i, y_i] \Pi_{k'=1}^p \tau_{k'})^r \tau_k \\ &= (\bar{\tau}_k (\Pi_{i=1}^g [x_i, y_i] \Pi_{k'=1}^p \tau_{k'})^r \tau_k)^{\psi(\phi)} \\ &= \bar{\tau}_k^{\psi(\phi)} (\Pi_{i=1}^g [x_i, y_i] \Pi_{k'=1}^p \tau_{k'})^r \tau_k^{\psi(\phi)}. \end{aligned}$$

Then, in $\Sigma_{g,1,p^{(d)}}$, $\bar{\tau}_k^{\psi(\phi)} \tau_k$ commutes with $(\Pi_{i=1}^g [x_i, y_i] \Pi_{k'=1}^p \tau_{k'})^r$. Recall $\Sigma_{g,1,p^{(d)}} = \Sigma_{g,1,p}/N_d$. Hence, $\Sigma_{g,1,p^{(d)}} \simeq \Sigma_{g,1,0} * \langle \tau_1, \tau_2, \dots, \tau_p \mid \tau_1^d, \tau_2^d, \dots, \tau_p^d \rangle$. Hence, $\bar{\tau}_k^{\psi(\phi)} \tau_k \in \langle \Pi_{i=1}^g [x_i, y_i] \Pi_{k'=1}^p \tau_{k'} \rangle$, and,

$$(2.9.1) \quad \tau_k^{\psi(\phi)} = (\Pi_{i=1}^g [x_i, y_i] \Pi_{k'=1}^p \tau_{k'})^{r'} \tau_k,$$

for some $r' \in \mathbb{Z}$. Recall $[\tau_k^{\psi(\phi)}] = [\tau_j]$, for some $1 \leq j \leq p$. If $(g, p) \neq (0, 1)$, and if $(g, p, d) \neq (0, 2, 2)$, then (2.9.1) implies $r' = 0$ and $\tau_k^{\psi(\phi)} = \tau_k$.

Fix $a \in \{x_i, y_i\}_{1 \leq i \leq g}$. Since H has finite index in $\Sigma_{g,1,p}$, there exists $s \in \mathbb{Z}$, $s \geq 1$, such that $a^s \in H$. If $\psi(\phi)|_{H/N_d} = 1$, then $(a^s)^{\psi(\phi)} = a^s$, and, $a^{\psi(\phi)} = a$.

Since $\Sigma_{g,1,p^{(d)}} \simeq \Sigma_{g,1,0} * \langle \tau_1, \tau_2, \dots, \tau_p \mid \tau_1^d, \tau_2^d, \dots, \tau_p^d \rangle$, $a^{\psi(\phi)} = a$ for all $a \in \{x_i, y_i\}_{1 \leq i \leq g}$, and, $\bar{\tau}_k^{\psi(\phi)} = \tau_k$ for all $1 \leq k \leq p$; we see $\psi(\phi) = 1$. By Theorem 2.6, $\phi = 1$. \square

Let $\phi \in \mathcal{AM}_{g,1,p}$. Suppose $\Sigma_{g',b,q}$ is ϕ -invariant. Then ϕ induces an automorphisms of $\Sigma_{g',b,q}$ by restriction. In $\Sigma_{g,1,p}$ we have

- (i) $\Pi_{i=1}^{g'} [\hat{x}_i, \hat{y}_i] \Pi_{l=1}^{(b-1)} \hat{z}_l \Pi_{k=1}^q \hat{t}_k = (\Pi_{i=1}^g [x_i, y_i] \Pi_{k=1}^p t_k)^c$;
- (ii) \hat{z}_l is conjugate to $(\Pi_{i=1}^g [x_i, y_i] \Pi_{k=1}^p t_k)^{-c}$, for all $1 \leq l \leq (b-1)$;
- (iii) \hat{t}_k is conjugate to t_j^d , $1 \leq j \leq p$, for all $1 \leq k \leq q$.

If we identify $\Sigma_{g',b,q}$ with $\Sigma_{g',1,(b-1)+q}$ by identifying \hat{z}_l with \hat{t}_l , for all $1 \leq l \leq (b-1)$, and \hat{t}_k with $\hat{t}_{(b-1)+k}$, for all $1 \leq k \leq q$; then the restriction of ϕ to $\Sigma_{g',1,(b-1)+q}$ lies inside $\mathcal{AM}_{g',1,(b-1)+q}$.

Let h be the homeomorphism of $S_{g,1,p}$ which fixes the boundary component pointwise and $h_* = \phi$. Since $\Sigma_{g',b,q}$ is ϕ -invariant, h lifts to a homeomorphism

\hat{h} of $S_{g',b,q}$ which fixes the b -th boundary component pointwise. Since \hat{h} may not fix the first $(b-1)$ boundary components pointwise, \hat{h} does not represent an element of $\mathcal{M}_{g',b,q}$, but it represents an element of $\mathcal{M}_{g',1,(b-1)+q}$, that is, we have to convert the first $(b-1)$ boundary components into punctures. If \hat{h} fixes the boundary components pointwise, we can conserve the first $(b-1)$ boundary components. Algebraically, if we want to have an element of $\mathcal{AM}_{g',b,q}$, we have to define the image of $\hat{e}_1, \hat{e}_2, \dots, \hat{e}_{(b-1)}$. Since $\Sigma_{g',b,q}$ is ϕ -invariant, we see ϕ induces an automorphism of $G = \Sigma_{g,1,p}/\Sigma_{g',b,q}$. If ϕ induces the identity of G , we can define an element of $\mathcal{AM}_{g',b,q}$ from ϕ .

Recall N_d is the normal closure in $\Sigma_{g,1,p}$ of $t_1^d, t_2^d, \dots, t_p^d$. By Lemma 2.7, N_d is the normal closure in $\Sigma_{g',b,q}$ of $\hat{t}_1, \hat{t}_2, \dots, \hat{t}_q$. Hence, $\Sigma_{g',b,0} = \Sigma_{g',b,q}/N_d$. We identify $\Sigma_{g',b,0}$ with $\Sigma_{g',1,(b-1)}$ by identifying \hat{z}_l with \hat{t}_l for all $1 \leq l \leq (b-1)$. Hence, $\Sigma_{g',1,(b-1)} = \Sigma_{g',b,q}/N_d$. Since $\Sigma_{g',b,q}$ is ϕ -invariant, by Proposition 2.9, there exists the restriction $\psi(\phi)|_{\Sigma_{g',1,(b-1)}} : \Sigma_{g',1,(b-1)} \rightarrow \Sigma_{g',1,(b-1)}$. Recall \hat{h} is a homeomorphism of $S_{g',b,q}$. Since $h_* = \phi$, we have $\hat{h}_* = \phi|_{\Sigma_{g',b,q}}$. Notice \hat{h} extends to a homeomorphism \hat{f} of $S_{g',b,0}$. Notice \hat{f} restricts to a homeomorphism of $S_{g',1,(b-1)} \subseteq S_{g',b,0}$. Since $\hat{h}_* = \phi|_{\Sigma_{g',b,q}}$ and $\Sigma_{g',1,(b-1)} = \Sigma_{g',b,q}/N_d$, the restriction of \hat{f} to $S_{g',1,(b-1)} \subseteq S_{g',b,0}$ induces the automorphism $\psi(\phi)|_{\Sigma_{g',1,(b-1)}}$.

We, now, can proof the algebraic analog of Theorem 1.1.

2.10 Theorem. *Suppose $(g,p) \neq (0,2)$. Let $\kappa : S_{g',b,0} \rightarrow S_{g,1,0}$ be a finite index regular cover with p branching points in $S_{g,1,0}$ which lift to q points in $S_{g',b,0}$. Suppose every branching point of $S_{g,1,0}$ lift to the same number of points in $S_{g',b,0}$. Let $\kappa' : S_{g',b,q} \rightarrow S_{g,1,p}$ be the corresponding unbranched cover. Let ϕ be an element of $\mathcal{AM}_{g,1,p}$. Suppose $\Sigma_{g',b,q}$ is ϕ -invariant. If $\psi(\phi)|_{\Sigma_{g',1,(b-1)}} = 1$, then $\phi = 1$.*

Proof. Since $\Sigma_{g,1,p^{(d)}} = \Sigma_{g,1,p}/N_d$, the natural homomorphism $\Sigma_{g,1,p} \rightarrow \Sigma_{g,1,p^{(d)}}$ restricts to the natural homomorphism $\Sigma_{g',b,q} \rightarrow \Sigma_{g',1,(b-1)}$.

Since $\psi : \mathcal{AM}_{g,1,p} \rightarrow \mathcal{AM}_{g,1,p^{(d)}}$ is given by the natural homomorphism $\Sigma_{g,1,p} \rightarrow \Sigma_{g,1,p^{(d)}}$, we see $\psi(\phi) : \Sigma_{g,1,p^{(d)}} \rightarrow \Sigma_{g,1,p^{(d)}}$ completes the following commutative square

$$\begin{array}{ccc} \Sigma_{g,1,p} & \xrightarrow{\phi} & \Sigma_{g,1,p} \\ \downarrow & & \downarrow \\ \Sigma_{g,1,p^{(d)}} & \xrightarrow{\psi(\phi)} & \Sigma_{g,1,p^{(d)}} \end{array}$$

where the vertical arrows are the natural homomorphisms. Notice

$\psi(\phi)|_{\Sigma_{g',1,(b-1)}} : \Sigma_{g',1,(b-1)} \rightarrow \Sigma_{g',1,(b-1)}$ completes the following commutative square

$$\begin{array}{ccc} \Sigma_{g',b,q} & \xrightarrow{\phi|_{\Sigma_{g',b,q}}} & \Sigma_{g',b,q} \\ \downarrow & & \downarrow \\ \Sigma_{g',1,(b-1)} & \xrightarrow{\psi(\phi)|_{\Sigma_{g',1,(b-1)}}} & \Sigma_{g',1,(b-1)} \end{array}$$

where the vertical arrows are the natural homomorphisms. By Proposition 2.9, if $\psi(\phi)|_{\Sigma_{g',1,(b-1)}} = 1$, then $\phi = 1$. \square

We state the algebraic analog of Corollary 1.2.

2.11 Corollary. *Suppose $(g, p) \neq (0, 2)$. Let $\kappa : S_{g',b,0} \rightarrow S_{g,1,0}$ be a finite index regular cover with p branching points in $S_{g,1,0}$ which lift to q points in $S_{g',b,0}$. Suppose every branching point of $S_{g,1,0}$ lift to the same number of points in $S_{g',b,0}$. Let $\kappa' : S_{g',b,q} \rightarrow S_{g,1,p}$ be the corresponding unbranched cover. If $\Sigma_{g',b,q}$ is $\mathcal{AM}_{g,1,p}$ -invariant, then $\mathcal{M}_{g,1,p}$ embeds in $\mathcal{M}_{g',1,(b-1)}$. In fact, the embedding is given by $\phi \mapsto \psi(\phi)|_{\Sigma_{g',1,(b-1)}}$.*

3 Examples

We fix g, p such that $(g, p) \neq (0, 2)$. Let \hat{S} be the universal cover of $S_{g,1,p}$. The fundamental group of $S_{g,1,p}$, denoted $\Sigma_{g,1,p}$, acts on \hat{S} . Let H be a subgroup of $\Sigma_{g,1,p}$ of index m . Suppose H is $\mathcal{AM}_{g,1,p}$ -invariant. The quotient space \hat{S}/H is an orientable surface, denoted $S_{g',b,q}$. We identify the fundamental group of $S_{g',b,q}$, denoted $\Sigma_{g',b,q}$, with H . The cover $\hat{S} \rightarrow S_{g,1,p}$ induces a cover $S_{g',b,q} \rightarrow S_{g,1,p}$ with group of deck transformation $G := \Sigma_{g,1,p}/\Sigma_{g',b,q}$. If $t_k \notin \Sigma_{g',b,q}$ for all $1 \leq k \leq p$, then the corresponding cover $S_{g',b,0} \rightarrow S_{g,1,0}$ has p branching points in $S_{g,1,0}$ which lift to q points in $S_{g',b,0}$. Since H is $\mathcal{AM}_{g,1,p}$ -invariant, it can be seen that every branching point of $S_{g,1,p}$ lifts to the same number of points in $S_{g',b,0}$. By Corollary 2.11, we have an embedding $\mathcal{AM}_{g,1,p} \hookrightarrow \mathcal{AM}_{g',1,(b-1)}$. By choosing an appropriated basis of H , we can compute elements in the image of this embedding from elements of $\mathcal{AM}_{g,1,p}$.

The first example is well-known. In the second example, we give a basis of H and compute elements in the image of the embedding.

Example 1. Let H be the kernel of the homomorphism $\Sigma_{0,1,p} \rightarrow \langle \tau \mid \tau^2 \rangle$ such that $t_k \mapsto \tau$ for all $1 \leq k \leq p$. It is standard to see that H is a free group of rank $2p - 1$ with basis $t_1^2, t_1 t_2, t_1 t_3, \dots, t_1 t_p, t_1 \bar{t}_2, t_1 \bar{t}_3, \dots, t_1 \bar{t}_p$. It is easy to see that H is invariant by the generators of $\mathcal{AM}_{0,1,p}$ given in (2.3.1). For $1 \leq k \leq p$, notice that $\varrho_k = t_k H$ has order 2 in $G := \Sigma_{0,1,p}/H \simeq \langle \tau \mid \tau^2 \rangle$. Hence, $\langle \varrho_k \rangle$ has index 1 in G and the k -th puncture in $S_{g,1,p}$ lifts to one puncture in $S_{g',b,q}$. Thus, $q = p$.

- (a). If p is odd, then $\prod_{k=1}^p t_k \notin H$ and $\varrho = \prod_{k=1}^p t_k H$ has order 2 in G . Hence, $\langle \varrho \rangle$ has index 1 in G and $b = 1$. Since $\Sigma_{g',b,q}$ has rank $2g' + b - 1 + q$ and H has rank $2p - 1$, we have $2g' + 1 - 1 + p = 2p - 1$ and $g' = (p - 1)/2$. Hence, $\mathcal{AM}_{0,1,p} \hookrightarrow \mathcal{AM}_{(p-1)/2,1,0}$, if p is odd.
- (b). If p is even, then $\prod_{k=1}^p t_k \in H$ and $\varrho = \prod_{k=1}^p t_k H$ has order 1 in G . Hence, $\langle \varrho \rangle$ has index 2 in G and we have $b = 2$. Since $\Sigma_{g',b,q}$ has rank $2g' + b - 1 + q$

and H has rank $2p - 1$, we have $2g' + 2 - 1 + p = 2p - 1$ and $g' = (p - 2)/2$. Hence, $\mathcal{AM}_{0,1,p} \hookrightarrow \mathcal{AM}_{(p-2)/2,1,1}$, if p is even.

Example 2. Let $\Sigma_{1,1,0} = \langle x, y \mid \rangle$. Let H be the kernel of the homomorphism $\Sigma_{1,1,0} \rightarrow \langle \tau_1 \mid \tau_1^2 \rangle \times \langle \tau_2 \mid \tau_2^2 \rangle$ such that $x \mapsto \tau_1$, $y \mapsto \tau_2$. It is standard to see that H is a free group of rank 5. It can be shown that H is a characteristic subgroup of $\Sigma_{1,1,0}$. Notice that $\varrho = [x, y]H$ has order 1 in $G := \Sigma_{1,1,0}/H \simeq \langle \tau_1 \mid \tau_1^2 \rangle \times \langle \tau_2 \mid \tau_2^2 \rangle$. Hence, $\langle \varrho \rangle$ has index 4 in G and $b = 4$. We have $p = 0$ and $q = 0$. Since $\Sigma_{g',b,q}$ has rank $2g' + b - 1 + q$ and H has rank 5, we have $2g' + 4 - 1 + 0 = 5$ and $g' = 1$. Hence, $\mathcal{AM}_{1,1,0} \hookrightarrow \mathcal{AM}_{1,1,3}$. We take the following basis of $\Sigma_{1,1,3}$: $\hat{x} = x^2$, $\hat{y} = y^2$, $\hat{t}_1 = (\bar{y} \bar{y} x x)^{\bar{x}} \bar{y}^2 x^2 y^2$, $\hat{t}_2 = (\bar{y} \bar{y} x x)^y$, $\hat{t}_3 = (\bar{y} \bar{y} x x)^{xy}$. It is well-known that $\mathcal{AM}_{1,1,0} = \langle \alpha, \beta \mid \alpha\beta\alpha = \beta\alpha\beta \rangle$, where

$$\alpha := \begin{cases} x & \mapsto \bar{y}x, \\ y & \mapsto y, \end{cases} \quad \beta := \begin{cases} x & \mapsto x, \\ y & \mapsto xy. \end{cases}$$

A straightforward computation shows that the image of α and β in $\mathcal{AM}_{1,1,3}$, denoted $\hat{\alpha}$ and $\hat{\beta}$, are

$$\hat{\alpha} := \begin{cases} \hat{x} & \mapsto \hat{y}^{-1} \hat{x} \hat{y} \hat{t}_2 \hat{t}_3 \hat{t}_2^{-1} \hat{y}^{-1}, \\ \hat{y} & \mapsto \hat{y}, \\ \hat{t}_1 & \mapsto \hat{t}_3^{\hat{t}_2^{-1} \hat{y}^{-1} \hat{x}^{-1} \hat{y}^{-1} \hat{x} \hat{y} \hat{t}_2 \hat{t}_3 \hat{t}_2^{-1}}, \\ \hat{t}_2 & \mapsto \hat{t}_2, \\ \hat{t}_3 & \mapsto \hat{t}_1^{\hat{t}_2 \hat{t}_3}, \end{cases} \quad \hat{\beta} := \begin{cases} \hat{x} & \mapsto \hat{x}, \\ \hat{y} & \mapsto \hat{x} \hat{y} \hat{t}_2, \\ \hat{t}_1 & \mapsto \hat{t}_1^{\hat{y}^{-1} \hat{x}^{-1} \hat{y} \hat{t}_2^{-1} \hat{y}^{-1} \hat{x} \hat{y} \hat{t}_2}, \\ \hat{t}_2 & \mapsto \hat{t}_3, \\ \hat{t}_3 & \mapsto \hat{t}_2^{\hat{y}^{-1} \hat{x} \hat{y} \hat{t}_2 \hat{t}_3}. \end{cases}$$

Example 3. Let $F_3 := \langle a_1, a_2, a_3 \mid \rangle$. Let H be the kernel of the homomorphism $F_3 \rightarrow \langle \tau_1 \mid \tau_1^2 \rangle \times \langle \tau_2 \mid \tau_2^2 \rangle \times \langle \tau_3 \mid \tau_3^2 \rangle$ such that $a_k \mapsto \tau_k$ for all $1 \leq k \leq 3$. It is standard to see that H is a free group of rank 17. It can be shown that H is a characteristic subgroup of F_3 .

- (a). We identify $\Sigma_{0,1,3}$ with F_3 by putting $t_k \leftrightarrow a_k$ for all $1 \leq k \leq 3$. Notice that $\varrho = t_1 t_2 t_3 H$ has order 2 in $G := \Sigma_{0,1,3}/H \simeq \langle \tau_1 \mid \tau_1^2 \rangle \times \langle \tau_2 \mid \tau_2^2 \rangle \times \langle \tau_3 \mid \tau_3^2 \rangle$. Hence, $\langle \varrho \rangle$ has index 4 in G and $b = 4$. On the other hand, for all $1 \leq k \leq 3$, $\varrho_k = t_k H$ has order 2 in G . Hence, for all $1 \leq k \leq 3$, $\langle \varrho_k \rangle$ has index 4 in G and the k -th puncture in $S_{0,1,3}$ lifts to 4 punctures in $S_{g',b,q}$. Thus, $q = 12$. Since $\Sigma_{g',b,q}$ has rank $2g' + b - 1 + q$ and H has rank 17, we have $2g' + 4 - 1 + 12 = 17$ and $g' = 1$. Hence, $\mathcal{AM}_{0,1,3} \hookrightarrow \mathcal{AM}_{1,1,3}$.
- (b). We identify $\Sigma_{1,1,1}$ with F_3 by putting $x \leftrightarrow a_1$, $y \leftrightarrow a_2$ and $t \leftrightarrow a_3$. Notice that $\varrho = [x, y]tH$ has order 2 in $G := \Sigma_{1,1,1}/H \simeq \langle \tau_1 \mid \tau_1^2 \rangle \times \langle \tau_2 \mid \tau_2^2 \rangle \times \langle \tau_3 \mid \tau_3^2 \rangle$. Hence, $\langle \varrho \rangle$ has index 4 in G and $b = 4$. On the other hand, $\varrho_1 = tH$ has order 2 in G . Hence, $\langle \varrho_1 \rangle$ has index 4 in G and the puncture in $S_{1,1,1}$ lifts to 4 punctures in $S_{g',b,q}$. Thus, $q = 4$. Since $\Sigma_{g',b,q}$ has rank $2g' + b - 1 + q$ and H has rank 17, we have $2g' + 4 - 1 + 4 = 17$ and $g' = 5$. Hence, $\mathcal{AM}_{1,1,1} \hookrightarrow \mathcal{AM}_{5,1,3}$.

4 Proof of Theorem 2.6

4.1 Definition. An element of $\Sigma_{g,1,p}$ is said to be *t-squarefree* if, in its reduced expression, no two consecutive terms in $\{t_k, \bar{t}_k\}_{1 \leq k \leq p}$ are equal; for example: $x_1 x_1 t_2 t_3$ is *t-squarefree*; $x_1 t_2 t_2 y_1$ is non-*t-squarefree*.

To proof Theorem 2.6 we need the following theorem.

4.2 Theorem. For every $\phi \in \mathcal{AM}_{g,1,p}$, the elements of $\{x_i^\phi, y_i^\phi\}_{1 \leq i \leq g} \cup \{t_k^\phi\}_{1 \leq k \leq p}$ are *t-squarefree*.

Proof. (of Theorem 2.6) If $p = 0$, then ψ is the identity and nothing needs to be said.

Suppose $p \geq 1$. Recall $\Sigma_{g,1,p^{(d)}} \simeq \Sigma_{g,1,0} * \langle \tau_1, \tau_2, \dots, \tau_p \mid \tau_1^d, \tau_2^d, \dots, \tau_p^d \rangle$. Let $a \in \{x_i, y_i\}_{1 \leq i \leq g} \cup \{t_k\}_{1 \leq k \leq p}$. If ϕ is an element of the kernel of $\psi : \mathcal{AM}_{g,1,p} \rightarrow \mathcal{AM}_{g,1,p^{(d)}}$, then a^ϕ and a have the same image in $\Sigma_{g,1,p^{(d)}}$. On the other hand, by Theorem 4.2, a^ϕ is *t-squarefree*. Hence, a^ϕ has the same normal form in $\Sigma_{g,1,p}$ as in $\Sigma_{g,1,p^{(d)}}$. Thus, $a^\phi = a$. \square

The rest of the paper is dedicated to proof Theorem 4.2. The proof is similar to [3, 7.6 Corollary]. Notice Theorem 4.2 is trivial if $p = 0$.

5 McCool's Groupoid

For the rest of the paper we suppose $p \geq 1$.

Let $n := 2g + p$, and, let F_n be the free group on X , where X is a set with n elements.

5.1 Notation. Let $w \in F_n$. In this section we will denote by $[w]$ the cyclic word of w .

5.2 Definitions. Let T be a set of words and cyclic words of F_n . Suppose the elements of T are reduced and cyclically reduced, respectively. We define the *Whitehead graph of T* as the graph with vertex set $X \cup \bar{X}$, and, one edge from $a \in X \cup \bar{X}$ to $b \in X \cup \bar{X}$ for every subword $\bar{a}b$ which appears in w or $[u]$, where w and $[u]$ are elements of T . We say that a is the initial vertex and b is the terminal vertex of the edges corresponding to the subword $\bar{a}b$. Repetitions have to be considered. For example, since the subword $\bar{a}b$ appears twice in $\bar{a}b\bar{a}b$, the Whitehead graph of $\{\bar{a}b\bar{a}b\}$ has 2 edges from a to b (and one edge from \bar{b} to \bar{a}). Notice that the cyclic word $[a]$ produces an edge from \bar{a} to a in the Whitehead graph.

We say that T is a *surface word set* if the Whitehead graph of T is an oriented segment, that is, the Whitehead graph of T is connected with exactly $2n - 1$ edges, every vertex but one is the *initial vertex* of exactly one edge, and, every vertex but one is the *terminal vertex* of exactly one edge.

5.3 Example. Let $F_4 := \langle a, b, c, d \mid \rangle$.

(i). Let $T := \{\overline{adc\bar{b}}, [\overline{db}], [\overline{ca}]\}$. The Whitehead graph of T is

$$\bar{a} \rightarrow \bar{c} \rightarrow \bar{b} \rightarrow \bar{d} \rightarrow c \rightarrow a \rightarrow d \rightarrow b.$$

Hence, T is a surface word set.

(ii). Let $T := \{\overline{adc\bar{b}}, \overline{db}, [\overline{ca}]\}$. The Whitehead graph of T is

$$\bar{a} \rightarrow \bar{c} \rightarrow \bar{b} \quad \bar{d} \rightarrow c \rightarrow a \rightarrow d \rightarrow b.$$

Hence, T is not a surface word set.

(iii). Let $T := \{\overline{adc\bar{b}}, dc, [\overline{db}], [\overline{ca}]\}$. The Whitehead graph of T is

$$\bar{a} \rightarrow \bar{c} \rightarrow \bar{b} \rightarrow \bar{d} \rightrightarrows c \rightarrow a \rightarrow d \rightarrow b.$$

Hence, T is not a surface word set.

We illustrate the following remarks with examples in $F_4 = \langle a, b, c, d \mid \rangle$.

5.4 Remarks. Let T be a surface word set.

(i) The Whitehead graph of T defines a sequence $(a_k)_{1 \leq k \leq 2n}$ which lists the element of $X \cup \overline{X}$ such that for all $1 \leq k \leq (2n - 1)$, the Whitehead graph of T has exactly one edge with initial vertex a_k and terminal vertex a_{k+1} , equivalently, $\overline{a_k a_{k+1}}$ is a subword of exactly one element of T . We say that $(a_k)_{1 \leq k \leq 2n}$ is the *associated sequence* of T .

In Example 5.3(i), the associated sequence of T is $(\bar{a}, \bar{c}, \bar{b}, \bar{d}, c, a, d, b)$.

(ii) We can recover T from the associated sequence of T . The process to recover T from its associated sequence is the invers process to construct the Whitehead graph. We give two examples below. From this process, it is easy to see that T has exactly one word, and, all other elements of T are cyclic words.

In F_4 , from the sequence $(a, b, c, d, \bar{a}, \bar{b}, \bar{c}, \bar{d})$ we have the surface word set $\{\overline{abcd\bar{a}\bar{b}\bar{c}\bar{d}}\}$, and, from the sequence $(a, b, c, d, \bar{d}, \bar{c}, \bar{b}, \bar{a})$ we have the surface word set $\{a, [\overline{b\bar{a}}], [\overline{c\bar{b}}], [\overline{d\bar{c}}], [\overline{d}]\}$.

(iii) Let p be the cardinality of T minus one. We say that T is a (g, p) -surface word set, where $g = (n - p)/2$. By induction on n , it can be seen that $n \geq p$ and $n - p$ is even. Hence, g is a non-negative integer.

5.5 Definition. Let $\phi \in \text{Aut}(F_n)$.

We say that ϕ is a *type-1 Nielsen automorphism* if ϕ restricts to a permutation of $X \cup \overline{X}$.

We say that ϕ is a *type-2 Nielsen automorphism* if there exists $a, b \in X \cup \overline{X}$ such that $a \neq b, \overline{b}$ and

$$\phi := \begin{cases} a \mapsto ab, \\ c \mapsto c \quad \text{for all } c \in X, c \neq a^{\pm 1}. \end{cases}$$

We denote ϕ by $(a \mapsto ab)$ or $(\overline{a} \mapsto \overline{b}\overline{a})$.

5.6 Definition. Let $\mathcal{G}_{g,p}$ be the groupoid with objects (g, p) -surface word sets, and, given T_1, T_2 two (g, p) -surface word sets

$$\text{Hom}(T_1, T_2) := \{\phi \in \text{Aut}(F_n) \mid T_1^\phi = T_2\},$$

where $T_1^\phi := \{w^\phi, [u^\phi] \mid w, [u] \in T_1\}$. Here, w^ϕ is reduced and $[u^\phi]$ is cyclically reduced. Hence, $[v] = [u^\phi]$ means that v and u^ϕ are conjugated.

We say that $(T_1, T_2, \phi) \in \text{Hom}(T_1, T_2)$ is a *type-1 Nielsen* of $\mathcal{G}_{g,p}$ if ϕ is a type-1 Nielsen automorphism. Similarly, for type-2 Nielsen automorphisms. We say that $(T_1, T_2, \phi) \in \text{Hom}(T_1, T_2)$ is a Nielsen if it is either a type-1 Nielsen or a type-2 Nielsen.

We illustrate the following remarks with examples in $F_4 = \langle a, b, c, d \mid \rangle$.

5.7 Remark. Let (T_1, T_2, ϕ) be a Nielsen of $\mathcal{G}_{g,p}$.

- (i) If (T_1, T_2, ϕ) is a type-1 Nielsen, then the associated sequence of T_2 is obtained from the associated sequence of T_1 by applying the permutation ϕ to every element of the sequence.

In F_4 , let $T_1 = \{a\overline{d}\overline{b}c, [\overline{a}b], [\overline{c}d]\}$. Notice the associated sequence of T_1 is $(a, b, c, d, \overline{b}, \overline{a}, \overline{d}, \overline{c})$. If $\phi := (a \mapsto \overline{b}, b \mapsto c, c \mapsto \overline{a}, d \mapsto \overline{d})$, then the associated sequence of T_2 is $(\overline{b}, c, \overline{a}, \overline{d}, \overline{c}, b, d, a)$.

- (ii) Suppose (T_1, T_2, ϕ) is a type-2 Nielsen. Then $\phi = (a_i \mapsto ba_i)$ for some $1 \leq i \leq 2n$, $b \in X \cup \overline{X}$ such that $a_i \neq b, \overline{b}$. Since in the Whitehead graph of T there are exactly $2n-1$ edges, there exists $w \in T_1$ or $[u] \in T_1$ such that applying ϕ to w or $[u]$ produces a cancellation. If the cancellation appears from the subword $\overline{a}_{i-1}a_i$, then $b = a_{i-1}$. If the cancellation appears from the subword $\overline{a}_i a_{i+1}$, then $b = a_{i+1}$. Hence, either $\phi = (a_i \mapsto a_{i-1}a_i)$ for some $2 \leq i \leq 2n$, $a_i \neq \overline{a}_{i-1}$; or $\phi = (\overline{a}_i \mapsto \overline{a}_i \overline{a}_{i+1})$ for some $1 \leq i \leq (2n-1)$, $a_i \neq \overline{a}_{i+1}$. In the former case the associated sequence of T_2 is obtained from the associated sequence of T_1 by moving a_i from immediately after a_{i-1} to immediately before \overline{a}_{i-1} . In the later case the associated sequence of T_2 is obtained from the associated sequence of T_1 by moving a_i from immediately before a_{i+1} to immediately after \overline{a}_{i+1} .

In F_4 , let $T_1 = \{\overline{abcd}\overline{ab\overline{cd}}\}$. Notice the associated sequence of T_1 is $(a, b, c, d, \overline{a}, \overline{b}, \overline{c}, \overline{d})$. If $\phi := (b \mapsto ab)$, then the associated sequence of T_2 is $(a, c, d, b, \overline{a}, \overline{b}, \overline{c}, \overline{d})$. In fact $(\overline{abcd}\overline{ab\overline{cd}})^{(b \mapsto ab)} = \overline{ab}\overline{acdb\overline{cd}}$. If $\phi := (\overline{a} \mapsto \overline{ab})$, then the associated sequence of T_2 is $(b, c, d, \overline{a}, \overline{b}, a, \overline{c}, \overline{d})$. In fact $(\overline{abcd}\overline{ab\overline{cd}})^{(\overline{a} \mapsto \overline{ab})} = \overline{babc\overline{d}}\overline{a}\overline{cd}$.

5.8 Remark. It is easy to see that $\{\Pi_{i=1}^g[x_i, y_i]\Pi_{k=1}^p t_k, [\overline{t}_1], [\overline{t}_2], \dots, [\overline{t}_p]\}$ is a (g, p) -surface word set of $\Sigma_{g,1,p}$. Its associated sequence is

$$(\overline{x}_1, y_1, x_1, \overline{y}_1, \overline{x}_2, y_2, x_2, \overline{y}_2, \dots, \overline{x}_g, y_g, x_g, \overline{y}_g, t_1, \overline{t}_1, t_2, \overline{t}_2, \dots, t_p, \overline{t}_p).$$

We say that $\{\Pi_{i=1}^g[x_i, y_i]\Pi_{k=1}^p t_k, [\overline{t}_1], [\overline{t}_2], \dots, [\overline{t}_p]\}$ is the standard (g, p) -surface word set of $\Sigma_{g,1,p}$.

5.9 Remark. $\mathcal{AM}_{g,1,p} = \text{Hom}(T, T)$, where T is the standard (g, p) -surface word set of $\Sigma_{g,1,p}$.

5.10 Theorem (McCool [15],[9]). $\mathcal{G}_{g,p}$ is generated by Nielsen elements.

6 Ends of free group

Let $n := 2g + p$ and let F_n be the free group on X , where $|X| = n$.

6.1 Notation. Let $\Pi_{i=1}^k a_i$ be the normal form for $w \in F_n$. Then we say that w has length k , denoted $|w| = k$. The set of elements of F_n whose normal forms have $\Pi_{i=1}^k a_i$ as an initial subword is denoted $(w\star)$; and, the set of elements of F_n whose normal forms have $\Pi_{i=1}^k a_i$ as a terminal subword is denoted $(\star w)$. The elements of $(w\star)$ are said to *begin with* w , and the elements of $(\star w)$ are said to *end with* w .

6.2 Review. An *end* of F_n is a sequence $(a_k)_{k \geq 1}$ in $X \cup \overline{X}$ such that, for each $k \geq 1$, $a_{k+1} \neq \overline{a}_k$. We represent $(a_k)_{k \geq 1}$ as a formal right-infinite reduced product, $\Pi_{k \geq 1} a_k = a_1 a_2 \dots$.

We denote the set of ends of F_n by ∂F_n .

For each $w \in F_n$, we define the *shadow* of w in ∂F_n to be

$$(w\blacktriangleleft) := \{(a_k)_{k \geq 1} \in \partial F_n \mid \Pi_{k=1}^{|w|} a_k = w\}.$$

Thus, for example, $(1\blacktriangleleft) = \partial F_n$.

6.3 Definition. Let T be a surface word set. We now give ∂F_n an ordering, $<_T$, with respect to T as follows. Let $(a_k)_{1 \leq k \leq 2n}$ be the associated sequence of T . Recall $(a_k)_{1 \leq k \leq 2n}$ is a listing of the elements of $X \cup \overline{X}$. For each $w \in F_n$, we assign an ordering, $<_T$, to a partition of $(w\blacktriangleleft)$ into $2n$ or $2n - 1$ subsets, depending as $w = 1$ or $w \neq 1$, as follows. We set

$$(a_1\blacktriangleleft) <_T (a_2\blacktriangleleft) <_T (a_3\blacktriangleleft) <_T \dots <_T (a_{2n-1}\blacktriangleleft) <_T (a_{2n}\blacktriangleleft).$$

If $1 \leq i \leq n$ and $w \in (\star \bar{a}_i)$, then we set

$$\begin{aligned} (wa_{i+1} \blacktriangleleft) &<_T (wa_{i+2} \blacktriangleleft) <_T (wa_{i+3} \blacktriangleleft) <_T \cdots \\ \cdots &<_T (wa_{2n-1} \blacktriangleleft) <_T (wa_{2n} \blacktriangleleft) <_T (wa_1 \blacktriangleleft) <_T (wa_2 \blacktriangleleft) <_T (wa_3 \blacktriangleleft) <_T \cdots \\ \cdots &<_T (wa_{i-2} \blacktriangleleft) <_T (wa_{i-1} \blacktriangleleft). \end{aligned}$$

Hence, for each $w \in F_n$, we have an ordering $<_T$ of a partition of $(w \blacktriangleleft)$ into $2n$ or $2n - 1$ subsets.

If $(b_k)_{k \geq 1}$ and $(c_k)_{k \geq 1}$ are two different ends, then there exists $j \in \mathbb{Z}, j \geq 0$, such that $(b_k)_{1 \leq k \leq j} = (c_k)_{1 \leq k \leq j}$ and $b_{j+1} \neq c_{j+1}$. Let $w = \prod_{k=1}^j b_k = \prod_{k=1}^j c_k$ in F_n . Then $(b_k)_{k \geq 1}$ and $(c_k)_{k \geq 1}$ lie in $(w \blacktriangleleft)$, but lie in different elements of the partition of $(w \blacktriangleleft)$ into $2n$ or $2n - 1$ subsets. We then order $(b_k)_{k \geq 1}$ and $(c_k)_{k \geq 1}$ using the order of the elements of the partition of $(w \blacktriangleleft)$ that they belong to. This completes the definition of the ordering $<_T$ of ∂F_n .

6.4 Remark. Let w be the non-cyclic element of T . In $(\partial F_n, <_T)$, the smallest element is w^∞ and the largest element is \bar{w}^∞ .

For example, in $F_4 = \langle a, b, c, d \mid \rangle$ we take the surface word set $T = \{a\bar{d} \bar{b}c, [\bar{a}b], [\bar{c}d]\}$. The associated sequence of T is $(a, b, c, d, \bar{b}, \bar{a}, \bar{d}, \bar{c})$. In $(\partial F_4, <_T)$, the smallest element is $(a\bar{d} \bar{b}c)^\infty$, and, the largest element is $(\bar{c}b\bar{d}\bar{a})^\infty$.

6.5 Notation. We denote by $<$ the order on $\partial \Sigma_{g,1,p}$ with respect to the standard (g, p) -surface word set of $\Sigma_{g,1,p}$.

6.6 Review. Let \hat{S} be the universal cover of $S_{g,1,p}$. Suppose $S_{g,1,p}$ has negative Euler characteristic, that is, $2g + p \geq 2$. Then \hat{S} can be identified with a region of the hyperbolic plane. Let $\partial \hat{S}$ be the boundary of \hat{S} . It is well-known that $\partial \hat{S}$ can be identified with $\mathbb{R} \cup \{\infty\}$. Let $*$ be the point in $\partial \hat{S}$ corresponding to ∞ by this identification. The identification between $\partial \hat{S}$ and $\mathbb{R} \cup \{\infty\}$ restricts to an identification between $\partial \hat{S} - \{*\}$ and \mathbb{R} . By work of Nielsen-Thurston [5], [16], there is an action of $\mathcal{M}_{g,1,p}$ on $\partial \hat{S}$ with a fixed point, which we can suppose to be $*$ $\in \partial \hat{S}$. Hence, there exists an action of $\mathcal{M}_{g,1,p}$ on \mathbb{R} . By [16], this action preserves the usual order of \mathbb{R} . Remark 5.9 and Proposition 6.7 give the analog statement for $\mathcal{AM}_{g,1,p}$ and $\partial \Sigma_{g,1,p}$.

Let $\phi \in \text{Aut}(F_n)$. It is proved in [5] that $(\prod_{k \geq 1} a_k)^\phi = \lim_{k \rightarrow \infty} (\prod_{i=1}^k a_i)^\phi$ defines a map $\partial F_n \rightarrow \partial F_n$, which we still denote by ϕ .

6.7 Proposition. *Let T_1, T_2 be surface word sets of F_n and $(T_1, T_2, \phi) \in \text{Hom}(T_1, T_2)$. Then $\phi : (\partial F_n, \leq_{T_1}) \rightarrow (\partial F_n, \leq_{T_2})$ respects the orderings.*

Proof. By Theorem 5.10, we can restrict ourselves to the case where (T_1, T_2, ϕ) is a Nielsen.

By Remark 5.7(i), the result is clear if (T_1, T_2, ϕ) is a type-1 Nielsen. Hence, we suppose (T_1, T_2, ϕ) is a type-2 Nielsen.

Let $(a_k)_{1 \leq k \leq 2n}$ be the associated sequence of T_1 . Then either $\phi = (a_i \mapsto a_{i-1}a_i)$ for some $2 \leq i \leq 2n$, $a_i \neq \bar{a}_{i-1}$; or, $\phi = (\bar{a}_i \mapsto \bar{a}_i\bar{a}_{i+1})$ for some $1 \leq i \leq (2n-1)$, $a_i \neq \bar{a}_{i+1}$.

Suppose $\phi = (a_i \mapsto a_{i-1}a_i)$ for some $2 \leq i \leq 2n$, $a_i \neq \bar{a}_{i-1}$.

The following correspondence by the action of $(a_i \mapsto a_{i-1}a_i)$ is clear.

$$\begin{array}{ccc}
& (a_i \mapsto a_{i-1}a_i) & \\
(\star \bar{a}_i a_{i-1}) & \longrightarrow & (\star \bar{a}_i), \\
(\star a_{i-1}) - (\star \bar{a}_i a_{i-1}) & \longrightarrow & (\star a_{i-1}), \\
(\star a_i) & \longrightarrow & (\star a_i), \\
(\star a_k) & \longrightarrow & (\star a_k), \quad a_k \neq a_{i-1}^{\pm 1}, a_i^{\pm 1}, \\
(\star \bar{a}_{i-1}) & \longrightarrow & (\star \bar{a}_{i-1}) - (\star \bar{a}_i \bar{a}_{i-1}), \\
(\star \bar{a}_i) & \longrightarrow & (\star \bar{a}_i \bar{a}_{i-1}).
\end{array}$$

The following correspondence by the action of $(a_i \mapsto a_{i-1}a_i)$ is clear.

$$\begin{array}{ccc}
& (a_i \mapsto a_{i-1}a_i) & \\
(a_{i-1} \blacktriangleleft) & \longrightarrow & (a_{i-1} \blacktriangleleft) - (a_{i-1}a_i \blacktriangleleft), \\
(a_i \blacktriangleleft) & \longrightarrow & (a_{i-1}a_i \blacktriangleleft), \\
(a_k \blacktriangleleft) & \longrightarrow & (a_k \blacktriangleleft), \quad a_k \neq a_{i-1}^{\pm 1}, a_i^{\pm 1}, \\
(\bar{a}_{i-1}a_i \blacktriangleleft) & \longrightarrow & (a_i \blacktriangleleft), \\
(\bar{a}_{i-1} \blacktriangleleft) - (\bar{a}_{i-1}a_i \blacktriangleleft) & \longrightarrow & (\bar{a}_{i-1} \blacktriangleleft), \\
(\bar{a}_i \blacktriangleleft) & \longrightarrow & (\bar{a}_i \blacktriangleleft).
\end{array}$$

From the first row of the first table and the second table we deduce the following table.

$$\begin{array}{ccc}
& (a_i \mapsto a_{i-1}a_i) & \\
(\star \bar{a}_i a_{i-1})(a_{i-1} \blacktriangleleft) & \longrightarrow & (\star \bar{a}_i)[(a_{i-1} \blacktriangleleft) - (a_{i-1}a_i \blacktriangleleft)], \\
(\star \bar{a}_i a_{i-1})(a_i \blacktriangleleft) & \longrightarrow & (\star \bar{a}_i)(a_{i-1}a_i \blacktriangleleft), \\
(\star \bar{a}_i a_{i-1})(a_k \blacktriangleleft) & \longrightarrow & (\star \bar{a}_i)(a_k \blacktriangleleft), \quad a_k \neq a_{i-1}^{\pm 1}, a_i^{\pm 1}, \\
(\star \bar{a}_i a_{i-1})(\bar{a}_i \blacktriangleleft) & \longrightarrow & (\star \bar{a}_i)(\bar{a}_i \blacktriangleleft).
\end{array}$$

Notice the cases $(\star \bar{a}_i a_{i-1})(\bar{a}_{i-1}a_i \blacktriangleleft)$ and $(\star \bar{a}_i a_{i-1})[(\bar{a}_{i-1} \blacktriangleleft) - (\bar{a}_{i-1}a_i \blacktriangleleft)]$ do not have to be considered since they are not in reduced form.

Let $\mathbf{e}, \mathbf{f} \in \partial F_n$ such that $\mathbf{e} = (w\bar{a}_i a_{i-1})\mathbf{e}'$, $\mathbf{f} = (w\bar{a}_i a_{i-1})\mathbf{f}'$ and the first letter of \mathbf{e}' is different from the first letter of \mathbf{f}' . Let $1 \leq j \leq 2n$ such that $a_j = \bar{a}_{i-1}$. By the third table, $\mathbf{e}^{(a_i \mapsto a_{i-1}a_i)} = (u\bar{a}_i)\mathbf{e}''$, $\mathbf{f}^{(a_i \mapsto a_{i-1}a_i)} = (u\bar{a}_i)\mathbf{f}''$ in reduced form. Let $(b_k)_{1 \leq k \leq 2n}$ be the associated sequence of T_2 . Recall $(b_k)_{1 \leq k \leq 2n}$ is obtained from $(a_k)_{1 \leq k \leq 2n}$ by moving a_i from immediately after a_{i-1} to immediately before $a_j = \bar{a}_{i-1}$. There are two cases according to $j < i-1$ or $i-1 < j$.

If $j < i - 1$, then

$$\begin{aligned} (b_k)_{1 \leq k \leq (j-1)} &= (a_k)_{1 \leq k \leq (j-1)}, \\ b_j &= a_i, \\ (b_k)_{(j+1) \leq k \leq i} &= (a_k)_{j \leq k \leq (i-1)}, \\ (b_k)_{(i+1) \leq k \leq 2n} &= (a_k)_{(i+1) \leq k \leq 2n}. \end{aligned}$$

The partition with respect to $(a_k)_{1 \leq k \leq 2n}$ of $(\bar{a}_j \blacktriangleleft) = (a_{i-1} \blacktriangleleft)$ is $(a_{j+1} \blacktriangleleft), (a_{j+2} \blacktriangleleft), \dots, (a_{i-1} \blacktriangleleft), (a_i \blacktriangleleft), (a_{i+1} \blacktriangleleft), \dots, (a_{2n} \blacktriangleleft), (a_1 \blacktriangleleft), (a_2 \blacktriangleleft), \dots, (a_{j-1} \blacktriangleleft)$. The partition with respect to $(b_k)_{1 \leq k \leq 2n}$ of $(\bar{a}_i \blacktriangleleft)$ is $(a_j \blacktriangleleft), (a_{j+1} \blacktriangleleft), \dots, (a_{i-1} \blacktriangleleft), (a_{i+1} \blacktriangleleft), (a_{i+2} \blacktriangleleft), \dots, (a_{2n} \blacktriangleleft), (a_1 \blacktriangleleft), (a_2 \blacktriangleleft), \dots, (a_{j-1} \blacktriangleleft)$. By the third table,

$$\begin{array}{ccc} & (a_i \mapsto a_{i-1}a_i) & \\ (w\bar{a}_i a_{i-1})(a_{j+1} \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_{j+1} \blacktriangleleft), \\ (w\bar{a}_i a_{i-1})(a_{j+2} \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_{j+2} \blacktriangleleft), \\ & \vdots & \\ (w\bar{a}_i a_{i-1})(a_{i-2} \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_{i-2} \blacktriangleleft), \\ (w\bar{a}_i a_{i-1})(a_{i-1} \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)[(a_{i-1} \blacktriangleleft) - (a_{i-1}a_i \blacktriangleleft)], \\ (w\bar{a}_i a_{i-1})(a_i \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_{i-1}a_i \blacktriangleleft), \\ (w\bar{a}_i a_{i-1})(a_{i+1} \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_{i+1} \blacktriangleleft), \\ & \vdots & \\ (w\bar{a}_i a_{i-1})(a_{2n} \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_{2n} \blacktriangleleft), \\ (w\bar{a}_i a_{i-1})(a_1 \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_1 \blacktriangleleft), \\ (w\bar{a}_i a_{i-1})(a_2 \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_2 \blacktriangleleft), \\ & \vdots & \\ (w\bar{a}_i a_{i-1})(a_{j-1} \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_{j-1} \blacktriangleleft). \end{array}$$

Since $a_j = \bar{a}_{i-1}$, the first column is ordered with respect to T_1 . On the other hand, $a_j = \bar{a}_{i-1}$ implies that the partition of $(u\bar{a}_i)(a_{i-1} \blacktriangleleft)$ with respect to T_2 ends with $(u\bar{a}_i)(a_{i-1}a_i \blacktriangleleft)$. Then, the second column of this table is ordered with respect to T_2 . Hence, if $(w\bar{a}_i a_{i-1})\mathbf{e}' <_{T_1} (w\bar{a}_i a_{i-1})\mathbf{f}'$ then $(u\bar{a}_i)\mathbf{e}'' <_{T_2} (u\bar{a}_i)\mathbf{f}''$.

If $i - 1 < j$, then

$$\begin{aligned} (b_k)_{1 \leq k \leq (i-1)} &= (a_k)_{1 \leq k \leq (i-1)} \\ (b_k)_{i \leq k \leq (j-2)} &= (a_k)_{(i+1) \leq k \leq (j-1)} \\ b_{j-1} &= a_i \\ (b_k)_{j \leq k \leq 2n} &= (a_k)_{j \leq k \leq 2n} \end{aligned}$$

The partition with respect to $(a_k)_{1 \leq k \leq 2n}$ of $(\bar{a}_j \blacktriangleleft) = (a_{i-1} \blacktriangleleft)$ is $(a_{j+1} \blacktriangleleft), (a_{j+2} \blacktriangleleft), \dots, (a_{2n} \blacktriangleleft), (a_1 \blacktriangleleft), (a_2 \blacktriangleleft), \dots, (a_{i-1} \blacktriangleleft), (a_i \blacktriangleleft), (a_{i+1} \blacktriangleleft), \dots, (a_{j-1} \blacktriangleleft)$. The partition with respect to $(b_k)_{1 \leq k \leq 2n}$ of $(\bar{a}_i \blacktriangleleft)$ is $(a_j \blacktriangleleft), (a_{j+1} \blacktriangleleft), \dots, (a_{2n} \blacktriangleleft),$

$(a_1 \blacktriangleleft), (a_2 \blacktriangleleft), \dots, (a_{i-1} \blacktriangleleft), (a_{i+1} \blacktriangleleft), (a_{i+2} \blacktriangleleft), \dots, (a_{j-1} \blacktriangleleft)$. By the third table,

$$\begin{array}{ccc}
& & (a_i \mapsto a_{i-1}a_i) \\
(w\bar{a}_i a_{i-1})(a_{j+1} \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_{j+1} \blacktriangleleft), \\
(w\bar{a}_i a_{i-1})(a_{j+2} \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_{j+2} \blacktriangleleft), \\
& \vdots & \\
(w\bar{a}_i a_{i-1})(a_{2n} \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_{2n} \blacktriangleleft), \\
(w\bar{a}_i a_{i-1})(a_1 \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_1 \blacktriangleleft), \\
(w\bar{a}_i a_{i-1})(a_2 \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_2 \blacktriangleleft), \\
& \vdots & \\
(w\bar{a}_i a_{i-1})(a_{i-2} \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_{i-2} \blacktriangleleft), \\
(w\bar{a}_i a_{i-1})(a_{i-1} \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)[(a_{i-1} \blacktriangleleft) - (a_{i-1}a_i \blacktriangleleft)], \\
(w\bar{a}_i a_{i-1})(a_i \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_{i-1}a_i \blacktriangleleft), \\
(w\bar{a}_i a_{i-1})(a_{i+1} \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_{i+1} \blacktriangleleft), \\
& \vdots & \\
(w\bar{a}_i a_{i-1})(a_{j-1} \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_{j-1} \blacktriangleleft).
\end{array}$$

Since $a_j = \bar{a}_{i-1}$, the first column is ordered with respect to T_1 . On the other hand, $a_j = \bar{a}_{i-1}$ implies that the partition of $(u\bar{a}_i)(a_{i-1} \blacktriangleleft)$ with respect to T_2 ends with $(u\bar{a}_i)(a_{i-1}a_i \blacktriangleleft)$. Then, the second column of this table is ordered with respect to T_2 . Hence, if $(w\bar{a}_i a_{i-1})\mathbf{e}' <_{T_1} (w\bar{a}_i a_{i-1})\mathbf{f}'$ then $(u\bar{a}_i)\mathbf{e}'' <_{T_2} (u\bar{a}_i)\mathbf{f}''$.

For every row of the first table, there is a case which needs to be considered. Similarly, in all these cases, it can be shown that if $\mathbf{e} <_{T_1} \mathbf{f}$, then $\mathbf{e}^{(a_i \mapsto a_{i-1}a_i)} <_{T_2} \mathbf{f}^{(a_i \mapsto a_{i-1}a_i)}$.

The case $\phi = (\bar{a}_i \mapsto \bar{a}_i \bar{a}_{i+1})$ for some $1 \leq i \leq (2n - 1)$, $a_i \neq \bar{a}_{i+1}$, is similar. \square

7 t -squarefreeness

Recall $2g + p = n$ and $\Sigma_{g,1,p}$ is the free group on $\{x_i, y_i\}_{1 \leq i \leq g} \cup \{t_k\}_{1 \leq k \leq p}$.

The following definition extends Definition 4.1 to $\Sigma_{g,1,p} \cup \partial\Sigma_{g,1,p}$.

7.1 Definition. An element of $\Sigma_{g,1,p} \cup \partial\Sigma_{g,1,p}$ is said to be t -squarefree if, in its reduced expression, no two consecutive terms in $\{t_k, \bar{t}_k\}_{1 \leq k \leq p}$ are equal.

7.2 Notation. Recall that if G is a group and $g_1, g_2, \dots, g_k \in G$, then $\prod_{i=1}^k g_i = g_1 g_2 \cdots g_k$. We introduce the notation $\prod_1^{i=k} g_i = g_k g_{k-1} \cdots g_1$.

In the standard surface word set, we denote

$$\begin{aligned}
\bar{z}_1 &= \prod_{i=1}^g [x_i, y_i] \prod_{k=1}^p t_k = [x_1, y_1][x_2, y_2] \cdots [x_g, y_g] t_1 t_2 \cdots t_p, \\
z_1 &= \prod_1^{k=p} \bar{t}_k \prod_1^{i=g} [y_i, x_i] = \bar{t}_p \bar{t}_{p-1} \cdots \bar{t}_1 [y_g, x_g][y_{g-1}, x_{g-1}] \cdots [y_1, x_1].
\end{aligned}$$

From Remark 6.4, the smallest element of $(\partial\Sigma_{g,1,p}, <)$ is \bar{z}_1^∞ and the largest element of $(\partial\Sigma_{g,1,p}, <)$ is z_1^∞ . We denote by $\min(\partial\Sigma_{g,1,p}) = \bar{z}_1^\infty$ and $\max(\partial\Sigma_{g,1,p}) = z_1^\infty$ these facts.

Given two ends $\mathbf{e}, \mathbf{f} \in \partial\Sigma_{g,1,p}$, we write

$$[\mathbf{e}, \mathbf{f}] := \{\mathbf{g} \in \partial\Sigma_{g,1,p} \mid \mathbf{e} \leq \mathbf{g} \leq \mathbf{f}\}.$$

7.3 Lemma. *Let $1 \leq k_0 \leq p$, $w \in \Sigma_{g,1,p} - (\star t_{k_0}) - (\star \bar{t}_{k_0})$ and $1 \leq i_0 \leq g$. Then, in $(\partial\Sigma_{g,1,p}, \leq)$, the following hold:*

(i). $wt_{k_0}\bar{w}(\bar{z}_1^\infty) \leq wt_{k_0}((\prod_{k=k_0}^p t_k \prod_{i=1}^g [x_i, y_i] \prod_{k=1}^{k_0-1} t_k)^\infty) = \min(wt_{k_0}t_{k_0} \blacktriangleleft);$

(ii). $\max(wt_{k_0}t_{k_0} \blacktriangleleft) < \min(w\bar{t}_{k_0}\bar{t}_{k_0} \blacktriangleleft);$

(iii). $\max(w\bar{t}_{k_0}\bar{t}_{k_0} \blacktriangleleft) = w\bar{t}_{k_0}((\prod_1^{k=k_0} \bar{t}_k \prod_1^{i=g} [y_i, x_i] \prod_{k_0+1}^{k=p} \bar{t}_k)^\infty) \leq w\bar{t}_{k_0}\bar{w}(z_1^\infty);$

(iv). $(wt_{k_0}t_{k_0} \blacktriangleleft) \cup (w\bar{t}_{k_0}\bar{t}_{k_0} \blacktriangleleft) \subseteq [wt_{k_0}\bar{w}(\bar{z}_1^\infty), w\bar{t}_{k_0}\bar{w}(z_1^\infty)];$

(v). *If $2g + p \geq 3$, then one of the following holds:*

(a) $\bar{t}_p(\bar{z}_1^\infty) > w\bar{t}_{k_0}\bar{w}(z_1^\infty);$

(b) $\bar{t}_p(\bar{z}_1^\infty) < wt_{k_0}\bar{w}(\bar{z}_1^\infty);$

and, hence, $\bar{t}_p(\bar{z}_1^\infty) \notin [wt_{k_0}\bar{w}(\bar{z}_1^\infty), w\bar{t}_{k_0}\bar{w}(z_1^\infty)];$

(vi). *If $a \in \{x_{i_0}, \bar{x}_{i_0}, y_{i_0}, \bar{y}_{i_0}\}$, then one of the following holds:*

(a). $a(z_1^\infty) > w\bar{t}_{k_0}\bar{w}(z_1^\infty);$

(b). $a(z_1^\infty) < wt_{k_0}\bar{w}(\bar{z}_1^\infty);$

and, hence, $a(z_1^\infty) \notin [wt_{k_0}\bar{w}(\bar{z}_1^\infty), w\bar{t}_{k_0}\bar{w}(z_1^\infty)].$

Proof. Recall $<$ is the ordering with respect to sequence the

$$(\bar{x}_1, y_1, x_1, \bar{y}_1, \bar{x}_2, y_2, x_2, \bar{y}_2, \dots, \bar{x}_g, y_g, x_g, \bar{y}_g, t_1, \bar{t}_1, t_2, \bar{t}_2, \dots, t_p, \bar{t}_p).$$

(i). It is straightforward to see that

$$wt_{k_0}((\prod_{k=k_0}^p t_k \prod_{i=1}^g [x_i, y_i] \prod_{k=1}^{k_0-1} t_k)^\infty) = \min(wt_{k_0}t_{k_0} \blacktriangleleft).$$

Let $a \in X \cup \bar{X}$ be such that $\bar{w}((\prod_{i=1}^g [x_i, y_i] \prod_{k=1}^p t_k)^\infty) \in (a \blacktriangleleft)$. Note $a \neq \bar{t}_{k_0}$.

If $a \neq t_{k_0}$, then $(wt_{k_0}a \blacktriangleleft) < (wt_{k_0}t_{k_0} \blacktriangleleft)$, and we have

$$wt_{k_0}\bar{w}(\bar{z}_1^\infty) = wt_{k_0}\bar{w}((\prod_{i=1}^g [x_i, y_i] \prod_{k=1}^p t_k)^\infty) < \min(wt_{k_0}t_{k_0}).$$

If $a = t_{k_0}$, then \bar{w} is completely canceled in $\bar{w}((\Pi_{i=1}^g[x_i, y_i]\Pi_{k=1}^p t_k)^\infty)$, and, moreover,

$$\begin{aligned} wt_{k_0}\bar{w}(\bar{z}_1^\infty) &= wt_{k_0}\bar{w}((\Pi_{i=1}^g[x_i, y_i]\Pi_{k=1}^p t_k)^\infty) \\ &= wt_{k_0}((\Pi_{k=k_0}^p t_k \Pi_{i=1}^g[x_i, y_i]\Pi_{k=1}^{k_0-1} t_k)^\infty) \\ &= \min(wt_{k_0} t_{k_0} \blacktriangleleft). \end{aligned}$$

(ii). It is clear.

(iii). It is straightforward to see that

$$\max(w\bar{t}_{k_0}\bar{t}_{k_0} \blacktriangleleft) = w\bar{t}_{k_0}((\Pi_1^{k=k_0}\bar{t}_k \Pi_1^{i=g}[y_i, x_i]\Pi_{k_0+1}^{k=p}\bar{t}_k)^\infty).$$

Let $a \in X \cup \bar{X}$ be such that $\bar{w}((\Pi_1^{k=p}\bar{t}_k \Pi_1^{i=g}[y_i, x_i])^\infty) \in (a \blacktriangleleft)$. Note $a \neq t_{k_0}$. If $a \neq \bar{t}_{k_0}$, then $(w\bar{t}_{k_0}\bar{t}_{k_0} \blacktriangleleft) < (w\bar{t}_{k_0}a \blacktriangleleft)$, and we have

$$\max(w\bar{t}_{k_0}\bar{t}_{k_0} \blacktriangleleft) < w\bar{t}_{k_0}\bar{w}((\Pi_1^{k=p}\bar{t}_k \Pi_1^{i=g}[y_i, x_i])^\infty) = w\bar{t}_{k_0}\bar{w}(z_1^\infty).$$

If $a = \bar{t}_{k_0}$, then \bar{w} is completely canceled in $\bar{w}((\Pi_1^{k=p}\bar{t}_k \Pi_1^{i=g}[y_i, x_i])^\infty)$, and, moreover,

$$\begin{aligned} w\bar{t}_{k_0}\bar{w}(z_1^\infty) &= w\bar{t}_{k_0}\bar{w}((\Pi_1^{k=p}\bar{t}_k \Pi_1^{i=g}[y_i, x_i])^\infty) \\ &= w\bar{t}_{k_0}((\Pi_1^{k=k_0}\bar{t}_k \Pi_1^{i=g}[y_i, x_i]\Pi_p^{k=k_0+1}\bar{t}_k)^\infty) \\ &= \max(w\bar{t}_{k_0}\bar{t}_{k_0} \blacktriangleleft). \end{aligned}$$

(iv). Follows from (i)-(iii).

(v). By (i)-(iii),

$$wt_{k_0}\bar{w}((\Pi_{i=1}^g[x_i, y_i]\Pi_{k=1}^p t_k)^\infty) < w\bar{t}_{k_0}\bar{w}((\Pi_1^{k=p}\bar{t}_k \Pi_1^{i=g}[y_i, x_i])^\infty).$$

Case 1. $w = 1$. Since $(\bar{t}_p \bar{x}_1 \blacktriangleleft) \cup (\bar{t}_p t_1 \blacktriangleleft) > (\bar{t}_{k_0} \bar{t}_p \blacktriangleleft)$, we see

$$\bar{t}_p(\bar{z}_1^\infty) = \bar{t}_p((\Pi_{i=1}^g[x_i, y_i]\Pi_{k=1}^p t_k)^\infty) > \bar{t}_{k_0}((\Pi_1^{k=p}\bar{t}_k \Pi_1^{i=g}[y_i, x_i])^\infty) = \bar{t}_{k_0}(z_1^\infty).$$

Thus, (a) holds.

Case 2. $w \notin (\bar{t}_p \star) \cup \{1\}$. Since $(\bar{t}_p \blacktriangleleft) > (w\bar{t}_{k_0} \blacktriangleleft)$, we see

$$\begin{aligned} \bar{t}_p(\bar{z}_1^\infty) &= \bar{t}_p((\Pi_{i=1}^g[x_i, y_i]\Pi_{k=1}^p t_k)^\infty) \\ &> w\bar{t}_{k_0}\bar{w}((\Pi_1^{k=p}\bar{t}_k \Pi_1^{i=g}[y_i, x_i])^\infty) = w\bar{t}_{k_0}\bar{w}(z_1^\infty). \end{aligned}$$

Thus, (a) holds.

Case 3. $w \in (\bar{t}_p \bar{t}_p \star)$. Since $(\bar{t}_p \bar{x}_1 \blacktriangleleft) \cup (\bar{t}_p t_1 \blacktriangleleft) > (w\bar{t}_{k_0} \blacktriangleleft)$, we see

$$\begin{aligned} \bar{t}_p(\bar{z}_1^\infty) &= \bar{t}_p((\Pi_{i=1}^g[x_i, y_i]\Pi_{k=1}^p t_k)^\infty) \\ &> w\bar{t}_{k_0}\bar{w}((\Pi_1^{k=p}\bar{t}_k \Pi_1^{i=g}[y_i, x_i])^\infty) = w\bar{t}_{k_0}\bar{w}(z_1^\infty). \end{aligned}$$

Thus, (a) holds.

Case 4. $w \in (\bar{t}_p \star) - (\bar{t}_p \bar{t}_p \star)$.

Here,

$$wt_{k_0} \bar{w}(\bar{z}_1^\infty) = wt_{k_0} \bar{w}((\Pi_{i=1}^g [x_i, y_i] \Pi_{k=1}^p t_k)^\infty) \in (wt_{k_0} \blacktriangleleft) \subset (\bar{t}_p \blacktriangleleft) - (\bar{t}_p \bar{t}_p \blacktriangleleft).$$

Hence,

$$\begin{aligned} \bar{t}_p((\Pi_{i=1}^g [x_i, y_i] \Pi_{k=1}^p t_k)^\infty) &= \min((\bar{t}_p \blacktriangleleft) - (\bar{t}_p \bar{t}_p \blacktriangleleft)) \\ &\leq wt_{k_0} \bar{w}((\Pi_{i=1}^g [x_i, y_i] \Pi_{k=1}^p t_k)^\infty). \end{aligned}$$

To prove (b) holds, it remains to show that

$$\bar{t}_p((\Pi_{i=1}^g [x_i, y_i] \Pi_{k=1}^p t_k)^\infty) \neq wt_{k_0} \bar{w}((\Pi_{i=1}^g [x_i, y_i] \Pi_{k=1}^p t_k)^\infty),$$

that is, $(\Pi_{i=1}^g [x_i, y_i] \Pi_{k=1}^p t_k)^\infty \neq t_p wt_{k_0} \bar{w}((\Pi_{i=1}^g [x_i, y_i] \Pi_{k=1}^p t_k)^\infty)$, that is, $t_p wt_{k_0} \bar{w} \notin \langle \Pi_{i=1}^g [x_i, y_i] \Pi_{k=1}^p t_k \rangle$. We can write $w = \bar{t}_p u$ where $u \notin (t_p \star)$. Then $t_p wt_{k_0} \bar{w} = ut_{k_0} \bar{u} t_p$, in normal form. Thus it suffices to show

$$ut_{k_0} \bar{u} t_p \notin \langle \Pi_{i=1}^g [x_i, y_i] \Pi_{k=1}^p t_k \rangle.$$

If $u = 1$, then $ut_{k_0} \bar{u} t_p \notin \langle \Pi_{i=1}^g [x_i, y_i] \Pi_{k=1}^p t_k \rangle$, since $2g + p \geq 3$.

If $u \neq 1$, then $ut_{k_0} \bar{u} t_p \notin \langle \Pi_{i=1}^g [x_i, y_i] \Pi_{k=1}^p t_k \rangle$, since $ut_{k_0} \bar{u} t_p$ does not lie in the submonoid of $\Sigma_{g,1,p}$ generated by $\Pi_{i=1}^g [x_i, y_i] \Pi_{k=1}^p t_k$, nor in the submonoid generated by $\Pi_1^{k=p} \bar{t}_k \Pi_1^{i=g} [y_i, x_i]$.

In all four cases (v) holds.

(vi). Let $a \in \{x_{i_0}, \bar{x}_{i_0}, y_{i_0}, \bar{y}_{i_0}\}$.

Case 1. $w = 1$. Since $(a \blacktriangleleft) < (t_{k_0} \blacktriangleleft)$, we see

$$a(z_1^\infty) = a((\Pi_1^{k=p} \bar{t}_k \Pi_1^{i=g} [y_i, x_i])^\infty) < t_{k_0}((\Pi_{i=1}^g [x_i, y_i] \Pi_{k=1}^p t_k)^\infty) = t_{k_0}(\bar{z}_1^\infty).$$

Thus, (b) holds.

Case 2. $w \notin (a \star) \cup \{1\}$.

If $(a \blacktriangleleft) > (w \blacktriangleleft)$, then $(a \blacktriangleleft) > (w \blacktriangleleft) \supset (w \bar{t}_{k_0} \blacktriangleleft)$ and

$$\begin{aligned} a(z_1^\infty) &= a((\Pi_1^{k=p} \bar{t}_k \Pi_1^{i=g} [y_i, x_i])^\infty) \\ &> w \bar{t}_{k_0} \bar{w}((\Pi_1^{k=p} \bar{t}_k \Pi_1^{i=g} [y_i, x_i])^\infty) = w \bar{t}_{k_0} \bar{w}(z_1^\infty). \end{aligned}$$

Thus, (a) holds.

If $(a \blacktriangleleft) < (w \blacktriangleleft)$, then $(a \blacktriangleleft) < (w \blacktriangleleft) \supset (wt_{k_0} \blacktriangleleft)$ and

$$\begin{aligned} a(z_1^\infty) &= a((\Pi_1^{k=p} \bar{t}_k \Pi_1^{i=g} [y_i, x_i])^\infty) \\ &< wt_{k_0} \bar{w}((\Pi_{i=1}^g [x_i, y_i] \Pi_{k=1}^p t_k)^\infty) = wt_{k_0} \bar{w}(\bar{z}_1^\infty). \end{aligned}$$

Thus, (b) holds.

Case 3. $w \in (a\bar{t}_p\star)$.

Since $a((\Pi_1^{k=p}\bar{t}_k\Pi_1^{i=g}[y_i, x_i])^\infty) = \max(a\bar{t}_p \blacktriangleleft)$, we see

$$\begin{aligned} a(z_1^\infty) &= a((\Pi_1^{k=p}\bar{t}_k\Pi_1^{i=g}[y_i, x_i])^\infty) \\ &\geq w\bar{t}_{k_0}\bar{w}((\Pi_1^{k=p}\bar{t}_k\Pi_1^{i=g}[y_i, x_i])^\infty) = w\bar{t}_{k_0}\bar{w}(z_1^\infty). \end{aligned}$$

To prove (a) holds, it remains to show that

$$a((\Pi_1^{k=p}\bar{t}_k\Pi_1^{i=g}[y_i, x_i])^\infty) \neq w\bar{t}_{k_0}\bar{w}((\Pi_1^{k=p}\bar{t}_k\Pi_1^{i=g}[y_i, x_i])^\infty),$$

that is, $(\Pi_1^{k=p}\bar{t}_k\Pi_1^{i=g}[y_i, x_i])^\infty \neq \bar{a}w\bar{t}_{k_0}\bar{w}((\Pi_1^{k=p}\bar{t}_k\Pi_1^{i=g}[y_i, x_i])^\infty)$, that is $\bar{a}w\bar{t}_{k_0}\bar{w} \notin \langle \Pi_1^{k=p}\bar{t}_k\Pi_1^{i=g}[y_i, x_i] \rangle$. We can write $w = a\bar{t}_p u$ where $u \notin (t_p\star)$. Then $\bar{a}w\bar{t}_{k_0}\bar{w} = \bar{t}_p u \bar{t}_{k_0} \bar{u} t_p \bar{a}$, in normal form. Thus it suffices to show that

$$\bar{t}_p u \bar{t}_{k_0} \bar{u} t_p \bar{a} \notin \langle \Pi_1^{k=p}\bar{t}_k\Pi_1^{i=g}[y_i, x_i] \rangle,$$

which is clear since $\bar{t}_p u \bar{t}_{k_0} \bar{u} t_p \bar{a}$ does not lie in the submonoid of $\Sigma_{g,1,p}$ generated by $\Pi_1^{k=p}\bar{t}_k\Pi_1^{i=g}[y_i, x_i]$, nor in the submonoid generated by $\Pi_{i=1}^g[x_i, y_i]\Pi_{k=1}^p t_k$.

Case 4. $w \in (a\star) - (a\bar{t}_p\star)$, $|w| \geq 2$.

If $(a\bar{t}_p \blacktriangleleft) > (w \blacktriangleleft)$, then $(a\bar{t}_p \blacktriangleleft) > (w \blacktriangleleft) \supset (w\bar{t}_{k_0} \blacktriangleleft)$ and

$$\begin{aligned} a(z_1^\infty) &= a((\Pi_1^{k=p}\bar{t}_k\Pi_1^{i=g}[y_i, x_i])^\infty) \\ &> w\bar{t}_{k_0}\bar{w}((\Pi_1^{k=p}\bar{t}_k\Pi_1^{i=g}[y_i, x_i])^\infty) = w\bar{t}_{k_0}\bar{w}(z_1^\infty). \end{aligned}$$

Thus, (a) holds.

If $(a\bar{t}_p \blacktriangleleft) < (w \blacktriangleleft)$, then $(a\bar{t}_p \blacktriangleleft) < (w \blacktriangleleft) \supset (wt_{k_0} \blacktriangleleft)$ and

$$\begin{aligned} a(z_1^\infty) &= a((\Pi_1^{k=p}\bar{t}_k\Pi_1^{i=g}[y_i, x_i])^\infty) \\ &< wt_{k_0}\bar{w}((\Pi_{i=1}^g[x_i, y_i]\Pi_{k=1}^p t_k)^\infty) = wt_{k_0}\bar{w}(\bar{z}_1^\infty). \end{aligned}$$

Thus, (b) holds.

Case 5. $w = a$.

Since $a(z_1^\infty) = \max(a\bar{t}_p \blacktriangleleft)$, $(a\bar{t}_p \blacktriangleleft) \supset (a\bar{t}_p\bar{y}_g\bar{x}_g \blacktriangleleft)$ and $(a\bar{t}_p\bar{y}_g\bar{x}_g \blacktriangleleft) > (a\bar{t}_{k_0}\bar{a}\bar{t}_p \blacktriangleleft)$, we see

$$\begin{aligned} a(z_1^\infty) &= a((\Pi_1^{k=p}\bar{t}_k\Pi_1^{i=g}[y_i, x_i])^\infty) \\ &> a\bar{t}_{k_0}\bar{a}((\Pi_1^{k=p}\bar{t}_k\Pi_1^{i=g}[y_i, x_i])^\infty) = a\bar{t}_{k_0}\bar{a}(z_1^\infty). \end{aligned}$$

Thus, (a) holds.

In all five cases (vi) holds. \square

7.4 Theorem. *If $2g + p \geq 3$ then, for each $\phi \in \mathcal{AM}_{g,1,p}$, the following hold:*

(i). $\bar{t}_p^\phi(\bar{z}_1^\infty)$ is a t -squarefree end,

(ii). for every $1 \leq i_0 \leq g$ and every $a \in \{x_{i_0}, \bar{x}_{i_0}, y_{i_0}, \bar{y}_{i_0}\}$, $a^\phi(z_1^\infty)$ is a t -squarefree end.

Proof. (i). Recall $\bar{z}_1 = \prod_{i=1}^g [x_i, y_i] \prod_{k=1}^p t_k$ and $z_1 = \prod_1^{k=p} \bar{t}_k \prod_1^{i=g} [y_i, x_i]$. By Lemma 7.3(v), $\bar{t}_p(\bar{z}_1^\infty)$ does not lie in

$$\bigcup_{u \in \{[t_k]\}_{1 \leq k \leq p}} [u(\bar{z}_1^\infty), \bar{u}(z_1^\infty)] (= \bigcup_{k=1}^p \bigcup_{w \in \Sigma_{g,1,p} - (\star t_k) - (\star \bar{t}_k)} [wt_k \bar{w}(\bar{z}_1^\infty), w \bar{t}_k \bar{w}(z_1^\infty)]).$$

Notice that ϕ respects each of the following sets:

$$\{[t_k]\}_{1 \leq k \leq p}; \quad \{\bar{z}_1^\infty\}; \quad \{z_1^\infty\}; \quad \text{and} \quad \bigcup_{u \in \{[t_k]\}_{1 \leq k \leq p}} [u(\bar{z}_1^\infty), \bar{u}(z_1^\infty)].$$

Hence, $(\bar{t}_p(\bar{z}_1^\infty))^\phi$ does not lie in $\bigcup_{u \in \{[t_k]\}_{1 \leq k \leq p}} [u(\bar{z}_1^\infty), \bar{u}(z_1^\infty)]$. By Lemma 7.3(iv),

$$\bigcup_{u \in \{[t_k]\}_{1 \leq k \leq p}} [u(\bar{z}_1^\infty), \bar{u}(z_1^\infty)] \supseteq \bigcup_{k=1}^p \bigcup_{w \in \Sigma_{g,1,p} - (\star t_k) - (\star \bar{t}_k)} ((wt_k t_k \blacktriangleleft) \cup (w \bar{t}_k \bar{t}_k \blacktriangleleft)).$$

Hence, $(\bar{t}_p(\bar{z}_1^\infty))^\phi$ does not lie in the right-hand side set either, and, hence, $(\bar{t}_p(\bar{z}_1^\infty))^\phi$ is a t -squarefree end. Since $(\bar{t}_p(\bar{z}_1^\infty))^\phi = \bar{t}_p^\phi(\bar{z}_1^\infty)$, the desired result holds.

(ii). The same proof as (i) using Lemma 7.3(vi) instead of Lemma 7.3(v). \square

Proof. (of Theorem 4.2) Recall (2.3.1). $\mathcal{AM}_{0,1,2} = \langle \sigma_1 \rangle$, and

$$t_2^{\mathcal{AM}_{0,1,2}} = \{t_2^{\sigma_1^{2m}}, t_2^{\sigma_1^{2m+1}} \mid m \in \mathbb{Z}\} = \{t_2^{(t_1 t_2)^m}, t_1^{(t_1 t_2)^m} \mid m \in \mathbb{Z}\}$$

Thus, every element of $t_2^{\mathcal{AM}_{0,1,2}}$ is t -squarefree.

Suppose, now, $2g + p \geq 3$. Let $1 \leq i_0 \leq g$ and $a \in \{x_{i_0}, y_{i_0}\}$. By Theorem 7.4(ii), $a^\phi(z_1^\infty) = a^\phi((\prod_1^{k=p} \bar{t}_k \prod_1^{i=g} [y_i, x_i])^\infty)$ is a t -squarefree end. Hence, either a^ϕ is t -squarefree or $a^\phi = ut_k t_k v$ in normal form, and $t_k v$ is canceled in $a^\phi(z_1^\infty) = ut_k t_k v(z_1^\infty)$; moreover $ut_k, t_k v$ are t -squarefree. By Theorem 7.4(ii),

$$\bar{a}^\phi(z_1^\infty) = \bar{a}^\phi((\prod_1^{k=p} \bar{t}_k \prod_1^{i=g} [y_i, x_i])^\infty)$$

is a t -squarefree end. Hence, $\bar{a}^\phi \neq \bar{v} \bar{t}_k \bar{t}_k \bar{u}$.

Since ϕ permutes the set $\{[t_k]\}_{1 \leq k \leq p}$, we can write $\bar{t}_p^\phi = \bar{t}_p^{w_p}$, where π is a permutation of $\{1, 2, \dots, p\}$ and $w_p \in \Sigma_{g,1,p} - (t_{p^\pi} \star) - (\bar{t}_{p^\pi} \star)$. It is not difficult to see that

$$\bar{t}_p^\phi(\bar{z}_1^\infty) = \bar{w}_p \bar{t}_{p^\pi} w_p ((\Pi_{i=1}^g [x_i, y_i] \Pi_{k=1}^p t_k)^\infty) \in (\bar{w}_p \blacktriangleleft).$$

By Theorem 7.4(i), $\bar{t}_p^\phi(\bar{z}_1^\infty)$ is a t -squarefree end. Hence, \bar{w}_p is t -squarefree.

Since \bar{w}_p is t -squarefree, $\bar{t}_p^\phi = \bar{w}_p \bar{t}_{p^\pi} w_p$ is also t -squarefree. Hence, t_p^ϕ is t -squarefree.

Suppose, now, $2g + p \geq 2$. Let $1 \leq k \leq p$. Since t_k is in the $\mathcal{AM}_{g,1,p}$ -orbit of t_p , we see t_k^ϕ is t -squarefree for all $\phi \in \mathcal{AM}_{g,1,p}$. \square

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References

- [1] Javier Aramayona, Christopher J. Leininger, Juan Souto *Injections of mapping class groups* *Geom. and Top.* **13**(2009), 2523–2541.
- [2] Norbert A’Campo, *Le groupe de monodromie du déploiement des singularités isolées de courbes planes I.* *Math. Ann.* **213**(1975), 1–32.
- [3] Lluís Bacardit and Warren Dicks, *Actions of the braid group, and new algebraic proofs of results of Dehornoy and Larue,* *Groups – Complexity – Cryptology*, **1**(2009), 77 – 129
- [4] Joan S. Birman and Hugh M. Hilden, *On isotopies of homeomorphisms of Riemann surfaces,* *Ann. of Math.* **97**(1973), 424–439.
- [5] Daryl Cooper, *Automorphisms of free groups have finitely generated fixed point sets,* *J.algebra* **111**(1987), 453–456.
- [6] John Crisp and Luis Paris, *Arin groups of type B and D,* *Adv. Geom.* **5**(2005), 607–636.
- [7] John Crisp and Luis Paris, *Representations of the braid group by automorphisms of groups, invariants of links, and Gar side groups,* *Pacific J. Math.* **221**(2005), 1–27.
- [8] Patrick Dehornoy, Ivan Baryshnikov, Dale Rolfsen and Bert West, *Why are braids orderable?*, *Panoramas et Synthèses* **14**, Soc. Math. France, Paris, 2002.
- [9] Warren Dicks and Edward Formanek, *Automorphism subgroups of finite index in algebraic mapping class groups,* *J. Algebra* **189**(1997), 58–89.
- [10] Warren Dicks and Edward Formanek, *Algebraic mapping-class groups of orientable surfaces with boundaries,* pp. 57–116, in: *Infinite groups: geometric, combinatorial and dynamical aspects* (eds. Laurent Bartholdi, Tulley Cherin-Silverstein, Tatiana Smirnov-Magnified, Andrej Auk), *Progress in Mathematics* **248**, Birkenstock Verlag, Basel, 2005.
Errata and addenda at: <http://mat.ab.cat/~dicks/Boundaries.HTML>
- [11] B. Farb and D. Margalit, *A primer book on mapping class groups,* http://www.mines.edu/fs_home/dlarue/papers/dm1.pdf
- [12] Nikolai V. Ivanov, *Mapping class groups,* p. 523 – 633 in *Handbook of geometric topology,* North-Holland, Amsterdam, 2002.
- [13] Roger C. Lyndon and Paul E. Schuss, *Combinatorial group theory,* Berget. Math. Grenada. **89**, Springer-Verlag, Berlin, 1977.
- [14] Wilhelm Magnus, *Über Automorphism en Von Fundamentalist Bernadette FlewäCheng,* *Math. Ann.* **109**(1934), 617–646.

- [15] James McCool, *Generating the mapping class group (an algebraic approach)*, Publ. Mat. **40**(1996/02), 457-468.
- [16] Hamish Short and Bert Wiest, *Orderings of mapping class groups after Thurston*, Enseign. Math. **46**(2000), 279–312.

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