A quasimartingale characterization of p stable type Banach spaces

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Abstract

We characterize Banach spaces $\mathcal{B}$ of stable-type p $(1 < p < 2)$ by the property that for every sequence $(\mathbf{X}_i)$ of $\mathcal{B}$-valued random variables, independent, centered and fulfilling some integrability assumption, the sequence $\left(\frac{\mathbf{X}_1+\cdots+\mathbf{X}_n}{n^{1/p}}\right)$ is a quasimartingale.

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1 Introduction

"How to characterize the regularity of a Banach space $(\mathcal{B}, \|\cdot\|)$ by the fact that a kind of classical strong law of large numbers (SLLN) holds for $\mathcal{B}$-valued random variables (r.v.)?" is a well know problem. Two kinds of regular Banach spaces – the spaces of Rademacher type p and the spaces of stable type p – have been nicely characterized in that way (see chapter 9 in [3]).

Here our goal will be to show that the SLLN characterization of spaces of stable type p $(1 < p < 2)$ can be made more precise in supposing that the "normalized sums" obeying the SLLN have even a quasimartingale behaviour.

To begin with, we recall some definitions.

In the sequel, $(\mathcal{B}, \|\cdot\|)$ will be a real separable Banach space, equipped with its Borel σ-field B. A $\mathcal{B}$-valued r.v. $\mathbf{X}$ is a measurable function defined on

a probability space $(\Omega, \mathcal{T}, \mathbb{P})$ with values in $(\mathcal{B}, B)$. Such a r.v. is said to be (strongly) integrable if $\mathbb{E}\|\mathbf{X}\| < +\infty$ and $\forall f \in \mathcal{B}'$, $\mathbb{E}f(\mathbf{X}) = 0$; this is denoted by $\mathbb{E}(\mathbf{X}) = 0$.

Let $p \geqslant 1$ be given. The weak-ℓ_p norm of a sequence $a := (a_1, \ldots, a_n)$ of real numbers is defined as follows :

$$\|a\|_{p,\infty} := \sup_{t>0} \left(t^p \operatorname{Card}(i : |a_i| > t)\right)^{\frac{1}{p}} = \sup_{k=1}^{n} a_k^* k^{1/p},$$

where $(a_1^*, \ldots, a_n^*)$ denotes the non-increasing rearrangement of the sequence $(|a_1|, \ldots, |a_n|)$.

Let now (ε_k) be a sequence of independent Rademacher random variables $\big($that is $\mathbb{P}(\varepsilon_k = 1) = \mathbb{P}(\varepsilon_k = -1) = \frac{1}{2}\big)$.

Rademacher type p spaces and stable type p spaces are defined as follows :

Definition 1.

1. Let $1 < p < 2$. The Banach space $(\mathcal{B}, \|\cdot\|)$ is of Rademacher type p if there exists a constant $c(p) > 0$ such that for every finite sequence (x_i) in $\mathcal{B}$:
$$\left(\mathbb{E}\left\|\sum_{i=1}^{n} \varepsilon_i x_i\right\|^p\right)^{\frac{1}{p}} \leqslant c(p)\left(\sum_{i=1}^{n} \|x_i\|^p\right)^{\frac{1}{p}}.$$
2. Let $1 \leqslant p \leqslant 2$. The Banach space $(\mathcal{B}, \|\cdot\|)$ is of stable type p if there exists a constant $C(p) > 0$ such that for every finite sequence (x_i) in $\mathcal{B}$:
$$\mathbb{E}\left\|\sum_{i=1}^{n} \varepsilon_i x_i\right\| \leqslant C(p)\, \|(\|x_1\|, \ldots, \|x_n\|)\|_{p,\infty}\,.$$

Remark 2.

1. If $(\mathcal{B}, \|\cdot\|)$ is of stable type p, then there exists $q > p$, such that $(\mathcal{B}, \|\cdot\|)$ is also of stable type q [5].
2. If $(\mathcal{B}, \|\cdot\|)$ is of stable type p, it is also of Rademacher type p [7].
3. The above definition of stable type is not the classical one (which involves standard stable r.v.), but an equivalent statement (see for instance [3], proposition 9.12) which will be used in our proofs.

Remark 3.
Let $(\mathcal{B}, \|\cdot\|)$ be a Banach space and $1 \leqslant p \leqslant 2$. If $\mathcal{B}$ is of (Rademacher) type p, there exists a constant $c(p)$ such that for every finite sequence $(\mathbf{X}_i)$ with $\|\mathbf{X}_i\| \in L^p$:

$$\mathbb{E}\left\|\sum_{i=1}^{n} \mathbf{X}_i\right\|^p \leqslant c(p) \sum_{i=1}^{n} \mathbb{E}\|\mathbf{X}_i\|^p. \tag{1}$$

The stable type p has been characterized in terms of Marcinkiewicz-Zygmund like SLLN :

Theorem 4 (Maurey-Pisier [5])**.**
Let $1 < p < 2$. The following two properties are equivalent :

1. *$(\mathcal{B}, \|\cdot\|)$ is of stable type p.*
2. *For every bounded sequence (x_i) in $\mathcal{B}$, the sequence $\left(\frac{1}{n^{1/p}} \sum_{k=1}^{n} \varepsilon_k x_k\right)$ converges a.s. to 0.*

Theorem 5 (Woyczynski [8])**.**
Let $1 < p < 2$. The following two properties are equivalent :

1. *$(\mathcal{B}, \|\cdot\|)$ is of stable type p.*
2. *For every sequence $(\mathbf{X}_i)$ of independent, strongly centered $\mathcal{B}$-valued r.v. for which there exists a nonnegative r.v. ξ with $\mathbb{E}\xi^p < +\infty$ such that :*

$$\exists c > 0,\ \forall t > 0,\ \forall i \in \mathbb{N}^*,\ \mathbb{P}(\|\mathbf{X}_i\| > t) \leqslant c\mathbb{P}(\xi > t),$$

the sequence $\left(\frac{\mathbf{S}_n}{n^{1/p}}\right)$ converges a.s. to 0, where, as usual, $\mathbf{S}_n := \mathbf{X}_1 + \cdots + \mathbf{X}_n$.

In this paper, we will prove a result in the same spirit as Woyczynski's result, but in which the SLLN behaviour of the sequence $\left(\frac{\mathbf{S}_n}{n^{1/p}}\right)$ is even a quasimartingale behaviour.

2 A quasimartingale characterization of spaces of stable type p.

We start this section by defining the quasimartingale behaviour of $\left(\frac{\mathbf{S}_n}{n^{1/p}}\right)$:

Definition 6.
Let $(\mathbf{X}_k)$ be a sequence of independent, strongly centered $(\mathcal{B}, \|\cdot\|)$-valued r.v. Denote $\mathbf{S}_n := \mathbf{X}_1 + \cdots + \mathbf{X}_n$ and $\mathcal{F}_n := \sigma(\mathbf{X}_1, \ldots, \mathbf{X}_n)$. Let $p \in]1, 2[$. The sequence $\left(\frac{\mathbf{S}_n}{n^{1/p}}, \mathcal{F}_n\right)$, or simplier $\left(\frac{\mathbf{S}_n}{n^{1/p}}\right)$, is a quasimartingale if :

$$\sum_{n=1}^{+\infty} \mathbb{E}\left\| \mathbb{E}\left(\frac{\mathbf{S}_{n+1}}{(n+1)^{1/p}} - \frac{\mathbf{S}_n}{n^{1/p}} \middle| \mathcal{F}_n \right) \right\| < +\infty. \tag{2}$$

Remark 7.
Since the r.v. $(\mathbf{X}_k)$ are independent and centered, condition (2) is equivalent to :

$$\sum_{n=1}^{+\infty} \frac{\mathbb{E}\|\mathbf{S}_n\|}{n^{1+1/p}} < +\infty.$$

Now we are able to state our characterization of spaces of stable type p $(1 < p < 2)$:

Theorem 8.
Let $1 < p < 2$. The following two properties are equivalent :

1. *$\mathcal{B}$ is a stable type p space;*
2. *For every sequence $(\mathbf{X}_n)$ of independent, strongly centered r.v., such that*

$$\int_0^{+\infty} g^{1/p}(t)dt < +\infty, \tag{3}$$

where

$$\forall t > 0,\ g(t) := \sup_{n \geqslant 1} \mathbb{P}(\|\mathbf{X}_n\| > t),$$

$\left(\frac{\mathbf{S}_n}{n^{1/p}}\right)$ is a quasimartingale.

Remark 9.

1. Condition (3) is not surprising : indeed, in the i.i.d. case, it can be written :

$$\int_0^{+\infty} \mathbb{P}^{1/p}(\|\mathbf{X}\| > t)dt < +\infty, \tag{4}$$

   which condition is necessary for $\left(\frac{\mathbf{S}_n}{n^{1/p}}\right)$ being a quasimartingale in every Banach space $\mathcal{B}$ (see [1], proposition 3).

2. Property (4) implies that $\mathbb{E}\|\mathbf{X}\|^p < +\infty$ (see [1], remark 4).

3. There exists a small class of r.v. $\mathbf{X}$ such that $\mathbb{E}\|\mathbf{X}\|^p < +\infty$ and for which (4) does not hold (see example 1 in [1]).

So – comparing theorems 5 and 8 – one sees that the price to pay for getting a quasimartingale behaviour for $\left(\frac{\mathbf{S}_n}{n^{1/p}}\right)$ instead of a simple a.s. convergence to 0, is to sharpen a little bit the hypothesis $\mathbb{E}\xi^p < +\infty$ of theorem 5.

Proof.
In the sequel, c_k will denote positive constants which precise value does not matter.

Let us show the implication $1 \Longrightarrow 2$.

First consider the special case where there exists $M > 0$ such that : $\forall t > M,\ g(t) = 0$. Then $\forall k,\ \|\mathbf{X}_k\| \leqslant M$ a.s. The space $\mathcal{B}$ being of Rademacher

type q for some $q > p$ by comments 1) and 2) following definition 1, and using relation (1), one has :

$$\frac{\mathbb{E}\|\mathbf{S}_n\|}{n^{1+1/p}} \leqslant \frac{c(q)}{n^{1+1/p}} \left(\sum_{k=1}^{n} \mathbb{E}\|\mathbf{X}_k\|^q \right)^{\frac{1}{q}} \leqslant \frac{c(q) M n^{\frac{1}{q}}}{n^{1+\frac{1}{p}}}$$

and the series having general terms $\frac{\mathbb{E}\|\mathbf{S}_n\|}{n^{1+\frac{1}{p}}}$ converges.

From now we suppose that $\forall t > 0, g(t) > 0$.

By a classical symmetrization argument it suffices to consider the case of symmetrically distributed r.v. $(\mathbf{X}_k)$. For showing that under condition (3) the series having general term $\frac{\mathbb{E}\|\mathbf{S}_n\|}{n^{1+\frac{1}{p}}}$ converges, one will split each r.v. $\mathbf{X}_1, \ldots, \mathbf{X}_n$ involved in the sum $\mathbf{S}_n$ into two parts $\mathbf{U}_{n,k}$ and $\mathbf{V}_{n,k}$ by truncating $\mathbf{X}_k$ at a suitable level v_n.

For defining v_n, one first notices that, g being decreasing, one has :

$$\sup_{t>0} t^p g(t) \leqslant \left(\int_0^{+\infty} g^{\frac{1}{p}}(x) dx \right)^p < +\infty.$$

Furthermore, multiplying the $\mathbf{X}_k$ by a suitable constant if necessary, one can suppose without loss of generality that :

$$\sup_{t>0} t^p g(t) \leqslant 1. \tag{5}$$

Now define $v_n := \inf \left(t > 0 | g(t) \leqslant \frac{1}{n} \right)$.

It follows from the definition of g and (5) that :

$$v_n \leqslant n^{1/p} \qquad \text{and} \qquad g(v_n) \leqslant \frac{1}{n}.$$

For every $n \in \mathbb{N}^*$ and $k = 1, \ldots, n$, one considers the following centered r.v. (by symmetry) :

$$\mathbf{U}_{n,k} := \mathbf{X}_k \mathbf{1}_{(\|\mathbf{X}_k\| \leqslant v_n)} \qquad \text{and} \qquad \mathbf{V}_{n,k} := \mathbf{X}_k \mathbf{1}_{(\|\mathbf{X}_k\| > v_n)},$$

and the associated sums :

$$\mathbf{A}_n := \sum_{k=1}^{n} \frac{\mathbf{U}_{n,k}}{n^{1+\frac{1}{p}}} \qquad \text{and} \qquad \mathbf{B}_n := \sum_{k=1}^{n} \frac{\mathbf{V}_{n,k}}{n^{1+\frac{1}{p}}}.$$

For showing that the series having general terms $\frac{\mathbb{E}\|\mathbf{S}_n\|}{n^{1+\frac{1}{p}}}$ converges, one will show that :

$$\sum_{n=1}^{+\infty} \mathbb{E}\|\mathbf{A}_n\| < +\infty \tag{6}$$

and

$$\sum_{n=1}^{+\infty} \mathbb{E}\|\mathbf{B}_n\| < +\infty. \tag{7}$$

We first prove (6).

By remark 1. following definition 1, there exists $q > p$ such that $(\mathcal{B}, \|\cdot\|)$ is also q–stable.

Now suppose that the r.v. $\mathbf{X}_k$ are defined on a probability space $(\Omega, \mathcal{T}, \mathbb{P})$ and consider (ε_k) a sequence of independent Rademacher r.v. defined on another probability space $(\Omega', \mathcal{T}', \mathbb{P}')$. By symmetry, one has :

$$\mathbb{E}\|A_n\| = \int_{\Omega} \left(\int_{\Omega'} \frac{1}{n^{1+\frac{1}{p}}} \left\| \sum_{k=1}^{n} \varepsilon_k(\omega') \mathbf{U}_{n,k}(\omega) \right\| d\mathbb{P}'(\omega') \right) d\mathbb{P}(\omega).$$

By application of the definition of the stable type q, one obtains :

$$\mathbb{E}\|A_n\| \leqslant \frac{C(q)}{n^{1+\frac{1}{p}}} \mathbb{E}\left(\|(\|\mathbf{U}_{n,1}\|, \ldots, \|\mathbf{U}_{n,n}\|)\|_{q,\infty} \right).$$

For bounding the tails of the weak-ℓ_p norm of a sequence of positive, independent r.v., we will use the following classical result due to Marcus Pisier [4] :

Lemma 10.
For positive valued, independent r.v. $\xi_1, \ldots, \xi_n$, one has :

$$\forall q \geqslant 1,\ \forall u > 0,\ \mathbb{P}\left(\|(\xi_1, \ldots \xi_n)\|_{q,\infty} > u \right) \leqslant \frac{2e}{u^q} \Delta(\xi_1, \ldots, \xi_n), \tag{8}$$

where $\Delta(\xi_1, \ldots, \xi_n) = \sup_{t>0} \left(t^q \sum_{k=1}^{n} \mathbb{P}(\xi_k > t) \right)$.

For simplicity, denote Δ_n the quantity $\Delta(\|\mathbf{U}_{n,1}\|, \ldots, \|\mathbf{U}_{n,n}\|)$, and notice that by application of lemma 10 :

$$\begin{aligned}
\mathbb{E}\left(\|(\|\mathbf{U}_{n,1}\|, \ldots, \|\mathbf{U}_{n,n}\|)\|_{q,\infty} \right) &= \int_0^{+\infty} \mathbb{P}\left(\|(\|\mathbf{U}_{n,1}\|, \ldots, \|\mathbf{U}_{n,n}\|)\|_{q,\infty} > u \right) du \\
&\leqslant \Delta_n^{\frac{1}{q}} + \int_{\Delta_n^{\frac{1}{q}}}^{+\infty} \frac{2e}{u^q} \Delta_n du \\
&\leqslant c_2 \Delta_n^{\frac{1}{q}}
\end{aligned}$$

So the proof of (6) reduces to the following lemma :

Lemma 11.

$$\sum_{n=1}^{+\infty} \frac{\Delta_n^{\frac{1}{q}}}{n^{1+\frac{1}{p}}} < +\infty$$

Proof of lemma 11 :

One first notices that :

$$\Delta_n \leqslant \sup_{t\leqslant v_n} t^q \sum_{k=1}^{n} \mathbb{P}(\|\mathbf{X}_k\| > t) \leqslant \sup_{t\leqslant v_n} nt^q g(t) \leqslant n\left(\int_0^{v_n} g^{\frac{1}{q}}(u)du\right)^q,$$

the last inequality following from the fact that g is decreasing.

For concluding the proof of lemma 11 it remains to check that the series with general term $a_n := \frac{1}{n^{1+\frac{1}{p}-\frac{1}{q}}} \int_0^{v_n} g^{\frac{1}{q}}(u)du$ converges.

First observe that :

$$\sum_{n=1}^{+\infty} a_n \leqslant \sum_{n=1}^{+\infty} \frac{1}{n^{1+\frac{1}{p}-\frac{1}{q}}} \sum_{j=0}^{n} \int_{v_j}^{v_{j+1}} g^{\frac{1}{q}}(u)du,$$

where $v_0 := 0$, and then exchange the summations in n and j :

$$\sum_{n=1}^{+\infty} a_n \leqslant c_3 \left(v_1 + \sum_{j=1}^{+\infty} \frac{1}{j^{\frac{1}{p}-\frac{1}{q}}} \left(\int_{v_j}^{v_{j+1}} g^{\frac{1}{q}}(u)du\right)\right),$$

so, by the definition of v_{j+1} :

$$\sum_{n=1}^{+\infty} a_n \leqslant c_4 \left(v_1 + \int_0^{+\infty} g^{\frac{1}{p}}(u)du\right),$$

which concludes the proof of lemma 11.

Now we are going to prove (7).

First notice the following chain of inequalities :

$$\begin{aligned}
\mathbb{E}\|\mathbf{B}_n\| &\leqslant \sum_{k=1}^{n} \frac{\mathbb{E}\|\mathbf{V}_{n,k}\|}{n^{1+\frac{1}{p}}} = \sum_{k=1}^{n} \int_0^{+\infty} \mathbb{P}\left(\frac{\|\mathbf{V}_{n,k}\|}{n^{1+\frac{1}{p}}} > u\right) du \\
&\leqslant \sum_{k=1}^{n} \frac{v_n}{n^{1+\frac{1}{p}}} \mathbb{P}(\|\mathbf{X}_k\| > v_n) + \frac{n}{n^{1+\frac{1}{p}}} \int_{v_n}^{+\infty} g(u)du \\
&\leqslant \frac{nv_n}{n^{1+\frac{1}{p}}} g(v_n) + \frac{1}{n^{\frac{1}{p}}} \int_{v_n}^{+\infty} g(u)du \leqslant \frac{v_n}{n^{1+\frac{1}{p}}} + \frac{1}{n^{\frac{1}{p}}} \int_{v_n}^{+\infty} g(u)du.
\end{aligned}$$

Therefore, for proving (7), it suffices to check that condition (3) implies the convergence of the two series with general terms $\frac{v_n}{n^{1+\frac{1}{p}}}$ and $\frac{1}{n^{\frac{1}{p}}} \int_{v_n}^{+\infty} g(u)du$.

Lemma 12.
If (3) is fulfilled, then $\sum_{n=1}^{+\infty} \frac{v_n}{n^{1+\frac{1}{p}}} < +\infty$.

Proof of lemma 12 :
For $j \in \mathbb{N}^*$, one denotes $t_j := \frac{v_j+v_{j+1}}{2}$. Then :

$$\int_0^{+\infty} g^{\frac{1}{p}}(u)du \geqslant \sum_{j=1}^{+\infty} \int_{v_j}^{t_j} g^{\frac{1}{p}}(t)dt \geqslant \sum_{j=1}^{+\infty} \frac{1}{(j+1)^{\frac{1}{p}}} \frac{v_{j+1}-v_j}{2}. \tag{9}$$

Now observe that :

$$\sum_{j=1}^{n-1} \frac{1}{(j+1)^{\frac{1}{p}}}(v_{j+1}-v_j) = -\frac{v_1}{2^{\frac{1}{p}}} + \sum_{j=2}^{n-1} v_j \left(\frac{1}{j^{\frac{1}{p}}} - \frac{1}{(j+1)^{\frac{1}{p}}} \right) + \frac{v_n}{n^{\frac{1}{p}}} \tag{10}$$

As :

$$\frac{v_n}{2} g^{\frac{1}{p}}(v_n) \leqslant \int_{\frac{v_n}{2}}^{v_n} g^{\frac{1}{p}}(u)du,$$

one gets $\lim\limits_{n\to+\infty} v_n g^{\frac{1}{p}}(v_n) = 0$ and also $\lim\limits_{n\to+\infty} \frac{v_n}{n^{\frac{1}{p}}} = 0$.

As $\frac{1}{j^{\frac{1}{p}}} - \frac{1}{(j+1)^{\frac{1}{p}}} \geqslant \frac{c_5}{j^{1+\frac{1}{p}}}$, it follows from (9) and (10) that the series having general term $\frac{v_n}{n^{1+\frac{1}{p}}}$ converges.

For completing the proof of the implication $1 \Longrightarrow 2$ of theorem 8, it remains to check :

Lemma 13.
Under (3), one has $\sum\limits_{n=1}^{+\infty} \frac{1}{n^{\frac{1}{p}}} \int_{v_n}^{+\infty} g(u)du < +\infty$.

Proof of lemma 13.
Let us write :

$$\alpha := \sum_{n=1}^{+\infty} \frac{1}{n^{\frac{1}{p}}} \int_{v_n}^{+\infty} g(u)du = \sum_{n=1}^{+\infty} \frac{1}{n^{\frac{1}{p}}} \sum_{j=n}^{+\infty} \int_{v_j}^{v_{j+1}} g(u)du.$$

By exchanging the summations in n and j, one gets :

$$\begin{aligned}\alpha &= \sum_{j=1}^{+\infty} \left(\int_{v_j}^{v_{j+1}} g(u)du \right) \sum_{n=1}^{j} \frac{1}{n^{\frac{1}{p}}} \leqslant c_6 \sum_{j=1}^{+\infty} \left(\int_{v_j}^{v_{j+1}} g^{\frac{1}{p}}(u)du \right) \frac{j^{1-\frac{1}{p}}}{j^{1-\frac{1}{p}}} \\ &\leqslant c_7 \int_0^{+\infty} g^{\frac{1}{p}}(u)du.\end{aligned}$$

Let us now show the converse implication $2 \Longrightarrow 1$ of theorem 8.
We first show a general property which is of independent interest :

Proposition 14.
Let $(\mathbf{Y}_n)$ *be a sequence of independent strongly centered r.v. with values in a*

general Banach space $(\mathcal{B}, \|\cdot\|)$. *Denote by* $\mathbf{T}_n$ *the sum* $\mathbf{Y}_1 + \ldots + \mathbf{Y}_n$ *and by* $\mathcal{G}_n$ *the* σ*-field* $\sigma(\mathbf{Y}_1, \ldots, \mathbf{Y}_n)$. *If* $\left(\frac{\mathbf{T}_n}{n^{\frac{1}{p}}}, \mathcal{G}_n\right)$ *is a quasimartingale, then* $\left(\frac{\mathbf{T}_n}{n^{\frac{1}{p}}}\right)$ *converges a.s. to 0.*

Proof of proposition 14 :

As noticed earlier, if $\left(\frac{\mathbf{T}_n}{n^{\frac{1}{p}}}\right)$ is a quasimartingale, then :

$$\sum_{n=1}^{+\infty} \frac{\mathbb{E}\|\mathbf{T}_n\|}{n^{1+\frac{1}{p}}} < +\infty. \tag{11}$$

By Jensen's inequality :

$$\forall N \in \mathbb{N}^*,\ \sum_{n=N}^{+\infty} \frac{\mathbb{E}\|\mathbf{T}_n\|}{n^{1+\frac{1}{p}}} \geqslant \mathbb{E}\|\mathbf{T}_N\| \sum_{n=N}^{+\infty} \frac{1}{n^{1+\frac{1}{p}}} \geqslant c_8 \frac{\mathbb{E}\|\mathbf{T}_N\|}{N^{\frac{1}{p}}},$$

so by (11),

$$\lim_{n\to+\infty} \frac{\mathbb{E}\|\mathbf{T}_n\|}{n^{\frac{1}{p}}} = 0. \tag{12}$$

By the conditionnal version of Jensen's inequality :

$$\forall n \in \mathbb{N}^*,\ \mathbb{E}\left(\|\mathbf{T}_{n+1}\||\mathcal{G}_n\right) \geqslant \|\mathbf{T}_n\|, \tag{13}$$

so :

$$\sum_{n=1}^{N} \mathbb{E}\left|\mathbb{E}\left(\frac{\|\mathbf{T}_{n+1}\|}{(n+1)^{\frac{1}{p}}} - \frac{\|\mathbf{T}_n\|}{n^{\frac{1}{p}}}\middle|\mathcal{G}_n\right)\right| \leqslant \sum_{n=1}^{N} \mathbb{E}\left|\mathbb{E}\left(\frac{\|\mathbf{T}_{n+1}\| - \|\mathbf{T}_n\|}{(n+1)^{\frac{1}{p}}}\middle|\mathcal{G}_n\right)\right| + c_9 \sum_{n=1}^{N} \frac{\mathbb{E}\|\mathbf{T}_n\|}{n^{1+\frac{1}{p}}}$$

and by (13) :

$$\sum_{n=1}^{N} \mathbb{E}\left|\mathbb{E}\left(\frac{\|\mathbf{T}_{n+1}\|}{(n+1)^{\frac{1}{p}}} - \frac{\|\mathbf{T}_n\|}{n^{\frac{1}{p}}}\middle|\mathcal{G}_n\right)\right| \leqslant \mathbb{E}\left(\frac{\|\mathbf{T}_{N+1}\|}{(N+1)^{\frac{1}{p}}}\right) + c_{10} \sum_{n=1}^{N} \frac{\mathbb{E}\|\mathbf{T}_n\|}{n^{1+\frac{1}{p}}}.$$

Finaly, by (11) and (12), the sequence $\left(\frac{\|\mathbf{T}_n\|}{n^{1+\frac{1}{p}}}\right)$ is a positive quasimartingale. Therefore, thanks to theorem 9.4 in [6], it converges a.s. to a limit, which, by (12) is necessary 0.

This concludes the proof of proposition 14.

Let us now come back to the proof of the implication (2) $\Longrightarrow$(1) of theorem 8.

Let (ε_k) be a sequence of independent Rademacher r.v. and (x_k) be a bounded sequence of elements in $\mathcal{B}$. Defining $M := \sup \|x_k\|$, $\mathbf{X}_k := \varepsilon_k x_k$, one gets :

$$\forall t > M,\ g(t) = \sup_k \mathbb{P}(\|\mathbf{X}_k\| > t) = 0,$$

so condition (3) holds. Therefore $\frac{1}{n^{\frac{1}{p}}} \sum_{k=1}^{n} \varepsilon_k x_k$ is a quasimartingale, which by proposition 14 converges a.s. to 0. The p-stability of the space $(\mathcal{B}, \|\cdot\|)$ then follows from theorem 4.

3 What happens when $p = 1$?

It is natural to wonder if the spaces of stable type 1 (see [2] for the definition of stable type 1) also admit a "quasimartingale characterization ". In fact it is the case, by theorem 6 in [2], which can be reformulated as follows :

Theorem 15. *Let $\mathcal{B}$ be a Banach space. The following two properties are equivalent :*

1. *$\mathcal{B}$ is of stable type 1.*
2. *For every sequence $(\mathbf{X}_n)$ of independent, strongly centered r.v., such that*

$$\int_0^{+\infty} g(t)dt < +\infty, \tag{14}$$

where

$$\forall t > 0,\ g(t) := \sup_{n \geqslant 1} \mathbb{P}(\|\mathbf{X}_n\| \ln(1 + \|\mathbf{X}_n\|) > t),$$

$\left(\frac{\mathbf{S}_n}{n}\right)$ is a quasimartingale.

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