



**HAL**  
open science

## Dual approach in non-linear fracture mechanics

Claude Stolz

► **To cite this version:**

Claude Stolz. Dual approach in non-linear fracture mechanics. *International Journal of Fracture*, 2010, 166, pp.135-143. 10.1007/s10704-010-9516-3 . hal-00489831

**HAL Id: hal-00489831**

**<https://hal.science/hal-00489831v1>**

Submitted on 26 Mar 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Dual approach in non-linear fracture mechanics

Claude Stolz

**Abstract** A dual approach in fracture mechanics based on complementary energy is proposed. The analysis of the dissipation shows that the thermodynamical force associated with the evolution of a crack is an energy release rate, form of which depends on the presence or not of mechanical discontinuities. This energy release rate is given as a integral based on free or complementary energy. The invariance of sintegrals is analysed and the obtained results in elastoplasticity are discussed. The energy release rate is determined in terms of potential energy or complementary energy, first in elasticity and secondly in elastoplasticity. Associated to these definitions, the law of propagation of the crack is chosen as a Griffith law and the propagation is governed by a normality rule. In this framework we formulate the evolution problem concerning crack propagation in an elastoplastic material. Variational formulations are obtained in terms of rate of displacement, of stresses and of crack length.

**Keywords**  $J$  Integrals · Fracture · Elastoplasticity · Eshelby tensors

---

C. Stolz (✉)  
Laboratoire de Mécanique des Solides, Ecole  
polytechnique, CNRS UMR7649, 91128 Palaiseau Cedex,  
France  
e-mail: stolz@lms.polytechnique.fr

C. Stolz  
LaMSID, EDF-CEA, CNRS UMR2832,  
92141 Clamart Cedex, France

## 1 Introduction

The study of crack propagation and stability of crack is an old problem in fracture mechanics. In brittle fracture, the Griffith's criterion is assumed, the stability analysis is classical and most investigations have been concerned with bi-dimensional problems of a crack, Its generalization to a system of linear cracks has been discussed. The Griffith's criterion has also an energetic significance, it has has been shown that the quasistatic propagation of planar cracks is a particular case of a class of processes time independent in solid mechanics like standard plasticity, damage and brittle fracture. The common features in the modelling of these processes is the applications of two fundamental concepts: the notion of energy at a given state and the normality rule expressed by Hill's principle of maximum dissipation. The purpose of the analysis is to study the evolution of a system of linear cracks inside an elastoplastic material.

Many papers are concerned with the formulation of the driving force associated to the propagation of a crack in in elastic-plastic material and discussion of  $J$  integrals (Nguyen and Stolz 1985; Simha et al. 2008; Maugin 1994). The interpretation in terms of dissipation is presneted and formulations of the evolution problem of a set of cracks in an elastoplastic boby are given in terms of variational inequalities.

Consider a body  $\Omega$  with a straight crack. Around the crack tip three domains determined by the distance from the tip are distinguished. In the nearest zone  $I$  all physical processes of rupture occur, that is the process

zone. This zone is embedded in a second domain where the mechanical fields are described by singular fields. And finally far away from the tip the mechanical fields become closer to the boundary conditions.

At macroscopic scale, the crack is represented by a line along  $Ox$ . The normal is  $\underline{e}_y$  in the plane, and  $\underline{e}_z$  is the normal to the plane. The crack length is  $l(t)$  and  $a = \dot{l}$  is the crack speed.

If the singularities of mechanical fields govern the crack propagation, it is not necessary to take into account the rupture process. This is an approximation which leads to a global approach of rupture. This description is powerful and constitutes the key point for describing classical fracture mechanics. In this case, the singularities represent the loading applied on the process zone.

In non linear mechanics, these notions must be revisited. The dissipation can be represented by several manner, depending on the local constitutive law and on the mechanisms of rupture. The dissipation can be concentrated in one point due to singularities or along a surface of discontinuities as in shock waves or formed inside a volume. Any situation depends on the characteristics of the material and on condition of propagation.

For example, the antiplane shear in stationary steady-state motion  $\underline{u} = w(x, y, t)\underline{e}_z = W(x - at, y)$  is governed by

$$(1 - M^2)W''_{,xx} + W''_{,yy} = 0 \quad (1)$$

where the Mach number  $M^2 = \frac{a^2}{\rho\mu}$  defines the nature of this equation. For small  $M < 1$  the equation is elliptic,  $W$  is singular and the dissipation is concentrated at the crack tip. For large value of  $M$  ( $M > 1$ ) the equation is hyperbolic, the gradient  $\nabla W$  has jump and the dissipation is concentrated along moving surface.

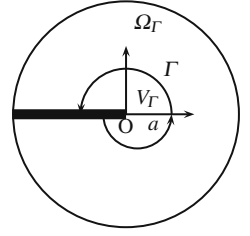
In elastoplasticity, the local equations are also hyperbolic, the same can be realized for some classes of hyperelastic material (Knowles 1977).

## 2 Characterization of the propagation

The main difficulty of the problem of propagation is the dependence of  $\Omega$  to the crack length and the presence of moving singularities or discontinuities due to the crack.

One possibility has been investigated in Destuynder and Djaoua (1981) by introducing a geometrical

**Fig. 1** Decomposition of  $\Omega$  in  $\Omega_\Gamma \cup V_\Gamma$



lagrangean description. Our description is based upon the concept of singularity transport (Nguyen and Stolz 1985; Stolz and Pradeilles-Duval 1996; Stolz 2004). Similar ideas are found in Gurtin and Podio-Guiugli (1996).

The nature of the singularity is preserved for a frame in motion with the crack tip.

The crack singularity is surrounded by a curve  $\Gamma$  delimiting a domain  $V_\Gamma$ . This domain moves with the position of the crack tip which is given by function  $l(t)$  (Fig. 1). All mechanical quantities are expressed in terms of the classical fixed coordinates outside  $V_\Gamma$  and in terms of moving coordinates inside  $V_\Gamma$ .

$$x = X - l(t), \quad y = Y. \quad (2)$$

Any mechanical quantity  $F$  has a time-derivative given by  $\overset{\circ}{f}$ , which represents the variation of  $f$  in the moving frame:

$$F(X, Y, t) = f(x, y, t), \quad \overset{\circ}{f} = \frac{\partial F}{\partial t} = -a \frac{\partial f}{\partial X} + \overset{\circ}{f} \quad (3)$$

In order to separate the contribution of the crack tip in the expression of dissipation, these definitions are applied to average quantities on the whole domain:

$$\mathcal{F} = \int_{\Omega} f \rho \, d\Omega = \int_{\Omega_\Gamma} f \rho \, d\Omega + \int_{V_\Gamma} f \rho \, d\Omega,$$

$$\frac{d}{dt} \mathcal{F} = \frac{d}{dt} \left( \int_{\Omega_\Gamma} f \rho \, d\Omega \right) + \int_{V_\Gamma} \overset{\circ}{f} \rho \, d\Omega.$$

$$0 = \frac{d}{dt} \left( \int_{\Omega_\Gamma} f \rho \, d\Omega \right) - \int_{\Omega_\Gamma} \overset{\circ}{f} \rho \, d\Omega + \int_{\Gamma} f \rho a n_x \, dS.$$

Let us introduce the notations:  $f_x = f \cdot \underline{e}_x$ ,  $\nabla_x f = \nabla f \cdot \underline{e}_x$ .

*Dissipation* The dissipation of the whole system can be rewritten as

$$D_m = \mathcal{P}_e - \frac{d}{dt} \int_{\Omega} w \rho \, d\Omega \geq 0, \quad (4)$$

where the power of external forces  $\mathcal{P}_e$  is given in term of local stresses, taking into account the conservation of the momentum:

$$\mathcal{P}_e = \int_{\partial\Omega} \underline{n} \cdot \underline{\sigma} \cdot \underline{v} \, dS. \quad (5)$$

Using the divergence theorem, this quantity is decomposed in two terms:

$$\int_{\partial\Omega} \underline{n} \cdot \underline{\sigma} \cdot \underline{v} \, dS = \int_{\Omega} \underline{\sigma} : \underline{\varepsilon}(\underline{v}) \, d\Omega + \int_{\Gamma} \underline{n} \cdot \underline{\sigma} \cdot \underline{v} \, dS. \quad (6)$$

Using now the decomposition of the volume ( $\Omega = \Omega_{\Gamma} \cup V_{\Gamma}$ ) the dissipation is written

$$\frac{d}{dt} \int_{\Omega} w \rho \, d\Omega = \frac{d}{dt} \left( \int_{\Omega_{\Gamma}} w \rho \, d\Omega \right) + \int_{V_{\Gamma}} \dot{w} \rho \, d\Omega, \quad (7)$$

and by application of the general relations, the dissipation is:

$$\frac{d}{dt} \int_{\Omega_{\Gamma}} w \rho \, d\Omega = \int_{\Omega_{\Gamma}} \dot{w} \rho \, d\Omega - \int_{\Gamma} \rho w n_x \, dS, \quad (8)$$

$$\int_{V_{\Gamma}} \dot{w} \rho \, d\Omega = \int_{\Gamma} \underline{n} \cdot \underline{\sigma} \cdot \underline{u} \, dS + o(R) \quad (9)$$

We have taken account of the traction-free boundary conditions  $\underline{\sigma} \cdot \underline{n} = 0$  along the crack lips. The quantity  $o(R)$  is regular and tends to zero with the radius of the volume  $V_{\Gamma}$ .

The dissipation is rewritten finally as

$$D_m = \int_{\Omega_{\Gamma}} (\underline{\sigma} : \underline{\varepsilon}(\underline{v}) - \rho \dot{w}) \, d\Omega + \int_{\Gamma} (\underline{n} \cdot \underline{\sigma} \cdot (\underline{v} - \underline{\dot{u}}) + \rho w \underline{n} \cdot \underline{e}_x) \, dS + o(R) \geq 0 \quad (10)$$

The displacement  $\underline{u}$  is continuous along the curve  $\Gamma$ . The condition of compatibility implies Hadamard relations on the rates:

$$[\underline{u}] = 0 \Rightarrow \underline{v} = \underline{\dot{u}} - \nabla_x \underline{u} \, a. \quad (11)$$

Thus the dissipation is decomposed in two terms: a volume part due to irreversibility of the constitutive behaviour and a surface term associated with the propagation of the crack:

$$D_m = \lim_{R \rightarrow 0} \int_{\Omega_{\Gamma}} (\underline{\sigma} : \underline{\varepsilon}(\underline{v}) - \rho \dot{w}) \, d\Omega + \lim_{R \rightarrow 0} \int_{\Gamma} (-\underline{n} \cdot \underline{\sigma} \cdot \nabla_x \underline{u} + \rho w n_x) \, dS \, a \geq 0. \quad (12)$$

We note  $J_{\Gamma}$  the curve integral

$$J_{\Gamma} = \int_{\Gamma} (-\underline{n} \cdot \underline{\sigma} \cdot \nabla_x \underline{u} + \rho w n_x) \, dS. \quad (13)$$

*Case of linear elasticity* In this case the local behaviour is reversible and the stresses are  $\underline{\sigma} = \rho \frac{\partial w}{\partial \underline{\varepsilon}}$ , then there is no dissipation in the volume:

$$\int_{\Omega_{\Gamma}} (\underline{\sigma} : \underline{\varepsilon}(\underline{v}) - \rho \dot{w}) \, d\Omega = 0. \quad (14)$$

When the curve  $\Gamma$  is reduced to the crack tip, the result is conserved:

$$\lim_{R \rightarrow 0} \int_{\Omega_{\Gamma}} (\underline{\sigma} : \underline{\varepsilon}(\underline{v}) - \rho \dot{w}) \, d\Omega = 0. \quad (15)$$

Thus the global dissipation contains only the contribution of the crack:

$$D_m = \lim_{R \rightarrow 0} \int_{\Gamma} (-\underline{n} \cdot \underline{\sigma} \cdot \nabla_x \underline{u} + \rho w n_x) \, dS \, a. \quad (16)$$

The thermodynamical driving force associated with the propagation is the free energy release rate  $\mathcal{G}$  defined by:

$$\mathcal{G} = \lim_{R \rightarrow 0} \int_{\Gamma} (-\underline{n} \cdot \underline{\sigma} \cdot \nabla_x \underline{u} + \rho w n_x) \, dS = \lim_{R \rightarrow 0} J_{\Gamma}. \quad (17)$$

Consider now a closed loop  $S$  inside a domain  $\Omega$ , and assume that the body forces are null. For an homogeneous linearly elastic material, the density  $\rho$  is uniform. The stresses satisfy both the equations of state and the conservation of the momentum:

$$\underline{\sigma} = \rho \frac{\partial w}{\partial \underline{\varepsilon}}, \quad \text{div } \underline{\sigma} = 0 \text{ over } \Omega. \quad (18)$$

Consider the integral  $C$

$$C = \int_S (\rho w n_k - \sigma_{ij} u_{i,k} n_j) \, dS, \quad (19)$$

then by divergence theorem the integral is equal to

$$\int_{\Omega_S} \frac{\partial(\rho w)}{\partial x_k} - \frac{\partial(\sigma_{ij} u_{i,k})}{\partial x_j} \, d\Omega = \int_{\Omega_S} (\sigma_{ij} \varepsilon_{ij,k} - \sigma_{ij} u_{i,jk} - \frac{\partial \sigma_{ij}}{\partial x_j} u_{i,k}) \, d\Omega.$$

Using now the conservation of momentum, we find that the integral  $C$  is null. From this result, it follows that the integral  $J_\Gamma$  is independent of the choice of the loop  $\Gamma$ , taking account of the free-traction conditions along the lips of the crack.

*Examples of non linear behaviour* For some classes of non linear material, the gradient of the displacement presents discontinuities along a surface  $S$ . In this case, the limit  $R \rightarrow 0$  must take these discontinuities into account.

In perfect plasticity (Nguyen and Stolz 1985) the release rate of energy appearing the dissipation is given by

$$J_S = \lim_{\Gamma \rightarrow S} \int_{\Gamma} (w(\varepsilon - \varepsilon_p) n_1 - \underline{n} \cdot \sigma \cdot \underline{u}_{,1}) ds \quad (20)$$

where  $S$  is the surface along which the gradient of displacement has discontinuities:

$$[\dot{\underline{u}}]_S + a[\underline{u}_{,1}]_S = 0 \quad (21)$$

$S$  is a shock surface or a shock line. In the moving frame, the derivative  $\dot{\underline{u}}$  is then continuous.

### 3 Energetical interpretation

We study the propagation of a straight crack in direction  $e_1$  in a continuum  $\Omega$ . On the boundary  $\partial\Omega_T$ , the stress  $\underline{T}^d$  is imposed and on the complementary part  $\partial\Omega_u$  the displacement is prescribed. The crack is stress free. For non linear materials, the strain energy  $w$  depends on the strain  $\varepsilon$  and on internal parameters  $\alpha$  which describe hardening behaviour. The potential energy for the system is:

$$\begin{aligned} \mathcal{E}(\underline{u}, \alpha, l, \underline{T}^d) = & \int_{\Omega} \rho w(\varepsilon(\underline{u}), \alpha) d\Omega \\ & - \int_{\partial\Omega_T} \underline{T}^d \cdot \underline{u} dS, \end{aligned} \quad (22)$$

and the dissipation is rewritten as

$$D_m = \mathcal{P}_e - \frac{d}{dt} \left( \mathcal{E} + \int_{\partial\Omega_T} \underline{T}^d \cdot \underline{u} dS \right) \geq 0. \quad (23)$$

Taking account of the derivation of the potential energy with respect to its arguments, we obtain:

$$\frac{d}{dt}(\mathcal{E}) = \frac{\partial \mathcal{E}}{\partial \underline{u}} \cdot \dot{\underline{u}} + \frac{\partial \mathcal{E}}{\partial \alpha} \cdot \dot{\alpha} + \frac{\partial \mathcal{E}}{\partial l} a + \frac{\partial \mathcal{E}}{\partial \underline{T}^d} \cdot \dot{\underline{T}}^d. \quad (24)$$

then

$$D_m = - \frac{\partial \mathcal{E}}{\partial \alpha} \cdot \dot{\alpha} - \frac{\partial \mathcal{E}}{\partial l} a \geq 0. \quad (25)$$

For the whole system, the crack length is playing the role of an internal parameter. The thermodynamical force associated with the propagation  $a$  is the release rate of energy  $\mathcal{G}$  obtained by the global state equation :

$$\mathcal{G} = - \frac{\partial \mathcal{E}}{\partial l}. \quad (26)$$

In dynamical evolution the potential energy of the whole system must be replaced by the Hamiltonian (Stolz and Pradeilles-Duval 1996).

### 4 Dual approach in linear elasticity

In classical brittle fracture, the constitutive behaviour is linear elastic. The strain energy  $w(\varepsilon)$  is a quadratic function of the strain  $\varepsilon(\underline{u})$  which is associated to the displacement  $\underline{u}$ :

$$w(\varepsilon) = \frac{1}{2} \varepsilon \cdot \mathbf{C} \cdot \varepsilon, \quad \varepsilon = \frac{1}{2} (\text{grad } \underline{u} + \text{grad}^t \underline{u}). \quad (27)$$

We denote by  $\sigma$  the value of the stress tensor associated with the strain  $\sigma = \mathbf{C} : \varepsilon$ .

The evolution law for the crack is based on the energy release rate  $\mathcal{G}$  and satisfies the normality rule  $\mathcal{G} - G_c \leq 0$ ,  $a \geq 0$ ,  $a(\mathcal{G} - G_c) = 0$ ; where  $G_c$  is a characteristic of the material. The energy release rate  $\mathcal{G}$  is given in terms of potential energy  $\mathcal{E}$

$$\mathcal{E}(\underline{u}, l) = \int_{\Omega(l)} w(\varepsilon) d\Omega - \int_{\partial\Omega_T} \underline{T}^d \cdot \underline{u} ds, \quad (28)$$

$$\mathcal{G} = - \frac{\partial \mathcal{E}}{\partial l} = \int_{\Gamma} (w(\varepsilon) n_1 - \underline{n} \cdot \sigma \cdot \underline{u}_{,1}) ds = J_\Gamma \quad (29)$$

A state of equilibrium for a given crack length  $l$  is defined by a displacement which minimizes  $\mathcal{E}$  among the set of displacements  $\underline{u}$  such that  $\underline{u} = \underline{u}^d$  along  $\partial\Omega_u$ . The equilibrium solution is  $\underline{u}^{sol}(l, \underline{u}^d, \underline{T}^d)$ .

In the dual approach, the complementary energy is defined in terms of stresses

$$\mathcal{E}^*(\sigma, l) = - \int_{\Omega(l)} w^*(\sigma) d\Omega + \int_{\partial\Omega_u} \underline{n} \cdot \sigma \cdot \underline{u}^d ds. \quad (30)$$

Then the release rate of energy is the Bui's integral (Bui 1973):

$$I_\Gamma = J_\Gamma = \int_{\Gamma} (-w^*(\sigma) n_1 + \underline{n} \cdot \sigma \cdot \underline{u}) ds. \quad (31)$$

The equality of the two integrals is due to the fact that at a state of equilibrium for the same crack length and the same boundary conditions, the potential energy and the complementary energy are equals. This results is obtained taking account of the classical relations

$$w^*(\sigma) + w(\varepsilon) = \sigma : \varepsilon, \quad \int_{V_c} \sigma : \varepsilon(\underline{u}) \, d\Omega = \int_{\mathcal{C}} \underline{n} \cdot \sigma \cdot \underline{u} \, ds. \quad (32)$$

**Lemma** For all stresses  $\tilde{\sigma}$  which satisfy the equilibrium equation ( $\text{div } \tilde{\sigma} = 0$ ), and for all strains  $\hat{\varepsilon}$  such that ( $2\hat{\varepsilon} = \text{grad } \hat{\underline{u}} + \text{grad}^t \hat{\underline{u}}$ ) and for all closed volume with boundary  $\mathcal{C}$ , we have:

$$\int_{\mathcal{C}} (\tilde{\sigma} : \hat{\varepsilon} n_i - \underline{n} \cdot \tilde{\sigma} \cdot \hat{\underline{u}} - \underline{n} \cdot \tilde{\sigma} \cdot \hat{\underline{u}}) \, ds = 0 \quad (33)$$

The two integrals  $I_\Gamma$  et  $J_\Gamma$  are independent of the closed curve  $\Gamma$ . This can be shown considering a close curve  $C$  composed by two curves  $\Gamma_1$  and  $\Gamma_2$  surrounding the crack tip ( $V_{\Gamma_1} \subset V_{\Gamma_2}$ ) and by the crack lips.

We can introduce the local Eshelby tensors  $p$  and  $p^*$ :

$$p = w \, \mathbf{1} - \sigma \cdot \nabla \underline{u}, \quad p^* = -w^* \, \mathbf{1} + \nabla \sigma \cdot \underline{u} \quad (34)$$

They satisfy

$$0 = \int_{V_c} \text{div } p \, d\Omega = \int_{\mathcal{C}} \underline{n} \cdot p \, ds,$$

$$0 = \int_{V_c} \text{div } p^* \, d\Omega = \int_{\mathcal{C}} \underline{n} \cdot p^* \, ds$$

Some other dual conservation laws can be formed as pointed out in [Lubarda and Markenscoff \(2007\)](#). For the equilibrium solution  $w(\varepsilon) = w^*(\sigma) = \frac{1}{2} \sigma : \varepsilon$  and then we obtain

$$p - p^* = \sigma : \varepsilon \, \mathbf{1} - (\sigma \cdot \nabla \underline{u} + \nabla \sigma \cdot \underline{u}),$$

$$p + p^* = \nabla \sigma \cdot \underline{u} - \sigma \cdot \nabla \underline{u}$$

On the crack lips, the contributions  $\underline{n} \cdot p \cdot \underline{e}_1$  and  $\underline{n} \cdot p^* \cdot \underline{e}_1$  vanish. This induces the invariance of integrals  $I_\Gamma$  and  $J_\Gamma$  with respect to the choice of  $\Gamma$ , the constant values of which are denoted by  $I$  and  $J$  respectively. Taking the properties of  $p$  and  $p^*$  and the lemma into account, for an equilibrium solution we obtain:

$$I + J = \int_{\Gamma} (\underline{n} \cdot \sigma \cdot \underline{u} - \underline{n} \cdot \sigma \cdot \underline{u}) \, ds \quad (35)$$

$$J - I = \int_{\Gamma} ((w + w^*) n_1 - \underline{n} \cdot \sigma \cdot \underline{u} - \underline{n} \cdot \sigma \cdot \underline{u}) \, ds = 0 \quad (36)$$

## 5 Dual approach in elastoplasticity

*Case of perfect plasticity* The local behaviour is given by the free energy  $w(\varepsilon - \varepsilon_p)$  where  $\varepsilon_p$  is the plastic strain, which evolution is governed by a normality rule associated with a convex function  $f(\sigma) \leq 0$  such that

$$f(\sigma) \leq 0, \quad \lambda > 0, \quad \varepsilon_p = \lambda \frac{\partial f}{\partial \sigma}, \quad \lambda f = 0. \quad (37)$$

The complementary energy for perfect plasticity is:

$$w^*(\sigma) = \sigma : (\varepsilon - \varepsilon^p) - w(\varepsilon - \varepsilon^p) \quad (38)$$

The definitions of both energies  $\mathcal{E}$  and  $\mathcal{E}^*$  are conserved with these local new densities. For a given field of internal parameter, the equilibrium solution minimizes the potential or the complementary energy, then it is obvious that

$$I_s = J_s = -\frac{\partial \mathcal{E}}{\partial l} = -\frac{\partial \mathcal{E}^*}{\partial l}, \quad (39)$$

where the thermodynamical force  $J_s$  is given by Eq. 20. Theses Eshelby's tensors  $p$  and  $p^*$  satisfy now

$$0 = \int_{V_c} \text{div } p \, d\omega + \int_{V_c} \sigma : \nabla \varepsilon^p \, d\omega$$

$$0 = \int_{V_c} \text{div } p^* \, d\omega - \int_{V_c} \nabla \sigma : \varepsilon^p \, d\omega$$

whatever the closed volume which does not contain discontinuities. Using the properties of  $p$  and  $p^*$  the two integrals

$$\mathcal{G} = J_\Gamma + \int_{V_\Gamma} \sigma : \varepsilon_{,1}^p \, d\Omega, \quad \mathcal{G}^* = I_\Gamma - \int_{V_\Gamma} \sigma_{,1} : \varepsilon^p \, d\Omega \quad (40)$$

are independ of  $\Gamma$ , and we find:

$$\begin{aligned}\mathcal{G} + \mathcal{G}^* &= 2J_s = \int_{\Gamma} (\underline{n} \cdot \sigma_{,1} \cdot \underline{u} - \underline{n} \cdot \sigma \cdot \underline{u}_{,1}) \, dS \\ &\quad + \int_{V_\Gamma} (\sigma : \varepsilon_{,1}^p - \sigma_{,1} : \varepsilon^p) \, d\Omega, \\ \mathcal{G} - \mathcal{G}^* &= 0 = \int_{\Gamma} ((w + w^*)n_1 - \underline{n} \cdot \sigma_{,1} \cdot \underline{u} - \underline{n} \cdot \sigma \cdot \underline{u}_{,1}) \, dS \\ &\quad + \int_{V_\Gamma} (\sigma : \varepsilon^p)_{,1} \, d\Omega.\end{aligned}$$

Adding these equations, take the limit  $\Gamma \rightarrow S$  to obtain

$$\mathcal{G} + \mathcal{G}^* = I_s + J_s = \int_S (\underline{n} \cdot [\sigma_{,1}]_S \cdot \underline{u} - \underline{n} \cdot \sigma \cdot [\underline{u}_{,1}]_S) \, ds, \quad (41)$$

therefore we have

$$I_s = \lim_{\Gamma \rightarrow S} \int_{\Gamma} (-w^*(\sigma)n_1 + \underline{n} \cdot \sigma_{,1} \cdot \underline{u}) \, ds. \quad (42)$$

This generalizes the integral of Bui to perfect plasticity. *Elastoplasticity with hardening* The local behaviour is defined by the local free energy  $w(\varepsilon, \alpha)$  where  $\alpha$  is a set of internal parameters, and  $w$  is assumed to be convex of it's argument. The thermodynamical forces associated with these state variables are the stresses  $\sigma$  and the internal forces  $A$

$$\sigma = \frac{\partial w}{\partial \varepsilon}, \quad A = -\frac{\partial w}{\partial \alpha}. \quad (43)$$

The complementary energy is defined as previously by  $w^*(\sigma, A) = \sigma : \varepsilon - A \cdot \alpha - w(\varepsilon, \alpha)$ . (44)

For the equilibrium solution the stress  $\sigma$  satisfy the equilibrium  $\text{div } \sigma = 0$  and the boundary conditions  $\underline{n} \cdot \sigma = \underline{T}^d$ . The internal forces  $A$  are inside the domain of reversibility.

The Eshelby's tensors  $p$  and  $p^*$  satisfy the properties

$$0 = \int_{V_c} \text{div } p \, d\Omega + \int_{V_c} A \cdot \nabla \alpha \, d\Omega,$$

$$0 = \int_{V_c} \text{div } p^* \, d\Omega - \int_{V_c} \nabla A \cdot \alpha \, d\Omega$$

and by a reasoning analogous to those of perfect plasticity

$$\mathcal{G} = J_\Gamma + \int_{V_\Gamma} A \cdot \alpha_{,1} \, d\Omega, \quad \mathcal{G}^* = I_\Gamma - \int_{V_\Gamma} A_{,1} : \alpha \, d\Omega. \quad (45)$$

The property  $\mathcal{G} = J_s$  is conserved.

When the energy  $w$  is a quadratic function, at the equilibrium solution we have ( $w = w^* = \frac{1}{2} \sigma : \varepsilon - \frac{1}{2} A : \alpha$ ) and we obtain the peculiar form:

$$\begin{aligned}\mathcal{G} &= \frac{1}{2} \int_{\Gamma} (\underline{n} \cdot \sigma_{,1} \cdot \underline{u} - \underline{n} \cdot \sigma \cdot \underline{u}_{,1}) \, dS \\ &\quad + \frac{1}{2} \int_{V_\Gamma} (A : \alpha_{,1} - A_{,1} : \alpha) \, d\Omega.\end{aligned}$$

## 6 Application to the evolution of cracks

For describing the evolution of cracks it is necessary to have an evolution law. The evolution of a crack is governed by a generalized Griffith's law

$$J_s \leq G_c, \quad a = 0, \quad J_s = G_c, \quad a \geq 0. \quad (46)$$

The evolution is governed by the normality rule: when  $J_s = G_c$  thus

$$\dot{J}_s (a - \mu) \geq 0, \quad \forall \mu \geq 0, \quad (47)$$

The rate  $\dot{J}_s$  must be characterized.

The volume  $V_\Gamma$  moves with the velocity  $a$ . The rate  $\overset{\circ}{f}$  is the rate of  $f$  in the moving frame  $V_\Gamma$ . The surface  $S$  has the same velocity, then it's normal velocity is  $a \underline{e}_1 \cdot \underline{n}$ . In the moving frame the variation of  $J_s$  is well defined:

$$\dot{\mathcal{G}} = \int_S [\sigma]_S : \nabla \overset{\circ}{\underline{u}} - \overset{\circ}{\sigma} : [\nabla \overset{\circ}{\underline{u}}]_S - [A \overset{\circ}{\alpha}]_S \, ds, \quad (48)$$

which can be rewritten as

$$\begin{aligned}\dot{\mathcal{G}} &= \int_{\Gamma} (\underline{n} \cdot \sigma_{,1} \cdot \overset{\circ}{\underline{u}} - \underline{n} \cdot \overset{\circ}{\sigma} \cdot \underline{u}_{,1}) \, ds \\ &\quad + \int_{V_\Gamma} (\overset{\circ}{A} : \alpha_{,1} - A_{,1} : \overset{\circ}{\alpha}) \, d\Omega.\end{aligned}$$

The last term contains only coupling terms between strain and hardening, as we can shown

$$\begin{aligned}-\overset{\circ}{A} : \alpha_{,1} + A_{,1} : \overset{\circ}{\alpha} \\ = \frac{\partial^2 w}{\partial \alpha \partial \varepsilon} \bullet (\overset{\circ}{\varepsilon} \otimes \alpha_{,1} - \overset{\circ}{\alpha} \otimes \varepsilon_{,1}) = \overset{\circ}{\sigma} : \varepsilon_{,1} - \sigma_{,1} : \overset{\circ}{\varepsilon}, \quad (49)\end{aligned}$$

then the rate is given by

$$\begin{aligned}\dot{\mathcal{G}} &= \int_{\Gamma} (\underline{n} \cdot \sigma_{,1} \cdot \overset{\circ}{\underline{u}} - \underline{n} \cdot \overset{\circ}{\sigma} \cdot \underline{u}_{,1}) \, ds \\ &\quad + \int_{V_\Gamma} (\overset{\circ}{\sigma} : \varepsilon_{,1} - \sigma_{,1} : \overset{\circ}{\varepsilon}) \, d\Omega.\end{aligned} \quad (50)$$

This generalizes the previous case obtained in [Nguyen and Stolz \(1985\)](#) for perfect plasticity.

## 7 On the rate boundary value problem

We introduce the rate  $\hat{f}$  such that

$$\hat{f} = \dot{f}, \text{ over } \Omega/V_\Gamma, \quad \hat{f} = \overset{\circ}{f}, \text{ over } V_\Gamma \quad (51)$$

The rate  $\hat{f}$  is discontinuous on  $\Gamma$  and satisfies the Hadamard condition

$$\hat{f} = \overset{\circ}{f} - \dot{l}f_{,1}; \quad [\hat{f}] + \dot{l}f_{,1} = 0. \quad (52)$$

A solution of the evolution problem satisfies the set of local equations

**Kinematics:** The strain rate is associated with the rate of displacement  $\hat{\varepsilon} = \frac{1}{2}(\nabla\hat{u} + \nabla\hat{u})^T$  over  $\Omega$ , the boundary conditions  $\hat{u} = \dot{u}^d$  over  $\partial\Omega_u$ , and the discontinuities  $[\hat{u}] + \dot{l}u_{,1} = 0$  along  $\Gamma$  are satisfied.

**Conservation of momentum:** The stresses satisfy  $\text{div } \hat{\sigma} = 0$  over  $\Omega$ , the boundary conditions  $\hat{\sigma} \cdot \underline{n} = \dot{T}^d$ , and the jump condition  $\underline{n} \cdot [\hat{\sigma}] + \dot{l}\sigma_{,1} \cdot \underline{n} = 0$  along  $\Gamma$ . The last equation preserves the continuity of the stress vector during the motion of the interface  $\Gamma$ . The constitutive law

$$\left[ \begin{array}{c} \dot{\sigma} \\ \text{dot } A \end{array} \right] = \left[ \begin{array}{cc} w''_{\varepsilon\varepsilon} & w''_{\varepsilon\alpha} \\ w''_{\alpha\varepsilon} & w''_{\alpha\alpha} \end{array} \right] \left[ \begin{array}{c} \dot{\varepsilon} \\ \dot{\alpha} \end{array} \right] \quad (53)$$

The internal parameters satisfy the normality rule associated with the domain of reversibility defined by the convex function  $f$ :

$$f(A) \leq 0, \quad \dot{\alpha} = \lambda N, \quad N = \frac{\partial f}{\partial A}, \quad \lambda \geq 0, \quad \lambda f(A) = 0 \quad (54)$$

The propagation law.

$$\dot{l}(\mathcal{G} - G_c) = 0, \quad \dot{l} \geq 0, \quad \dot{J}_s = \dot{I}_s = 0. \quad (55)$$

The second derivative of the local energy is noted by  $C = W''_{\varepsilon\varepsilon}$ ,  $L = W''_{\varepsilon\alpha}$ ,  $Z = W''_{\alpha\alpha}$ . The local behaviour is hypoelastic: the potential  $U(\dot{\varepsilon})$  is defined

$$\dot{\sigma} = \frac{\partial U}{\partial \dot{\varepsilon}}, \quad U(\dot{\varepsilon}) = \frac{1}{2} \dot{\varepsilon} : C : \dot{\varepsilon} - \frac{1}{2} \frac{\langle N^T \cdot L^T \cdot \dot{\varepsilon} \rangle^2}{N^T \cdot Z \cdot N} \quad (56)$$

or the complementary potential is  $U^*(\dot{\sigma})$ :

$$\dot{\varepsilon} = \frac{\partial U^*}{\partial \dot{\sigma}}, \quad (57)$$

$$U^*(\dot{\sigma}) = \frac{1}{2} \dot{\sigma} \cdot C^{-1} : \dot{\sigma} - \frac{1}{2} \frac{\langle N^T \cdot L^T \cdot C^{-1} \cdot \dot{\sigma} \rangle^2}{N^T \cdot L^T \cdot C^{-1} \cdot L \cdot N - N^T \cdot Z \cdot N}$$

*Remark* We take the derivatives  $\hat{f}$  as variables. This field presents jumps along  $\Gamma$

$$[\hat{\varepsilon}] + a [u_{,1}] = 0, \quad [\hat{\sigma}] \cdot \underline{n} + a \underline{n} \cdot [\sigma_{,1}] = 0. \quad (58)$$

In  $V_\Gamma$ , the local behaviour is given by

$$\hat{\varepsilon} - a \varepsilon_{,1} = \frac{\partial U^*}{\partial \hat{\sigma}}(\hat{\sigma} - a \sigma_{,1}), \quad (59)$$

$$\hat{\sigma} - a \sigma_{,1} = \frac{\partial U}{\partial \hat{\varepsilon}}(\hat{\varepsilon} - a \varepsilon_{,1}) \quad (60)$$

We introduce now the overall functional  $\mathcal{F}$  associated to the Hill's potential  $U$

$$\begin{aligned} \mathcal{F} = & \int_{\Omega/V_\Gamma} U(\hat{\varepsilon}) \, d\Omega + \int_{V_\Gamma} U(\hat{\varepsilon} - a \varepsilon_{,1}) \, d\Omega \\ & + \int_{V_\Gamma} (a \sigma_{,1} : \hat{\varepsilon} - \frac{1}{2} a^2 \sigma_{,1} : \varepsilon_{,1}) \, d\Omega \\ & + \int_{\Gamma} (-a \underline{n} \cdot \sigma_{,1} \cdot \hat{u}^- + \frac{1}{2} a^2 \underline{n} \cdot \sigma_{,1} \cdot u_{,1}) \, dS \\ & - \int_{\partial\Omega_T} \dot{T}^d \cdot \hat{u} \, dS \end{aligned}$$

*Property of the solution* The solution of the evolution problem  $(\hat{u}, a)$  satisfies the variational inequalities

$$\frac{\partial \mathcal{F}}{\partial \hat{u}}(\hat{u} - \tilde{u}) + \frac{\partial \mathcal{F}}{\partial a}(a - \tilde{l}) \leq 0 \quad (61)$$

among the set of admissible fields

$$\begin{aligned} \mathcal{H} = & \{(\tilde{u}, \tilde{l}) / [\tilde{u}] + \tilde{l} u_{,1} = 0, \text{ along } \Gamma, \\ & \tilde{l} \geq 0, \quad \tilde{u} = \dot{u}^d, \text{ over } \partial\Omega_u\} \end{aligned} \quad (62)$$

*Elements of proof:* The variations of  $\mathcal{F}$  are given by:

$$\begin{aligned} \delta \mathcal{F} = & \int_{\Omega/V_\Gamma} \hat{\sigma} : \delta \hat{\varepsilon} \, d\Omega + \int_{V_\Gamma} (\hat{\sigma} - a \sigma_{,1}) : (\delta \hat{\varepsilon} - \delta l \varepsilon_{,1}) \, d\Omega \\ & + \int_{V_\Gamma} (\delta l \sigma_{,1} : \hat{\varepsilon} + a \sigma_{,1} \delta \hat{\varepsilon} - a \delta l \sigma_{,1} : \varepsilon_{,1}) \, d\Omega \\ & + \int_{\Gamma} (-\delta l \underline{n} \cdot \sigma_{,1} \cdot \hat{u}^- - a \underline{n} \cdot \sigma_{,1} \cdot \delta \hat{u}^- \\ & + a \delta l \underline{n} \cdot \sigma_{,1} \cdot u_{,1}) \, dS - \int_{\partial\Omega_T} \dot{T}^d \cdot \delta \hat{u} \, dS \end{aligned}$$

After integration by part, we obtain

$$\begin{aligned} \delta \mathcal{F} = & \int_{\Omega} -\text{div } \hat{\sigma} \cdot \delta \hat{u} \, d\Omega + \int_{\Gamma} \underline{n} \cdot [\hat{\sigma} \cdot \delta \hat{u}] \, dS \\ & + \int_{V_\Gamma} \delta l (\sigma_{,1} \hat{\varepsilon} - \hat{\sigma} : \varepsilon_{,1}) \, d\Omega + \int_{\Gamma} (-\delta l \underline{n} \cdot \sigma_{,1} \cdot \hat{u}^- \\ & - a \underline{n} \cdot \sigma_{,1} \cdot \delta \hat{u}^- + a \delta l \underline{n} \cdot \sigma_{,1} \cdot u_{,1}) \, dS \end{aligned}$$



The conservation of momentum is recovered  $\text{div } \hat{\sigma} = 0$  over  $\Omega$ . Using the jump conditions  $[\delta \underline{u}] + \delta l \underline{u}_{,1} = 0$ , finally the variations are reduced to

$$\begin{aligned} \delta \mathcal{F} = & \int_{\Gamma} \underline{n} \cdot ([\hat{\sigma}] + a \sigma_{,1}) \cdot \delta \underline{\hat{u}}^- \, dS \\ & + \delta l \left[ \int_{V_{\Gamma}} \sigma_{,1} : \hat{\varepsilon} - \hat{\sigma} : \varepsilon_{,1} \, d\Omega \right. \\ & \left. + \int_{\Gamma} \underline{n} \cdot \hat{\sigma}^- \cdot \underline{u}_{,1} - \underline{n} \cdot \sigma_{,1} \cdot \underline{\hat{u}}^- \, dS \right] \end{aligned}$$

thus, the jump conditions for the stresses and the law of propagation by application of the inequality are recovered.

*Dual formulation* We introduce the functional  $\mathcal{F}^*$  associated with the potential  $U^*$ :

$$\begin{aligned} \mathcal{F}^* = & \int_{\Omega/V_{\Gamma}} -U^*(\hat{\sigma}) \, d\Omega - \int_{V_{\Gamma}} U^*(\hat{\sigma} - a \sigma_{,1}) \, d\Omega \\ & + \int_{V_{\Gamma}} (-a \hat{\sigma} : \varepsilon_{,1} + \frac{j^2}{2} \sigma_{,1} : \varepsilon_{,1}) \, d\Omega \\ & + \int_{\Gamma} (a \underline{n} \cdot \hat{\sigma}^- \cdot \underline{u}_{,1} - \frac{j^2}{2} \underline{n} \cdot \sigma_{,1} \cdot \underline{u}_{,1}) \, ds \\ & + \int_{\partial\Omega_u} \underline{n} \cdot \hat{\sigma} \cdot \underline{\hat{u}}^d \, ds \end{aligned}$$

$\mathcal{F}^*$  is defined among the set of admissible fields  $\mathcal{K}^*$ :

$$\begin{aligned} \mathcal{K}^* = & \{(\tilde{\sigma}, \tilde{l}) / \text{div } \tilde{\sigma} = 0, \text{ over } \Omega, \\ & \tilde{\sigma} \cdot \underline{n} = \dot{T}^d \text{ along } \partial\Omega_{\Gamma}, \\ & \underline{n} \cdot [\tilde{\sigma}] + \tilde{l} \underline{n} \cdot \sigma_{,1} = 0, \text{ along } \Gamma \quad \tilde{l} \geq 0\} \end{aligned} \quad (63)$$

*Property* A solution of the evolution problem satisfies the variational inequalities

$$\frac{\partial \mathcal{F}^*}{\partial \hat{\sigma}} \cdot (\hat{\sigma} - \tilde{\sigma}) + \frac{\partial \mathcal{F}^*}{\partial \tilde{l}} (a - \tilde{l}) \leq 0 \quad (64)$$

among the set  $\mathcal{K}^*$  of admissibles fields  $(\tilde{\sigma}, \tilde{l})$ .

*Elements of proof* The variations of  $\mathcal{F}^*$  satisfy:

$$\begin{aligned} \delta \mathcal{F}^* = & - \int_{\Omega/V_{\Gamma}} \delta \sigma : \hat{\varepsilon} \, d\Omega \\ & - \int_{V_{\Gamma}} (\delta \sigma - \delta l \sigma_{,1}) : (\hat{\varepsilon} - a \varepsilon_{,1}) \, d\Omega \\ & - \int_{V_{\Gamma}} (\delta l \varepsilon_{,1} : \hat{\sigma} + a \varepsilon_{,1} : \delta \sigma - a \delta l \sigma_{,1} : \varepsilon_{,1}) \, d\Omega \\ & + \int_{\Gamma} (\delta l \underline{n} \cdot \hat{\sigma}^- \cdot \underline{u}_{,1} + a \underline{n} \cdot \delta \sigma^- \cdot \underline{u}_{,1} - a \delta l \underline{n} \cdot \sigma_{,1} \cdot \underline{u}_{,1}) \, dS \end{aligned}$$

After integration by part and reduction, as  $\delta \sigma$  satisfies the conservation of momentum the strain rate  $\hat{\varepsilon}$  derives from a displacement  $\hat{u}$  and

$$2\hat{\varepsilon} = \text{grad } \hat{u}^T + \text{grad } \underline{\hat{u}}. \quad (65)$$

Using this property, the variations of  $\mathcal{F}^*$  are now

$$\begin{aligned} \delta \mathcal{F}^* = & \int_{\Gamma} \underline{n} \cdot [\delta \sigma \cdot \underline{\hat{u}}] \, dS - \int_{V_{\Gamma}} \delta l (\sigma_{,1} : \hat{\varepsilon} - \hat{\sigma} : \varepsilon_{,1}) \, d\Omega \\ & + \int_{\Gamma} \delta l (\underline{n} \cdot \hat{\sigma}^- \cdot \underline{u}_{,1} - a \underline{n} \cdot \sigma_{,1} \cdot \underline{u}_{,1}) \\ & + a \underline{n} \cdot \delta \sigma^- \cdot \underline{u}_{,1} \, dS \end{aligned}$$

Considering the jump condition

$$\underline{n} \cdot \delta \sigma^+ - \underline{n} \cdot \delta \sigma^- + \delta l \underline{n} \cdot \sigma_{,1} = 0$$

we obtain:

$$\begin{aligned} \delta \mathcal{F}^* = & \int_{\Gamma} \underline{n} \cdot \delta \sigma^- \cdot ([\underline{\hat{u}}] + a \underline{u}_{,1}) \, dS \\ = & \delta l \left( \int_{\Gamma} \underline{n} \cdot \hat{\sigma}^- \cdot \underline{u}_{,1} - \sigma_{,1} : \underline{\hat{u}}^- \, dS \right. \\ & \left. + \int_{V_{\Gamma}} \hat{\sigma} : \varepsilon_{,1} - \sigma_{,1} : \hat{\varepsilon} \, d\Omega \right) \end{aligned}$$

Thus the jump conditions for the rate of the displacement and the law of propagation for the crack, using of the variational inequality are recovered.

## 8 Conclusion

We have generalized the integral of Bui to elastoplastic materials. We have connected this integral to the

release rate of potential and complementary energy in case of elastoplastic behaviour with hardening.

With these definitions, the formulations of the evolution of cracks are determined by variational inequalities as shown in [Stolz \(2008a,b\)](#). The methods are identical as proposed in [\(Nguyen and Stolz 1985; Nguyen et al. 1990\)](#) and allows stability and bifurcation analysis of a system of cracks.

## References

- Bui HD (1973) Dualité entre les intégrales indépendantes du contour dans la théorie des solides fissurés. *Comptes Rendus Académie des Sciences Paris, Série A* 376:1425–1428
- Destuynder P, Djaoua M (1981) Sur une interprétation mathématique de l'intégrale de Rice en théorie de la rupture fragile. *Math Appl Sci* 3:70–77
- Gurtin ME, Podio-Guiugli P (1996) Configurational forces and the basic law for crack propagation. *J Mech Phys Solids* 44:905–927
- Knowles JK (1977) The finite antiplane shear field near the tip of a crack for a class of incompressible elastic solids. *Int J Fracture* 13:611–639
- Lubarda VA, Markenscoff X (2007) Dual conservation integrals and energy release rates. *Int J Solids Struct* 44:4079–4091
- Maugin GA (1994) Eshelby stress in elastoplasticity and ductile fracture. *Int J Plasticity* 10:393–408
- Nguyen QS (1985) Critère de propagation en rupture ductile. *Comptes Rendus Académie des Sciences Paris* 301:567–570
- Nguyen QS, Stolz C, Debruyne G (1990) Energy methods in fracture mechanics: stability, bifurcation and second variations. *Eur J Mech, A Solids* 9:157–173
- Nguyen QS, Stolz C (1985) Sur le problème d'évolution en vitesse de propagation de fissure et de déplacement en rupture fragile ou ductile. *Comptes Rendus Académie des Sciences Paris* 301:661–664
- Simha NK, Fisher FD, Shan GX, Chen CR, Kolednik O (2008) J-Integral and crack driving force in elastic-plastic materials. *J Mech Phys Solids* 56:2876–2895
- Stolz C, Pradeilles-Duval RM (1996) Approche énergétique de la propagation dynamique de discontinuités mécaniques. *Comptes Rendus Académie des Sciences Paris, Série IIB* 322:525–532
- Stolz C (2004) Energy methods in non linear mechanics, lecture notes 11. Centre of Excellence for Advanced Materials and Structures. Poland: IPPT, Polish Academy of Sciences
- Stolz C (2008) Sur le problème de propagation de fissure en élastoplasticité: approches primale et duale. *C.R. Mécanique* 336:500–505
- Stolz C (2008) Intégrale duale en mécanique de la rupture. *C.R. Mécanique* 336:434–439