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# Embedding mapping-class groups of orientable surfaces with one boundary component

Lluís Bacardit\*

## Abstract

We denote by  $S_{g,b,p}$  an orientable surface of genus  $g$  with  $b$  boundary components and  $p$  punctures. We construct homomorphisms from the mapping-class groups of  $S_{g,1,p}$  to the mapping-class groups of  $S_{g',1,(b-1)}$ , where  $b \geq 1$ . Our main result is that these homomorphisms are injective. A particular case of these homomorphisms is a well-known embedding of the braid group on  $p$  strands into the mapping-class group of  $S_{(p-2)/2,1,1}$  if  $p$  is even, or into the mapping-class group of  $S_{(p-1)/2,1,0}$  if  $p$  is odd. We give a short proof of a theorem of Birman and Hilden [4] for surfaces with one boundary component.

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## 1 Main results

We fix non-negative integers  $g, p$  and a positive integer  $b$ . We denote by  $S_{g,b,p}$  an orientable surface of genus  $g$  with  $b$  boundary components and  $p$  punctures.

Our main theorem is the following.

**1.1 Theorem.** *Suppose  $(g, p) \neq (0, 2)$ . Let  $\kappa : S_{g',b,0} \rightarrow S_{g,1,0}$  be a finite index regular cover with  $p$  branching points in  $S_{g,1,0}$  which lift to  $q$  points in  $S_{g',b,0}$ . Suppose every branching point of  $S_{g,1,0}$  lift to the same number of points in  $S_{g',b,0}$ . Let  $\kappa' : S_{g',b,q} \rightarrow S_{g,1,p}$  be the corresponding unbranched cover. Let  $h$  be a homeomorphism of  $S_{g,1,p}$  which fixes the boundary component pointwise. Suppose  $h$  lifts to  $S_{g',b,q}$ . Let  $\hat{h}$  be the lift of  $h$  which fixes the  $b$ -th boundary component pointwise. Let  $\hat{f}$  be the extension of  $\hat{h}$  to  $S_{g',b,0}$ . If the restriction*

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of  $\hat{f}$  to  $S_{g',1,(b-1)} \subseteq S_{g',b,0}$  is isotopic to the identity relative to the boundary component of  $S_{g',1,(b-1)}$ , then  $h$  is isotopic to the identity relative to the boundary component of  $S_{g,1,p}$ .

Let  $\mathcal{M}_{g,b,p}$  be the mapping-class group of  $S_{g,b,p}$  relative to the boundary components. That is,  $\mathcal{M}_{g,b,p}$  is the group of homeomorphisms of  $S_{g,b,p}$  which fix the boundary components pointwise modulo isotopy relative to the boundary components of  $S_{g,b,p}$ . Since  $b \geq 1$ , we are restricting ourselves to orientation-preserving homeomorphisms of  $S_{g,b,p}$ .

The following result is immediate from Theorem 1.1.

**1.2 Corollary.** *Suppose  $(g, p) \neq (0, 2)$ . Let  $\kappa : S_{g',b,0} \rightarrow S_{g,1,0}$  be a finite index regular cover with  $p$  branching points in  $S_{g,1,0}$  which lift to  $q$  points in  $S_{g',b,0}$ . Suppose every branching point of  $S_{g,1,0}$  lift to the same number of points in  $S_{g',b,0}$ . Let  $\kappa' : S_{g',b,q} \rightarrow S_{g,1,p}$  be the corresponding unbranched cover. If every homeomorphism of  $S_{g,1,p}$  which fixes the boundary component pointwise lifts to a homeomorphism of  $S_{g',b,q}$ , then  $\mathcal{M}_{g,1,p}$  embeds in  $\mathcal{M}_{g',1,(b-1)}$ .*

Suppose  $p \geq 3$ . If  $p$  is odd, there exists a well-known index 2 regular cover  $\kappa : S_{(p-1)/2,1,0} \rightarrow S_{0,1,0}$  with  $p$  branching points in  $S_{0,1,0}$  which lift to  $p$  branching points in  $S_{(p-1)/2,1,0}$ . Every homeomorphism of  $S_{0,1,p}$  which fixes the boundary component pointwise lifts to a homeomorphism of  $S_{(p-1)/2,1,p}$  which fixes the boundary component pointwise. Corollary 1.2 says that  $\mathcal{M}_{0,1,p}$  embeds in  $\mathcal{M}_{(p-1)/2,1,0}$ . If  $p$  is even, there exists a well-known index 2 regular cover  $\kappa : S_{(p-2)/2,2,0} \rightarrow S_{0,1,0}$  with  $p$  branching points in  $S_{0,1,0}$  which lift to  $p$  branching points in  $S_{(p-2)/2,2,0}$ . Every homeomorphism of  $S_{0,1,p}$  which fixes the boundary component pointwise lifts to a homeomorphism of  $S_{(p-2)/2,2,p}$  which fixes the second boundary component pointwise. Corollary 1.2 says that  $\mathcal{M}_{0,1,p}$  embeds in  $\mathcal{M}_{(p-2)/2,1,1}$ .

From Theorem 1.1 we can prove the following theorem, which is an analog for surfaces with one boundary component of a theorem of Birman and Hilden [4, Theorem 2].

**1.3 Theorem.** *Suppose  $(g, p) \neq (0, 2)$ . Let  $\kappa : S_{g',b,0} \rightarrow S_{g,1,0}$  be a finite index regular cover with  $p$  branching points in  $S_{g,1,0}$ . Suppose every branching point of  $S_{g,1,0}$  lift to the same number of points in  $S_{g',b,0}$ . Let  $\hat{f}$  be a homeomorphism of  $S_{g',b,0}$  which fixes the  $b$ -th boundary component pointwise and preserves the fibers of  $\kappa : S_{g',b,0} \rightarrow S_{g,1,0}$ . Then  $\hat{f}$  induces a homeomorphism  $f$  of  $S_{g,1,0}$  such that  $\kappa\hat{f} = f\kappa$ . If  $\hat{f}$  is isotopic to the identity relative to the  $b$ -th boundary component, then  $f$  is isotopic to the identity relative to the boundary.*

*Proof.* It is a general fact that if  $\hat{f}$  preserves the fibers of  $\kappa : S_{g',b,0} \rightarrow S_{g,1,0}$ , then  $\hat{f}$  induces a homeomorphism  $f$  of  $S_{g,1,0}$  such that  $\kappa\hat{f} = f\kappa$ . In particular,  $f$  sends branching points to branching points.

Let  $\kappa' : S_{g',b,q} \rightarrow S_{g,1,p}$  be the corresponding unbranched cover. Since  $f$  sends branching points to branching points,  $f$  restricts to a homeomorphism  $h$  of  $S_{g,1,p}$ . Let  $\hat{h}$  be the lift of  $h$  which fixes the  $b$ -th boundary component of  $S_{g',b,q}$  pointwise. Notice  $\hat{h}$  extends to a homeomorphism of  $S_{g',b,0}$ . This extension of  $\hat{h}$  coincides with  $\hat{f}$ . If  $\hat{f}$  is isotopic to the identity relative to the  $b$ -th boundary component, then the restriction of  $\hat{f}$  to  $S_{g',1,(b-1)} \subseteq S_{g',b,0}$  is isotopy to the identity relative to the boundary component of  $S_{g',1,(b-1)}$ . Then, by Theorem 1.1,  $h$  is isotopic to the identity relative to the boundary component of  $S_{g,1,p}$ . This isotopy extends to an isotopy relative to the boundary component of  $S_{g,1,0}$  from  $f$  to the identity.  $\square$

## 2 The algebraic analog

Instead of dealing with  $S_{g,b,p}$  and homeomorphisms of  $S_{g,b,p}$  which fix the boundary components pointwise, we will deal with the fundamental group of  $S_{g,b,p}$ , denoted  $\pi_1(S_{g,b,p})$ . Since  $b \geq 1$ , we choose the base point of  $\pi_1(S_{g,b,p})$  in the  $b$ -th boundary component. In this way, a homeomorphism of  $S_{g,b,p}$  which fixes the boundary components pointwise induces an automorphism of  $\pi_1(S_{g,b,p})$ .

**2.1 Notation.** Let  $G$  be a group and let  $g, h$  be elements of  $G$ .

We denote by  $\bar{g}$  the invers of  $g$ . We denote by  $g^h$  the conjugated of  $g$  by  $h$ , that is,  $g^h = \bar{h}gh$ . We denote by  $[g]$  the conjugacy class of  $G$ , that is,  $[g] = \{g^a \mid a \in G\}$ . We denote by  $[g, h]$  the element  $\bar{g}hgh$ . Let  $g_1, g_2, \dots, g_k$  be elements of  $G$ . We denote by  $\prod_{i=1}^k g_i$  the product  $g_1g_2 \cdots g_k$ .

We denote by  $\text{Aut}(G)$  the automorphism group of  $G$ , and, by  $\text{Out}(G)$  the group of extern automorphism of  $G$ . Given  $\phi \in \text{Aut}(G)$ , we denote by  $g^\phi$  the image of  $g$  by  $\phi$ .

**2.2 Notation.** Let  $\Sigma_{g,b,p}$  be the rank  $2g + (b-1) + p$  free group with generating set  $\{x_i, y_i\}_{1 \leq i \leq g} \cup \{z_l\}_{1 \leq l \leq (b-1)} \cup \{t_k\}_{1 \leq k \leq p}$ . We view  $\Sigma_{g,b,p}$  as a presentation of  $\pi_1(S_{g,b,p}, *)$ , the fundamental group of  $S_{g,b,p}$  based at a point  $*$  in the  $b$ -th boundary component. In addition, for every  $1 \leq l \leq (b-1)$ ,  $z_l$  represents a loop around the  $l$ -th boundary component; for every  $1 \leq k \leq p$ ,  $t_k$  represents a loop around the  $k$ -th puncture, and  $(\prod_{i=1}^g [x_i, y_i] \prod_{l=1}^{b-1} z_l \prod_{k=1}^p t_k)^{-1}$  represents a loop around the  $b$ -th boundary component. Note that, if  $p = 0$ , there is no puncture in  $S_{g,b,p} = S_{g,b,0}$ , and  $\prod_{k=1}^p t_k = 1$ .

Let  $f$  be a homeomorphism of  $S_{g,b,p}$  which fixes the boundary components pointwise. Then  $f$  induces an automorphism  $f_*$  of  $\Sigma_{g,b,p}$  which fixes the set of conjugacy classes of  $t_1, t_2, \dots, t_p$ . Since  $f$  fixes the boundary components of  $S_{g,b,p}$  pointwise, we see that  $f_*$  fixes  $(\prod_{i=1}^g [x_i, y_i] \prod_{l=1}^{b-1} z_l \prod_{k=1}^p t_k)^{-1}$  and the conjugacy class of  $z_l$ , for all  $1 \leq l \leq b-1$ . Two isotopic homeomorphisms of  $S_{g,b,p}$  induce the same automorphism of  $\Sigma_{g,b,p}$ . Recall we consider isotopies relative to the boundary components. Notice the Dehn twist with respect a loop around a boundary component is isotopic to the identity, but it is not isotopic to the

identity relative to the boundary. To capture this fact, we associate to  $f$  an automorphism of  $\Sigma_{g,b,p} * \langle e_1, e_2, \dots, e_{(b-1)} \mid \rangle$  which maps  $\Sigma_{g,b,p}$  to itself and respects the following sets

$$(2.2.1) \quad \begin{aligned} & \text{(i)} \quad \{ \Pi_{i=1}^g [x_i, y_i] \Pi_{l=1}^{b-1} z_l \Pi_{k=1}^p t_k \}, \\ & \text{(ii)} \quad \{ \bar{z}_1^{e_1} \}, \{ \bar{z}_2^{e_2} \}, \dots, \{ \bar{z}_{(b-1)}^{e_{(b-1)}} \}, \\ & \text{(iii)} \quad \{ [\bar{t}_k] \}_{1 \leq k \leq p}. \end{aligned}$$

Recall  $z_l$  represents a loop around the  $l$ -th boundary component which is based at a point in the  $b$ -boundary component. For every  $1 \leq l \leq (b-1)$ , we view  $e_l$  as an arc from the base point in the  $b$ -th boundary component to a chosen point in the  $l$ -th boundary component. We view  $\bar{e}_l z_l e_l = z_l^{e_l}$  as a loop around the  $l$ -boundary component and based at the chosen point in the  $l$ -boundary component. Since the homeomorphism  $f$  fixes the  $l$ -boundary component pointwise, the automorphism  $f_*$  fixes  $\bar{z}_l^{e_l}$ . For example, the Dehn twist with respect to the loop represented by  $z_l^{e_l}$  gives the following automorphism of  $\Sigma_{g,b,p} * \langle e_1, e_2, \dots, e_{(b-1)} \mid \rangle$

$$\begin{cases} e_l & \mapsto z_l e_l, \\ a & \mapsto a, \quad a \in \{x_i, y_i\}_{1 \leq i \leq g} \cup \{t_k\}_{1 \leq k \leq p} \cup \{z_l\}_{1 \leq l \leq b} \cup \{e_{l'}\}_{1 \leq l' \leq b, l' \neq l}. \end{cases}$$

**2.3 Definition.** We denote by  $\mathcal{AM}_{g,b,p}$  the subgroup of  $\text{Aut}(\Sigma_{g,b,p} * \langle e_1, e_2, \dots, e_{b-1} \mid \rangle)$  consisting of all the automorphisms of  $\Sigma_{g,b,p} * \langle e_1, e_2, \dots, e_{(b-1)} \mid \rangle$  which map  $\Sigma_{g,b,p}$  to itself and respect the sets of (2.2.1).

We call  $\mathcal{AM}_{g,b,p}$  the *algebraic mapping-class group* of a surface of genus  $g$  with  $b$  boundary components and  $p$  punctures,  $S_{g,b,p}$ .

The mapping-class group of  $S_{g,b,p}$ , denoted  $\mathcal{M}_{g,b,p}$ , is defined as the group of homeomorphisms of  $S_{g,b,p}$  modulo isotopy relative to the boundary components. The above discussion shows that there exists a map  $\mathcal{M}_{g,b,p} \rightarrow \mathcal{AM}_{g,b,p}$ . We sketch the proof that this map is an isomorphism.

Since  $S_{g,0,p}$  has empty boundary, we cannot suppose that the basepoint of  $\Pi_1(S_{g,0,p})$  is fixed by homeomorphisms of  $S_{g,0,p}$ . A discussion as above shows that there exists a map  $\mathcal{M}_{g,0,p} \rightarrow \text{Out}(\Pi_1(S_{g,0,p}))$ . The Dehn-Nielsen-Baer theorem says that this map is an isomorphism onto the subgroup of  $\text{Out}(\Pi_1(S_{g,0,p}))$  consisting of classes of automorphisms which respect the set  $\{[t_k]\}_{1 \leq k \leq p}$ . See, for example, [11, Theorem 3.6], [12, Theorem 2.9.A]. In particular,  $\mathcal{M}_{g,0,p} \leq \text{Out}(\Pi_1(S_{g,0,p}))$ .

We say that  $S_{g,b-1,p+1}$  is obtained from  $S_{g,b,p}$  by converting a boundary component into a puncture if  $S_{g,b-1,p+1} \subset S_{g,b,p}$  and  $S_{g,b,p} - S_{g,b-1,p+1}$  is homeomorphic to a circle. If  $b \geq 1$ , by converting all the boundary components of  $S_{g,b,p}$  into punctures, from the Dehn-Nielsen-Baer theorem we can deduce  $\mathcal{M}_{g,b,p} \simeq \mathcal{AM}_{g,b,p}$ , [10, Theorem 9.6]. See [10] for a background on algebraic mapping-class groups, with some changes of notation.

For  $(g, b) = (0, 1)$  and  $p \geq 1$ ,  $\mathcal{AM}_{0,1,p}$  is isomorphic to the  $p$ -string braid group. We have  $\mathcal{AM}_{0,1,p} = \langle \sigma_1, \sigma_2, \dots, \sigma_{p-1} \rangle$ , where for all  $1 \leq i \leq (p-1)$ ,  $\sigma_i \in \text{Aut}(\Sigma_{0,1,p})$  is defined by

$$(2.3.1) \quad \sigma_i := \begin{cases} t_i & \mapsto t_{i+1}, \\ t_{i+1} & \mapsto t_i^{t_{i+1}}, \\ t_k & \mapsto t_k, \end{cases} \quad \text{for all } 1 \leq k \leq p, k \neq i, i+1.$$

Let  $d \in \mathbb{Z}$ ,  $d \geq 2$ .

**2.4 Notation.** Let  $N_d$  be the normal closure of  $t_1, t_2, \dots, t_p$  in  $\Sigma_{g,1,p}$ . We denote by  $\Sigma_{g,1,p^{(d)}}$  the group  $\Sigma_{g,1,p}/N_d$ . For every  $1 \leq k \leq p$ , we denote by  $\tau_k$  the image of  $t_k$  by the natural homomorphism  $\Sigma_{g,1,p} \rightarrow \Sigma_{g,1,p^{(d)}}$ .

Notice that if  $p = 0$ , then  $N_d = 1$  and  $\Sigma_{g,1,p^{(d)}} = \Sigma_{g,1,p}$ .

**2.5 Definition.** Let  $\mathcal{AM}_{g,1,p^{(d)}}$  denote the group of all automorphisms of  $\Sigma_{g,1,p^{(d)}}$  that respect the sets

$$\{\Pi_{i=1}^g[x_i, y_i] \Pi_{l=1}^p \tau_l\}, \quad \{\{\bar{t}_k\}\}_{1 \leq k \leq p}.$$

Since the elements of  $\mathcal{AM}_{g,1,p}$  respect the set  $\{\{\bar{t}_k\}\}_{1 \leq k \leq p}$ , the natural homomorphism  $\Sigma_{g,1,p} \rightarrow \Sigma_{g,1,p^{(d)}}$  induces a natural homomorphism

$$\psi : \mathcal{AM}_{g,1,p} \rightarrow \mathcal{AM}_{g,1,p^{(d)}}.$$

If  $p = 0$ , then  $\Sigma_{g,1,p} = \Sigma_{g,1,p^{(d)}}$  and  $\psi$  is the identity.

**2.6 Theorem.** *The homomorphism  $\psi : \mathcal{AM}_{g,1,p} \rightarrow \mathcal{AM}_{g,1,p^{(d)}}$  is injective.*

We proof Theorem 2.6 in Section 4.

Let  $\kappa : S_{g',b,0} \rightarrow S_{g,1,0}$  be an index  $m$  regular cover with  $p$  branching points in  $S_{g,1,0}$  which lift to  $q$  points in  $S_{g',b,0}$ . Suppose every branching point of  $S_{g,1,0}$  lift to the same number of points in  $S_{g',b,0}$ . Notice that  $q = 0$  if and only if  $p = 0$ . Let  $\kappa' : S_{g',b,q} \rightarrow S_{g,1,p}$  be the corresponding unbranched cover. We identify  $\Sigma_{g',b,q}$  with  $\kappa'_*(\Sigma_{g',b,q})$ . Hence,  $\Sigma_{g',b,q}$  is a normal subgroup of  $\Sigma_{g,1,p}$  of index  $m$ . We set  $G := \Sigma_{g,1,p}/\Sigma_{g',b,q}$  the group of deck transformations.

We put  $\varrho = \Pi_{i=1}^g[x_i, y_i] \Pi_{k=1}^p t_k \Sigma_{g',b,q} \in G$ . Let  $c$  be the order of  $\varrho$  in  $G$ . Since  $\varrho^c = 1$  in  $G$ , we see that  $(\Pi_{i=1}^g[x_i, y_i] \Pi_{k=1}^p t_k)^c \in \Sigma_{g',b,q}$ . Notice that  $(\Pi_{i=1}^g[x_i, y_i] \Pi_{k=1}^p t_k)^{-c}$  represents a loop around the  $b$ -th boundary component. We take a basis  $\{\hat{x}_i, \hat{y}_i\}_{1 \leq i \leq g'} \cup \{\hat{z}_l\}_{1 \leq l \leq (b-1)} \cup \{\hat{t}_k\}_{1 \leq k \leq q}$  of  $\Sigma_{g',b,q}$  such that

$$\Pi_{i=1}^{g'}[\hat{x}_i, \hat{y}_i] \Pi_{l=1}^{b-1} \hat{z}_l \Pi_{k=1}^q \hat{t}_k = (\Pi_{i=1}^g[x_i, y_i] \Pi_{k=1}^p t_k)^c.$$

Recall  $G$  has cardinality  $m$ . The subgroup  $\langle \varrho \rangle \leq G$  has index  $b = m/c$ . For every  $1 \leq l \leq b-1$ , we take  $w_l \in \Sigma_{g,1,p} - \Sigma_{g',b,q}$  such that

$$\hat{z}_l = \bar{w}_l (\Pi_{i=1}^g[x_i, y_i] \Pi_{k=1}^p t_k)^{-c} w_l.$$

We put  $\rho_l = w_l \Sigma_{g',b,q} \in G$ . Then  $G = \langle \varrho \rangle \rho_1 \cup \langle \varrho \rangle \rho_2 \cdots \cup \langle \varrho \rangle \rho_{(b-1)} \cup \langle \varrho \rangle$ . That is, the boundary components of  $S_{g',b,p}$  are image by deck transformations of the  $b$ -th boundary component.

For every  $1 \leq k \leq p$ , we put  $\varrho_k = t_k \Sigma_{g',b,q} \in G$ . Let  $d_k$  be the order of  $\varrho_k$  in  $G$ . Since  $t_k$  corresponds to a branching point,  $t_k \notin \Sigma_{g',b,q}$  and  $d_k \geq 2$ . Since  $\varrho_k^{d_k} = 1$  in  $G$ , we see that  $t_k^{d_k} \in \Sigma_{g',b,q}$ . Notice that  $t_k^{d_k}$  represents a loop around a lift of the  $k$ -th puncture of  $S_{g,1,p}$ . The subgroup  $\langle \varrho_k \rangle$  has index  $m/d_k$  in  $G$ . Since all the branching point of  $S_{g,1,0}$  lift to the same number of points in  $S_{g',b,0}$ ,  $m/d_1 = m/d_k$  for all  $2 \leq k \leq p$ . Hence,  $d_1 = d_k$  for all  $2 \leq k \leq p$ . Let  $d = d_1$ . We have  $G = \langle \varrho_k \rangle \rho_{1,k} \cup \langle \varrho_k \rangle \rho_{2,k} \cup \cdots \cup \langle \varrho_k \rangle \rho_{m/d,k}$ , where  $\rho_{j,k} = u_{j,k} \Sigma_{g',b,q} \in G$  for all  $1 \leq j \leq m/d$ . Notice that  $(t_k^d)^{u_{1,k}}, (t_k^d)^{u_{2,k}}, \dots, (t_k^d)^{u_{m/d,k}}$  represent loops around the  $m/d$  lifts of the  $k$ -th puncture. We choose  $u_{1,k}, u_{2,k}, \dots, u_{m/d,k} \in \Sigma_{g,1,p}$  such that  $\{(t_k^d)^{u_{1,k}}, (t_k^d)^{u_{2,k}}, \dots, (t_k^d)^{u_{m/d,k}}\} \subseteq \{\hat{t}_1, \hat{t}_2, \dots, \hat{t}_q\}$ . Then

$$(2.6.1) \quad \{\hat{t}_1, \hat{t}_2, \dots, \hat{t}_q\} = \bigcup_{k=1}^p \{(t_k^d)^{u_{1,k}}, (t_k^d)^{u_{2,k}}, \dots, (t_k^d)^{u_{m/d,k}}\}.$$

Recall  $N_d$  is the normal closure of  $t_1^d, t_2^d, \dots, t_p^d$  in  $\Sigma_{g,1,p}$ .

**2.7 Lemma.** *With the above notation,  $N_d$  is equal to the normal closure of  $\hat{t}_1, \hat{t}_2, \dots, \hat{t}_q$  in  $\Sigma_{g',b,q}$ .*

*Proof.* By (2.6.1), the normal closure of  $\hat{t}_1, \hat{t}_2, \dots, \hat{t}_q$  in  $\Sigma_{g',b,q}$  is a subgroup of  $N_d$ .

Let  $1 \leq k \leq p$  and  $w \in \Sigma_{g,1,p}$ . By (2.6.1), it is enough to proof  $(t_k^d)^w = (t_k^d)^{u_{j,k}v}$  for some  $1 \leq j \leq (m/d)$  and  $v \in \Sigma_{g',b,q}$ . Recall  $G = \Sigma_{g,1,p}/\Sigma_{g',b,q}$ ,  $\varrho_k = t_k \Sigma_{g',b,q} \in G$  and  $G = \langle \varrho_k \rangle \rho_{1,k} \cup \langle \varrho_k \rangle \rho_{2,k} \cup \cdots \cup \langle \varrho_k \rangle \rho_{m/d,k}$ , where  $\rho_{j,k} = u_{j,k} \Sigma_{g',b,q} \in G$  for all  $1 \leq j \leq (m/d)$ . Let  $1 \leq j \leq (m/d)$  such that  $w \Sigma_{g',b,q} \in \langle \varrho_k \rangle \rho_{j,k}$ . Let  $1 \leq r \leq d$  such that  $w \Sigma_{g',b,q} = \varrho_k^r \rho_{j,k} = t_k^r u_{j,k} \Sigma_{g',b,q}$ . Then  $w = t_k^r u_{j,k} v$ , for some  $v \in \Sigma_{g',b,q}$  and  $(t_k^d)^w = (t_k^d)^{t_k^r u_{j,k} v} = (t_k^d)^{u_{j,k} v}$ .  $\square$

Recall  $\Sigma_{g,1,p}/N_d = \Sigma_{g,1,p^{(d)}}$ , and for every  $1 \leq k \leq p$ , we denote by  $\tau_k$  the image of  $t_k$  by the natural homomorphism  $\Sigma_{g,1,p} \rightarrow \Sigma_{g,1,p^{(d)}}$ .

**2.8 Notation.** Let  $H \leq \Sigma_{g,1,p}$  be a normal subgroup of finite index such that  $N_d \leq H$ . Notice  $H/N_d \leq \Sigma_{g,1,p^{(d)}}$ . We set

$$\mathcal{AM}_{g,1,p}(H) = \{\phi \in \mathcal{AM}_{g,1,p} \mid H^\phi = H\},$$

and

$$\mathcal{AM}_{g,1,p^{(d)}}(H/N_d) = \{\tilde{\phi} \in \mathcal{AM}_{g,1,p^{(d)}} \mid (H/N_d)^{\tilde{\phi}} = H/N_d\}.$$

**2.9 Proposition.** *Suppose  $(g, p, d) \neq (0, 2, 2)$ . Let  $H \leq \Sigma_{g,1,p}$  be a normal subgroup of finite index such that  $N_d \leq H$ . Let  $\phi \in \mathcal{AM}_{g,1,p}(H)$ . Then  $\psi(\phi) \in \mathcal{AM}_{g,1,p^{(d)}}(H/N_d)$ . If  $\psi(\phi)|_{H/N_d} = 1$ , then  $\phi = 1$ .*

*Proof.* Since  $N_d$  and  $H$  are  $\phi$ -invariant, we see  $H/N_d$  is  $\psi(\phi)$ -invariant. Since  $\psi(\phi) \in \mathcal{AM}_{g,1,p^{(d)}}$ , we have  $\psi(\phi) \in \mathcal{AM}_{g,1,p}(H/N_d)$

Since  $H$  has finite index in  $\Sigma_{g,1,p}$ , there exists  $r \in \mathbb{Z}$ ,  $r \geq 1$ , such that

$$(\Pi_{i=1}^g [x_i, y_i] \Pi_{k=1}^p t_k)^r \in H.$$

Fix  $1 \leq k \leq p$ . Since  $H$  is normal in  $\Sigma_{g,1,p}$ , we see

$$\bar{t}_k (\Pi_{i=1}^g [x_i, y_i] \Pi_{k'=1}^p t_{k'})^r t_k \in H.$$

If  $\psi(\phi)|_{H/N_d} = 1$ , in  $\Sigma_{g,1,p^{(d)}}$ ,

$$\begin{aligned} & \bar{\tau}_k (\Pi_{i=1}^g [x_i, y_i] \Pi_{k'=1}^p \tau_{k'})^r \tau_k \\ &= (\bar{\tau}_k (\Pi_{i=1}^g [x_i, y_i] \Pi_{k'=1}^p \tau_{k'})^r \tau_k)^{\psi(\phi)} \\ &= \bar{\tau}_k^{\psi(\phi)} (\Pi_{i=1}^g [x_i, y_i] \Pi_{k'=1}^p \tau_{k'})^r \tau_k^{\psi(\phi)}. \end{aligned}$$

Then, in  $\Sigma_{g,1,p^{(d)}}$ ,  $\bar{\tau}_k^{\psi(\phi)} \tau_k$  commutes with  $(\Pi_{i=1}^g [x_i, y_i] \Pi_{k'=1}^p \tau_{k'})^r$ . Recall  $\Sigma_{g,1,p^{(d)}} = \Sigma_{g,1,p}/N_d$ . Hence,  $\Sigma_{g,1,p^{(d)}} \simeq \Sigma_{g,1,0} * \langle \tau_1, \tau_2, \dots, \tau_p \mid \tau_1^d, \tau_2^d, \dots, \tau_p^d \rangle$ . Hence,  $\bar{\tau}_k^{\psi(\phi)} \tau_k \in \langle \Pi_{i=1}^g [x_i, y_i] \Pi_{k'=1}^p \tau_{k'} \rangle$ , and,

$$(2.9.1) \quad \tau_k^{\psi(\phi)} = (\Pi_{i=1}^g [x_i, y_i] \Pi_{k'=1}^p \tau_{k'})^{r'} \tau_k,$$

for some  $r' \in \mathbb{Z}$ . Recall  $[\tau_k^{\psi(\phi)}] = [\tau_j]$ , for some  $1 \leq j \leq p$ . If  $(g, p) \neq (0, 1)$ , and if  $(g, p, d) \neq (0, 2, 2)$ , then (2.9.1) implies  $r' = 0$  and  $\tau_k^{\psi(\phi)} = \tau_k$ .

Fix  $a \in \{x_i, y_i\}_{1 \leq i \leq g}$ . Since  $H$  has finite index in  $\Sigma_{g,1,p}$ , there exists  $s \in \mathbb{Z}$ ,  $s \geq 1$ , such that  $a^s \in H$ . If  $\psi(\phi)|_{H/N_d} = 1$ , then  $(a^s)^{\psi(\phi)} = a^s$ , and,  $a^{\psi(\phi)} = a$ .

Since  $\Sigma_{g,1,p^{(d)}} \simeq \Sigma_{g,1,0} * \langle \tau_1, \tau_2, \dots, \tau_p \mid \tau_1^d, \tau_2^d, \dots, \tau_p^d \rangle$ ,  $a^{\psi(\phi)} = a$  for all  $a \in \{x_i, y_i\}_{1 \leq i \leq g}$ , and,  $\bar{\tau}_k^{\psi(\phi)} = \tau_k$  for all  $1 \leq k \leq p$ ; we see  $\psi(\phi) = 1$ . By Theorem 2.6,  $\phi = 1$ .  $\square$

Let  $\phi \in \mathcal{AM}_{g,1,p}$ . Suppose  $\Sigma_{g',b,q}$  is  $\phi$ -invariant. Then  $\phi$  induces an automorphisms of  $\Sigma_{g',b,q}$  by restriction. In  $\Sigma_{g,1,p}$  we have

- (i)  $\Pi_{i=1}^{g'} [\hat{x}_i, \hat{y}_i] \Pi_{l=1}^{(b-1)} \hat{z}_l \Pi_{k=1}^q \hat{t}_k = (\Pi_{i=1}^g [x_i, y_i] \Pi_{k=1}^p t_k)^c$ ;
- (ii)  $\hat{z}_l$  is conjugate to  $(\Pi_{i=1}^g [x_i, y_i] \Pi_{k=1}^p t_k)^{-c}$ , for all  $1 \leq l \leq (b-1)$ ;
- (iii)  $\hat{t}_k$  is conjugate to  $t_j^d$ ,  $1 \leq j \leq p$ , for all  $1 \leq k \leq q$ .

If we identify  $\Sigma_{g',b,q}$  with  $\Sigma_{g',1,(b-1)+q}$  by identifying  $\hat{z}_l$  with  $\hat{t}_l$ , for all  $1 \leq l \leq (b-1)$ , and  $\hat{t}_k$  with  $\hat{t}_{(b-1)+k}$ , for all  $1 \leq k \leq q$ ; then the restriction of  $\phi$  to  $\Sigma_{g',1,(b-1)+q}$  lies inside  $\mathcal{AM}_{g',1,(b-1)+q}$ .

Let  $h$  be the homeomorphism of  $S_{g,1,p}$  which fixes the boundary component pointwise and  $h_* = \phi$ . Since  $\Sigma_{g',b,q}$  is  $\phi$ -invariant,  $h$  lifts to a homeomorphism



$\hat{h}$  of  $S_{g',b,q}$  which fixes the  $b$ -th boundary component pointwise. Since  $\hat{h}$  may not fix the first  $(b-1)$  boundary components pointwise,  $\hat{h}$  does not represent an element of  $\mathcal{M}_{g',b,q}$ , but it represents an element of  $\mathcal{M}_{g',1,(b-1)+q}$ , that is, we have to convert the first  $(b-1)$  boundary components into punctures. If  $\hat{h}$  fixes the boundary components pointwise, we can conserve the first  $(b-1)$  boundary components. Algebraically, if we want to have an element of  $\mathcal{AM}_{g',b,q}$ , we have to define the image of  $\hat{e}_1, \hat{e}_2, \dots, \hat{e}_{(b-1)}$ . Since  $\Sigma_{g',b,q}$  is  $\phi$ -invariant, we see  $\phi$  induces an automorphism of  $G = \Sigma_{g,1,p}/\Sigma_{g',b,q}$ . If  $\phi$  induces the identity of  $G$ , we can define an element of  $\mathcal{AM}_{g',b,q}$  from  $\phi$ .

Recall  $N_d$  is the normal closure in  $\Sigma_{g,1,p}$  of  $t_1^d, t_2^d, \dots, t_p^d$ . By Lemma 2.7,  $N_d$  is the normal closure in  $\Sigma_{g',b,q}$  of  $\hat{t}_1, \hat{t}_2, \dots, \hat{t}_q$ . Hence,  $\Sigma_{g',b,0} = \Sigma_{g',b,q}/N_d$ . We identify  $\Sigma_{g',b,0}$  with  $\Sigma_{g',1,(b-1)}$  by identifying  $\hat{z}_l$  with  $\hat{t}_l$  for all  $1 \leq l \leq (b-1)$ . Hence,  $\Sigma_{g',1,(b-1)} = \Sigma_{g',b,q}/N_d$ . Since  $\Sigma_{g',b,q}$  is  $\phi$ -invariant, by Proposition 2.9, there exists the restriction  $\psi(\phi)|_{\Sigma_{g',1,(b-1)}} : \Sigma_{g',1,(b-1)} \rightarrow \Sigma_{g',1,(b-1)}$ . Recall  $\hat{h}$  is a homeomorphism of  $S_{g',b,q}$ . Since  $h_* = \phi$ , we have  $\hat{h}_* = \phi|_{\Sigma_{g',b,q}}$ . Notice  $\hat{h}$  extends to a homeomorphism  $\hat{f}$  of  $S_{g',b,0}$ . Notice  $\hat{f}$  restricts to a homeomorphism of  $S_{g',1,(b-1)} \subseteq S_{g',b,0}$ . Since  $\hat{h}_* = \phi|_{\Sigma_{g',b,q}}$  and  $\Sigma_{g',1,(b-1)} = \Sigma_{g',b,q}/N_d$ , the restriction of  $\hat{f}$  to  $S_{g',1,(b-1)} \subseteq S_{g',b,0}$  induces the automorphism  $\psi(\phi)|_{\Sigma_{g',1,(b-1)}}$ .

We, now, can proof the algebraic analog of Theorem 1.1.

**2.10 Theorem.** *Suppose  $(g,p) \neq (0,2)$ . Let  $\kappa : S_{g',b,0} \rightarrow S_{g,1,0}$  be a finite index regular cover with  $p$  branching points in  $S_{g,1,0}$  which lift to  $q$  points in  $S_{g',b,0}$ . Suppose every branching point of  $S_{g,1,0}$  lift to the same number of points in  $S_{g',b,0}$ . Let  $\kappa' : S_{g',b,q} \rightarrow S_{g,1,p}$  be the corresponding unbranched cover. Let  $\phi$  be an element of  $\mathcal{AM}_{g,1,p}$ . Suppose  $\Sigma_{g',b,q}$  is  $\phi$ -invariant. If  $\psi(\phi)|_{\Sigma_{g',1,(b-1)}} = 1$ , then  $\phi = 1$ .*

*Proof.* Since  $\Sigma_{g,1,p^{(d)}} = \Sigma_{g,1,p}/N_d$ , the natural homomorphism  $\Sigma_{g,1,p} \rightarrow \Sigma_{g,1,p^{(d)}}$  restricts to the natural homomorphism  $\Sigma_{g',b,q} \rightarrow \Sigma_{g',1,(b-1)}$ .

Since  $\psi : \mathcal{AM}_{g,1,p} \rightarrow \mathcal{AM}_{g,1,p^{(d)}}$  is given by the natural homomorphism  $\Sigma_{g,1,p} \rightarrow \Sigma_{g,1,p^{(d)}}$ , we see  $\psi(\phi) : \Sigma_{g,1,p^{(d)}} \rightarrow \Sigma_{g,1,p^{(d)}}$  completes the following commutative square

$$\begin{array}{ccc} \Sigma_{g,1,p} & \xrightarrow{\phi} & \Sigma_{g,1,p} \\ \downarrow & & \downarrow \\ \Sigma_{g,1,p^{(d)}} & \xrightarrow{\psi(\phi)} & \Sigma_{g,1,p^{(d)}} \end{array}$$

where the vertical arrows are the natural homomorphisms. Notice  $\psi(\phi)|_{\Sigma_{g',1,(b-1)}} : \Sigma_{g',1,(b-1)} \rightarrow \Sigma_{g',1,(b-1)}$  completes the following commutative square

$$\begin{array}{ccc} \Sigma_{g',b,q} & \xrightarrow{\phi|_{\Sigma_{g',b,q}}} & \Sigma_{g',b,q} \\ \downarrow & & \downarrow \\ \Sigma_{g',1,(b-1)} & \xrightarrow{\psi(\phi)|_{\Sigma_{g',1,(b-1)}}} & \Sigma_{g',1,(b-1)} \end{array}$$

where the vertical arrows are the natural homomorphisms. By Proposition 2.9, if  $\psi(\phi)|_{\Sigma_{g',1,(b-1)}} = 1$ , then  $\phi = 1$ .  $\square$

We state the algebraic analog of Corollary 1.2.

**2.11 Corollary.** *Suppose  $(g, p) \neq (0, 2)$ . Let  $\kappa : S_{g',b,0} \rightarrow S_{g,1,0}$  be a finite index regular cover with  $p$  branching points in  $S_{g,1,0}$  which lift to  $q$  points in  $S_{g',b,0}$ . Suppose every branching point of  $S_{g,1,0}$  lift to the same number of points in  $S_{g',b,0}$ . Let  $\kappa' : S_{g',b,q} \rightarrow S_{g,1,p}$  be the corresponding unbranched cover. If  $\Sigma_{g',b,q}$  is  $\mathcal{AM}_{g,1,p}$ -invariant, then  $\mathcal{M}_{g,1,p}$  embeds in  $\mathcal{M}_{g',1,(b-1)}$ . In fact, the embedding is given by  $\phi \mapsto \psi(\phi)|_{\Sigma_{g',1,(b-1)}}$ .*

### 3 Examples

We fix  $g, p$  such that  $(g, p) \neq (0, 2)$ . Let  $\hat{S}$  be the universal cover of  $S_{g,1,p}$ . The fundamental group of  $S_{g,1,p}$ , denoted  $\Sigma_{g,1,p}$ , acts on  $\hat{S}$ . Let  $H$  be a subgroup of  $\Sigma_{g,1,p}$  of index  $m$ . Suppose  $H$  is  $\mathcal{AM}_{g,1,p}$ -invariant. The quotient space  $\hat{S}/H$  is an orientable surface, denoted  $S_{g',b,q}$ . We identify the fundamental group of  $S_{g',b,q}$ , denoted  $\Sigma_{g',b,q}$ , with  $H$ . The cover  $\hat{S} \rightarrow S_{g,1,p}$  induces a cover  $S_{g',b,q} \rightarrow S_{g,1,p}$  with group of deck transformation  $G := \Sigma_{g,1,p}/\Sigma_{g',b,q}$ . If  $t_k \notin \Sigma_{g',b,q}$  for all  $1 \leq k \leq p$ , then the corresponding cover  $S_{g',b,0} \rightarrow S_{g,1,0}$  has  $p$  branching points in  $S_{g,1,0}$  which lift to  $q$  points in  $S_{g',b,0}$ . Since  $H$  is  $\mathcal{AM}_{g,1,p}$ -invariant, it can be seen that every branching point of  $S_{g,1,p}$  lifts to the same number of points in  $S_{g',b,0}$ . By Corollary 2.11, we have an embedding  $\mathcal{AM}_{g,1,p} \hookrightarrow \mathcal{AM}_{g',1,(b-1)}$ . By choosing an appropriated basis of  $H$ , we can compute elements in the image of this embedding from elements of  $\mathcal{AM}_{g,1,p}$ .

The first example is well-known. In the second example, we give a basis of  $H$  and compute elements in the image of the embedding.

**Example 1.** Let  $H$  be the kernel of the homomorphism  $\Sigma_{0,1,p} \rightarrow \langle \tau \mid \tau^2 \rangle$  such that  $t_k \mapsto \tau$  for all  $1 \leq k \leq p$ . It is standard to see that  $H$  is a free group of rank  $2p - 1$  with basis  $t_1^2, t_1 t_2, t_1 t_3, \dots, t_1 t_p, t_1 \bar{t}_2, t_1 \bar{t}_3, \dots, t_1 \bar{t}_p$ . It is easy to see that  $H$  is invariant by the generators of  $\mathcal{AM}_{0,1,p}$  given in (2.3.1). For  $1 \leq k \leq p$ , notice that  $\varrho_k = t_k H$  has order 2 in  $G := \Sigma_{0,1,p}/H \simeq \langle \tau \mid \tau^2 \rangle$ . Hence,  $\langle \varrho_k \rangle$  has index 1 in  $G$  and the  $k$ -th puncture in  $S_{g,1,p}$  lifts to one puncture in  $S_{g',b,q}$ . Thus,  $q = p$ .

- (a). If  $p$  is odd, then  $\prod_{k=1}^p t_k \notin H$  and  $\varrho = \prod_{k=1}^p t_k H$  has order 2 in  $G$ . Hence,  $\langle \varrho \rangle$  has index 1 in  $G$  and  $b = 1$ . Since  $\Sigma_{g',b,q}$  has rank  $2g' + b - 1 + q$  and  $H$  has rank  $2p - 1$ , we have  $2g' + 1 - 1 + p = 2p - 1$  and  $g' = (p - 1)/2$ . Hence,  $\mathcal{AM}_{0,1,p} \hookrightarrow \mathcal{AM}_{(p-1)/2,1,0}$ , if  $p$  is odd.
- (b). If  $p$  is even, then  $\prod_{k=1}^p t_k \in H$  and  $\varrho = \prod_{k=1}^p t_k H$  has order 1 in  $G$ . Hence,  $\langle \varrho \rangle$  has index 2 in  $G$  and we have  $b = 2$ . Since  $\Sigma_{g',b,q}$  has rank  $2g' + b - 1 + q$

and  $H$  has rank  $2p - 1$ , we have  $2g' + 2 - 1 + p = 2p - 1$  and  $g' = (p - 2)/2$ . Hence,  $\mathcal{AM}_{0,1,p} \hookrightarrow \mathcal{AM}_{(p-2)/2,1,1}$ , if  $p$  is even.

**Example 2.** Let  $\Sigma_{1,1,0} = \langle x, y \mid \rangle$ . Let  $H$  be the kernel of the homomorphism  $\Sigma_{1,1,0} \rightarrow \langle \tau_1 \mid \tau_1^2 \rangle \times \langle \tau_2 \mid \tau_2^2 \rangle$  such that  $x \mapsto \tau_1$ ,  $y \mapsto \tau_2$ . It is standard to see that  $H$  is a free group of rank 5. It can be shown that  $H$  is a characteristic subgroup of  $\Sigma_{1,1,0}$ . Notice that  $\varrho = [x, y]H$  has order 1 in  $G := \Sigma_{1,1,0}/H \simeq \langle \tau_1 \mid \tau_1^2 \rangle \times \langle \tau_2 \mid \tau_2^2 \rangle$ . Hence,  $\langle \varrho \rangle$  has index 4 in  $G$  and  $b = 4$ . We have  $p = 0$  and  $q = 0$ . Since  $\Sigma_{g',b,q}$  has rank  $2g' + b - 1 + q$  and  $H$  has rank 5, we have  $2g' + 4 - 1 + 0 = 5$  and  $g' = 1$ . Hence,  $\mathcal{AM}_{1,1,0} \hookrightarrow \mathcal{AM}_{1,1,3}$ . We take the following basis of  $\Sigma_{1,1,3}$ :  $\hat{x} = x^2$ ,  $\hat{y} = y^2$ ,  $\hat{t}_1 = (\bar{y} \bar{y} x x)^{\bar{x}} \bar{y}^2 x^2 y^2$ ,  $\hat{t}_2 = (\bar{y} \bar{y} x x)^y$ ,  $\hat{t}_3 = (\bar{y} \bar{y} x x)^{xy}$ . It is well-known that  $\mathcal{AM}_{1,1,0} = \langle \alpha, \beta \mid \alpha\beta\alpha = \beta\alpha\beta \rangle$ , where

$$\alpha := \begin{cases} x & \mapsto \bar{y}x, \\ y & \mapsto y, \end{cases} \quad \beta := \begin{cases} x & \mapsto x, \\ y & \mapsto xy. \end{cases}$$

A straightforward computation shows that the image of  $\alpha$  and  $\beta$  in  $\mathcal{AM}_{1,1,3}$ , denoted  $\hat{\alpha}$  and  $\hat{\beta}$ , are

$$\hat{\alpha} := \begin{cases} \hat{x} & \mapsto \hat{y}^{-1} \hat{x} \hat{y} \hat{t}_2 \hat{t}_3 \hat{t}_2^{-1} \hat{y}^{-1}, \\ \hat{y} & \mapsto \hat{y}, \\ \hat{t}_1 & \mapsto \hat{t}_3^{\hat{t}_2^{-1} \hat{y}^{-1} \hat{x}^{-1} \hat{y}^{-1} \hat{x} \hat{y} \hat{t}_2 \hat{t}_3 \hat{t}_2^{-1}}, \\ \hat{t}_2 & \mapsto \hat{t}_2, \\ \hat{t}_3 & \mapsto \hat{t}_1^{\hat{t}_2 \hat{t}_3}, \end{cases} \quad \hat{\beta} := \begin{cases} \hat{x} & \mapsto \hat{x}, \\ \hat{y} & \mapsto \hat{x} \hat{y} \hat{t}_2, \\ \hat{t}_1 & \mapsto \hat{t}_1^{\hat{y}^{-1} \hat{x}^{-1} \hat{y} \hat{t}_2^{-1} \hat{y}^{-1} \hat{x} \hat{y} \hat{t}_2}, \\ \hat{t}_2 & \mapsto \hat{t}_3, \\ \hat{t}_3 & \mapsto \hat{t}_2^{\hat{y}^{-1} \hat{x} \hat{y} \hat{t}_2 \hat{t}_3}. \end{cases}$$

**Example 3.** Let  $F_3 := \langle a_1, a_2, a_3 \mid \rangle$ . Let  $H$  be the kernel of the homomorphism  $F_3 \rightarrow \langle \tau_1 \mid \tau_1^2 \rangle \times \langle \tau_2 \mid \tau_2^2 \rangle \times \langle \tau_3 \mid \tau_3^2 \rangle$  such that  $a_k \mapsto \tau_k$  for all  $1 \leq k \leq 3$ . It is standard to see that  $H$  is a free group of rank 17. It can be shown that  $H$  is a characteristic subgroup of  $F_3$ .

- (a). We identify  $\Sigma_{0,1,3}$  with  $F_3$  by putting  $t_k \leftrightarrow a_k$  for all  $1 \leq k \leq 3$ . Notice that  $\varrho = t_1 t_2 t_3 H$  has order 2 in  $G := \Sigma_{0,1,3}/H \simeq \langle \tau_1 \mid \tau_1^2 \rangle \times \langle \tau_2 \mid \tau_2^2 \rangle \times \langle \tau_3 \mid \tau_3^2 \rangle$ . Hence,  $\langle \varrho \rangle$  has index 4 in  $G$  and  $b = 4$ . On the other hand, for all  $1 \leq k \leq 3$ ,  $\varrho_k = t_k H$  has order 2 in  $G$ . Hence, for all  $1 \leq k \leq 3$ ,  $\langle \varrho_k \rangle$  has index 4 in  $G$  and the  $k$ -th puncture in  $S_{0,1,3}$  lifts to 4 punctures in  $S_{g',b,q}$ . Thus,  $q = 12$ . Since  $\Sigma_{g',b,q}$  has rank  $2g' + b - 1 + q$  and  $H$  has rank 17, we have  $2g' + 4 - 1 + 12 = 17$  and  $g' = 1$ . Hence,  $\mathcal{AM}_{0,1,3} \hookrightarrow \mathcal{AM}_{1,1,3}$ .
- (b). We identify  $\Sigma_{1,1,1}$  with  $F_3$  by putting  $x \leftrightarrow a_1$ ,  $y \leftrightarrow a_2$  and  $t \leftrightarrow a_3$ . Notice that  $\varrho = [x, y]tH$  has order 2 in  $G := \Sigma_{1,1,1}/H \simeq \langle \tau_1 \mid \tau_1^2 \rangle \times \langle \tau_2 \mid \tau_2^2 \rangle \times \langle \tau_3 \mid \tau_3^2 \rangle$ . Hence,  $\langle \varrho \rangle$  has index 4 in  $G$  and  $b = 4$ . On the other hand,  $\varrho_1 = tH$  has order 2 in  $G$ . Hence,  $\langle \varrho_1 \rangle$  has index 4 in  $G$  and the puncture in  $S_{1,1,1}$  lifts to 4 punctures in  $S_{g',b,q}$ . Thus,  $q = 4$ . Since  $\Sigma_{g',b,q}$  has rank  $2g' + b - 1 + q$  and  $H$  has rank 17, we have  $2g' + 4 - 1 + 4 = 17$  and  $g' = 5$ . Hence,  $\mathcal{AM}_{1,1,1} \hookrightarrow \mathcal{AM}_{5,1,3}$ .

## 4 Proof of Theorem 2.6

**4.1 Definition.** An element of  $\Sigma_{g,1,p}$  is said to be *t-squarefree* if, in its reduced expression, no two consecutive terms in  $\{t_k, \bar{t}_k\}_{1 \leq k \leq p}$  are equal; for example:  $x_1 x_1 t_2 t_3$  is *t-squarefree*;  $x_1 t_2 t_2 y_1$  is non-*t-squarefree*.

To proof Theorem 2.6 we need the following theorem.

**4.2 Theorem.** For every  $\phi \in \mathcal{AM}_{g,1,p}$ , the elements of  $\{x_i^\phi, y_i^\phi\}_{1 \leq i \leq g} \cup \{t_k^\phi\}_{1 \leq k \leq p}$  are *t-squarefree*.

*Proof.* (of Theorem 2.6) If  $p = 0$ , then  $\psi$  is the identity and nothing needs to be said.

Suppose  $p \geq 1$ . Recall  $\Sigma_{g,1,p^{(d)}} \simeq \Sigma_{g,1,0} * \langle \tau_1, \tau_2, \dots, \tau_p \mid \tau_1^d, \tau_2^d, \dots, \tau_p^d \rangle$ . Let  $a \in \{x_i, y_i\}_{1 \leq i \leq g} \cup \{t_k\}_{1 \leq k \leq p}$ . If  $\phi$  is an element of the kernel of  $\psi : \mathcal{AM}_{g,1,p} \rightarrow \mathcal{AM}_{g,1,p^{(d)}}$ , then  $a^\phi$  and  $a$  have the same image in  $\Sigma_{g,1,p^{(d)}}$ . On the other hand, by Theorem 4.2,  $a^\phi$  is *t-squarefree*. Hence,  $a^\phi$  has the same normal form in  $\Sigma_{g,1,p}$  as in  $\Sigma_{g,1,p^{(d)}}$ . Thus,  $a^\phi = a$ .  $\square$

The rest of the paper is dedicated to proof Theorem 4.2. The proof is similar to [3, 7.6 Corollary]. Notice Theorem 4.2 is trivial if  $p = 0$ .

## 5 McCool's Groupoid

For the rest of the paper we suppose  $p \geq 1$ .

Let  $n := 2g + p$ , and, let  $F_n$  be the free group on  $X$ , where  $X$  is a set with  $n$  elements.

**5.1 Notation.** Let  $w \in F_n$ . In this section we will denote by  $[w]$  the cyclic word of  $w$ .

**5.2 Definitions.** Let  $T$  be a set of words and cyclic words of  $F_n$ . Suppose the elements of  $T$  are reduced and cyclically reduced, respectively. We define the *Whitehead graph of  $T$*  as the graph with vertex set  $X \cup \bar{X}$ , and, one edge from  $a \in X \cup \bar{X}$  to  $b \in X \cup \bar{X}$  for every subword  $\bar{a}b$  which appears in  $w$  or  $[u]$ , where  $w$  and  $[u]$  are elements of  $T$ . We say that  $a$  is the initial vertex and  $b$  is the terminal vertex of the edges corresponding to the subword  $\bar{a}b$ . Repetitions have to be considered. For example, since the subword  $\bar{a}b$  appears twice in  $\bar{a}b\bar{a}b$ , the Whitehead graph of  $\{\bar{a}b\bar{a}b\}$  has 2 edges from  $a$  to  $b$  (and one edge from  $\bar{b}$  to  $\bar{a}$ ). Notice that the cyclic word  $[a]$  produces an edge from  $\bar{a}$  to  $a$  in the Whitehead graph.

We say that  $T$  is a *surface word set* if the Whitehead graph of  $T$  is an oriented segment, that is, the Whitehead graph of  $T$  is connected with exactly  $2n - 1$  edges, every vertex but one is the *initial vertex* of exactly one edge, and, every vertex but one is the *terminal vertex* of exactly one edge.

**5.3 Example.** Let  $F_4 := \langle a, b, c, d \mid \rangle$ .

(i). Let  $T := \{\overline{adc\bar{b}}, [\overline{db}], [\overline{ca}]\}$ . The Whitehead graph of  $T$  is

$$\bar{a} \rightarrow \bar{c} \rightarrow \bar{b} \rightarrow \bar{d} \rightarrow c \rightarrow a \rightarrow d \rightarrow b.$$

Hence,  $T$  is a surface word set.

(ii). Let  $T := \{\overline{adc\bar{b}}, \overline{db}, [\overline{ca}]\}$ . The Whitehead graph of  $T$  is

$$\bar{a} \rightarrow \bar{c} \rightarrow \bar{b} \quad \bar{d} \rightarrow c \rightarrow a \rightarrow d \rightarrow b.$$

Hence,  $T$  is not a surface word set.

(iii). Let  $T := \{\overline{adc\bar{b}}, dc, [\overline{db}], [\overline{ca}]\}$ . The Whitehead graph of  $T$  is

$$\bar{a} \rightarrow \bar{c} \rightarrow \bar{b} \rightarrow \bar{d} \rightrightarrows c \rightarrow a \rightarrow d \rightarrow b.$$

Hence,  $T$  is not a surface word set.

We illustrate the following remarks with examples in  $F_4 = \langle a, b, c, d \mid \rangle$ .

**5.4 Remarks.** Let  $T$  be a surface word set.

(i) The Whitehead graph of  $T$  defines a sequence  $(a_k)_{1 \leq k \leq 2n}$  which lists the element of  $X \cup \overline{X}$  such that for all  $1 \leq k \leq (2n - 1)$ , the Whitehead graph of  $T$  has exactly one edge with initial vertex  $a_k$  and terminal vertex  $a_{k+1}$ , equivalently,  $\overline{a_k a_{k+1}}$  is a subword of exactly one element of  $T$ . We say that  $(a_k)_{1 \leq k \leq 2n}$  is the *associated sequence* of  $T$ .

In Example 5.3(i), the associated sequence of  $T$  is  $(\bar{a}, \bar{c}, \bar{b}, \bar{d}, c, a, d, b)$ .

(ii) We can recover  $T$  from the associated sequence of  $T$ . The process to recover  $T$  from its associated sequence is the invers process to construct the Whitehead graph. We give two examples below. From this process, it is easy to see that  $T$  has exactly one word, and, all other elements of  $T$  are cyclic words.

In  $F_4$ , from the sequence  $(a, b, c, d, \bar{a}, \bar{b}, \bar{c}, \bar{d})$  we have the surface word set  $\{\overline{abcd\bar{a}\bar{b}\bar{c}\bar{d}}\}$ , and, from the sequence  $(a, b, c, d, \bar{d}, \bar{c}, \bar{b}, \bar{a})$  we have the surface word set  $\{a, [\overline{b\bar{a}}], [\overline{c\bar{b}}], [\overline{d\bar{c}}], [\overline{d}]\}$ .

(iii) Let  $p$  be the cardinality of  $T$  minus one. We say that  $T$  is a  $(g, p)$ -surface word set, where  $g = (n - p)/2$ . By induction on  $n$ , it can be seen that  $n \geq p$  and  $n - p$  is even. Hence,  $g$  is a non-negative integer.

**5.5 Definition.** Let  $\phi \in \text{Aut}(F_n)$ .

We say that  $\phi$  is a *type-1 Nielsen automorphism* if  $\phi$  restricts to a permutation of  $X \cup \overline{X}$ .

We say that  $\phi$  is a *type-2 Nielsen automorphism* if there exists  $a, b \in X \cup \overline{X}$  such that  $a \neq b, \overline{b}$  and

$$\phi := \begin{cases} a \mapsto ab, \\ c \mapsto c \quad \text{for all } c \in X, c \neq a^{\pm 1}. \end{cases}$$

We denote  $\phi$  by  $(a \mapsto ab)$  or  $(\overline{a} \mapsto \overline{b}\overline{a})$ .

**5.6 Definition.** Let  $\mathcal{G}_{g,p}$  be the groupoid with objects  $(g, p)$ -surface word sets, and, given  $T_1, T_2$  two  $(g, p)$ -surface word sets

$$\text{Hom}(T_1, T_2) := \{\phi \in \text{Aut}(F_n) \mid T_1^\phi = T_2\},$$

where  $T_1^\phi := \{w^\phi, [u^\phi] \mid w, [u] \in T_1\}$ . Here,  $w^\phi$  is reduced and  $[u^\phi]$  is cyclically reduced. Hence,  $[v] = [u^\phi]$  means that  $v$  and  $u^\phi$  are conjugated.

We say that  $(T_1, T_2, \phi) \in \text{Hom}(T_1, T_2)$  is a *type-1 Nielsen* of  $\mathcal{G}_{g,p}$  if  $\phi$  is a type-1 Nielsen automorphism. Similarly, for type-2 Nielsen automorphisms. We say that  $(T_1, T_2, \phi) \in \text{Hom}(T_1, T_2)$  is a Nielsen if it is either a type-1 Nielsen or a type-2 Nielsen.

We illustrate the following remarks with examples in  $F_4 = \langle a, b, c, d \mid \rangle$ .

**5.7 Remark.** Let  $(T_1, T_2, \phi)$  be a Nielsen of  $\mathcal{G}_{g,p}$ .

- (i) If  $(T_1, T_2, \phi)$  is a type-1 Nielsen, then the associated sequence of  $T_2$  is obtained from the associated sequence of  $T_1$  by applying the permutation  $\phi$  to every element of the sequence.

In  $F_4$ , let  $T_1 = \{a\overline{d}\overline{b}c, [\overline{a}b], [\overline{c}d]\}$ . Notice the associated sequence of  $T_1$  is  $(a, b, c, d, \overline{b}, \overline{a}, \overline{d}, \overline{c})$ . If  $\phi := (a \mapsto \overline{b}, b \mapsto c, c \mapsto \overline{a}, d \mapsto \overline{d})$ , then the associated sequence of  $T_2$  is  $(\overline{b}, c, \overline{a}, \overline{d}, \overline{c}, b, d, a)$ .

- (ii) Suppose  $(T_1, T_2, \phi)$  is a type-2 Nielsen. Then  $\phi = (a_i \mapsto ba_i)$  for some  $1 \leq i \leq 2n$ ,  $b \in X \cup \overline{X}$  such that  $a_i \neq b, \overline{b}$ . Since in the Whitehead graph of  $T$  there are exactly  $2n-1$  edges, there exists  $w \in T_1$  or  $[u] \in T_1$  such that applying  $\phi$  to  $w$  or  $[u]$  produces a cancellation. If the cancellation appears from the subword  $\overline{a}_{i-1}a_i$ , then  $b = a_{i-1}$ . If the cancellation appears from the subword  $\overline{a}_i a_{i+1}$ , then  $b = a_{i+1}$ . Hence, either  $\phi = (a_i \mapsto a_{i-1}a_i)$  for some  $2 \leq i \leq 2n$ ,  $a_i \neq \overline{a}_{i-1}$ ; or  $\phi = (\overline{a}_i \mapsto \overline{a}_i \overline{a}_{i+1})$  for some  $1 \leq i \leq (2n-1)$ ,  $a_i \neq \overline{a}_{i+1}$ . In the former case the associated sequence of  $T_2$  is obtained from the associated sequence of  $T_1$  by moving  $a_i$  from immediately after  $a_{i-1}$  to immediately before  $\overline{a}_{i-1}$ . In the later case the associated sequence of  $T_2$  is obtained from the associated sequence of  $T_1$  by moving  $a_i$  from immediately before  $a_{i+1}$  to immediately after  $\overline{a}_{i+1}$ .

In  $F_4$ , let  $T_1 = \{\overline{abcd}\overline{ab\overline{cd}}\}$ . Notice the associated sequence of  $T_1$  is  $(a, b, c, d, \overline{a}, \overline{b}, \overline{c}, \overline{d})$ . If  $\phi := (b \mapsto ab)$ , then the associated sequence of  $T_2$  is  $(a, c, d, b, \overline{a}, \overline{b}, \overline{c}, \overline{d})$ . In fact  $(\overline{abcd}\overline{ab\overline{cd}})^{(b \mapsto ab)} = \overline{ab}\overline{acdb\overline{cd}}$ . If  $\phi := (\overline{a} \mapsto \overline{ab})$ , then the associated sequence of  $T_2$  is  $(b, c, d, \overline{a}, \overline{b}, a, \overline{c}, \overline{d})$ . In fact  $(\overline{abcd}\overline{ab\overline{cd}})^{(\overline{a} \mapsto \overline{ab})} = \overline{babc\overline{d}}\overline{a}\overline{cd}$ .

**5.8 Remark.** It is easy to see that  $\{\Pi_{i=1}^g[x_i, y_i]\Pi_{k=1}^p t_k, [\overline{t}_1], [\overline{t}_2], \dots, [\overline{t}_p]\}$  is a  $(g, p)$ -surface word set of  $\Sigma_{g,1,p}$ . Its associated sequence is

$$(\overline{x}_1, y_1, x_1, \overline{y}_1, \overline{x}_2, y_2, x_2, \overline{y}_2, \dots, \overline{x}_g, y_g, x_g, \overline{y}_g, t_1, \overline{t}_1, t_2, \overline{t}_2, \dots, t_p, \overline{t}_p).$$

We say that  $\{\Pi_{i=1}^g[x_i, y_i]\Pi_{k=1}^p t_k, [\overline{t}_1], [\overline{t}_2], \dots, [\overline{t}_p]\}$  is the standard  $(g, p)$ -surface word set of  $\Sigma_{g,1,p}$ .

**5.9 Remark.**  $\mathcal{AM}_{g,1,p} = \text{Hom}(T, T)$ , where  $T$  is the standard  $(g, p)$ -surface word set of  $\Sigma_{g,1,p}$ .

**5.10 Theorem** (McCool [15],[9]).  $\mathcal{G}_{g,p}$  is generated by Nielsen elements.

## 6 Ends of free group

Let  $n := 2g + p$  and let  $F_n$  be the free group on  $X$ , where  $|X| = n$ .

**6.1 Notation.** Let  $\Pi_{i=1}^k a_i$  be the normal form for  $w \in F_n$ . Then we say that  $w$  has length  $k$ , denoted  $|w| = k$ . The set of elements of  $F_n$  whose normal forms have  $\Pi_{i=1}^k a_i$  as an initial subword is denoted  $(w\star)$ ; and, the set of elements of  $F_n$  whose normal forms have  $\Pi_{i=1}^k a_i$  as a terminal subword is denoted  $(\star w)$ . The elements of  $(w\star)$  are said to *begin with*  $w$ , and the elements of  $(\star w)$  are said to *end with*  $w$ .

**6.2 Review.** An *end* of  $F_n$  is a sequence  $(a_k)_{k \geq 1}$  in  $X \cup \overline{X}$  such that, for each  $k \geq 1$ ,  $a_{k+1} \neq \overline{a}_k$ . We represent  $(a_k)_{k \geq 1}$  as a formal right-infinite reduced product,  $\Pi_{k \geq 1} a_k = a_1 a_2 \dots$ .

We denote the set of ends of  $F_n$  by  $\partial F_n$ .

For each  $w \in F_n$ , we define the *shadow* of  $w$  in  $\partial F_n$  to be

$$(w\blacktriangleleft) := \{(a_k)_{k \geq 1} \in \partial F_n \mid \Pi_{k=1}^{|w|} a_k = w\}.$$

Thus, for example,  $(1\blacktriangleleft) = \partial F_n$ .

**6.3 Definition.** Let  $T$  be a surface word set. We now give  $\partial F_n$  an ordering,  $<_T$ , with respect to  $T$  as follows. Let  $(a_k)_{1 \leq k \leq 2n}$  be the associated sequence of  $T$ . Recall  $(a_k)_{1 \leq k \leq 2n}$  is a listing of the elements of  $X \cup \overline{X}$ . For each  $w \in F_n$ , we assign an ordering,  $<_T$ , to a partition of  $(w\blacktriangleleft)$  into  $2n$  or  $2n - 1$  subsets, depending as  $w = 1$  or  $w \neq 1$ , as follows. We set

$$(a_1\blacktriangleleft) <_T (a_2\blacktriangleleft) <_T (a_3\blacktriangleleft) <_T \dots <_T (a_{2n-1}\blacktriangleleft) <_T (a_{2n}\blacktriangleleft).$$

If  $1 \leq i \leq n$  and  $w \in (\star \bar{a}_i)$ , then we set

$$\begin{aligned} (wa_{i+1} \blacktriangleleft) &<_T (wa_{i+2} \blacktriangleleft) <_T (wa_{i+3} \blacktriangleleft) <_T \cdots \\ \cdots &<_T (wa_{2n-1} \blacktriangleleft) <_T (wa_{2n} \blacktriangleleft) <_T (wa_1 \blacktriangleleft) <_T (wa_2 \blacktriangleleft) <_T (wa_3 \blacktriangleleft) <_T \cdots \\ \cdots &<_T (wa_{i-2} \blacktriangleleft) <_T (wa_{i-1} \blacktriangleleft). \end{aligned}$$

Hence, for each  $w \in F_n$ , we have an ordering  $<_T$  of a partition of  $(w \blacktriangleleft)$  into  $2n$  or  $2n - 1$  subsets.

If  $(b_k)_{k \geq 1}$  and  $(c_k)_{k \geq 1}$  are two different ends, then there exists  $j \in \mathbb{Z}, j \geq 0$ , such that  $(b_k)_{1 \leq k \leq j} = (c_k)_{1 \leq k \leq j}$  and  $b_{j+1} \neq c_{j+1}$ . Let  $w = \prod_{k=1}^j b_k = \prod_{k=1}^j c_k$  in  $F_n$ . Then  $(b_k)_{k \geq 1}$  and  $(c_k)_{k \geq 1}$  lie in  $(w \blacktriangleleft)$ , but lie in different elements of the partition of  $(w \blacktriangleleft)$  into  $2n$  or  $2n - 1$  subsets. We then order  $(b_k)_{k \geq 1}$  and  $(c_k)_{k \geq 1}$  using the order of the elements of the partition of  $(w \blacktriangleleft)$  that they belong to. This completes the definition of the ordering  $<_T$  of  $\partial F_n$ .

**6.4 Remark.** Let  $w$  be the non-cyclic element of  $T$ . In  $(\partial F_n, <_T)$ , the smallest element is  $w^\infty$  and the largest element is  $\bar{w}^\infty$ .

For example, in  $F_4 = \langle a, b, c, d \mid \rangle$  we take the surface word set  $T = \{a\bar{d} \bar{b}c, [\bar{a}b], [\bar{c}d]\}$ . The associated sequence of  $T$  is  $(a, b, c, d, \bar{b}, \bar{a}, \bar{d}, \bar{c})$ . In  $(\partial F_4, <_T)$ , the smallest element is  $(a\bar{d} \bar{b}c)^\infty$ , and, the largest element is  $(\bar{c}b\bar{d}\bar{a})^\infty$ .

**6.5 Notation.** We denote by  $<$  the order on  $\partial \Sigma_{g,1,p}$  with respect to the standard  $(g, p)$ -surface word set of  $\Sigma_{g,1,p}$ .

**6.6 Review.** Let  $\hat{S}$  be the universal cover of  $S_{g,1,p}$ . Suppose  $S_{g,1,p}$  has negative Euler characteristic, that is,  $2g + p \geq 2$ . Then  $\hat{S}$  can be identified with a region of the hyperbolic plane. Let  $\partial \hat{S}$  be the boundary of  $\hat{S}$ . It is well-known that  $\partial \hat{S}$  can be identified with  $\mathbb{R} \cup \{\infty\}$ . Let  $*$  be the point in  $\partial \hat{S}$  corresponding to  $\infty$  by this identification. The identification between  $\partial \hat{S}$  and  $\mathbb{R} \cup \{\infty\}$  restricts to an identification between  $\partial \hat{S} - \{*\}$  and  $\mathbb{R}$ . By work of Nielsen-Thurston [5], [16], there is an action of  $\mathcal{M}_{g,1,p}$  on  $\partial \hat{S}$  with a fixed point, which we can suppose to be  $*$   $\in \partial \hat{S}$ . Hence, there exists an action of  $\mathcal{M}_{g,1,p}$  on  $\mathbb{R}$ . By [16], this action preserves the usual order of  $\mathbb{R}$ . Remark 5.9 and Proposition 6.7 give the analog statement for  $\mathcal{AM}_{g,1,p}$  and  $\partial \Sigma_{g,1,p}$ .

Let  $\phi \in \text{Aut}(F_n)$ . It is proved in [5] that  $(\prod_{k \geq 1} a_k)^\phi = \lim_{k \rightarrow \infty} (\prod_{i=1}^k a_i)^\phi$  defines a map  $\partial F_n \rightarrow \partial F_n$ , which we still denote by  $\phi$ .

**6.7 Proposition.** *Let  $T_1, T_2$  be surface word sets of  $F_n$  and  $(T_1, T_2, \phi) \in \text{Hom}(T_1, T_2)$ . Then  $\phi : (\partial F_n, \leq_{T_1}) \rightarrow (\partial F_n, \leq_{T_2})$  respects the orderings.*

*Proof.* By Theorem 5.10, we can restrict ourselves to the case where  $(T_1, T_2, \phi)$  is a Nielsen.

By Remark 5.7(i), the result is clear if  $(T_1, T_2, \phi)$  is a type-1 Nielsen. Hence, we suppose  $(T_1, T_2, \phi)$  is a type-2 Nielsen.



Let  $(a_k)_{1 \leq k \leq 2n}$  be the associated sequence of  $T_1$ . Then either  $\phi = (a_i \mapsto a_{i-1}a_i)$  for some  $2 \leq i \leq 2n$ ,  $a_i \neq \bar{a}_{i-1}$ ; or,  $\phi = (\bar{a}_i \mapsto \bar{a}_i\bar{a}_{i+1})$  for some  $1 \leq i \leq (2n-1)$ ,  $a_i \neq \bar{a}_{i+1}$ .

Suppose  $\phi = (a_i \mapsto a_{i-1}a_i)$  for some  $2 \leq i \leq 2n$ ,  $a_i \neq \bar{a}_{i-1}$ .

The following correspondence by the action of  $(a_i \mapsto a_{i-1}a_i)$  is clear.

$$\begin{array}{ccc}
& (a_i \mapsto a_{i-1}a_i) & \\
(\star \bar{a}_i a_{i-1}) & \longrightarrow & (\star \bar{a}_i), \\
(\star a_{i-1}) - (\star \bar{a}_i a_{i-1}) & \longrightarrow & (\star a_{i-1}), \\
(\star a_i) & \longrightarrow & (\star a_i), \\
(\star a_k) & \longrightarrow & (\star a_k), \quad a_k \neq a_{i-1}^{\pm 1}, a_i^{\pm 1}, \\
(\star \bar{a}_{i-1}) & \longrightarrow & (\star \bar{a}_{i-1}) - (\star \bar{a}_i \bar{a}_{i-1}), \\
(\star \bar{a}_i) & \longrightarrow & (\star \bar{a}_i \bar{a}_{i-1}).
\end{array}$$

The following correspondence by the action of  $(a_i \mapsto a_{i-1}a_i)$  is clear.

$$\begin{array}{ccc}
& (a_i \mapsto a_{i-1}a_i) & \\
(a_{i-1} \blacktriangleleft) & \longrightarrow & (a_{i-1} \blacktriangleleft) - (a_{i-1}a_i \blacktriangleleft), \\
(a_i \blacktriangleleft) & \longrightarrow & (a_{i-1}a_i \blacktriangleleft), \\
(a_k \blacktriangleleft) & \longrightarrow & (a_k \blacktriangleleft), \quad a_k \neq a_{i-1}^{\pm 1}, a_i^{\pm 1}, \\
(\bar{a}_{i-1}a_i \blacktriangleleft) & \longrightarrow & (a_i \blacktriangleleft), \\
(\bar{a}_{i-1} \blacktriangleleft) - (\bar{a}_{i-1}a_i \blacktriangleleft) & \longrightarrow & (\bar{a}_{i-1} \blacktriangleleft), \\
(\bar{a}_i \blacktriangleleft) & \longrightarrow & (\bar{a}_i \blacktriangleleft).
\end{array}$$

From the first row of the first table and the second table we deduce the following table.

$$\begin{array}{ccc}
& (a_i \mapsto a_{i-1}a_i) & \\
(\star \bar{a}_i a_{i-1})(a_{i-1} \blacktriangleleft) & \longrightarrow & (\star \bar{a}_i)[(a_{i-1} \blacktriangleleft) - (a_{i-1}a_i \blacktriangleleft)], \\
(\star \bar{a}_i a_{i-1})(a_i \blacktriangleleft) & \longrightarrow & (\star \bar{a}_i)(a_{i-1}a_i \blacktriangleleft), \\
(\star \bar{a}_i a_{i-1})(a_k \blacktriangleleft) & \longrightarrow & (\star \bar{a}_i)(a_k \blacktriangleleft), \quad a_k \neq a_{i-1}^{\pm 1}, a_i^{\pm 1}, \\
(\star \bar{a}_i a_{i-1})(\bar{a}_i \blacktriangleleft) & \longrightarrow & (\star \bar{a}_i)(\bar{a}_i \blacktriangleleft).
\end{array}$$

Notice the cases  $(\star \bar{a}_i a_{i-1})(\bar{a}_{i-1}a_i \blacktriangleleft)$  and  $(\star \bar{a}_i a_{i-1})[(\bar{a}_{i-1} \blacktriangleleft) - (\bar{a}_{i-1}a_i \blacktriangleleft)]$  do not have to be considered since they are not in reduced form.

Let  $\mathbf{e}, \mathbf{f} \in \partial F_n$  such that  $\mathbf{e} = (w\bar{a}_i a_{i-1})\mathbf{e}'$ ,  $\mathbf{f} = (w\bar{a}_i a_{i-1})\mathbf{f}'$  and the first letter of  $\mathbf{e}'$  is different from the first letter of  $\mathbf{f}'$ . Let  $1 \leq j \leq 2n$  such that  $a_j = \bar{a}_{i-1}$ . By the third table,  $\mathbf{e}^{(a_i \mapsto a_{i-1}a_i)} = (u\bar{a}_i)\mathbf{e}''$ ,  $\mathbf{f}^{(a_i \mapsto a_{i-1}a_i)} = (u\bar{a}_i)\mathbf{f}''$  in reduced form. Let  $(b_k)_{1 \leq k \leq 2n}$  be the associated sequence of  $T_2$ . Recall  $(b_k)_{1 \leq k \leq 2n}$  is obtained from  $(a_k)_{1 \leq k \leq 2n}$  by moving  $a_i$  from immediately after  $a_{i-1}$  to immediately before  $a_j = \bar{a}_{i-1}$ . There are two cases according to  $j < i-1$  or  $i-1 < j$ .

If  $j < i - 1$ , then

$$\begin{aligned} (b_k)_{1 \leq k \leq (j-1)} &= (a_k)_{1 \leq k \leq (j-1)}, \\ b_j &= a_i, \\ (b_k)_{(j+1) \leq k \leq i} &= (a_k)_{j \leq k \leq (i-1)}, \\ (b_k)_{(i+1) \leq k \leq 2n} &= (a_k)_{(i+1) \leq k \leq 2n}. \end{aligned}$$

The partition with respect to  $(a_k)_{1 \leq k \leq 2n}$  of  $(\bar{a}_j \blacktriangleleft) = (a_{i-1} \blacktriangleleft)$  is  $(a_{j+1} \blacktriangleleft), (a_{j+2} \blacktriangleleft), \dots, (a_{i-1} \blacktriangleleft), (a_i \blacktriangleleft), (a_{i+1} \blacktriangleleft), \dots, (a_{2n} \blacktriangleleft), (a_1 \blacktriangleleft), (a_2 \blacktriangleleft), \dots, (a_{j-1} \blacktriangleleft)$ . The partition with respect to  $(b_k)_{1 \leq k \leq 2n}$  of  $(\bar{a}_i \blacktriangleleft)$  is  $(a_j \blacktriangleleft), (a_{j+1} \blacktriangleleft), \dots, (a_{i-1} \blacktriangleleft), (a_{i+1} \blacktriangleleft), (a_{i+2} \blacktriangleleft), \dots, (a_{2n} \blacktriangleleft), (a_1 \blacktriangleleft), (a_2 \blacktriangleleft), \dots, (a_{j-1} \blacktriangleleft)$ . By the third table,

$$\begin{array}{ccc} & (a_i \mapsto a_{i-1}a_i) & \\ (w\bar{a}_i a_{i-1})(a_{j+1} \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_{j+1} \blacktriangleleft), \\ (w\bar{a}_i a_{i-1})(a_{j+2} \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_{j+2} \blacktriangleleft), \\ & \vdots & \\ (w\bar{a}_i a_{i-1})(a_{i-2} \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_{i-2} \blacktriangleleft), \\ (w\bar{a}_i a_{i-1})(a_{i-1} \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)[(a_{i-1} \blacktriangleleft) - (a_{i-1}a_i \blacktriangleleft)], \\ (w\bar{a}_i a_{i-1})(a_i \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_{i-1}a_i \blacktriangleleft), \\ (w\bar{a}_i a_{i-1})(a_{i+1} \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_{i+1} \blacktriangleleft), \\ & \vdots & \\ (w\bar{a}_i a_{i-1})(a_{2n} \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_{2n} \blacktriangleleft), \\ (w\bar{a}_i a_{i-1})(a_1 \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_1 \blacktriangleleft), \\ (w\bar{a}_i a_{i-1})(a_2 \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_2 \blacktriangleleft), \\ & \vdots & \\ (w\bar{a}_i a_{i-1})(a_{j-1} \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_{j-1} \blacktriangleleft). \end{array}$$

Since  $a_j = \bar{a}_{i-1}$ , the first column is ordered with respect to  $T_1$ . On the other hand,  $a_j = \bar{a}_{i-1}$  implies that the partition of  $(u\bar{a}_i)(a_{i-1} \blacktriangleleft)$  with respect to  $T_2$  ends with  $(u\bar{a}_i)(a_{i-1}a_i \blacktriangleleft)$ . Then, the second column of this table is ordered with respect to  $T_2$ . Hence, if  $(w\bar{a}_i a_{i-1})\mathbf{e}' <_{T_1} (w\bar{a}_i a_{i-1})\mathbf{f}'$  then  $(u\bar{a}_i)\mathbf{e}'' <_{T_2} (u\bar{a}_i)\mathbf{f}''$ .

If  $i - 1 < j$ , then

$$\begin{aligned} (b_k)_{1 \leq k \leq (i-1)} &= (a_k)_{1 \leq k \leq (i-1)} \\ (b_k)_{i \leq k \leq (j-2)} &= (a_k)_{(i+1) \leq k \leq (j-1)} \\ b_{j-1} &= a_i \\ (b_k)_{j \leq k \leq 2n} &= (a_k)_{j \leq k \leq 2n} \end{aligned}$$

The partition with respect to  $(a_k)_{1 \leq k \leq 2n}$  of  $(\bar{a}_j \blacktriangleleft) = (a_{i-1} \blacktriangleleft)$  is  $(a_{j+1} \blacktriangleleft), (a_{j+2} \blacktriangleleft), \dots, (a_{2n} \blacktriangleleft), (a_1 \blacktriangleleft), (a_2 \blacktriangleleft), \dots, (a_{i-1} \blacktriangleleft), (a_i \blacktriangleleft), (a_{i+1} \blacktriangleleft), \dots, (a_{j-1} \blacktriangleleft)$ . The partition with respect to  $(b_k)_{1 \leq k \leq 2n}$  of  $(\bar{a}_i \blacktriangleleft)$  is  $(a_j \blacktriangleleft), (a_{j+1} \blacktriangleleft), \dots, (a_{2n} \blacktriangleleft),$

$(a_1 \blacktriangleleft), (a_2 \blacktriangleleft), \dots, (a_{i-1} \blacktriangleleft), (a_{i+1} \blacktriangleleft), (a_{i+2} \blacktriangleleft), \dots, (a_{j-1} \blacktriangleleft)$ . By the third table,

$$\begin{array}{ccc}
& (a_i \mapsto a_{i-1}a_i) & \\
(w\bar{a}_i a_{i-1})(a_{j+1} \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_{j+1} \blacktriangleleft), \\
(w\bar{a}_i a_{i-1})(a_{j+2} \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_{j+2} \blacktriangleleft), \\
& \vdots & \\
(w\bar{a}_i a_{i-1})(a_{2n} \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_{2n} \blacktriangleleft), \\
(w\bar{a}_i a_{i-1})(a_1 \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_1 \blacktriangleleft), \\
(w\bar{a}_i a_{i-1})(a_2 \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_2 \blacktriangleleft), \\
& \vdots & \\
(w\bar{a}_i a_{i-1})(a_{i-2} \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_{i-2} \blacktriangleleft), \\
(w\bar{a}_i a_{i-1})(a_{i-1} \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)[(a_{i-1} \blacktriangleleft) - (a_{i-1}a_i \blacktriangleleft)], \\
(w\bar{a}_i a_{i-1})(a_i \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_{i-1}a_i \blacktriangleleft), \\
(w\bar{a}_i a_{i-1})(a_{i+1} \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_{i+1} \blacktriangleleft), \\
& \vdots & \\
(w\bar{a}_i a_{i-1})(a_{j-1} \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_{j-1} \blacktriangleleft).
\end{array}$$

Since  $a_j = \bar{a}_{i-1}$ , the first column is ordered with respect to  $T_1$ . On the other hand,  $a_j = \bar{a}_{i-1}$  implies that the partition of  $(u\bar{a}_i)(a_{i-1} \blacktriangleleft)$  with respect to  $T_2$  ends with  $(u\bar{a}_i)(a_{i-1}a_i \blacktriangleleft)$ . Then, the second column of this table is ordered with respect to  $T_2$ . Hence, if  $(w\bar{a}_i a_{i-1})\mathbf{e}' <_{T_1} (w\bar{a}_i a_{i-1})\mathbf{f}'$  then  $(u\bar{a}_i)\mathbf{e}'' <_{T_2} (u\bar{a}_i)\mathbf{f}''$ .

For every row of the first table, there is a case which needs to be considered. Similarly, in all these cases, it can be shown that if  $\mathbf{e} <_{T_1} \mathbf{f}$ , then  $\mathbf{e}^{(a_i \mapsto a_{i-1}a_i)} <_{T_2} \mathbf{f}^{(a_i \mapsto a_{i-1}a_i)}$ .

The case  $\phi = (\bar{a}_i \mapsto \bar{a}_i \bar{a}_{i+1})$  for some  $1 \leq i \leq (2n - 1)$ ,  $a_i \neq \bar{a}_{i+1}$ , is similar.  $\square$

## 7 $t$ -squarefreeness

Recall  $2g + p = n$  and  $\Sigma_{g,1,p}$  is the free group on  $\{x_i, y_i\}_{1 \leq i \leq g} \cup \{t_k\}_{1 \leq k \leq p}$ .

The following definition extends Definition 4.1 to  $\Sigma_{g,1,p} \cup \partial\Sigma_{g,1,p}$ .

**7.1 Definition.** An element of  $\Sigma_{g,1,p} \cup \partial\Sigma_{g,1,p}$  is said to be  $t$ -squarefree if, in its reduced expression, no two consecutive terms in  $\{t_k, \bar{t}_k\}_{1 \leq k \leq p}$  are equal.

**7.2 Notation.** Recall that if  $G$  is a group and  $g_1, g_2, \dots, g_k \in G$ , then  $\Pi_{i=1}^k g_i = g_1 g_2 \cdots g_k$ . We introduce the notation  $\Pi_1^{i=k} g_i = g_k g_{k-1} \cdots g_1$ .

In the standard surface word set, we denote

$$\begin{aligned}
\bar{z}_1 &= \Pi_{i=1}^g [x_i, y_i] \Pi_{k=1}^p t_k = [x_1, y_1][x_2, y_2] \cdots [x_g, y_g] t_1 t_2 \cdots t_p, \\
z_1 &= \Pi_1^{k=p} \bar{t}_k \Pi_1^{i=g} [y_i, x_i] = \bar{t}_p \bar{t}_{p-1} \cdots \bar{t}_1 [y_g, x_g][y_{g-1}, x_{g-1}] \cdots [y_1, x_1].
\end{aligned}$$

From Remark 6.4, the smallest element of  $(\partial\Sigma_{g,1,p}, <)$  is  $\bar{z}_1^\infty$  and the largest element of  $(\partial\Sigma_{g,1,p}, <)$  is  $z_1^\infty$ . We denote by  $\min(\partial\Sigma_{g,1,p}) = \bar{z}_1^\infty$  and  $\max(\partial\Sigma_{g,1,p}) = z_1^\infty$  these facts.

Given two ends  $\mathbf{e}, \mathbf{f} \in \partial\Sigma_{g,1,p}$ , we write

$$[\mathbf{e}, \mathbf{f}] := \{\mathbf{g} \in \partial\Sigma_{g,1,p} \mid \mathbf{e} \leq \mathbf{g} \leq \mathbf{f}\}.$$

**7.3 Lemma.** *Let  $1 \leq k_0 \leq p$ ,  $w \in \Sigma_{g,1,p} - (\star t_{k_0}) - (\star \bar{t}_{k_0})$  and  $1 \leq i_0 \leq g$ . Then, in  $(\partial\Sigma_{g,1,p}, \leq)$ , the following hold:*

(i).  $wt_{k_0}\bar{w}(\bar{z}_1^\infty) \leq wt_{k_0}((\prod_{k=k_0}^p t_k \prod_{i=1}^g [x_i, y_i] \prod_{k=1}^{k_0-1} t_k)^\infty) = \min(wt_{k_0}t_{k_0} \blacktriangleleft);$

(ii).  $\max(wt_{k_0}t_{k_0} \blacktriangleleft) < \min(w\bar{t}_{k_0}\bar{t}_{k_0} \blacktriangleleft);$

(iii).  $\max(w\bar{t}_{k_0}\bar{t}_{k_0} \blacktriangleleft) = w\bar{t}_{k_0}((\prod_1^{k=k_0} \bar{t}_k \prod_1^{i=g} [y_i, x_i] \prod_{k_0+1}^{k=p} \bar{t}_k)^\infty) \leq w\bar{t}_{k_0}\bar{w}(z_1^\infty);$

(iv).  $(wt_{k_0}t_{k_0} \blacktriangleleft) \cup (w\bar{t}_{k_0}\bar{t}_{k_0} \blacktriangleleft) \subseteq [wt_{k_0}\bar{w}(\bar{z}_1^\infty), w\bar{t}_{k_0}\bar{w}(z_1^\infty)];$

(v). *If  $2g + p \geq 3$ , then one of the following holds:*

(a)  $\bar{t}_p(\bar{z}_1^\infty) > w\bar{t}_{k_0}\bar{w}(z_1^\infty);$

(b)  $\bar{t}_p(\bar{z}_1^\infty) < wt_{k_0}\bar{w}(\bar{z}_1^\infty);$

and, hence,  $\bar{t}_p(\bar{z}_1^\infty) \notin [wt_{k_0}\bar{w}(\bar{z}_1^\infty), w\bar{t}_{k_0}\bar{w}(z_1^\infty)];$

(vi). *If  $a \in \{x_{i_0}, \bar{x}_{i_0}, y_{i_0}, \bar{y}_{i_0}\}$ , then one of the following holds:*

(a).  $a(z_1^\infty) > w\bar{t}_{k_0}\bar{w}(z_1^\infty);$

(b).  $a(z_1^\infty) < wt_{k_0}\bar{w}(\bar{z}_1^\infty);$

and, hence,  $a(z_1^\infty) \notin [wt_{k_0}\bar{w}(\bar{z}_1^\infty), w\bar{t}_{k_0}\bar{w}(z_1^\infty)].$

*Proof.* Recall  $<$  is the ordering with respect to sequence the

$$(\bar{x}_1, y_1, x_1, \bar{y}_1, \bar{x}_2, y_2, x_2, \bar{y}_2, \dots, \bar{x}_g, y_g, x_g, \bar{y}_g, t_1, \bar{t}_1, t_2, \bar{t}_2, \dots, t_p, \bar{t}_p).$$

(i). It is straightforward to see that

$$wt_{k_0}((\prod_{k=k_0}^p t_k \prod_{i=1}^g [x_i, y_i] \prod_{k=1}^{k_0-1} t_k)^\infty) = \min(wt_{k_0}t_{k_0} \blacktriangleleft).$$

Let  $a \in X \cup \bar{X}$  be such that  $\bar{w}((\prod_{i=1}^g [x_i, y_i] \prod_{k=1}^p t_k)^\infty) \in (a \blacktriangleleft)$ . Note  $a \neq \bar{t}_{k_0}$ .

If  $a \neq t_{k_0}$ , then  $(wt_{k_0}a \blacktriangleleft) < (wt_{k_0}t_{k_0} \blacktriangleleft)$ , and we have

$$wt_{k_0}\bar{w}(\bar{z}_1^\infty) = wt_{k_0}\bar{w}((\prod_{i=1}^g [x_i, y_i] \prod_{k=1}^p t_k)^\infty) < \min(wt_{k_0}t_{k_0}).$$

If  $a = t_{k_0}$ , then  $\bar{w}$  is completely canceled in  $\bar{w}((\Pi_{i=1}^g[x_i, y_i]\Pi_{k=1}^p t_k)^\infty)$ , and, moreover,

$$\begin{aligned} wt_{k_0}\bar{w}(\bar{z}_1^\infty) &= wt_{k_0}\bar{w}((\Pi_{i=1}^g[x_i, y_i]\Pi_{k=1}^p t_k)^\infty) \\ &= wt_{k_0}((\Pi_{k=k_0}^p t_k \Pi_{i=1}^g[x_i, y_i]\Pi_{k=1}^{k_0-1} t_k)^\infty) \\ &= \min(wt_{k_0} t_{k_0} \blacktriangleleft). \end{aligned}$$

(ii). It is clear.

(iii). It is straightforward to see that

$$\max(w\bar{t}_{k_0}\bar{t}_{k_0} \blacktriangleleft) = w\bar{t}_{k_0}((\Pi_1^{k=k_0}\bar{t}_k \Pi_1^{i=g}[y_i, x_i]\Pi_{k_0+1}^{k=p}\bar{t}_k)^\infty).$$

Let  $a \in X \cup \bar{X}$  be such that  $\bar{w}((\Pi_1^{k=p}\bar{t}_k \Pi_1^{i=g}[y_i, x_i])^\infty) \in (a \blacktriangleleft)$ . Note  $a \neq t_{k_0}$ . If  $a \neq \bar{t}_{k_0}$ , then  $(w\bar{t}_{k_0}\bar{t}_{k_0} \blacktriangleleft) < (w\bar{t}_{k_0}a \blacktriangleleft)$ , and we have

$$\max(w\bar{t}_{k_0}\bar{t}_{k_0} \blacktriangleleft) < w\bar{t}_{k_0}\bar{w}((\Pi_1^{k=p}\bar{t}_k \Pi_1^{i=g}[y_i, x_i])^\infty) = w\bar{t}_{k_0}\bar{w}(z_1^\infty).$$

If  $a = \bar{t}_{k_0}$ , then  $\bar{w}$  is completely canceled in  $\bar{w}((\Pi_1^{k=p}\bar{t}_k \Pi_1^{i=g}[y_i, x_i])^\infty)$ , and, moreover,

$$\begin{aligned} w\bar{t}_{k_0}\bar{w}(z_1^\infty) &= w\bar{t}_{k_0}\bar{w}((\Pi_1^{k=p}\bar{t}_k \Pi_1^{i=g}[y_i, x_i])^\infty) \\ &= w\bar{t}_{k_0}((\Pi_1^{k=k_0}\bar{t}_k \Pi_1^{i=g}[y_i, x_i]\Pi_p^{k=k_0+1}\bar{t}_k)^\infty) \\ &= \max(w\bar{t}_{k_0}\bar{t}_{k_0} \blacktriangleleft). \end{aligned}$$

(iv). Follows from (i)-(iii).

(v). By (i)-(iii),

$$wt_{k_0}\bar{w}((\Pi_{i=1}^g[x_i, y_i]\Pi_{k=1}^p t_k)^\infty) < w\bar{t}_{k_0}\bar{w}((\Pi_1^{k=p}\bar{t}_k \Pi_1^{i=g}[y_i, x_i])^\infty).$$

**Case 1.**  $w = 1$ . Since  $(\bar{t}_p \bar{x}_1 \blacktriangleleft) \cup (\bar{t}_p t_1 \blacktriangleleft) > (\bar{t}_{k_0} \bar{t}_p \blacktriangleleft)$ , we see

$$\bar{t}_p(\bar{z}_1^\infty) = \bar{t}_p((\Pi_{i=1}^g[x_i, y_i]\Pi_{k=1}^p t_k)^\infty) > \bar{t}_{k_0}((\Pi_1^{k=p}\bar{t}_k \Pi_1^{i=g}[y_i, x_i])^\infty) = \bar{t}_{k_0}(z_1^\infty).$$

Thus, (a) holds.

**Case 2.**  $w \notin (\bar{t}_p \star) \cup \{1\}$ . Since  $(\bar{t}_p \blacktriangleleft) > (w\bar{t}_{k_0} \blacktriangleleft)$ , we see

$$\begin{aligned} \bar{t}_p(\bar{z}_1^\infty) &= \bar{t}_p((\Pi_{i=1}^g[x_i, y_i]\Pi_{k=1}^p t_k)^\infty) \\ &> w\bar{t}_{k_0}\bar{w}((\Pi_1^{k=p}\bar{t}_k \Pi_1^{i=g}[y_i, x_i])^\infty) = w\bar{t}_{k_0}\bar{w}(z_1^\infty). \end{aligned}$$

Thus, (a) holds.

**Case 3.**  $w \in (\bar{t}_p \bar{t}_p \star)$ . Since  $(\bar{t}_p \bar{x}_1 \blacktriangleleft) \cup (\bar{t}_p t_1 \blacktriangleleft) > (w\bar{t}_{k_0} \blacktriangleleft)$ , we see

$$\begin{aligned} \bar{t}_p(\bar{z}_1^\infty) &= \bar{t}_p((\Pi_{i=1}^g[x_i, y_i]\Pi_{k=1}^p t_k)^\infty) \\ &> w\bar{t}_{k_0}\bar{w}((\Pi_1^{k=p}\bar{t}_k \Pi_1^{i=g}[y_i, x_i])^\infty) = w\bar{t}_{k_0}\bar{w}(z_1^\infty). \end{aligned}$$

Thus, (a) holds.

**Case 4.**  $w \in (\bar{t}_p \star) - (\bar{t}_p \bar{t}_p \star)$ .

Here,

$$wt_{k_0} \bar{w}(\bar{z}_1^\infty) = wt_{k_0} \bar{w}((\Pi_{i=1}^g[x_i, y_i] \Pi_{k=1}^p t_k)^\infty) \in (wt_{k_0} \blacktriangleleft) \subset (\bar{t}_p \blacktriangleleft) - (\bar{t}_p \bar{t}_p \blacktriangleleft).$$

Hence,

$$\begin{aligned} \bar{t}_p((\Pi_{i=1}^g[x_i, y_i] \Pi_{k=1}^p t_k)^\infty) &= \min((\bar{t}_p \blacktriangleleft) - (\bar{t}_p \bar{t}_p \blacktriangleleft)) \\ &\leq wt_{k_0} \bar{w}((\Pi_{i=1}^g[x_i, y_i] \Pi_{k=1}^p t_k)^\infty). \end{aligned}$$

To prove (b) holds, it remains to show that

$$\bar{t}_p((\Pi_{i=1}^g[x_i, y_i] \Pi_{k=1}^p t_k)^\infty) \neq wt_{k_0} \bar{w}((\Pi_{i=1}^g[x_i, y_i] \Pi_{k=1}^p t_k)^\infty),$$

that is,  $(\Pi_{i=1}^g[x_i, y_i] \Pi_{k=1}^p t_k)^\infty \neq t_p wt_{k_0} \bar{w}((\Pi_{i=1}^g[x_i, y_i] \Pi_{k=1}^p t_k)^\infty)$ , that is,  $t_p wt_{k_0} \bar{w} \notin \langle \Pi_{i=1}^g[x_i, y_i] \Pi_{k=1}^p t_k \rangle$ . We can write  $w = \bar{t}_p u$  where  $u \notin (t_p \star)$ . Then  $t_p wt_{k_0} \bar{w} = ut_{k_0} \bar{u} t_p$ , in normal form. Thus it suffices to show

$$ut_{k_0} \bar{u} t_p \notin \langle \Pi_{i=1}^g[x_i, y_i] \Pi_{k=1}^p t_k \rangle.$$

If  $u = 1$ , then  $ut_{k_0} \bar{u} t_p \notin \langle \Pi_{i=1}^g[x_i, y_i] \Pi_{k=1}^p t_k \rangle$ , since  $2g + p \geq 3$ .

If  $u \neq 1$ , then  $ut_{k_0} \bar{u} t_p \notin \langle \Pi_{i=1}^g[x_i, y_i] \Pi_{k=1}^p t_k \rangle$ , since  $ut_{k_0} \bar{u} t_p$  does not lie in the submonoid of  $\Sigma_{g,1,p}$  generated by  $\Pi_{i=1}^g[x_i, y_i] \Pi_{k=1}^p t_k$ , nor in the submonoid generated by  $\Pi_1^{k=p} \bar{t}_k \Pi_1^{i=g} [y_i, x_i]$ .

In all four cases (v) holds.

(vi). Let  $a \in \{x_{i_0}, \bar{x}_{i_0}, y_{i_0}, \bar{y}_{i_0}\}$ .

**Case 1.**  $w = 1$ . Since  $(a \blacktriangleleft) < (t_{k_0} \blacktriangleleft)$ , we see

$$a(z_1^\infty) = a((\Pi_1^{k=p} \bar{t}_k \Pi_1^{i=g} [y_i, x_i])^\infty) < t_{k_0}((\Pi_{i=1}^g[x_i, y_i] \Pi_{k=1}^p t_k)^\infty) = t_{k_0}(\bar{z}_1^\infty).$$

Thus, (b) holds.

**Case 2.**  $w \notin (a \star) \cup \{1\}$ .

If  $(a \blacktriangleleft) > (w \blacktriangleleft)$ , then  $(a \blacktriangleleft) > (w \blacktriangleleft) \supset (w \bar{t}_{k_0} \blacktriangleleft)$  and

$$\begin{aligned} a(z_1^\infty) &= a((\Pi_1^{k=p} \bar{t}_k \Pi_1^{i=g} [y_i, x_i])^\infty) \\ &> w \bar{t}_{k_0} \bar{w}((\Pi_1^{k=p} \bar{t}_k \Pi_1^{i=g} [y_i, x_i])^\infty) = w \bar{t}_{k_0} \bar{w}(z_1^\infty). \end{aligned}$$

Thus, (a) holds.

If  $(a \blacktriangleleft) < (w \blacktriangleleft)$ , then  $(a \blacktriangleleft) < (w \blacktriangleleft) \supset (wt_{k_0} \blacktriangleleft)$  and

$$\begin{aligned} a(z_1^\infty) &= a((\Pi_1^{k=p} \bar{t}_k \Pi_1^{i=g} [y_i, x_i])^\infty) \\ &< wt_{k_0} \bar{w}((\Pi_{i=1}^g[x_i, y_i] \Pi_{k=1}^p t_k)^\infty) = wt_{k_0} \bar{w}(\bar{z}_1^\infty). \end{aligned}$$

Thus, (b) holds.

**Case 3.**  $w \in (a\bar{t}_p\star)$ .

Since  $a((\Pi_1^{k=p}\bar{t}_k\Pi_1^{i=g}[y_i, x_i])^\infty) = \max(a\bar{t}_p \blacktriangleleft)$ , we see

$$\begin{aligned} a(z_1^\infty) &= a((\Pi_1^{k=p}\bar{t}_k\Pi_1^{i=g}[y_i, x_i])^\infty) \\ &\geq w\bar{t}_{k_0}\bar{w}((\Pi_1^{k=p}\bar{t}_k\Pi_1^{i=g}[y_i, x_i])^\infty) = w\bar{t}_{k_0}\bar{w}(z_1^\infty). \end{aligned}$$

To prove (a) holds, it remains to show that

$$a((\Pi_1^{k=p}\bar{t}_k\Pi_1^{i=g}[y_i, x_i])^\infty) \neq w\bar{t}_{k_0}\bar{w}((\Pi_1^{k=p}\bar{t}_k\Pi_1^{i=g}[y_i, x_i])^\infty),$$

that is,  $(\Pi_1^{k=p}\bar{t}_k\Pi_1^{i=g}[y_i, x_i])^\infty \neq \bar{a}w\bar{t}_{k_0}\bar{w}((\Pi_1^{k=p}\bar{t}_k\Pi_1^{i=g}[y_i, x_i])^\infty)$ , that is  $\bar{a}w\bar{t}_{k_0}\bar{w} \notin \langle \Pi_1^{k=p}\bar{t}_k\Pi_1^{i=g}[y_i, x_i] \rangle$ . We can write  $w = a\bar{t}_p u$  where  $u \notin (t_p\star)$ . Then  $\bar{a}w\bar{t}_{k_0}\bar{w} = \bar{t}_p u \bar{t}_{k_0} \bar{u} t_p \bar{a}$ , in normal form. Thus it suffices to show that

$$\bar{t}_p u \bar{t}_{k_0} \bar{u} t_p \bar{a} \notin \langle \Pi_1^{k=p}\bar{t}_k\Pi_1^{i=g}[y_i, x_i] \rangle,$$

which is clear since  $\bar{t}_p u \bar{t}_{k_0} \bar{u} t_p \bar{a}$  does not lie in the submonoid of  $\Sigma_{g,1,p}$  generated by  $\Pi_1^{k=p}\bar{t}_k\Pi_1^{i=g}[y_i, x_i]$ , nor in the submonoid generated by  $\Pi_{i=1}^g[x_i, y_i]\Pi_{k=1}^p t_k$ .

**Case 4.**  $w \in (a\star) - (a\bar{t}_p\star)$ ,  $|w| \geq 2$ .

If  $(a\bar{t}_p \blacktriangleleft) > (w \blacktriangleleft)$ , then  $(a\bar{t}_p \blacktriangleleft) > (w \blacktriangleleft) \supset (w\bar{t}_{k_0} \blacktriangleleft)$  and

$$\begin{aligned} a(z_1^\infty) &= a((\Pi_1^{k=p}\bar{t}_k\Pi_1^{i=g}[y_i, x_i])^\infty) \\ &> w\bar{t}_{k_0}\bar{w}((\Pi_1^{k=p}\bar{t}_k\Pi_1^{i=g}[y_i, x_i])^\infty) = w\bar{t}_{k_0}\bar{w}(z_1^\infty). \end{aligned}$$

Thus, (a) holds.

If  $(a\bar{t}_p \blacktriangleleft) < (w \blacktriangleleft)$ , then  $(a\bar{t}_p \blacktriangleleft) < (w \blacktriangleleft) \supset (wt_{k_0} \blacktriangleleft)$  and

$$\begin{aligned} a(z_1^\infty) &= a((\Pi_1^{k=p}\bar{t}_k\Pi_1^{i=g}[y_i, x_i])^\infty) \\ &< wt_{k_0}\bar{w}((\Pi_{i=1}^g[x_i, y_i]\Pi_{k=1}^p t_k)^\infty) = wt_{k_0}\bar{w}(\bar{z}_1^\infty). \end{aligned}$$

Thus, (b) holds.

**Case 5.**  $w = a$ .

Since  $a(z_1^\infty) = \max(a\bar{t}_p \blacktriangleleft)$ ,  $(a\bar{t}_p \blacktriangleleft) \supset (a\bar{t}_p\bar{y}_g\bar{x}_g \blacktriangleleft)$  and  $(a\bar{t}_p\bar{y}_g\bar{x}_g \blacktriangleleft) > (a\bar{t}_{k_0}\bar{a}\bar{t}_p \blacktriangleleft)$ , we see

$$\begin{aligned} a(z_1^\infty) &= a((\Pi_1^{k=p}\bar{t}_k\Pi_1^{i=g}[y_i, x_i])^\infty) \\ &> a\bar{t}_{k_0}\bar{a}((\Pi_1^{k=p}\bar{t}_k\Pi_1^{i=g}[y_i, x_i])^\infty) = a\bar{t}_{k_0}\bar{a}(z_1^\infty). \end{aligned}$$

Thus, (a) holds.

In all five cases (vi) holds.  $\square$

**7.4 Theorem.** *If  $2g + p \geq 3$  then, for each  $\phi \in \mathcal{AM}_{g,1,p}$ , the following hold:*

(i).  $\bar{t}_p^\phi(\bar{z}_1^\infty)$  is a  $t$ -squarefree end,

(ii). for every  $1 \leq i_0 \leq g$  and every  $a \in \{x_{i_0}, \bar{x}_{i_0}, y_{i_0}, \bar{y}_{i_0}\}$ ,  $a^\phi(z_1^\infty)$  is a  $t$ -squarefree end.

*Proof.* (i). Recall  $\bar{z}_1 = \prod_{i=1}^g [x_i, y_i] \prod_{k=1}^p t_k$  and  $z_1 = \prod_1^{k=p} \bar{t}_k \prod_1^{i=g} [y_i, x_i]$ . By Lemma 7.3(v),  $\bar{t}_p(\bar{z}_1^\infty)$  does not lie in

$$\bigcup_{u \in \{[t_k]\}_{1 \leq k \leq p}} [u(\bar{z}_1^\infty), \bar{u}(z_1^\infty)] (= \bigcup_{k=1}^p \bigcup_{w \in \Sigma_{g,1,p} - (\star t_k) - (\star \bar{t}_k)} [wt_k \bar{w}(\bar{z}_1^\infty), w \bar{t}_k \bar{w}(z_1^\infty)]).$$

Notice that  $\phi$  respects each of the following sets:

$$\{[t_k]\}_{1 \leq k \leq p}; \quad \{\bar{z}_1^\infty\}; \quad \{z_1^\infty\}; \quad \text{and} \quad \bigcup_{u \in \{[t_k]\}_{1 \leq k \leq p}} [u(\bar{z}_1^\infty), \bar{u}(z_1^\infty)].$$

Hence,  $(\bar{t}_p(\bar{z}_1^\infty))^\phi$  does not lie in  $\bigcup_{u \in \{[t_k]\}_{1 \leq k \leq p}} [u(\bar{z}_1^\infty), \bar{u}(z_1^\infty)]$ . By Lemma 7.3(iv),

$$\bigcup_{u \in \{[t_k]\}_{1 \leq k \leq p}} [u(\bar{z}_1^\infty), \bar{u}(z_1^\infty)] \supseteq \bigcup_{k=1}^p \bigcup_{w \in \Sigma_{g,1,p} - (\star t_k) - (\star \bar{t}_k)} ((wt_k t_k \blacktriangleleft) \cup (w \bar{t}_k \bar{t}_k \blacktriangleleft)).$$

Hence,  $(\bar{t}_p(\bar{z}_1^\infty))^\phi$  does not lie in the right-hand side set either, and, hence,  $(\bar{t}_p(\bar{z}_1^\infty))^\phi$  is a  $t$ -squarefree end. Since  $(\bar{t}_p(\bar{z}_1^\infty))^\phi = \bar{t}_p^\phi(\bar{z}_1^\infty)$ , the desired result holds.

(ii). The same proof as (i) using Lemma 7.3(vi) instead of Lemma 7.3(v).  $\square$

*Proof.* (of Theorem 4.2) Recall (2.3.1).  $\mathcal{AM}_{0,1,2} = \langle \sigma_1 \rangle$ , and

$$t_2^{\mathcal{AM}_{0,1,2}} = \{t_2^{\sigma_1^{2m}}, t_2^{\sigma_1^{2m+1}} \mid m \in \mathbb{Z}\} = \{t_2^{(t_1 t_2)^m}, t_1^{(t_1 t_2)^m} \mid m \in \mathbb{Z}\}$$

Thus, every element of  $t_2^{\mathcal{AM}_{0,1,2}}$  is  $t$ -squarefree.

Suppose, now,  $2g + p \geq 3$ . Let  $1 \leq i_0 \leq g$  and  $a \in \{x_{i_0}, y_{i_0}\}$ . By Theorem 7.4(ii),  $a^\phi(z_1^\infty) = a^\phi((\prod_1^{k=p} \bar{t}_k \prod_1^{i=g} [y_i, x_i])^\infty)$  is a  $t$ -squarefree end. Hence, either  $a^\phi$  is  $t$ -squarefree or  $a^\phi = ut_k t_k v$  in normal form, and  $t_k v$  is canceled in  $a^\phi(z_1^\infty) = ut_k t_k v(z_1^\infty)$ ; moreover  $ut_k, t_k v$  are  $t$ -squarefree. By Theorem 7.4(ii),

$$\bar{a}^\phi(z_1^\infty) = \bar{a}^\phi((\prod_1^{k=p} \bar{t}_k \prod_1^{i=g} [y_i, x_i])^\infty)$$

is a  $t$ -squarefree end. Hence,  $\bar{a}^\phi \neq \bar{v} \bar{t}_k \bar{t}_k \bar{u}$ .



Since  $\phi$  permutes the set  $\{[t_k]\}_{1 \leq k \leq p}$ , we can write  $\bar{t}_p^\phi = \bar{t}_{p^\pi}^{w_p}$ , where  $\pi$  is a permutation of  $\{1, 2, \dots, p\}$  and  $w_p \in \Sigma_{g,1,p} - (t_{p^\pi} \star) - (\bar{t}_{p^\pi} \star)$ . It is not difficult to see that

$$\bar{t}_p^\phi(\bar{z}_1^\infty) = \bar{w}_p \bar{t}_{p^\pi} w_p ((\Pi_{i=1}^g [x_i, y_i] \Pi_{k=1}^p t_k)^\infty) \in (\bar{w}_p \blacktriangleleft).$$

By Theorem 7.4(i),  $\bar{t}_p^\phi(\bar{z}_1^\infty)$  is a  $t$ -squarefree end. Hence,  $\bar{w}_p$  is  $t$ -squarefree.

Since  $\bar{w}_p$  is  $t$ -squarefree,  $\bar{t}_p^\phi = \bar{w}_p \bar{t}_{p^\pi} w_p$  is also  $t$ -squarefree. Hence,  $t_p^\phi$  is  $t$ -squarefree.

Suppose, now,  $2g + p \geq 2$ . Let  $1 \leq k \leq p$ . Since  $t_k$  is in the  $\mathcal{AM}_{g,1,p}$ -orbit of  $t_p$ , we see  $t_k^\phi$  is  $t$ -squarefree for all  $\phi \in \mathcal{AM}_{g,1,p}$ .  $\square$

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LLUÍS BACARDIT,  
INSTITUT DE MATHÉMATIQUES DE BOURGOGNE  
UNIVERSITÉ DE BOURGOGNE  
UMR 5584 DU CNRS, BP 47870  
21078 DIJON CEDEX  
FRANCE

*E-mail address:* `Lluís.Bacardit@u-bourgogne.fr`