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Serguei Barannikov

CNRS

02/06/2010

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$$\hbar\Delta S + \frac{1}{2}\{S, S\} = 0,$$

$S = \sum_{g \geq 0, i} \hbar^{2g-1+i} S_{g,i}$  where

$$S_{g,i} \in \text{Symm}^i(C_\lambda[1-d]), \quad C_\lambda = (\oplus_{j=0}^{\infty} ((V[1]^{\otimes j})^\vee)^{\mathbb{Z}/j\mathbb{Z}})$$

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$$\int \exp \widehat{S}(X, \Lambda) dX$$

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- In the case of the algebra  $e \cdot e = e$ , the answer is the matrix Airy integral

$$\int \exp\left(\frac{1}{6} \text{Tr}(Y^3) - \frac{1}{2} \text{Tr}(\Lambda Y^2)\right) dY$$

# The A-infinity equivariant matrix integrals ([B2],2006)

- The asymptotic expansion as  $\Lambda \rightarrow \infty \Rightarrow$  sum over stable ribbon graphs  $\Rightarrow$  cohomology classes in  $H^*(\overline{\mathcal{M}}_{g,n}^K)$  (in  $H^*(\overline{\mathcal{M}}_{g,n}^K, \mathcal{L})$  for odd  $d$ )



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- My  $A_\infty$  equivariant matrix integrals give an integration framework in the noncommutative (derived algebraic) geometry, particularly adopted to the equation  $\{m_{A_\infty}, m_{A_\infty}\} = 0$

$$\int \exp \widehat{S}(X, \Lambda) \widehat{\varphi} dX$$

$$\varphi \in \text{Ker}(\hbar\Delta + \{S, \cdot\}), \quad \varphi_{g,i} \in \text{Symm}^i(C_\lambda[1-d])$$

# CY complex projective variety ( $g=0$ calculations)

- $M$ - projective manifold/ $\mathbb{C}$ ,  $c_1(T_M) = 0$

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- $$\Omega(t, \hbar) = \int_M \exp(\gamma)\omega$$

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- for  $\gamma^W$ , normalized via a filtration  $W$  opposite to  $F^{\text{Hodge}}$  ([B5]):

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$$C_{kij}(t) = \partial^3(\text{genus} = 0 \text{ GW-potential of } M^{\text{mirror}})$$

# Noncommutative Hodge structures ([B5])

- The class  $[\exp(\gamma^W)\omega] = \Omega(t, \hbar)$  is obtained as intersection

$$\Omega(t, \hbar) = \mathcal{L}(t) \cap (\text{Affine space}(W))$$

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(implies  $\text{tt}^*$ -equations, remarkably  $\mathcal{D}_{\frac{\partial}{\partial \hbar}}$ -modules over  $\mathbb{A}^1$  with similar properties appeared many years ago in works of Birkhoff, Malgrange, K.Saito and M.Saito)

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- Over moduli space of complex structures

$$\mathcal{L}(t) = \sum_r (F^{\text{Hodge}})^r \hbar^{-r} [[\hbar]]$$

## Noncommutative Hodge structures ([B5]), cont'd

- $\mathcal{L}(t)$  corresponds via HKR and formality isomorphisms for  $C^*(A, A) + k[\zeta, \frac{\partial}{\partial \zeta}]$ -module  $C_*(A)$  to

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$$\frac{\partial}{\partial t} HC^-(A_t) \subset \hbar^{-1} HC^-(A_t), \quad \frac{\partial}{\partial t} - \text{Getzler flat connection on } HP(A_t)$$

where

$$rk_{C[[\hbar]]} HC^-(A) = rk_{C((\hbar))} HP$$

assumed, i.e. the degeneration of nc Hodge -to-De Rham spectral sequence, proven (Kaledin) for  $A$ -smooth and compact,  $Z_+$ -graded, then  $HC^- \subset HP$ ,



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- Real structure on  $HP$  in the case of arbitrary  $A_\infty$ -algebra?

# Noncommutative Batalin-Vilkovisky operator ([B1])

- $V$   $-\mathbb{Z}/2\mathbb{Z}$  graded vector space,  $l$ - scalar product on  $V^\vee$  of degree  $d$ ,

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$$C_\lambda = (\bigoplus_{j=0}^{\infty} ((V[1]^{\otimes j})^\vee)^{\mathbb{Z}/j\mathbb{Z}})$$

- Define the noncommutative BV differential on  $F$  via

$$\begin{aligned} \Delta(x_{\rho_1} \dots x_{\rho_r})_\lambda(x_{\tau_1} \dots x_{\tau_t})_\lambda &= \\ &= \sum_{p,q} (-1)^\varepsilon l_{\rho_p \tau_q} (x_{\rho_1} \dots x_{\rho_{p-1}} x_{\tau_{q+1}} \dots x_{\tau_{q-1}} x_{\rho_{p+1}} \dots x_{\rho_r})_\lambda + \\ &\sum_{p \pm 1 \neq q} (-1)^{\tilde{\varepsilon}} l_{\rho_p \rho_q} (x_{\rho_1} \dots x_{\rho_{p-1}} x_{\rho_{q+1}} \dots x_{\rho_r})_\lambda (x_{\rho_{p+1}} \dots x_{\rho_{q-1}})_\lambda (x_{\tau_1} \dots x_{\tau_t})_\lambda \\ &\sum_{p \pm 1 \neq q} (-1)^{\tilde{\varepsilon}} l_{\tau_p \tau_q} (x_{\rho_1} \dots x_{\rho_r})_\lambda (x_{\tau_1} \dots x_{\tau_{p-1}} x_{\tau_{q+1}} \dots x_{\tau_t})_\lambda (x_{\tau_{p+1}} \dots x_{\tau_{q-1}})_\lambda \end{aligned}$$

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- $\Delta_1 + \hbar\Delta_2 \rightarrow$  non-commutative Batalin–Vilkovisky geometry, stable ribbon graphs, compactified moduli spaces  $H_*(\overline{\mathcal{M}}_{g,n}^K)$  ([B1])



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- $\Delta_1 + \hbar\Delta_2 \rightarrow$  non-commutative Batalin–Vilkovisky geometry, stable ribbon graphs, compactified moduli spaces  $H_*(\overline{\mathcal{M}}_{g,n}^K)$  ([B1])
- $\text{Ker}\Delta_1 + \Delta_2 = \text{Im}\Delta_1 + \Delta_2$

## Solutions to nc BV equation

- Conjecture ([B1]). Counting of holomorphic curves  $(\Sigma, \partial\Sigma, p_i) \rightarrow (M, \coprod L_i, \oplus H_*(L_i \cap L_j))$ , with  $\mathbb{Z}/2\mathbb{Z}$ -graded local systems, gives solution to the nc-BV equations.

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- Theorem ([B6]). Summation over ribbon graphs  $\rightarrow$  solution to the nc Batalin-Vilkovisky equation from dg-associative algebras (summation over trees  $\rightarrow$  A-infinity algebra structure)

# Strange associative superalgebra with odd trace and psi-classes.

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- Similarly, with even scalar product and an *odd derivation*, in particular for  $gl(N|N)$  and  $I = [\Xi, \cdot]$ ,  $\Xi \in gl(N|N)_{\text{odd}}$ .

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