Converse Lyapunov Theorems for Switched Systems in Banach and Hilbert Spaces

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CONVERSE LYAPUNOV THEOREMS FOR SWITCHED SYSTEMS IN BANACH AND HILBERT SPACES

FALK M. HANTE AND MARIO SIGALOTTI

Abstract. We consider switched systems on Banach and Hilbert spaces governed by strongly continuous one-parameter semigroups of linear evolution operators. We provide necessary and sufficient conditions for their global exponential stability, uniform with respect to the switching signal, in terms of the existence of a Lyapunov function common to all modes.

1. INTRODUCTION

It is well known that the existence of a *common Lyapunov function* is necessary and sufficient for the global uniform asymptotic stability of finite-dimensional switched dynamical systems [12]. In the *linear* finite-dimensional case, the existence of a common Lyapunov function is actually equivalent to global uniform exponential stability [15] and, provided that the system has finitely many modes, the Lyapunov function can be taken polyhedral or polynomial [2, 3, 7]. A special role in the switched control literature has been played by common quadratic Lyapunov functions, since their existence can be tested rather efficiently (see the surveys [11, 19] and the references therein). It is known, however, that not only the existence of a common quadratic Lyapunov function is not necessary for the global uniform exponential stability of a linear switched system with finitely many modes, but also that there exists no uniform upper bound on the minimal degree of a common polynomial Lyapunov function [13].

The scope of this paper is to understand the relationships between global uniform exponential stability and the existence of a common Lyapunov function for infinite-dimensional switched systems of the type

$$\begin{cases} \dfrac{d}{dt}\,x(t) = A_{\sigma(t)}x(t), \quad t>0 \\ \quad x(0) = x \in X \end{cases} \tag{1}$$

where each A_j is a (possibly unbounded) operator generating a strongly continuous semigroup $T_j(t)$ on a Banach space X and $\sigma(\cdot)$ belongs to the class of piecewise constant switching signals with values in an index set Q.

Such systems provide a convenient design paradigm for modeling a wide variety of complex processes comprising distributed parameters, see [9] and the references therein for examples in the context of networked transport systems.

Except for special cases, that is, when X has a Hilbert structure and the infinitesimal generators A_j commute pairwise [18], when the switching signals satisfy a dwell-time constraint [14] or when A_j is a linear convection-reaction operator with reflecting boundary conditions [1], global uniform exponential stability of systems such as (1) has not been investigated (up to our knowledge).

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The characterization of exponential stability for a single linear dynamical system on Banach and Hilbert spaces dates back to Datko [6] and Pazy [16] and has, since then, seen a broad range of applications in control theory for partial differential equations (see, for instance, [21]). However, we recall that exponential stability of all subsystems (with $\sigma(t) \equiv j$ fixed in (1)) is of course necessary but not sufficient for the global uniform exponential stability with respect to all possible switching laws $\sigma(\cdot)$. This is a classical result for the finite dimensional case and we give an infinite dimensional variant with interesting destabilizing properties in Example 1 below.

Our starting point will be a switching system of the general form

$$\begin{cases} x(t_{k+1}) = T_{\sigma(t_k)}(t_{k+1} - t_k)x(t_k), \quad k \in \mathbb{N}, \\ \quad x(0) = x \in X, \end{cases} \tag{2}$$

where $\sigma\colon [0,\infty) \to Q$ is a piecewise constant right-continuous switching signal with switching times $0 = t_0 < t_1 < \cdots < t_k < \cdots$. Each $t \mapsto T_j(t)$, $j \in Q$, is a strongly continuous semigroup on a Banach space X. If $t \in (t_k, t_{k+1})$, then $x(t) = T_{\sigma(t_k)}(t - t_k)x(t_k)$. In particular, our results apply to switched dynamical systems such as (1), even with the infinitesimal generators A_j not sharing a common domain and also in the case of infinitely many available modes Q. (The semigroup formulation (2) can be seen as a choice of the notion of solution to the Cauchy problem (1).)

The construction of a common Lyapunov function, under the assumption that global uniform exponential stability holds true, follows the same lines as in finite dimension. In particular, a possible choice of the Lyapunov function is

$$V(x) = \sup\left\{ \int_0^\infty \|x(t)\|^2 dt : x(\cdot) \text{ solution to (2) for some } \sigma \right\}.$$

(Alternatively, one could take $V(x) = \int_0^\infty \sup_{\sigma(\cdot)} \|x(t)\|^2 dt$, as done in [10].)

Then one proves that the left Dini derivatives of V at x along the evolution of each semigroup T_j are bounded from above by $-\|x\|^2$. Moreover, $V(x) \le C\|x\|^2$ for some $C > 0$ independent of x. The converse implication is not true, i. e., even if some nonnegative function V satisfies these two properties, global uniform exponential stability may fail. This is illustrated in Remark 4 where an explicit counterexample is discussed.

The converse implication becomes true if we add the requirement that the system is *globally uniformly exponentially bounded*, i. e., there exist $K, \mu > 0$ such that $\|x(t)\| \le Ke^{\mu t}\|x(0)\|$ for every $t > 0$, $\sigma(\cdot)$ and $x(0)$. We obtain in this way a first characterization: global uniform exponential stability is equivalent to the existence of a Lyapunov function satisfying the properties mentioned above together with global uniform exponential boundedness (Theorem 3).

A second result is obtained by dropping the global uniform exponential boundedness and by strengthening the requirements on V, namely, by asking it to be comparable with respect to the squared norm, that is,

$$c\|x\|^2 \le V(x) \le C\|x\|^2, \quad x \in X,$$

for some $c, C > 0$ independent of x (Theorem 6).

From the point of view of applications, Theorem 3, imposing less conditions on V, is better suited for proving the global uniform exponential stability of a switched system (although the uniform exponential growth boundedness needs also be proved), while Theorem 6 provides more information on a globally uniformly exponentially stable switched system.

In the case of a single mode ($Q = \{0\}$), it was observed by Pazy [16] that $x \mapsto \int_0^\infty \|T_0(t)x\|^2\, dt$ defines a Lyapunov function that is comparable with the

squared norm if and only if T_0 extends to an exponentially stable strongly continuous group. Theorem 6 states that, even if T_0 does not admit an extension to a group, a Lyapunov function comparable with the squared norm can still be found (see Remark 8).

Concerning the regularity of the Lyapunov functions obtained through the construction described above we prove that they are always convex and continuous. In the special case in which X is a Hilbert space, we also prove the Fréchet directional differentiability of V and we establish a characterization of the directional Fréchet derivatives.

The paper is organized as follows. In Section 2 we provide the two different necessary and sufficient conditions for global uniform exponential stability in terms of the existence of a common Lyapunov function for switched systems whose dynamics are described by strongly continuous semigroups on Banach spaces (Theorems 3 and 6). We also discuss the possible redundancy of the two-points condition equivalent to global uniform exponential stability appearing in the statement of Theorem 3, showing that point i) cannot be removed (Remark 4). Section 2 ends by the proof of the Fréchet differentiability of the common Lyapunov function when X is a Hilbert space (Proposition 9). In Section 3 we suggest some possible future research direction.

2. CONVERSE THEOREMS

By $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$ we denote the set of natural, rational and real numbers, respectively. Further, let X be a Banach space, $\mathcal{L}(X)$ be the space of bounded linear operators on X, Q be a countable set and, for all $j \in Q$, let $t \mapsto T_j(t) \in \mathcal{L}(X)$, $t \geq 0$ be a strongly continuous semigroup.

We wish to investigate the qualitative behavior of

$$x(t) = T_{\sigma(\cdot)}(t)x \tag{3}$$

for $x \in X$, where $\sigma\colon [0,\infty) \to Q$ is a piecewise constant switching signal and

$$T_{\sigma(\cdot)}(t) = T_{j_p}(t-\tau_p)T_{j_{p-1}}(\tau_p - \tau_{p-1}) \cdots T_{j_1}(\tau_1) \tag{4}$$

for $\sigma(\cdot)$ equal to j_k on (τ_{k-1}, τ_k) for $k = 1, \ldots, p+1$ and

$$0 = \tau_0 < \tau_1 < \cdots < \tau_{p+1} = t.$$

In particular, we wish to study the asymptotic behavior of $x(t)$ as t tends to $+\infty$, uniformly with respect to the switching law $\sigma(\cdot)$ in the set Σ of all piecewise constant switching signals. We note that, for any given $\sigma(\cdot) \in \Sigma$, the operator $T_{\sigma(\cdot)}(t) \in \mathcal{L}(X)$ is strongly continuous with respect to t ($\lim_{t \downarrow t_0} \|T_{\sigma(\cdot)}(t) - T_{\sigma(\cdot)}(t_0)\| = 0$) and satisfies

$$T_{\sigma(\cdot)}(t+s) = T_{\sigma_s(\cdot)}(t)T_{\sigma(\cdot)}(s) \tag{5}$$

for some switching signal $\sigma_s(\cdot) \in \Sigma$ depending on s, but in general, $T_{\sigma(\cdot)}(t)$ does not satisfy the semigroup property, i. e., equation (5) with $\sigma_s(\cdot)$ replaced by $\sigma(\cdot)$ (independently of s).

For a function $V\colon X \to [0,\infty)$ we define the generalized derivative

$$\underline{L}_j V(x) = \liminf_{t \downarrow 0} \frac{V(T_j(t)x) - V(x)}{t}, \tag{6}$$

noting the possibility that $|\underline{L}_j V(x)| = \infty$ for some $x \in X$ and $j \in Q$. Further, we call a switched system (3) (completely determined by $\{T_j\}_{j \in Q}$) *globally uniformly exponentially stable* when there exist constants $K \geq 1$ and $\mu > 0$ such that

$$\|T_{\sigma(\cdot)}(t)\|_{\mathcal{L}(X)} \leq Ke^{-\mu t}, \quad t \geq 0,\ \sigma(\cdot)\text{-uniformly.} \tag{7}$$

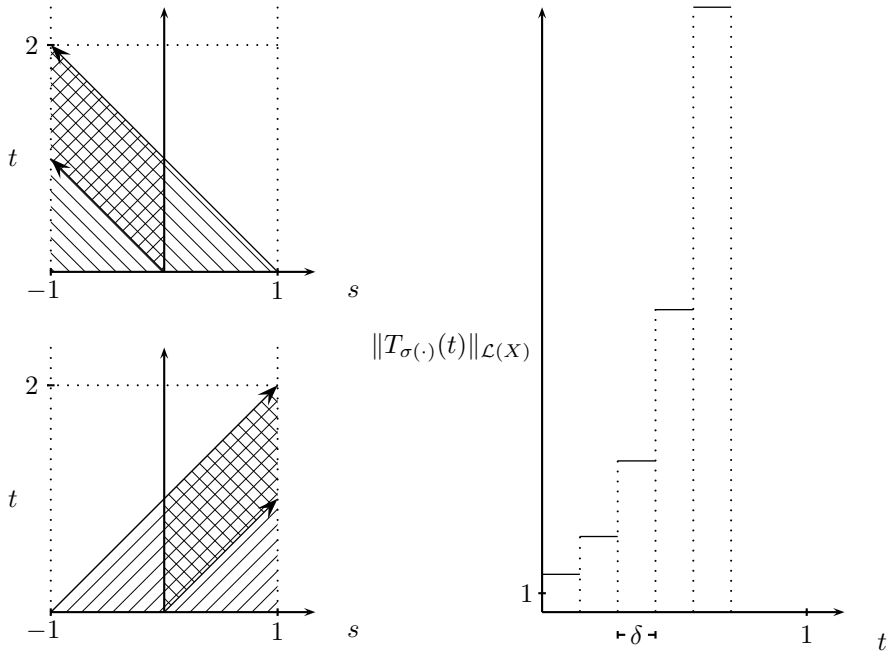

FIGURE 1. Illustration of $T_j(t)$, $j = 1, 2$, (left) and the blow up of the operator norm of $T_{\sigma(\cdot)}(t)$ (right) in Example 1.

It is clear that (7) implies

$$\|x(t)\|_X \leq Ke^{-\mu t}\|x\|_X, \quad t \geq 0$$

globally for all $x \in X$ and uniformly for all $\sigma(\cdot) \in \Sigma$ justifying the terminology. We point out that (7) implies *strong attractivity at the origin*, i. e.,

$$\lim_{t\to\infty} \|T_{\sigma(\cdot)}(t)x\|_X = 0, \quad x \in X,\ \sigma(\cdot) \in \Sigma, \tag{8}$$

and *uniform stability*, i. e.,

$$\begin{cases} \text{for all } \varepsilon > 0 \text{ there exists a } \delta > 0, \\ \text{independent of } \sigma(\cdot), \text{ such that } \|x\|_X < \delta \text{ implies} \\ \quad \|T_{\sigma(\cdot)}(t)x\|_X < \varepsilon,\ t \geq 0,\ \sigma(\cdot)\text{-uniformly} \end{cases} \tag{9}$$

but that the converse implication is false in general, even for a single mode: As a counterexample it suffices to take the left translation semigroup defined by

$$(T(t)f)(s) := f(s + t)$$

on the Lebesgue space $X = L^1(\mathbb{R}_+)$. This is in contrast to the equivalence of (8) and (7) when X is a n-dimensional real coordinate space, Q is finite, and $T_j(t)$ is given by the matrix exponential $e^{A_j t}$ for some real $n\times n$-matrix A_j, as a consequence of Fenichel's Uniformity Lemma (see, for instance, [5, §5.2]).

Before turning our attention to necessary and sufficient conditions for global uniform exponential stability, we give an example of a switched system exhibiting illustrative instability properties, though the subsystems are exponentially stable.

Example 1. Consider the bimodal system $\{T_j(t)\}_{j=1,2}$ with $T_j(t)$ defined on the Lebesgue space $X = L^1(-1,1)$ by

$$(T_1(t)f)(s) = \begin{cases} 2f(s+t), & s \in [-1, 1-t] \cap [-t, 0] \\ f(s+t), & s \in [-1, 1-t] \setminus [-t, 0] \\ 0, & s \in (1-t, 1] \end{cases}$$

and

$$(T_2(t)f)(s) = \begin{cases} 2f(s-t), & s \in [-1+t, 1] \cap [0, t] \\ f(s-t), & s \in [-1+t, 1] \setminus [0, t] \\ 0, & s \in [-1, -1+t). \end{cases}$$

Notice that both $T_1(\cdot)$ and $T_2(\cdot)$ are nilpotent semigroups, since $T_1(t) = T_2(t) = 0$ for $t \geq 2$. In particular, each of them is exponentially stable.

It is easy to see that for suitable switching signals $\sigma(\cdot) \in \Sigma$, e. g., switching at $\tau_k = k\delta$, $k \in \mathbb{N}$, for a fixed $\delta < 1$,

$$\|T_{\sigma(\cdot)}(t)\|_{\mathcal{L}(X)} \to +\infty \quad \text{as } t \to +\infty.$$

In fact, the speed of blow-up is not uniformly exponentially bounded over the set of all possible $\sigma(\cdot)$, i. e., for any fixed $t > 0$, we have

$$\|T_{\sigma(\cdot)}(t)\|_{\mathcal{L}(X)} \geq 2^{\lceil \frac{t}{\delta} \rceil} \to +\infty \quad \text{as } \delta \to 0,$$

with $\lceil \tau \rceil = \min\{k \in \mathbb{N} : \tau \leq k\}$ for $\tau > 0$ (see Figure 1). This can be seen by taking $L^1(-1,1)$-functions f of norm one, identically constant near $x = 0$ on progressively smaller intervals and zero elsewhere. □

The following lemma, related to the blow-up phenomenon illustrated in Example 1 is a variant of a result obtained in [20] in the framework of strongly continuous semigroups. While extending the property to switched system of the form (3), the proof given in [20] should be modified in order to replace the semigroup property by (5). We include the modified proof for the sake of completeness.

Lemma 2. *Assume that*

i) *there exist constants $M \geq 1$ and $\omega > 0$ such that*

$$\|T_{\sigma(\cdot)}(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}, \quad t \geq 0, \ \sigma(\cdot)\text{-uniformly};$$

ii) *there exists a constant $C > 0$ and some $p \in [1, \infty)$ such that*

$$\int_0^\infty \|T_{\sigma(\cdot)}(t)x\|_X^p \, dt \leq C\|x\|_X^p, \quad x \in X, \ \sigma(\cdot)\text{-uniformly}.$$

Then, there exist constants $K \geq 1$ and $\mu > 0$ such that

$$\|T_{\sigma(\cdot)}(t)\|_{\mathcal{L}(X)} \leq Ke^{-\mu t}, \quad t \geq 0, \ \sigma(\cdot)\text{-uniformly}.$$

Proof. First, we show that under the assumptions i) and ii), for every $x \in X$, there exists a constant $C_x > 0$ such that

$$\|T_{\sigma(\cdot)}(t)x\|_X \leq C_x, \ t \geq 0, \quad \sigma(\cdot)\text{-uniformly} \tag{10}$$

and that, for all $\sigma(\cdot)$ and for all $x \in X$,

$$\lim_{t \to +\infty} \|T_{\sigma(\cdot)}(t)x\|_X = 0. \tag{11}$$

To this end, let $t > \frac{1}{\omega}$ and set $\Delta(t) = [t - \frac{1}{\omega}, t]$. Observe that, for every $\sigma(\cdot)$ and every $\tau \in \Delta(t)$, there exists a $\sigma_\tau(\cdot)$ such that

$$\begin{aligned} \|T_{\sigma(\cdot)}(t)x\|_X &= \|T_{\sigma_\tau(\cdot)}(t-\tau)T_{\sigma(\cdot)}(\tau)x\|_X \\ &\leq \|T_{\sigma_\tau(\cdot)}(t-\tau)\|_{\mathcal{L}(X)}\|T_{\sigma(\cdot)}(\tau)x\|_X. \end{aligned} \tag{12}$$

Moreover, by assumption i) and by definition of $\Delta(t)$, we have

$$\|T_{\sigma_\tau(\cdot)}(t-\tau)\|_{\mathcal{L}(X)} \leq Me^{\omega(t-\tau)} \leq Me^{\omega\frac{1}{\omega}} = Me, \tag{13}$$

yielding

$$\|T_{\sigma(\cdot)}(\tau)x\|_X \geq \frac{\|T_{\sigma(\cdot)}(t)x\|_X}{Me}, \quad \tau \in \Delta(t). \tag{14}$$

Now suppose (10) does not hold. Then, there exist $x \in X$, a sequence of switching signals $(\sigma_i(\cdot))_{i\in\mathbb{N}}$ in Σ and a sequence of times $(t_i)_{i\in\mathbb{N}}$ such that

$$\delta_i = \|T_{\sigma_i(\cdot)}(t_i)x\|_X \to +\infty \quad \text{as } i \to +\infty. \tag{15}$$

Assumption i) guarantees that t_i is diverging. Without loss of generality, $t_i > \frac{1}{\omega}$ for every $i \in \mathbb{N}$.

For $\tau \in \Delta(t_i)$, (14) yields

$$\|T_{\sigma_i(\cdot)}(\tau)x\|_X \geq \frac{\delta_i}{Me}, \quad \tau \in \Delta(t_i). \tag{16}$$

Hence, using (16) and again the size of $\Delta(t_i)$, we obtain from (15)

$$\begin{aligned}\int_0^\infty \|T_{\sigma_i(\cdot)}(\tau)x\|_X^p \, d\tau &\geq \int_{\Delta(t_i)} \|T_{\sigma_i(\cdot)}(\tau)x\|_X^p \, d\tau \\ &\geq \left(\frac{\delta_i}{Me}\right)^p \frac{1}{\omega} \to \infty\end{aligned}$$

as $i \to \infty$. This contradicts assumption ii). Hence, (10) holds true.

Next, suppose (11) does not hold. Then, there exist $x \in X$, $\sigma(\cdot) \in \Sigma$, $\delta > 0$ and a diverging sequence of times $(t_i)_{i\in\mathbb{N}}$ such that

$$\|T_{\sigma(\cdot)}(t_i)x\|_X \geq \delta \quad \text{for all } i. \tag{17}$$

Without loss of generality $t_i > t_{i-1} + \frac{1}{\omega}$ for every $i \in \mathbb{N}$ with $t_0 = 0$. For $\tau \in \Delta(t_i)$, (14) yields

$$\|T_{\sigma(\cdot)}(\tau)x\|_X \geq \frac{\delta}{Me}, \quad \tau \in \Delta(t_i). \tag{18}$$

Hence, using (16) and again the size of $\Delta(t_i)$, we obtain

$$\begin{aligned}\int_0^\infty \|T_{\sigma(\cdot)}(\tau)x\|_X^p \, d\tau &\geq \sum_{i=1}^\infty \int_{\Delta(t_i)} \|T_{\sigma(\cdot)}(\tau)x\|_X^p \, d\tau \\ &\geq \left(\frac{\delta}{Me}\right)^p \sum_{i=1}^\infty \frac{1}{\omega} = \infty.\end{aligned}$$

This again contradicts assumption ii) and hence (11) holds true.

Let

$$t_{x,\sigma(\cdot)}(\rho) = \max\{t : \|T_{\sigma(\cdot)}(t)x\|_X \geq \rho\|x\|_X, \ 0 \leq s \leq t\}.$$

By (11), $t_{x,\sigma(\cdot)}(\rho)$ is finite (and positive) for every $\sigma(\cdot)$ and $x \in X \setminus \{0\}$. By strong continuity,

$$\|T_{\sigma(\cdot)}(t_{x,\sigma(\cdot)}(\rho))x\|_X = \rho\|x\|_X.$$

Using assumption ii),

$$\begin{aligned}t_{x,\sigma(\cdot)}(\rho)\rho^p\|x\|_X^p &\leq \int_0^{t_{x,\sigma(\cdot)}(\rho)} \|T_{\sigma(\cdot)}(t)x\|_X^p \, dt \\ &\leq \int_0^\infty \|T_{\sigma(\cdot)}(t)x\|_X^p \, dt \leq C\|x\|_X^p\end{aligned}$$

whereby,

$$t_{x,\sigma(\cdot)}(\rho) \leq \frac{C}{\rho^p} =: t_0, \text{ independent of } \sigma(\cdot).$$

By the principle of uniform boundedness, (10) implies that there exists a constant $k > 0$ such that

$$\|T_{\sigma(\cdot)}(t)\|_{\mathcal{L}(X)} \leq k, \quad t \geq 0,\ \sigma(\cdot)\text{-uniformly.} \tag{19}$$

Hence, for $t > t_0$, we have

$$\begin{aligned}\|T_{\sigma(\cdot)}(t)x\|_X &\leq \sup_{\tilde{\sigma}(\cdot)} \|T_{\tilde{\sigma}(\cdot)}(t - t_{x,\sigma(\cdot)}(\rho))\|_{\mathcal{L}(X)}\rho\|x\|_X \\ &\leq k\rho\|x\|_X, \quad \sigma(\cdot)\text{-uniformly.}\end{aligned}$$

Choose $\rho > 0$ such that $\beta := k\rho < 1$, so that

$$\|T_{\sigma(\cdot)}(t)x\|_X \leq \beta\|x\|_X, \quad t \geq t_0,\ \sigma(\cdot)\text{-uniformly.}$$

Finally, let $t_1 > t_0$ be fixed and let $t = nt_1 + s$, $0 \leq s < t_1$. Then,

$$\begin{aligned}\|T_{\sigma(\cdot)}(t)\|_{\mathcal{L}(X)} &\leq \sup_{\tilde{\sigma}(\cdot),\,\hat{\sigma}(\cdot)} \|T_{\tilde{\sigma}(\cdot)}(s)\|_{\mathcal{L}(X)}\|T_{\hat{\sigma}(\cdot)}(nt_1)\|_{\mathcal{L}(X)} \\ &\leq k\left(\sup_{\hat{\sigma}(\cdot)} \|T_{\hat{\sigma}(\cdot)}(t_1)\|_{\mathcal{L}(X)}\right)^n \\ &\leq k\beta^n \leq Ke^{-\mu t},\ t \geq 0,\ \sigma(\cdot)\text{-uniformly}\end{aligned}$$

with $K = \frac{C}{\beta}$ and $\mu = -\left(\frac{1}{t_1}\right)\ln\beta > 0$. □

Lemma 2 allows us to prove the first of the converse Lyapunov theorems that are the object of this paper.

Theorem 3. *The following conditions:*

i) *there exist constants $M \geq 1$ and $\omega > 0$ such that*

$$\|T_{\sigma(\cdot)}(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}, \quad t \geq 0,\ \sigma(\cdot)\text{-uniformly,} \tag{20}$$

ii) *there exists a continuous convex function $V\colon X \to [0,\infty)$ such that*

$$V(x) \leq C\|x\|_X^2 \tag{21}$$

for a constant $C > 0$ and

$$\underline{L}_j V(x) \leq -\|x\|_X^2, \quad \text{for all } j \in Q, \tag{22}$$

with $\underline{L}_j V(x)$ defined as in (6)

are necessary and sufficient for the existence of constants $K \geq 1$ and $\mu > 0$ such that

$$\|T_{\sigma(\cdot)}(t)\|_{\mathcal{L}(X)} \leq Ke^{-\mu t}, \quad t \geq 0,\ \sigma(\cdot)\text{-uniformly.} \tag{23}$$

Proof. Assume that the conditions i) and ii) hold. For all $\sigma(\cdot) \in \Sigma$, $x \in X$ and for $t \geq 0$ small enough so that the restriction of $\sigma(\cdot)$ to the interval $[0,t]$ is constant, we have

$$0 \leq V(T_{\sigma(\cdot)}(t)x) \leq V(x) - \int_0^t \|T_{\sigma(\cdot)}(\tau)x\|_X^2\, d\tau$$

as it follows from (22) and [17, §VI.7] (see also [8]). Thus, for all $\sigma(\cdot)$ and $x \in X$,

$$\int_0^\infty \|T_{\sigma(\cdot)}(\tau)x\|_X^2\, d\tau \leq V(x) \leq C\|x\|_X^2. \tag{24}$$

The uniform exponential decay (23) now follows from (20) and (24), thanks to Lemma 2 with $p = 2$.

Conversely, assume that (23) holds for some constants $K \geq 1$ and $\mu > 0$. Then, (20) holds for $M = K$ and arbitrary $\omega > 0$. Define $V\colon X \to [0, \infty)$ by

$$V(x) = \sup_{\sigma(\cdot)\in\Sigma} \int_0^\infty \|T_{\sigma(\cdot)}(t)x\|_X^2 \, dt. \tag{25}$$

Then, by assumption, $V(x)$ satisfies

$$V(x) \leq \sup_{\sigma(\cdot)} \int_0^\infty K^2 e^{-2\mu t} \|x\|_X^2 \, dt = \frac{K^2}{2\mu} \|x\|_X^2,$$

establishing (21) with $C = \frac{K^2}{2\mu}$. In particular, V is well-posed. Moreover, since each

$$x \mapsto \int_0^\infty \|T_{\sigma(\cdot)}(t)x\|_X^2 \, dt$$

is convex, it follows that $x \mapsto V(x)$ is convex. We are left to prove that V is continuous and satisfies (22).

In order to verify the continuity of V, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X converging to x in X. By definition of V,

$$\int_0^\infty \|T_{\sigma(\cdot)}(t)x_n\|_X^2 \, dt \leq V(x_n), \tag{26}$$

for all $\sigma(\cdot) \in \Sigma$. So taking the $\liminf$ over $n \in \mathbb{N}$ in (26) on both sides and using the continuity of $T_{\sigma(\cdot)}(t)$ for all $t \geq 0$, we have

$$\int_0^\infty \|T_{\sigma(\cdot)}(t)x\|_X^2 \, dt \leq \liminf_{n\to\infty} \int_0^\infty \|T_{\sigma(\cdot)}(t)x_n\|_X^2 \, dt \leq \liminf_{n\to\infty} V(x_n). \tag{27}$$

Taking the sup over $\sigma(\cdot)$ in (27) then yields

$$V(x) = \sup_{\sigma(\cdot)} \int_0^\infty \|T_{\sigma(\cdot)}(t)x\|_X^2 \, dt \leq \liminf_{n\to\infty} V(x_n),$$

proving that V is lower semi-continuous. On the other hand, for a fixed $\varepsilon > 0$, there exist $\sigma_\varepsilon(\cdot) \in \Sigma$ such that

$$\begin{aligned} V(x_n) - \frac{\varepsilon}{2} &< \int_0^\infty \|T_{\sigma_\varepsilon(\cdot)}(t)x_n\|^2 \, dt \\ &\leq (1+m) \int_0^\infty \|T_{\sigma_\varepsilon(\cdot)}(t)(x_n - x)\|^2 \, dt + \left(1 + \frac{1}{m}\right) \int_0^\infty \|T_{\sigma_\varepsilon(\cdot)}(t)x\|^2 \, dt, \end{aligned}$$

for any $m > 0$. Notice that, by definition of V,

$$\int_0^\infty \|T_{\sigma_\varepsilon(\cdot)}(t)(x_n - x)\|^2 \, dt \leq V(x_n - x) \leq C\|x_n - x\|_X^2,$$

$$\int_0^\infty \|T_{\sigma_\varepsilon(\cdot)}(t)x\|^2 \, dt \leq V(x).$$

Thus, for any $m > 0$, we have

$$V(x_n) - \frac{\varepsilon}{2} < C(1+m)\|x_n - x\|_X^2 + \left(1 + \frac{1}{m}\right) V(x). \tag{28}$$

In particular, choosing m such that $(1+\frac{1}{m})V(x) < V(x) + \frac{\varepsilon}{4}$ and taking n sufficiently large, so that $(1+m)C\|x_n - x\|_X^2 \leq \frac{\varepsilon}{4}$, we have from (28)

$$V(x_n) < V(x) + \varepsilon, \quad n \text{ sufficienty large.}$$

This implies the upper semi-continuity of V. Resuming, we proved the continuity of V.

Finally, fixing $t > 0$, $j \in Q$ and letting

$$\Sigma_{t,j} = \{\sigma(\cdot) \in \Sigma : \sigma|_{[0,t]} \equiv j\}$$

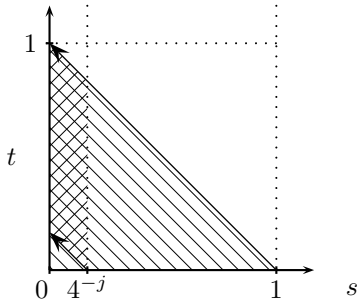

FIGURE 2. Illustration of $T_j(t)$, $j \in Q$, of the example in Remark 4.

be the set of switching signals whose restriction to the interval $[0,t]$ is constantly equal to j, we have, since $\Sigma_{t,j} \subseteq \Sigma$,

$$\begin{aligned} V(x) &\geq \sup_{\sigma(\cdot)\in\Sigma_{t,j}} \int_0^\infty \|T_{\sigma(\cdot)}(\tau)x\|_X^2 \, d\tau \\ &= \int_0^t \|T_j(\tau)x\|_X^2 \, d\tau + \sup_{\sigma(\cdot)\in\Sigma_{t,j}} \int_t^\infty \|T_{\sigma(\cdot)}(\tau)x\|_X^2 \, d\tau. \end{aligned} \tag{29}$$

Moreover, thanks to (5) and the invariance of Σ by time-shift,

$$\begin{aligned} V(T_j(t)x) &= \sup_{\sigma(\cdot)\in\Sigma} \int_0^\infty \|T_{\sigma(\cdot)}(\tau)T_j(t)x\|_X^2 \, d\tau \\ &= \sup_{\sigma(\cdot)\in\Sigma} \int_t^\infty \|T_{\sigma(\cdot)}(\tau-t)T_j(t)x\|_X^2 \, d\tau \\ &= \sup_{\sigma(\cdot)\in\Sigma_{t,j}} \int_t^\infty \|T_{\sigma(\cdot)}(\tau)x\|_X^2 \, d\tau. \end{aligned}$$

This and (29) yield

$$V(T_j(t)x) - V(x) \leq -\int_0^t \|T_j(\tau)\|_X^2 \, d\tau,$$

for all $j \in Q$ and $t > 0$. Therefore

$$\begin{aligned} \underline{L}_j V(x) &= \liminf_{t\downarrow 0} \frac{V(T_j(t)x) - V(x)}{t} \\ &\leq -\limsup_{t\downarrow 0} \frac{1}{t} \int_0^t \|T_j(\tau)x\|_X^2 \, d\tau = -\|x\|_X^2 \end{aligned}$$

for all $j \in Q$, establishing (22). □

Remark 4. We show here, through an example, that condition i) appearing in the statement of Theorem 3 cannot be removed.

Consider the family of semigroups $\{T_j(t)\}_{j\in Q}$ with $Q = \mathbb{N}$ and $T_j(t)$ defined on the Lebesgue space $X = L^p(0,1)$, $p \in [1,\infty)$, by

$$\left(T_j(t)f\right)(s) = \begin{cases} 2^{\frac{1}{p}} f(s+t), & s \in [0, 1-t] \cap [0, 4^{-j}) \\ f(s+t), & s \in [0, 1-t] \setminus [0, 4^{-j}) \\ 0, & s \in (1-t, 1], \end{cases} \tag{30}$$

cf. also Figure 2. Notice that, for all $j \in Q$, $T_j(\cdot)$ is a nilpotent semigroup, since $\|T_j(t)f\|_X = 0$ for $t > 1$. In particular, each of them is exponentially stable. Moreover, for all $\sigma(\cdot) \in \Sigma$, we have

$$\left|\left(T_{\sigma(\cdot)}(t)f\right)(s)\right| \leq 2^{\frac{k}{p}}|f(s+t)|, \quad \text{with } k = \#\{l \in \mathbb{N} : s < 4^{-l} \leq s+t\}.$$

In particular,

$$\left|\left(T_{\sigma(\cdot)}(t)f\right)(s)\right| \leq 2^{\frac{k}{p}}|f(s+t)|, \quad \text{if } 4^{-k-1} \leq s < 4^{-k},$$

yielding

$$\begin{aligned}
\int_0^\infty \|T_{\sigma(\cdot)}(t)f\|_X^p &= \int_0^1 \int_0^{1-t} \left|\left(T_{\sigma(\cdot)}(t)f\right)(s)\right|^p \, ds\, dt \\
&= \int_0^1 \int_0^{1-s} \left|\left(T_{\sigma(\cdot)}(t)f\right)(s)\right|^p \, dt\, ds \\
&\leq \sum_{k=0}^{\infty} \int_{4^{-k-1}}^{4^{-k}} 2^k \left(\int_0^{1-s} |f(s+t)|^p \, dt\right) ds \\
&\leq \sum_{k=0}^{\infty} \left(\frac{1}{4^k} - \frac{1}{4^{k+1}}\right) 2^k \int_0^1 |f(t)|^p \, dt = \frac{3}{2}\|f\|_X^p.
\end{aligned}$$

Hence, defining $V(x)$ as in (25), we have

$$V(x) = \sup_{\sigma(\cdot)} \int_0^\infty \|T_{\sigma(\cdot)}(t)x\|_X^2 \, dt \leq \frac{3}{2}\|x\|_X^2$$

and, by the same arguments as in the proof of Theorem 3,

$$\underline{L}_j V(x) \leq -\|x\|_X^2, \quad \text{for all } j \in Q,$$

so condition ii) in Theorem 3 holds with $C = \frac{3}{2}$.

Nevertheless, for a sequence of switching signals $(\sigma_n(\cdot))_{n\in\mathbb{N}} \subset \Sigma$ with switching times $\tau_k = \frac{1}{4^k}$ and modes $j_k = k+1$, $0 \leq k \leq n$, we have for functions $\mathbb{1}_{[s,1]}$ of L^p norm one concentrated on the interval $[s,1]$,

$$T_{\sigma_n(\cdot)}(1-\epsilon)\mathbb{1}_{[s,1]} = 2^{\frac{n}{p}}\mathbb{1}_{[s-1+\epsilon,\epsilon]}, \quad \text{if } 1 \geq s \geq 1-\epsilon > 1-4^{-n}.$$

Therefore, for $\epsilon < 4^{-n}$,

$$\begin{aligned}
\|T_{\sigma_n(\cdot)}(1-\epsilon)\|_{\mathcal{L}(X)} &= \sup_{\|f\|_X=1} \|T_{\sigma_n(\cdot)}(1-\epsilon)f\|_X \\
&\geq \lim_{s\uparrow 1} \|T_{\sigma_n(\cdot)}(1-\epsilon)\mathbb{1}_{[s,1]}\|_X = 2^{\frac{n}{p}}.
\end{aligned}$$

Hence,

$$\sup\{\|T_{\sigma_n(\cdot)}(1-\epsilon)\|_{\mathcal{L}(X)} : \epsilon \in [0,1],\ n \in \mathbb{N}\} = +\infty \tag{31}$$

violating any uniform bound of the form (23).

This example also shows that assumption i) appearing in Lemma 2 is necessary for the validity of the lemma. □

If one wishes to conclude that a switched system is globally uniformly exponentially stable, Theorem 3 requires the knowledge of a nonnegative function $V(\cdot)$ satisfying (21), (22) for all modes $j \in Q$ and the knowledge of a global uniform exponential bound (20). As an alternative, we will, in Theorem 6, obtain the existence of a function $V(\cdot)$ that is equivalent to the squared norm in X and allows to conclude global uniform exponential stability without knowledge of a global uniform exponential bound of the type (20). The latter is not obvious, since the Lyapunov

function $V(\cdot)$ constructed in the proof of Theorem 3 (see definition (25)) is in general not equivalent to the squared norm in X, i. e., in general $V(\cdot)$ does not satisfy a lower bound of the form

$$c\|x\|_X^2 \leq V(x), \quad x \in X \tag{32}$$

for a constant $c > 0$. Such a lower bound always holds, on the contrary, when X has finite dimension, as it is exploited in [15, 7]. The bound (32) may fail to hold even in the case of a single strongly continuous semigroup, as it is the case, for instance, of the semigroups $T_1(\cdot)$ and $T_2(\cdot)$ introduced in Example 1. (For a characterization of exponentially stable strongly continuous semigroups whose Lyapunov function defined as in (25) satisfies (32), see [16].)

In order to obtain a Lyapunov function equivalent to the squared norm for infinite dimensional switched systems, we make use of the following lemma imposing a stronger assumption on the family of semigroups $T_j(\cdot)$.

Lemma 5. *Assume that there exists $j^* \in Q$ such that $T_{j^*}(\cdot)$ can be extended to a group of bounded linear operators on X. Moreover, assume that there exist constants $K \geq 1$ and $\mu > 0$ such that*

$$\|T_{\sigma(\cdot)}(t)\|_{\mathcal{L}(X)} \leq Ke^{-\mu t}, \quad t \geq 0, \ \sigma(\cdot)\text{-uniformly.} \tag{33}$$

Then there exists a continuous convex function $V\colon X \to [0,\infty)$ such that

$$c\|x\|_X^2 \leq V(x) \leq C\|x\|_X^2 \tag{34}$$

for constants $c, C > 0$ and

$$\underline{L}_j V(x) \leq -\|x\|_X^2, \quad \text{for all } j \in Q, \tag{35}$$

with $\underline{L}_j V(x)$ defined as in (6).

Proof. Assume that (33) holds for some constants $K \geq 1$ and $\mu > 0$ independent of $\sigma(\cdot)$. Define $V(\cdot)$ by (25). As seen in the proof of Theorem 3, (33) guarantees that V is convex, continuos, satisfies (35) and that there exist $C > 0$ such that $V(x) \leq C\|x\|_X^2$.

The remaining bound $c\|x\|^2 \leq V(x)$ for some constant $c > 0$ is a consequence of the assumption that $T_{j^*}(\cdot)$ can be extended to a group. Indeed, $T_{j^*}(t)$ is then invertible for every $t \geq 0$ and satisfies $\|T_{j^*}(t)x\|_X \geq (\|T_{j^*}(-t)\|_{\mathcal{L}(X)})^{-1}\|x\|_X$ (cf. [16]). Hence

$$\begin{aligned} V(x) &= \sup_{\sigma(\cdot)} \int_0^\infty \|T_{\sigma(\cdot)}(t)x\|_X^2 \, dt \geq \int_0^\infty \|T_{j^*}(t)x\|_X^2 \\ &\geq \int_0^\infty (\|T_{j^*}(-t)\|_{\mathcal{L}(X)})^{-2} \, dt \|x\|_X^2 = c\|x\|_X^2 \end{aligned}$$

with $c = \int_0^\infty (\|T_{j^*}(-t)\|_{\mathcal{L}(X)})^{-2} \, dt < \infty$. □

We can now state and prove our second converse Lyapunov theorem.

Theorem 6. *The existence of a continuous convex function $V\colon X \to [0,\infty)$, such that*

$$c\|x\|_X^2 \leq V(x) \leq C\|x\|_X^2 \tag{36}$$

for constants $c, C > 0$ and

$$\underline{L}_j V(x) \leq -\|x\|_X^2, \quad \text{for all } j \in Q, \tag{37}$$

with $\underline{L}_j V(x)$ defined as in (6) is necessary and sufficient for the existence of constants $K \geq 1$ and $\mu > 0$ such that

$$\|T_{\sigma(\cdot)}(t)\|_{\mathcal{L}(X)} \leq Ke^{-\mu t}, \quad t \geq 0, \ \sigma(\cdot)\text{-uniformly.} \tag{38}$$

Proof. Assume that there exists a function $V\colon X \to [0,\infty)$ such that (36) and (37) hold for $c, C > 0$. Then, for all $x \in X$, $\sigma(\cdot) \in \Sigma$, and $t \geq 0$,

$$c\|T_{\sigma(\cdot)}(t)x\|_X^2 \leq V(T_{\sigma(\cdot)}(t)x) \leq V(x) - \int_0^t \|T_{\sigma(\cdot)}(\tau)x\|_X^2 \, d\tau,$$

so, using again (36) and dividing by c,

$$\|T_{\sigma(\cdot)}(t)x\|_X^2 \leq \frac{C}{c}\|x\|_X^2 - \frac{1}{c}\int_0^t \|T_{\sigma(\cdot)}(\tau)x\|_X^2 \, d\tau.$$

From Gronwall's Lemma, we obtain

$$\|T_{\sigma(\cdot)}(t)x\|_X^2 \leq \frac{C}{c}e^{-\frac{1}{c}t}\|x\|_X^2, \quad t \geq 0, \ \sigma(\cdot)\text{-uniformly}.$$

Hence,

$$\|T_{\sigma(\cdot)}(t)\|_{\mathcal{L}(X)} \leq Ke^{-\mu t}, \quad t \geq 0, \ \sigma(\cdot)\text{-uniformly},$$

for the constants $K = \sqrt{\frac{C}{c}} \geq 1$ and $\mu = \frac{1}{2c} > 0$, establishing (38).

Conversely, assume that (38) holds for $K \geq 1$ and $\mu > 0$ and consider the switched system with $\sigma^*(\cdot)$ taking values in $Q^* = Q \cup \{j^*\}$, where

$$T_{j^*}(t) = e^{-\mu t} I \tag{39}$$

and $I\colon X \to X$ denotes the identity. Then,

$$T_{j^*}(t)T_j(s) = T_j(s)T_{j^*}(t), \quad t, s \geq 0, \ j \in Q.$$

Moreover, for $t^* = |\{\tau \in [0,t] : \sigma^*(\tau) = j^*\}|$ and some $\sigma(\cdot)$ just taking values in Q,

$$\begin{aligned}\|T_{\sigma^*(\cdot)}\|_{\mathcal{L}(X)} &= \|T_{\sigma(\cdot)}(t-t^*)e^{-\mu t^*}\|_{\mathcal{L}(X)} \leq Ke^{-\mu(t-t^*)}e^{-\mu t^*} \\ &= Ke^{-\mu t}, \qquad t \geq 0, \ \sigma(\cdot)\text{-uniformly}\end{aligned} \tag{40}$$

where we have used (38). The existence of a continuous convex function $V\colon X \to [0,\infty)$ and constants $c, C > 0$ such that (36) and (37) hold now follows from (40) and Lemma 5, noting that (39) actually defines a group. □

Remark 7. Both Theorems 3 and 6 provide equivalent conditions to the global uniform exponential stability of a linear switched system on a Banach space in terms of the existence of a common Lyapunov function. We already noticed that a Lyapunov function $V(\cdot)$ satisfying condition ii) of the statement of Theorem 3 does not necessarily satisfy the stronger conditions appearing in the statement of Theorem 6.

Hence, Theorem 3 is better suited for proving the global uniform exponential stability of a switched system (although the uniform exponential growth boundedness should also be proved), while Theorem 6 provides more information on a switched system that is known to be globally uniformly exponentially stable, by tightening the properties satisfied by $V(\cdot)$.

In the same spirit—following the same proofs presented above—we can characterize the global uniform exponential stability of a switched system by (a priori) loosening the hypotheses on $V(\cdot)$, replacing inequalities (22) and (37) by

$$\underline{L}_j V(x) \leq -\kappa\|x\|_X^2, \quad \text{for all } j \in Q, \text{ for some } \kappa > 0.$$

We can also remove the hypotheses of continuity and convexity on V. Conversely, we can (a priori) tighten the same hypotheses replacing $\underline{L}_j V(x)$ by

$$\bar{L}_j V(x) = \limsup_{t \downarrow 0} \frac{V(T_j(t)x) - V(x)}{t}.$$ □

Remark 8. In the case of a single strongly continuous semigroup $T(\cdot)$, Theorem 6 shows that its global exponential stability is equivalent to the existence of a continuous convex Lyapunov function comparable with the squared norm of X.

When $T(\cdot)$ is globally exponentially stable, such a Lyapunov function can be obtained by the construction suggested in the proof of Theorem 6.

A more explicit expression for V is obtained following the alternative construction of a common Lyapunov function proposed in [10], where (25) is replaced by

$$V(x) = \int_0^\infty \sup_{\sigma(\cdot)\in\Sigma} \|T_{\sigma(\cdot)}(t)x\|_X^2 \, dt. \tag{41}$$

In the case of a single mode and following the strategy of augmenting Q by adding a group corresponding to a diagonal operator, we get

$$V(x) = \int_0^\infty \left(\max_{s\in[0,\tau]} e^{2\mu(s-\tau)} \|T(s)x\|_X^2 \right) d\tau,$$

for any fixed $\mu > 0$.

It should be noticed that, although not stated in [10], the definition of V given in (41) identifies a function which is continuous and convex. The proof of this fact can be rather easily obtained by adapting the proof of Theorem 3. □

The following proposition allows to improve the regularity of the common Lyapunov function in the case in which X is a Hilbert space. The proof adapts similar arguments presented in [4, §4.3.1].

Proposition 9. *Let X be a Hilbert space with scalar product $\langle\cdot,\cdot\rangle$ and assume that there exist constants $K \geq 1$ and $\mu > 0$ such that*

$$\|T_{\sigma(\cdot)}(t)\|_{\mathcal{L}(X)} \leq Ke^{-\mu t}, \quad t \geq 0, \ \sigma(\cdot)\text{-uniformly.} \tag{42}$$

Then, there exists a weakly-$*$-compact subset $\mathcal{B}$ of self-adjoint operators in $\mathcal{L}(X)$ such that the function $V\colon X \to \mathbb{R}$ given by

$$V(x) = \max_{B\in\mathcal{B}} \langle x, Bx\rangle \tag{43}$$

satisfies condition ii) of Theorem 3. In particular, V is convex and directionally differentiable in the sense of Fréchet and its derivative in the direction $\psi \in X$ is given by

$$DV(x)(\psi) = \max_{\hat{B}\in\mathcal{S}(x)} 2\langle \psi, \hat{B}x\rangle, \tag{44}$$

where

$$\mathcal{S}(x) = \arg\max_{B\in\mathcal{B}} \langle x, Bx\rangle. \tag{45}$$

Proof. For a fixed $\sigma(\cdot) \in \Sigma$ and all $t \geq 0$, let $T^*_{\sigma(\cdot)}(t) \in \mathcal{L}(X)$ be the adjoint operator of $T_{\sigma(\cdot)}(t) \in \mathcal{L}(X)$, uniquely defined by

$$\langle T_{\sigma(\cdot)}(t)x, x'\rangle = \langle x, T^*_{\sigma(\cdot)}(t)x'\rangle, \quad x, x' \in X.$$

If $T_{\sigma(\cdot)}(t)$ has the expression given in (4), then

$$T^*_{\sigma(\cdot)}(t) = T^*_{j_1}(\tau_1)\cdots T^*_{j_{p-1}}(\tau_p - \tau_{p-1})T^*_{j_p}(t - \tau_p),$$

where $T^*_j(t)$ denotes the adjoint semigroup of $T_j(t)$, $t \geq 0$, $j \in Q$.

Assuming that there exist constants $K \geq 1$ and $\mu > 0$, independent of $\sigma(\cdot)$, such that (42) holds, the operator $B_{\sigma(\cdot)}\colon X \to X$, given by

$$B_{\sigma(\cdot)}x = \int_0^\infty T^*_{\sigma(\cdot)}(t)T_{\sigma(\cdot)}(t)x \, dt,$$

is linear, self-adjoint, and satisfies

$$\|B_{\sigma(\cdot)}x\|_X \leq \int_0^\infty \|T^*_{\sigma(\cdot)}(t)\|_{\mathcal{L}(X)}\|T_{\sigma(\cdot)}(t)\|_{\mathcal{L}(X)}\|x\|_X\,dt$$
$$\leq \int_0^\infty K^2 e^{-2\mu t}\|x\|_X\,dt = \frac{K^2}{2\mu}\|x\|_X,$$

where we have used (42). In particular, $B_{\sigma(\cdot)} \in \mathcal{L}(X)$ for all $\sigma(\cdot) \in \Sigma$ and

$$\|B_{\sigma(\cdot)}\|_{\mathcal{L}(X)} \leq \frac{K^2}{2\mu}, \quad \sigma(\cdot)\text{-uniformly.} \tag{46}$$

Therefore, the set

$$\mathcal{B} = \{B \in \mathcal{L}(X) : \text{ there exists a sequence } (\sigma_n(\cdot))_{n\in\mathbb{N}} \subset \Sigma \text{ such that } B_{\sigma_n(\cdot)} \overset{*}{\rightharpoonup} B\}$$

is weakly-$*$-compact, as it follows from the Banach-Alaoglu theorem. Moreover, $\mathcal{B}$ consists of self-adjoint operators.

Notice that if $B_{\sigma_n(\cdot)} \overset{*}{\rightharpoonup} B$ and $(x_n, y_n) \to (x, y)$ in $X \times X$, then

$$\lim_{n\to\infty} \langle x_n, B_{\sigma_n(\cdot)}y_n\rangle = \langle x, By\rangle. \tag{47}$$

Define V as in (43). The maximization makes sense because of (47) (with $x_n = y_n = x$) and of the compactness of $\mathcal{B}$. Moreover,

$$V(x) = \max_{B\in\mathcal{B}}\langle x, Bx\rangle = \sup_{\sigma(\cdot)\in\Sigma}\langle x, B_{\sigma(\cdot)}x\rangle = \sup_{\sigma(\cdot)\in\Sigma}\langle x, \int_0^\infty T^*_{\sigma(\cdot)}(t)T_{\sigma(\cdot)}(t)x\,dt\rangle$$
$$= \sup_{\sigma(\cdot)\in\Sigma}\int_0^\infty \langle x, T^*_{\sigma(\cdot)}(t)T_{\sigma(\cdot)}(t)x\rangle dt = \sup_{\sigma(\cdot)}\int_0^\infty \|T_{\sigma(\cdot)}(t)x\|^2\,dt,$$

for all $x \in X$. Comparing with (25), we recover that V satisfies the condition ii) of Theorem 3. In particular, V is convex and continuous.

In order to verify the directional differentiability in the sense of Fréchet and to prove (44), first observe that the map $x \mapsto \langle x, Bx\rangle$ is differentiable on X in the sense of Fréchet for all $B \in \mathcal{L}(X)$. For B self-adjoint, the derivative is given by $2\langle\cdot, Bx\rangle$. Fix now some $x_0 \in X$ and define

$$\Phi(B, x) = 2\|Bx - Bx_0\|_X.$$

We claim that

$$\lim_{x\to x_0}\Phi(B, x) = 0, \quad \text{uniformly with respect to } B \in \mathcal{B}. \tag{48}$$

Indeed, suppose by contradiction that there exists some $\varepsilon > 0$, a sequence $(x_n)_{n\in\mathbb{N}}$ converging to x_0 in X and a sequence $(B_n)_{n\in\mathbb{N}}$ in $\mathcal{B}$ such that

$$\Phi(B_n, x_n) > \varepsilon \quad \text{for all } n \in \mathbb{N}. \tag{49}$$

Since $\mathcal{B}$ is weakly-$*$-compact, there exist $B \in \mathcal{B}$ and a subsequence $n(k)$ such that

$$B_{n(k)} \overset{*}{\rightharpoonup} B \quad \text{as } k \to \infty.$$

Then, thanks to (47),

$$\Phi(B_{n(k)}, x_{n(k)}) = 0,$$

contradicting (49).

The Mean Value Theorem then gives

$$|\langle x_0 + \psi, B(x_0 + \psi)\rangle - \langle x_0, Bx_0\rangle - 2\langle\psi, Bx_0\rangle|$$
$$\leq \|\psi\|_X \int_0^1 \Phi(B, x_0 + \xi\psi)\,d\xi \leq \varepsilon\|\psi\|_X \tag{50}$$

for $\varepsilon > 0$, ψ sufficiently close to zero, and uniformly with respect to $B \in \mathcal{B}$, as it follows from (48).

Let $\mathcal{S}(\cdot)$ be defined as in (45). Notice that $\mathcal{S}(x)$ is weak-$*$-compact for every $x \in X$. For any $\hat{B} \in \mathcal{S}(x_0)$, $V(x_0) = \langle x_0, \hat{B}x_0\rangle$ and $V(x_0+\psi) \geq \langle x_0+\psi, \hat{B}(x_0+\psi)\rangle$. Thus, using (50),

$$V(x_0+\psi) - V(x_0) \geq \max_{\hat{B}\in\mathcal{S}(x_0)} 2\langle \psi, \hat{B}x_0\rangle - \varepsilon\|\psi\|_X.$$

In order to prove (44), it therefore remains to show that for all $\varepsilon > 0$ and ψ close to zero,

$$V(x_0+\psi) - V(x_0) \leq \max_{\hat{B}\in\mathcal{S}(x_0)} 2\langle \psi, \hat{B}x_0\rangle + \varepsilon\|\psi\|_X. \tag{51}$$

So, suppose (51) not to hold. Then there exist $\varepsilon > 0$ and a sequence $(\psi_n)_{n\in\mathbb{N}}$ in X converging to zero such that

$$V(x_0+\psi_n) - V(x_0) > \max_{\hat{B}\in\mathcal{S}(x_0)} 2\langle \psi_n, \hat{B}x_0\rangle + \varepsilon\|\psi_n\|_X. \tag{52}$$

By definition of V, there exist $B_0, \hat{B}_n \in \mathcal{B}$ such that

$$\begin{aligned} V(x_0+\psi_n) - V(x_0) &= \langle x_0+\psi_n, \hat{B}_n(x_0+\psi_n)\rangle - \langle x_0, B_0x_0\rangle \\ &\leq \langle x_0+\psi_n, \hat{B}_n(x_0+\psi_n)\rangle - \langle x_0, \hat{B}_nx_0\rangle. \end{aligned}$$

Again by the Mean Value Theorem,

$$\begin{aligned} &|\langle x_0+\psi_n, \hat{B}_n(x_0+\psi_n)\rangle - \langle x_0, \hat{B}_nx_0\rangle - 2\langle \psi_n, \hat{B}_nx_0\rangle| \\ &\qquad \leq \|\psi_n\|_X \int_0^1 \Phi(\hat{B}_n, x_0+\psi_n)\, dt \leq \frac{\varepsilon}{2}\|\psi_n\|_X \end{aligned}$$

for all n large enough. Thus,

$$V(x_0+\psi_n) - V(x_0) \leq 2\langle \psi_n, \hat{B}_nx_0\rangle + \frac{\varepsilon}{2}\|\psi_n\|_X.$$

Using one more time the weak-$*$-compactness of $\mathcal{B}$, there exist a subsequence $n(k)$ such that $\hat{B}_{n(k)} \overset{*}{\rightharpoonup} \hat{B}$ for some $\hat{B} \in \mathcal{B}$. Moreover, by continuity of V and because of (47), $\hat{B} \in \mathcal{S}(x_0)$. Hence,

$$\begin{aligned} V(x_0+\psi_{n(k)}) - V(x_0) &\leq 2\langle \psi_{n(k)}, \hat{B}x_0\rangle + 2\langle \psi_{n(k)}, (\hat{B}_{n(k)} - \hat{B})x_0\rangle + \frac{\varepsilon}{2}\|\psi_{n(k)}\|_X \\ &\leq \max_{B\in\mathcal{S}(x_0)} 2\langle \psi_{n(k)}, Bx_0\rangle + \frac{3}{4}\varepsilon\|\psi_{n(k)}\| \end{aligned}$$

where we used that $2\langle \psi_{n(k)}, (\hat{B}_{n(k)} - \hat{B})x_0\rangle \leq \frac{\varepsilon}{4}\|\psi_{n(k)}\|_X$ for k sufficiently large. This contradicts (52) and completes the proof. □

3. CONCLUSIONS AND FUTURE WORKS

We presented necessary and sufficient conditions for a (possibly infinite) family of semigroups to be globally uniformly exponentially stable for arbitrary switching signals $\sigma(\cdot)$, in terms of the existence of a common Lyapunov-function. In particular, our results apply to switched dynamical systems such as (1), involving (possibly unbounded) operators on a Banach space X.

We have shown that the existence of a positive continuous function $V(\cdot)$, decaying uniformly along trajectories and either bounded from above by a multiple of the squared norm in presence of a uniform exponential growth bound for the switched system or comparable with the squared norm is equivalent to the switched system being globally uniformly exponentially stable. The latter generalizes a well-known result for switched linear dynamical systems in $\mathbb{R}^n$, $n \in \mathbb{N}$.

As an application, our results answer for example the question of existence of a common Lyapunov function for the switched linear hyperbolic system with reflecting boundaries considered in [1].

The common Lyapunov function is shown to be Fréchet directionally differentiable in the case in which X is a Hilbert space.

Future work involves the investigation of further regularity properties for $V(\cdot)$. For the finite-dimensional case, it is for instance known that $V(\cdot)$ can be taken polyhedral or polynomial [2, 3, 7]. It would be interesting to recover results in the same direction for infinite dimensional switched systems.

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