



**HAL**  
open science

## Constitutive law gap functionals to solve Cauchy problem for a linear elliptic PDE: a review

Thouraya Baranger, Stéphane Andrieux

► **To cite this version:**

Thouraya Baranger, Stéphane Andrieux. Constitutive law gap functionals to solve Cauchy problem for a linear elliptic PDE: a review. 2010. hal-00489572v2

**HAL Id: hal-00489572**

**<https://hal.science/hal-00489572v2>**

Preprint submitted on 10 Jun 2010

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Constitutive law gap functionals to solve Cauchy problem for a linear elliptic PDE: a review

Thouraya N. Baranger<sup>1</sup> and Stéphane Andrieux<sup>2</sup>

<sup>1</sup> Universit de Lyon, CNRS, Universit Lyon 1, LaMCoS UMR5259, INSA-Lyon, F-69621, Villeurbanne, France.

<sup>2</sup> LaMSID, UMR CNRS-EDF 2832, Clamart, France.

**Abstract.** This paper describes a general method based on minimizing constitutive law gap functional in order to solve the Cauchy problem for a linear elliptic PDE. This functional measures the gap between the solutions of two well-posed problems. Each of these problems has one of the Cauchy data as known boundary condition: Dirichlet or Neumann, and on the boundary where the data is lacking, unknown Robin boundary conditions  $\eta + \alpha\tau$  and  $\eta + \beta\tau$  are imposed, respectively. The data  $\eta$  and  $\tau$  have to be identified and  $(\alpha, \beta)$  are positives scalars parameters controlling the functional behavior. This approach generalizes that presented in Andrieux *et al* [2] and encompasses various methods proposed in the literature. According to the values of  $\alpha$  and  $\beta$  when they tend toward 0 or  $\infty$ , there are two groups of methods: the first group includes those which depends on only one unknown data ( $\eta$ ,  $\tau$  or  $\eta + \alpha\tau$ ). The second group includes those which depend on two unknown data  $\eta$  and  $\tau$ . Then, the equivalence between Euler-Lagrange conditions for the constitutive law functionals and interfacial operators usually used in the Domain Decomposition field is shown. Using the Hadamard example we analyse analytically the behavior of these operators as functions of the parameters  $(\alpha, \beta)$ . Then, the derivatives of the functional are given using adjoint fields which are parametrized by the same parameters. Finally, numerical examples are given to illustrate the behavior of these methods, which are not function of the parameters  $(\alpha, \beta)$  but also of the regularity of the Cauchy data and the overall geometry of the domain.

## 1. Introduction

Consider a solid body  $\Omega$ , given a flux  $\Phi$  and the corresponding temperature  $T$  on  $\Gamma_m$ , one wants to recover the corresponding flux and temperature on the remaining part of the boundary  $\Gamma_u$ , where  $\Gamma_m$  and  $\Gamma_u$  constitute a partition of the whole boundary  $\partial\Omega$ . The problem is therefore set as follows:

Find  $(\varphi, \mathcal{T})$  on  $\Gamma_u$  such that there exists a field  $u$  satisfying:

$$\begin{cases} \nabla \cdot k(x) \nabla u = 0 & \text{in } \Omega \\ k(x) \nabla u \cdot n = \Phi & \text{on } \Gamma_m \\ u = T & \text{on } \Gamma_m \end{cases} \quad (1)$$

where the conductivity field  $k(x)$  is real positive analytic in  $L^\infty(\Omega)$ . This problem is known since Hadamard [16] to be ill-posed in the sense that the dependence of  $u$  and consequently of  $(\varphi, \mathcal{T})$  on the data  $(\Phi, T)$  is known to be not continuous.

We propose, in this paper, to identify the lacking data  $(\varphi, \mathcal{T})$  by minimizing a constitutive law gap function which generalizes the one introduced in [2]. Then, Robin (or Fourier) boundary conditions are defined on the  $\Gamma_u$  part of the boundary. The aim is to study if better numerical behavior can be observed with special values of the Robin parameters. We restrict ourselves here to elliptic operators although a similar approach can be applied to parabolic or hyperbolic ones [1]. Other elliptic operators describing various physical phenomena has been addressed in [5, 6, 7, 13, 14, 17].

This paper is organized as follows: In section 2, after a background on the literature dealing with Cauchy problem, two mixed well-posed problems are defined by splitting the overspecified data on  $\Gamma_m$  and Robin boundary conditions are introduced on the boundary  $\Gamma_u$ . The latter are parametrized by two positive real constants  $\alpha$  and  $\beta$ . Then, the boundary condition identification problem is defined as an optimization one with constraints, where the objective functional is a constitutive law gap function. This function quantifies the energy gap between two fields solution of the well-posed problems defined above, which constitute the constraints of the optimization problem. Hence, particular cases are outlined when  $\alpha$  and  $\beta$  tend toward limit values 0 and  $\infty$ . In section 3, we present an equivalent formulation based on domain decomposition strategy. Then, we show for all the cases outlined in section 3, that the Euler-Lagrange conditions for the constitutive law functionals and interfacial operators are equivalent. Hadamard example is presented in section 4, in order to illustrate the behavior of the operators introduced above as functions of the Robin parameters  $\alpha$  and  $\beta$ . In section 5, the evaluation of the derivatives of the constitutive law gap is given by using adjoint methods. Numerical examples are presented in section 6, to illustrate the behavior of the methods when the geometry or/and the boundary data on  $\Gamma_m$  and  $\Gamma_u$  present singularities or discontinuities. Finally a conclusion is given.

## 2. Boundary conditions identification

Consider the above Cauchy problem (1). Assuming that the data  $(\Phi, T)$  are compatible, which means that this pair is indeed the trace and normal trace of a unique harmonic function  $u$ , extending the data means finding  $(\varphi, \mathcal{T})$  such as:

$$\begin{cases} \nabla \cdot k(x) \nabla u = 0 & \text{in } \Omega, \\ u = T, \quad k(x) \nabla u \cdot n = \Phi & \text{on } \Gamma_m, \\ u = \mathcal{T}, \quad k(x) \nabla u \cdot n = \varphi & \text{on } \Gamma_u \end{cases} \quad (2)$$

The question now is how to reconstruct numerically the pair  $(\varphi, \mathcal{T})$ . In practical problems, data is not expected to be compatible, since data errors can occur from errors in measurements. The ill-posedness in Hadamard's sense shows up - dramatically - when one tries to approximate a given data  $(\Phi, T)$ : it is possible to approach it as closely as desired on  $\Gamma_m$  by traces of a single harmonic function, the "surprise" being a hectic behavior of this function on the remaining part of the boundary. This behavior can be understood by the fact that the compatible data are dense in the space of incompatible ones, which makes hopeless the natural idea of least square fitting of the incompatible data by the compatible ones. Regularization procedures are therefore required to treat the data completion problem [22, 12, 11]. There are several approaches to regularize such ill-posedness. Some of them transform the ill-posed problem into a well-posed one by adding a penalty term or by mollifying the data in order to avoid data oscillations. Tikhonov like methods use the penalty approach. Another class of rough but usually efficient regularizing techniques try to solve the ill-posed problem iteratively and choose a suitable stopping criteria, for instance L-curve based criteria. In the approach proposed here, the introduction of two distinct fields, each of them meeting only one of the over-specified data, turns out to avoid the need of a regularization procedure for the resolution of the data completion problem, when the noise rate remains reasonable, see [5, 6, 7]. Using separately the two boundary conditions on  $\Gamma_m$  has also been used in the algorithm proposed by Kozlov *et al* [18] and analysed by Baumeister *et al* [8] in a general framework, where again no regularization procedure is cast into the resolution method.

We will restrict ourselves, throughout the paper for the setting to the case where the boundary  $\partial\Omega$  consists of two closed manifolds of class  $C^2$  such that  $\partial\Omega = \Gamma_m \cup \Gamma_u$ . The following results remain true for less smooth boundaries and when  $\Gamma_m, \Gamma_u$  have contact points. However, for sake of simplicity, we have chosen the above framework. As already mentioned, the pairs of compatible data are dense in the set of all possible data pairs. For this known result we refer to Fursikov [15] and to a preceding paper [3], where the mentioned proofs are adapted to our settings.

**Lemma 2.1.** (i) For a fixed  $T$  in  $H^{1/2}(\Gamma_m)$ , the set of data  $\Phi$  for which there exists a function  $u$  in  $H^1(\Omega)$ , satisfying the Cauchy problem (1) is everywhere dense in  $H^{-1/2}(\Gamma_m)$ .

(ii) For a fixed  $\Phi$  in  $H^{-1/2}(\Gamma_m)$ , the set of data  $T$  for which there exists a function  $u$  in  $H^1(\Omega)$ , satisfying the Cauchy problem (1) is everywhere dense in  $H^{1/2}(\Gamma_m)$ .

Observe that, when the complete data are available on  $\Gamma$ , we have an overspecified boundary value problem

$$\begin{cases} \nabla \cdot k(x) \nabla u = 0 & \text{in } \Omega, \\ u = T, \quad k(x) \nabla u \cdot n = \Phi & \text{on } \Gamma_m, \\ u = \mathcal{T}, \quad k(x) \nabla u \cdot n = \varphi & \text{on } \Gamma_u \end{cases} \quad (3)$$

The approach followed here generalizes the one given in [3]. It follows two steps: consider for a given pair  $(\eta, \tau) \in H^{-\frac{1}{2}}(\Gamma_u) \times H^{\frac{1}{2}}(\Gamma_u)$  the following two families of mixed well posed problems

$$\begin{cases} \nabla \cdot k(x) \nabla u_1 = 0 & \text{in } \Omega \\ u_1 = T & \text{on } \Gamma_m \\ k(x) \nabla u_1 \cdot n + \alpha u_1 = \eta + \alpha \tau & \text{on } \Gamma_u \end{cases} \quad (4)$$

$$\begin{cases} \nabla \cdot k(x) \nabla u_2 = 0 & \text{in } \Omega \\ k(x) \nabla u_2 \cdot n = \Phi & \text{on } \Gamma_m \\ k(x) \nabla u_2 \cdot n + \beta u_2 = \eta + \beta \tau & \text{on } \Gamma_u \end{cases} \quad (5)$$

We denote by  $\alpha$  and  $\beta$  two non-negative real coefficients. This condition ensures that problems (3) and (4) are well-posed. Using a  $H^1$  semi-norm the constitutive law gap functional is defined as a measure of the pseudo-energy gap between the two above fields  $u_1$  and  $u_2$  as follows:

$$E_{\alpha\beta}(\eta, \tau) = \frac{1}{2} \int_{\Omega} k(x) (\nabla u_1 - \nabla u_2) \cdot (\nabla u_1 - \nabla u_2) \quad (6)$$

This functional is positive and quadratic. Indeed,  $u_1$  and  $u_2$  are obviously equal when the pair  $(\eta, \tau)$  on the boundary  $\Gamma_u$  meets the actual compatible data pair  $(\varphi, \mathcal{T}) \in H^{-1/2} \times H^{1/2}$  on the boundary  $\Gamma_u$ , then:

$$E_{\alpha,\beta}(\eta, \tau) = 0 = \min_{\eta, \tau} E_{\alpha\beta}(\eta, \tau)$$

Thanks to the uniqueness of the Cauchy problem solution we can state that the data completion problem can be achieved through the minimization one:

$$(\varphi, \mathcal{T}) = \arg \min_{\eta, \tau} E_{\alpha\beta}(\eta, \tau) \quad (7)$$

Using Green theorem this functional can be expressed as a boundary control:

$$\begin{aligned} E_{\alpha\beta}(\eta, \tau) &= \frac{1}{2} \int_{\Gamma_m} (k(x) \nabla u_1 \cdot n - \Phi)(T - u_2) \\ &\quad + \frac{1}{2} \int_{\Gamma_u} k(x) (\nabla u_1 - \nabla u_2) \cdot n (u_1 - u_2) \end{aligned} \quad (8)$$

Note that the boundary conditions defined on  $\Gamma_u$  in the problems (4) and (5) degenerate when the coefficients  $\alpha$  and  $\beta$  are equal to particular values as shown on the table 1. So

we can define two approaches differing by the number of unknown field on  $\Gamma_u$ . In the first approach there is only one unknown boundary field, and, happens when  $\alpha = \beta$ . Three different single field methods can be outlined: in the first one, when  $\alpha = \beta = 0$ , the unknown boundary data is the Neumann boundary condition  $\eta$ , in the second approach, when  $\alpha = \beta = \infty$ , the unknown data is the Dirichlet boundary condition  $\tau$  and in the last one, when  $0 < \alpha = \beta < \infty$ , the unknown data is the Robin boundary condition  $v = \eta + \alpha\tau$ . The second approach is based on two unknown fields  $(\eta, \tau)$  and includes the other cases where  $\alpha \neq \beta$ .

**Table 1.** Boundary conditions on  $\Gamma_u$  as function of the parameters  $\alpha$  and  $\beta$

	$\alpha = 0$	$\alpha = \infty$	$0 < \alpha < \infty$
$\beta = 0$	$\nabla u_1.n = \eta$ $\nabla u_2.n = \eta$	$u_1 = \tau$ $\nabla u_2.n = \eta$	$\nabla u_1.n + \alpha u_1 = \eta + \alpha\tau$ $\nabla u_2.n = \eta$
$\beta = \infty$	$\nabla u_1.n = \eta$ $u_2 = \tau$	$u_1 = \tau$ $u_2 = \tau$	$\nabla u_1.n + \alpha u_1 = \eta + \alpha\tau$ $u_2 = \tau$
$0 < \beta < \infty$	$\nabla u_1.n = \eta$ $\nabla u_2.n + \beta u_2 = \eta + \beta\tau$	$u_1 = \tau$ $\nabla u_2.n + \beta u_2 = \eta + \beta\tau$	$\nabla u_1.n + \alpha u_1 = \eta + \alpha\tau$ $\nabla u_2.n + \beta u_2 = \eta + \beta\tau$ if $\alpha = \beta$ then: $\nabla u_1.n + \alpha u_1 = v$ $\nabla u_2.n + \alpha u_2 = v$

### 2.1. A review

This general setting leads to an interesting interpretation for different values of  $\alpha$  and  $\beta$ . In [3], we deal with the case where  $\alpha = 0$  and  $\beta = +\infty$ . This two fields (i.e.  $(\eta, \tau) \in H^{-1/2}(\Gamma_u) \times H^{1/2}(\Gamma_u)$ ) approach has been numerically explored and turned out to be efficient and robust. Other authors, [4, 9, 10] have explored mathematically and numerically the cases with one single field when  $\alpha = \beta = +\infty$  and its dual form when  $\alpha = \beta = 0$ . One may, however, wants to know which approach is more efficient. Notice that when  $\alpha = \beta = +\infty$ , the first optimality condition of our optimization process lead to the variational form of the well-known Steklov-Poincar method borrowed to the Domain Decomposition field, see [19, 21]. This fact has been already pointed out in [3, 9]. Moreover, when  $\alpha = \beta = 0$  we find the so-called dual Steklov-Poincar operator method. An other alternative form with single field formulation is the Alternating Direction Iterative method, which consists of solving two minimization problem where each problem depends on only one field, such that:

$$\eta_k = \arg \min_{\eta} E_{\alpha\beta}(\eta, \tau_{k-1}) \quad (9)$$

and

$$\tau_k = \arg \min_{\tau} E_{\alpha\beta}(\eta_k, \tau) \quad (10)$$

As it will be shown later, this last formulation turns out to be the KMF's (Kozlov, Maz'ya and Fomin) process described in [3]. Let us outline that, from the continuous point of view, all these methods are equivalent. In fact, they are all based on the introduction of two fields  $u_1$  and  $u_2$  dealing separately with the over-specified data, and the search for missing data by equalizing the two fields. At the discrete level, all these problems are of course ill-posed but some of them are expected to be better conditioned than others.

## 2.2. Two fields approaches

Two fields approach can be set up if  $0 \leq \alpha \neq \beta \leq \infty$  the constitutive law gap functional can be expressed as follows.

$$E_{\alpha\beta}(\eta, \tau) = \frac{1}{2} \int_{\Gamma_m} (k(x)\nabla u_1 \cdot n - \Phi)(T - u_2) + \frac{1}{2} \int_{\Gamma_u} (\alpha(\tau - u_1) - \beta(\tau - u_2))(u_1 - u_2) \quad (11)$$

with,  $u_1$  and  $u_2$  are the solution of (4) and (5). Let us observe that, for compatible data, the following lemma is straightforward.

**Lemma 2.2.** *(Characterization of the  $u_1$  and  $u_2$  fields at the minimum)*

*If  $(\Phi, T)$  is a compatible pair, there exists a pair  $(\varphi, \mathcal{T})$  solution of the Cauchy problem such that:*

$$(\varphi, \mathcal{T}) = \arg \min_{\eta, \tau} E_{\alpha\beta}(\eta, \tau) \text{ and } E_{\alpha\beta}(\varphi, \mathcal{T}) = 0$$

*Hence, when the functional reaches its minimum, the fields  $u_1$  and  $u_2$  verify:  $\nabla u_1 = \nabla u_2$  in  $\Omega$ , which is equivalent to:*

$$\begin{cases} u_1 = u_2 + K \text{ on } \Gamma_u \\ k(x)\nabla u_1 \cdot n = k(x)\nabla u_2 \cdot n \text{ on } \Gamma_u \end{cases} \quad (12)$$

*Where  $K$  is a real constant. From (12) and the boundary conditions defined in (4) and (5), we can deduce :*

$$u = u_1 = \tau + \frac{\beta}{\alpha - \beta} K \quad (13)$$

$$\nabla u \cdot n = \eta + \alpha(\tau - u) = \eta + \frac{\alpha\beta}{\alpha - \beta} K \quad (14)$$

The general approach built with the energy and the Robin's boundary conditions can lead to many two fields methods by setting extreme values for  $\alpha$  and  $\beta$ . These methods are outlined hereafter:

- (i) if  $\alpha = 0$  and  $\beta = \infty$ , the boundary conditions defined on  $\Gamma_u$  becomes Neumann boundary condition in (4), whereas, it becomes a Dirichlet one in (5) and we have

two well-posed mixed problems. Then, we denote the functional  $E_{0\infty}$  by  $E_{ND}$  which reduces to:

$$E_{ND}(\eta, \tau) = \int_{\Gamma_u} (\eta - k(x)\nabla u_2 \cdot n)(u_1 - \tau) + \int_{\Gamma_m} (k(x)\nabla u_1 \cdot n - \Phi)(T - u_2) \quad (15)$$

When  $E_{ND}$  reaches its minimum we have  $u = u_1 = \tau - K$  and  $\nabla u \cdot n = \eta$ . This case has been widely studied in [2].

- (ii) if  $\alpha = 0$  and  $\beta$  is finite and non zero, the boundary condition on  $\Gamma_u$  in (4) becomes a Neumann one. Then we denote the functional  $E_{0\beta}$  by  $E_{N\beta}$  which reduces to:

$$E_{N\beta}(\eta, \tau) = \int_{\Gamma_u} \beta(u_2 - \tau)(u_1 - u_2) + \int_{\Gamma_m} (k(x)\nabla u_1 \cdot n - \Phi)(T - u_2) \quad (16)$$

When the functional  $E_{N\beta}$  reaches its minimum we have  $u = u_1 = \tau - K$  and  $\nabla u \cdot n = \eta$ .

- (iii) if  $\alpha = \infty$  and  $\beta$  is finite and non zero, the boundary condition on  $\Gamma_u$  in (4) becomes a Dirichlet one. Then we denote the functional  $E_{\infty\beta}$  by  $E_{D\beta}$  which reduces to:

$$E_{D\beta}(\eta, \tau) = \int_{\Gamma_u} \frac{1}{\beta} (k(x)\nabla u_2 \cdot n - \eta) k(x)\nabla (u_1 - u_2) \cdot n + \int_{\Gamma_m} (k(x)\nabla u_1 \cdot n - \Phi)(T - u_2) \quad (17)$$

When the functional  $E_{D\beta}$  reaches its minimum we have  $u = u_1 = \tau$  and  $\nabla u \cdot n = \eta$ .

Similar results can be achieved by switching  $\alpha$  and  $\beta$  in the above cases. The only difference lies in the additional conditions required on  $u_2$  and the compatibility condition on the flux in the case where (4) is a Neumann problem. In conclusion, to avoid supplementary constraints on the fields  $u_2$ , the cases where  $\beta = 0$  should be avoided.

### 2.3. Single field approaches: $\alpha = \beta$

The single field approach can be set up when  $\alpha = \beta$ . Three different cases can be distinguished:

- (i) The Neumann approach is obtained for  $\alpha = \beta = 0$ . The unknown boundary data is the Neumann boundary condition  $\eta$ . We denote the functional  $E_{00}$  by  $E_N$  which can be expressed as boundary control on  $\Gamma_m$ :

$$E_N(\eta) = \int_{\Gamma_m} (k(x)\nabla u_1 \cdot n - \Phi)(T - u_2) \quad (18)$$

- (ii) The Dirichlet approach is obtained for  $\alpha = \beta = \infty$ . The unknown boundary data is the Dirichlet boundary condition  $\tau$ . We denote the functional  $E_{\infty\infty}$  by  $E_D$  which can be expressed as boundary control on  $\Gamma_m$ :

$$E_D(\eta) = \int_{\Gamma_m} (k(x)\nabla u_1 \cdot n - \Phi)(T - u_2) \quad (19)$$

- (iii) The Robin approach is obtained for  $0 < \alpha = \beta < \infty$ . The auxiliary unknown boundary data is the Robin boundary condition  $v = \eta + \alpha\tau$ . We denote the functional  $E_{\alpha\alpha}$  by  $E_\alpha$  which can be expressed as follows:

$$E_\alpha(v) = \int_{\Gamma_m} (k(x)\nabla u_1 \cdot n - \Phi)(T - u_2) - \int_{\Gamma_u} \frac{k(x)}{\alpha} (\nabla u_1 \cdot n - \nabla u_2 \cdot n)^2 \quad (20)$$

or equivalently:

$$E_\alpha(v) = \int_{\Gamma_m} (k(x)\nabla u_1 \cdot n - \Phi)(T - u_2) - \int_{\Gamma_u} \alpha(u_1 - u_2)^2 \quad (21)$$

Here too, we observe that, for compatible data, the following lemma is straightforward.

**Lemma 2.3.** *(Characterization of  $u_1$  and  $u_2$  fields at the minimum)*

*If  $(\Phi, T)$  is a compatible pair, there exists a pair  $(\varphi, \mathcal{T})$  solution of the Cauchy problem such that:*

$$\varphi = \arg \min_{\eta} E_N(\eta) \text{ and } E_N(\varphi) = 0$$

$$\mathcal{T} = \arg \min_{\tau} E_D(\tau) \text{ and } E_D(\mathcal{T}) = 0$$

$$(\varphi + \alpha\mathcal{T}) = \arg \min_v E_\alpha(v) \text{ and } E_\alpha(\varphi + \alpha\mathcal{T}) = 0$$

*Hence, when these functionals reach their minimum, the fields  $u_1$  and  $u_2$  verify:  $\nabla u_1 = \nabla u_2$  in  $\Omega$ , which is equivalent to:*

$$\begin{cases} u_1 = u_2 + K \text{ on } \Gamma_u \\ k(x)\nabla u_1 \cdot n = k(x)\nabla u_2 \cdot n \text{ on } \Gamma_u \end{cases} \quad (22)$$

*Where  $K$  is a constant. This constant is undetermined for the first case (i), because the second problem (5) is Neumann problem. For the second and third cases  $K = 0$ .*

To highlight the properties of the different parametrization of the constitutive law functionals, we will now derive the first optimality or Euler-Lagrange conditions as an interfacial equation on the boundary  $\Gamma_u$  where the data is unknown.

### 3. Euler-Lagrange conditions and interfacial operators.

In the literature there are methods to equalize the two fields  $u_1$  and  $u_2$  on  $\Gamma_u$  using constraints conditions known as interface conditions. These methods are issued from the domain decomposition field, see Quarteroni *et al* [21]. However, it should be noted that, here there is only one domain and a boundary while in the domain decomposition methods, the interfaces are located between subdomains. Nevertheless, in our problem,  $\Gamma_u$  plays the role of the interface.

We will prove hereafter that these pseudo interface conditions are equivalent to the first optimality condition (12) of the minimization problem (6). First different interface operators are defined on  $\Gamma_u$ , then for each case previously emphasized a proof is developed. To begin with, we consider the following mixed boundary value problems:

$$\begin{cases} \nabla \cdot k(x)\nabla w_1^o = 0 & \text{in } \Omega \\ w_1^o = 0 & \text{on } \Gamma_m \\ k(x)\nabla w_1^o \cdot n + \alpha w_1^o = v & \text{on } \Gamma_u \end{cases} \text{ and } \begin{cases} \nabla \cdot k(x)\nabla w_1^* = 0 & \text{in } \Omega \\ w_1^* = T & \text{on } \Gamma_m \\ k(x)\nabla w_1^* \cdot n + \alpha w_1^* = 0 & \text{on } \Gamma_u \end{cases} \quad (23)$$

$$\left\{ \begin{array}{ll} \nabla \cdot k(x) \nabla w_2^o = 0 & \text{in } \Omega \\ k(x) \nabla w_2^o \cdot n = 0 & \text{on } \Gamma_m \\ k(x) \nabla w_2^o \cdot n + \beta w_2^o = v & \text{on } \Gamma_u \end{array} \right. \text{ and } \left\{ \begin{array}{ll} \nabla \cdot k(x) \nabla w_2^* = 0 & \text{in } \Omega \\ k(x) \nabla w_2^* \cdot n = \Phi & \text{on } \Gamma_m \\ k(x) \nabla w_2^* \cdot n + \beta w_2^* = 0 & \text{on } \Gamma_u \end{array} \right. \quad (24)$$

Given  $v$ , for each  $i = 1, 2$ ,  $w_i^o$  will be denoted by  $\mathcal{H}_i(v)$ . We denoted also  $\mathcal{R}_1(T)$  and  $\mathcal{R}_2(\Phi)$  instead of  $w_1^*$  and  $w_2^*$ . Thanks to the linearity of the problems we can state that:

$$u_1 = \mathcal{H}_1(v) + \mathcal{R}_1(T) \text{ and } u_2 = \mathcal{H}_2(v) + \mathcal{R}_2(\Phi)$$

Now, for each  $i = 1, 2$  we define the Robin to Neumann operators  $S_i$  and the Robin to Dirichlet operators  $\tilde{S}_i$  as follows:

$$\begin{array}{ll} S_i : H^{-1/2}(\Gamma_u) & \longrightarrow H^{-1/2}(\Gamma_u) \\ v & \longmapsto k(x) \nabla (\mathcal{H}_i(v)) \cdot n \end{array} \quad (25)$$

$$\begin{array}{ll} \tilde{S}_i : H^{-1/2}(\Gamma_u) & \longrightarrow H^{1/2}(\Gamma_u) \\ v & \longmapsto \mathcal{H}_i(v) \end{array} \quad (26)$$

The operators  $(S_1, \tilde{S}_1)$  and  $(S_2, \tilde{S}_2)$  depend on the parameters  $\alpha$  and  $\beta$  respectively. In the following these operators will be applied to  $v$  fields expressed as  $\eta + \alpha\tau$  or  $\eta + \beta\tau$  to deal with the problems (4) and (5) respectively, with  $(\eta, \tau) \in H(\Gamma_u)^{1/2} \times H(\Gamma_u)^{-1/2}$ . Remark now, that for  $\alpha = 0$  and  $\beta = 0$ , the Robin boundary conditions on  $\Gamma_u$  become Neumann ones. Then, the operators  $S_1$  and  $S_2$  are the identity operator  $I$  and the operators  $\tilde{S}_1$  and  $\tilde{S}_2$  are the well-known Poincar-Steklov operators. However, for  $\alpha = \infty$  and  $\beta = \infty$  the Robin boundary conditions on  $\Gamma_u$  becomes Dirichlet ones. The operators  $S_1$  and  $S_2$  are redefined as follows and are the classical Steklov-Poincar operators:

$$\begin{array}{ll} S_1 : H^{1/2}(\Gamma_u) & \longrightarrow H^{-1/2}(\Gamma_u) \\ \tau & \longmapsto k(x) \nabla (\mathcal{H}_1(\tau)) \cdot n \end{array} \quad (27)$$

$$\begin{array}{ll} S_2 : H^{1/2}(\Gamma_u) & \longrightarrow H^{-1/2}(\Gamma_u) \\ \tau & \longmapsto k(x) \nabla (\mathcal{H}_2(\tau)) \cdot n \end{array} \quad (28)$$

while the operators  $\tilde{S}_1$  and  $\tilde{S}_2$  are the identity operator  $I$ .

**Lemma 3.1.** *If  $0 < \alpha \neq \beta < \infty$ , the interface conditions on  $\Gamma_u$  are:*

$$u_1 = u_2 + K \quad (29)$$

$$k(x) \nabla u_1 \cdot n = k(x) \nabla u_2 \cdot n \quad (30)$$

Using the operators defined above, the interface conditions can be expressed as follows:

$$\underbrace{\begin{bmatrix} \tilde{S}_2 - \tilde{S}_1 & \beta \tilde{S}_2 - \alpha \tilde{S}_1 \\ S_1 - S_2 & \alpha S_1 - \beta S_2 \end{bmatrix}}_{S_{\alpha\beta}} \begin{Bmatrix} \eta \\ \tau \end{Bmatrix} = \begin{Bmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{Bmatrix} \quad (31)$$

where

$$\mathcal{X}_1 = \mathcal{R}_1(T) - \mathcal{R}_2(\Phi) - K \quad (32)$$

$$\mathcal{X}_2 = -k(x) \nabla \mathcal{R}_1(T) \cdot n + k(x) \nabla \mathcal{R}_2(\Phi) \cdot n \quad (33)$$

The interface conditions stated in the above Lemma become for the different cases outlined in the table 1:

(i) if  $\alpha = 0$  and  $\beta = \infty$  then  $u = \tau - K$ ,  $\nabla u \cdot n = \eta$  and:

$$\underbrace{\begin{bmatrix} -\tilde{S}_1 & I \\ I & -S_2 \end{bmatrix}}_{S_{ND}} \begin{Bmatrix} \eta \\ \tau \end{Bmatrix} = \begin{Bmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{Bmatrix}$$

(ii) if  $\alpha = \beta = 0$  then  $u = u_1 = u_2 + K$ ,  $\nabla u \cdot n = \eta$  and:

$$S_N \eta = (\tilde{S}_2 - \tilde{S}_1) \eta = \mathcal{X}_1$$

(iii) if  $\alpha = \beta = \infty$  then  $K = 0$ ,  $u = u_1 = u_2 = \tau$ ,  $\nabla u \cdot n = \nabla u_1 \cdot n$  and:

$$S_D \tau = (S_1 - S_2) \tau = \mathcal{X}_2$$

(iv) if  $0 < \alpha = \beta < \infty$  then  $v = \eta + \alpha \tau$ ,  $K = 0$ ,  $u = u_1 = u_2 = \tau$ ,  $\nabla u \cdot n = \eta$  and:

$$\mathcal{S}_\alpha v = (\tilde{S}_2 - \tilde{S}_1) v = \mathcal{X}_1$$

(v) if  $\alpha = 0$  and  $0 < \beta < \infty$  then  $k(x) \nabla u \cdot n = \eta$ ,  $u = u_1 = \tau - K$  and:

$$\underbrace{\begin{bmatrix} \tilde{S}_2 - \tilde{S}_1 & \beta \tilde{S}_2 \\ I - S_2 & -\beta S_2 \end{bmatrix}}_{S_{N\beta}} \begin{Bmatrix} \eta \\ \tau \end{Bmatrix} = \begin{Bmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{Bmatrix}$$

(vi) if  $\alpha = \infty$  and  $0 < \beta < \infty$  then  $K = 0$ ,  $k(x) \nabla u \cdot n = \eta$ ,  $u = u_1 = u_2 = \tau$  and:

$$\underbrace{\begin{bmatrix} \tilde{S}_2 - I & \beta \tilde{S}_2 \\ S_1 - S_2 & -\beta S_2 \end{bmatrix}}_{S_{D\beta}} \begin{Bmatrix} \eta \\ \tau \end{Bmatrix} = \begin{Bmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{Bmatrix}$$

**Remark.** Notice that the operator  $\mathcal{S}_{\alpha\beta}$  is symmetric.

### 3.1. Case $0 < \alpha \neq \beta < +\infty$

In this section we show that the optimality conditions stated in the Lemma 2.2 are equivalent to the interface conditions stated in the Lemma 3.1. We consider the functional defined by (6) and the fields  $\mathcal{H}_1$  and  $\mathcal{H}_2$  which are the solutions of the two first well posed problems defined by (23) and (24). These fields depend linearly on  $\eta$  and  $\tau$ .

**Lemma 3.2.** The first optimality condition of the functional  $E_{\alpha\beta}$  reads:

$$(\tilde{S}_2 - \tilde{S}_1) \eta + (\beta \tilde{S}_2 - \alpha \tilde{S}_1) \tau = \mathcal{X}_1 \quad (34)$$

$$(S_1 - S_2) \eta + (\alpha S_1 - \beta S_2) \tau = \mathcal{X}_2 \quad (35)$$

*Proof.* Let us recall that  $u_1$  and  $u_2$  depend linearly on the variable  $\eta$  and  $\tau$ , and  $v = \eta + \alpha\tau$  for  $\mathcal{H}_1$  and  $v = \eta + \beta\tau$  for  $\mathcal{H}_2$ , then:

$$\begin{aligned}\frac{\partial E_{\alpha\beta}(\eta, \tau)}{\partial \eta} \cdot \delta\eta &= \int_{\Omega} k(x) (\nabla u_1(\eta, \tau) - \nabla u_2(\eta, \tau)) (\nabla \mathcal{H}_1(\delta\eta) - \nabla \mathcal{H}_2(\delta\eta)) \\ \frac{\partial E_{\alpha\beta}(\eta, \tau)}{\partial \tau} \cdot \delta\tau &= \int_{\Omega} k(x) (\nabla u_1(\eta, \tau) - \nabla u_2(\eta, \tau)) (\alpha \nabla \mathcal{H}_1(\delta\tau) - \beta \nabla \mathcal{H}_2(\delta\tau))\end{aligned}$$

Which can be written as follows:

$$\begin{aligned}\frac{\partial E_{\alpha\beta}(\eta, \tau)}{\partial \eta} \cdot \delta\eta &= \int_{\partial\Omega} k(x) (\nabla u_1(\eta, \tau) - \nabla u_2(\eta, \tau)) \cdot n \mathcal{H}_1(\delta\eta) - \nabla \mathcal{H}_2(\delta\eta) \cdot n (u_1(\eta, \tau) - u_2(\eta, \tau)) \\ \frac{\partial E_{\alpha\beta}(\eta, \tau)}{\partial \tau} \cdot \delta\tau &= \int_{\partial\Omega} \alpha k(x) (\nabla u_1(\eta, \tau) - \nabla u_2(\eta, \tau)) \cdot n \mathcal{H}_1(\delta\tau) - \beta \nabla \mathcal{H}_2(\delta\tau) \cdot n (u_1(\eta, \tau) - u_2(\eta, \tau))\end{aligned}$$

using the properties of the fields  $\mathcal{H}_1$  and  $\mathcal{H}_2$  we obtain:

$$\begin{aligned}\frac{\partial E_{\alpha\beta}(\eta, \tau)}{\partial \eta} \cdot \delta\eta &= \int_{\Gamma_u} k(x) \nabla (u_1(\eta, \tau) - u_2(\eta, \tau)) \cdot n \mathcal{H}_1(\delta\eta) - \nabla \mathcal{H}_2(\delta\eta) \cdot n (u_1(\eta, \tau) - u_2(\eta, \tau)) \\ \frac{\partial E_{\alpha\beta}(\eta, \tau)}{\partial \tau} \cdot \delta\tau &= \int_{\Gamma_u} \alpha k(x) \nabla (u_1(\eta, \tau) - u_2(\eta, \tau)) \cdot n \mathcal{H}_1(\delta\tau) - \beta \nabla \mathcal{H}_2(\delta\tau) \cdot n (u_1(\eta, \tau) - u_2(\eta, \tau))\end{aligned}$$

The stationarity condition leads to:

$$\left\{ \begin{array}{l} \forall \delta\eta \in H^{-1/2} \\ \int_{\Gamma_u} k(x) (\nabla u_1(\eta, \tau) - \nabla u_2(\eta, \tau)) \cdot n \mathcal{H}_1(\delta\eta) = 0 \\ \int_{\Gamma_u} (u_1(\eta, \tau) - u_2(\eta, \tau)) k(x) \nabla \mathcal{H}_2(\delta\eta) \cdot n = 0 \\ \forall \delta\tau \in H^{1/2} \\ \int_{\Gamma_u} \alpha k(x) (\nabla u_1(\eta, \tau) - \nabla u_2(\eta, \tau)) \cdot n \mathcal{H}_1(\delta\tau) = 0 \\ \int_{\Gamma_u} \beta (u_1(\eta, \tau) - u_2(\eta, \tau)) k(x) \nabla \mathcal{H}_2(\delta\tau) \cdot n = 0 \end{array} \right. \quad (36)$$

Here we introduce the following Lemma:

**Lemma 3.3.** *Consider two functions:  $(\psi, \xi) \in H^{1/2}(\Gamma_u) \times H^{-1/2}(\Gamma_u)$  with  $\int_{\Gamma_u} \xi = 0$ , there exists a pair  $(\varpi_\psi, \varpi_\xi) \in H^{-1/2}(\Gamma_u) \times H^{-1/2}(\Gamma_u)$ .*

$$\left\{ \begin{array}{ll} \nabla \cdot k(x) \nabla w_1 = 0 & \text{in } \Omega \\ w_1 = 0 & \text{on } \Gamma_m \\ k(x) \nabla w_1 \cdot n + \alpha w_1 = \varpi_\psi & \text{on } \Gamma_u \end{array} \right. \text{ and } \left\{ \begin{array}{ll} \nabla \cdot k(x) \nabla w_2 = 0 & \text{in } \Omega \\ k(x) \nabla w_2 \cdot n = 0 & \text{on } \Gamma_m \\ k(x) \nabla w_2 \cdot n + \beta w_2 = \varpi_\xi & \text{on } \Gamma_u \end{array} \right. \quad (37)$$

With  $w_1 = \psi$  and  $k(x) \nabla w_2 \cdot n = \xi$ .

The proof of this Lemma is detailed in the annexe. Using the above Lemma, we denote by  $\psi$  and  $\xi$  respectively  $\mathcal{H}_1(\delta\eta)$  and  $\nabla \mathcal{H}_2(\delta\tau) \cdot n$ , then we can state that:

$$\int_{\Gamma_u} k(x) (\nabla u_1(\eta, \tau) - \nabla u_2(\eta, \tau)) \cdot n \psi = 0 \quad \forall \psi \in H^{1/2}(\Gamma_u) \quad (38)$$

$$\int_{\Gamma_u} (u_1(\eta, \tau) - u_2(\eta, \tau)) \xi = 0 \quad \forall \xi \in H^{-1/2}(\Gamma_u) \quad (39)$$

This leads to:

$$u_1(\eta, \tau) = u_2(\eta, \tau) + K \quad \text{and} \quad \nabla k(x) u_1(\eta, \tau) \cdot n = \nabla k(x) u_2(\eta, \tau) \cdot n$$

Then, using the operators defined above, we obtain:

$$(\tilde{S}_2 - \tilde{S}_1)\eta + (\beta\tilde{S}_2 - \alpha\tilde{S}_1)\tau = \mathcal{R}_1(T) - \mathcal{R}_2(\Phi) - K$$

and

$$(S_1 - S_2)\eta + (\alpha S_1 - \beta S_2)\tau = -k(x)\nabla\mathcal{R}_1(T).n + k(x)\nabla\mathcal{R}_2(\Phi).n$$

□

The first optimality conditions are exactly the interface condition stated in Lemma 3.1.

$$\begin{bmatrix} \tilde{S}_2 - \tilde{S}_1 & \beta\tilde{S}_2 - \alpha\tilde{S}_1 \\ S_1 - S_2 & \alpha S_1 - \beta S_2 \end{bmatrix} \begin{Bmatrix} \eta \\ \tau \end{Bmatrix} = \begin{Bmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{Bmatrix} \quad (40)$$

3.2. Case:  $\alpha = 0$  and  $\beta = +\infty$

In this case,  $\eta$  and  $\tau$  denote the unknown values of  $k(x)\nabla u.n$  and the trace of  $u$  on  $\Gamma_u$ . The mixed boundary value problems defined by (23) and (24) become:

$$\begin{cases} \nabla.k(x)\nabla w_1^o = 0 & \text{in } \Omega \\ w_1^o = 0 & \text{on } \Gamma_m \\ k(x)\nabla w_1^o.n = \eta & \text{on } \Gamma_u \end{cases} \quad \text{and} \quad \begin{cases} \nabla.k(x)\nabla w_1^* = 0 & \text{in } \Omega \\ w_1^* = T & \text{on } \Gamma_m \\ k(x)\nabla w_1^*.n = 0 & \text{on } \Gamma_u \end{cases} \quad (41)$$

$$\begin{cases} \nabla.k(x)\nabla w_2^o = 0 & \text{in } \Omega \\ k(x)\nabla w_2^o.n = 0 & \text{on } \Gamma_m \\ w_2^o = \tau & \text{on } \Gamma_u \end{cases} \quad \text{and} \quad \begin{cases} \nabla.k(x)\nabla w_2^* = 0 & \text{in } \Omega \\ k(x)\nabla w_2^*.n = \Phi & \text{on } \Gamma_m \\ w_2^* = 0 & \text{on } \Gamma_u \end{cases} \quad (42)$$

The Cauchy problem (1) is solved, if and only if  $u_1 = u_2 + K$  and  $\nabla u_1(\eta).n = \nabla u_2(\tau).n$  on  $\Gamma_u$  which are the interface conditions defined in (12).

**Lemma 3.4.** *The first optimality condition of the functional  $E_{ND}$  reads:*

$$\begin{bmatrix} -\tilde{S}_1 & I \\ I & -S_2 \end{bmatrix} \begin{Bmatrix} \eta \\ \tau \end{Bmatrix} = \begin{Bmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{Bmatrix}$$

*Proof.* The proof follows the same steps as that of Lemma 3.5 and 3.6. In this case the function depends on the pair  $(\eta, \tau)$ :

$$E_{ND}(\eta, \tau) = \frac{1}{2} \int_{\Omega} k(x) (\nabla u_1(\eta) - \nabla u_2(\tau)) (\nabla u_1(\eta) - \nabla u_2(\tau))$$

Here, two optimality conditions have to be satisfied:

$$\begin{aligned} \frac{\partial E_{ND}(\eta, \tau)}{\partial \eta} . \delta \eta &= \int_{\Omega} k(x) (\nabla u_1(\eta) - \nabla u_2(\tau)) \nabla \mathcal{H}_1(\delta \eta) \\ &= \int_{\Gamma_u} k(x) (\nabla u_1(\eta) - \nabla u_2(\tau)) .n . \mathcal{H}_1(\delta \eta) \text{ because } \mathcal{H}_1(\delta \eta) = 0 \text{ on } \Gamma_m \end{aligned}$$

$$\begin{aligned}
 \frac{\partial E_{ND}(\eta, \tau)}{\partial \eta} \cdot \delta \eta = 0 \quad \forall \delta \eta &\implies \int_{\Gamma_u} k(x) (\nabla u_1(\eta) - \nabla u_2(\tau)) \cdot n = 0 \\
 &\implies \nabla u_1(\eta) \cdot n = \nabla u_2(\tau) \cdot n \\
 &\implies I\eta - S_2\tau = -k(x)\nabla\mathcal{R}_1(T) \cdot n + k(x)\nabla\mathcal{R}_2(\Phi) \cdot n
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial E_{ND}(\eta, \tau)}{\partial \tau} \cdot \delta \tau &= - \int_{\Omega} k(x) (\nabla u_1(\eta) - \nabla u_2(\tau)) \nabla \mathcal{H}_2(\delta \tau) \\
 &= - \int_{\Gamma_u} (u_1(\eta) - u_2(\tau)) \cdot k(x) \nabla \mathcal{H}_2(\delta \tau) \cdot n \text{ because } \nabla \mathcal{H}_2(\delta \tau) \cdot n = 0 \text{ on } \Gamma_m
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial E_{ND}(\eta, \tau)}{\partial \tau} \cdot \delta \tau = 0 \quad \forall \delta \tau &\implies u_1(\eta) - u_2(\tau) = K \text{ because } \int_{\Gamma_u} k(x) \nabla \mathcal{H}_2(\delta \tau) \cdot n = 0 \\
 &\implies (\tilde{S}_1\eta - I\tau) = -\mathcal{R}_1(T) + \mathcal{R}_2(\Phi) + K
 \end{aligned}$$

□

Therefore the optimality condition is equivalent to the interfacial equation defined in the case 1 of the Lemma 3.1:

$$\begin{bmatrix} -\tilde{S}_1 & I \\ I & -S_2 \end{bmatrix} \begin{Bmatrix} \eta \\ \tau \end{Bmatrix} = \begin{Bmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{Bmatrix}$$

### 3.3. The case $\alpha = \beta = +\infty$

This case correspond to the so-called Cauchy-Poincar-Steklov method, see Quarteroni *et al* [21] and Ben Belgacem *et al* [9]. The unknown boundary condition on  $\Gamma_u$  is the Dirichlet one denoted by  $\tau$ . The mixed boundary value problems defined by (23) and (24) become:

$$\begin{cases} \nabla \cdot k(x) \nabla w_1^o = 0 & \text{in } \Omega \\ w_1^o = 0 & \text{on } \Gamma_m \\ w_1^o = \tau & \text{on } \Gamma_u \end{cases} \text{ and } \begin{cases} \nabla \cdot k(x) \nabla w_1^* = 0 & \text{in } \Omega, \\ w_1^* = T & \text{on } \Gamma_m, \\ w_1^* = 0 & \text{on } \Gamma_u \end{cases} \quad (43)$$

$$\begin{cases} \nabla \cdot k(x) \nabla w_2^o = 0 & \text{in } \Omega \\ k(x) \nabla w_2^o \cdot n = 0 & \text{on } \Gamma_m \\ w_2^o = \tau & \text{on } \Gamma_u \end{cases} \text{ and } \begin{cases} \nabla \cdot k(x) \nabla w_2^* = 0 & \text{in } \Omega, \\ k(x) \nabla w_2^* \cdot n = \Phi & \text{on } \Gamma_m, \\ w_2^* = 0 & \text{on } \Gamma_u \end{cases} \quad (44)$$

Then  $\mathcal{H}_i$  for  $i = 1, 2$  are function of  $\tau$  only. The Cauchy problem (1) is solved, if and only if  $\nabla u_1 \cdot n = \nabla u_2 \cdot n$  on  $\Gamma_u$  which is the second optimality condition defined in 2.2, the first one is satisfied by the definition of  $\tau$ .

**Lemma 3.5.** *The first optimality condition of the functional  $E_D$  reads:*

$$S(\tau) = (S_1 - S_2)\tau = \mathcal{X}_2 = -k(x)\nabla\mathcal{R}_1(T) \cdot n + k(x)\nabla\mathcal{R}_2(\Phi) \cdot n$$

*Proof.* Let us recall that  $E_D$  depends only on the variable  $\tau$ :

$$E_D(\tau) = \frac{1}{2} \int_{\Omega} k(x) (\nabla u_1(\tau) - \nabla u_2(\tau)) (\nabla u_1(\tau) - \nabla u_2(\tau))$$

$$\begin{aligned} \frac{\partial E_D(\tau)}{\partial \tau} \cdot \delta \tau &= \int_{\Omega} (\nabla u_1(\tau) - \nabla u_2(\tau)) (\nabla \mathcal{H}_1(\delta \tau) - \nabla \mathcal{H}_2(\delta \tau)) \\ &= \int_{\Omega} (\nabla u_1(\tau) - \nabla u_2(\tau)) \cdot \nabla \mathcal{H}_1(\delta \tau) - \int_{\Omega} (\nabla u_1(\tau) - \nabla u_2(\tau)) \cdot \nabla \mathcal{H}_2(\delta \tau) \\ &= \int_{\partial \Omega} (\nabla u_1(\tau) - \nabla u_2(\tau)) \cdot n \cdot \mathcal{H}_1(\delta \tau) - \int_{\partial \Omega} (u_1(\tau) - u_2(\tau)) \cdot \nabla \mathcal{H}_2(\delta \tau) \cdot n \end{aligned} \quad (45)$$

one has:

$$\begin{aligned} \int_{\Gamma_u} (u_1(\tau) - u_2(\tau)) \cdot \nabla \mathcal{H}_2(\delta \tau) \cdot n &= 0 \text{ because } u_1 - u_2 = 0 \\ \int_{\Gamma_m} (u_1(\tau) - u_2(\tau)) \cdot \nabla \mathcal{H}_2(\delta \tau) \cdot n &= 0 \text{ because } \nabla \mathcal{H}_2(\delta \tau) \cdot n = 0 \\ \int_{\Gamma_m} (\nabla u_1(\tau) - \nabla u_2(\tau)) \cdot n \cdot \mathcal{H}_1(\delta \tau) &= 0 \text{ because } \mathcal{H}_1(\delta \tau) = 0 \end{aligned}$$

Then:

$$\begin{aligned} \frac{\partial E_D(\tau)}{\partial \tau} \cdot \delta \tau &= \int_{\Gamma_u} (\nabla u_1(\tau) - \nabla u_2(\tau)) \cdot n \cdot \mathcal{H}_1(\delta \tau) \\ \frac{\partial E_D(\tau)}{\partial \tau} \cdot \delta \tau = 0 \quad \forall \delta \tau &\implies (\nabla u_1(\tau) - \nabla u_2(\tau)) \cdot n = 0 \\ &\implies \nabla u_1 \cdot n = \nabla u_2 \cdot n \\ &\implies (S_1 - S_2)\tau = \mathcal{X}_2 = -k(x)\nabla \mathcal{R}_1(T) \cdot n + k(x)\nabla \mathcal{R}_2(\Phi) \cdot n \end{aligned}$$

□

Therefore the optimality condition is equivalent to the interfacial equation defined in the case 3 of the Lemma 3.1:

$$(S_1 - S_2)\tau = \mathcal{X}_2.$$

### 3.4. The case $\alpha = \beta = 0$

This case corresponds to the so-called Neumann to Neumann Steklov-Poincaré method. The unknown boundary condition on  $\Gamma_u$  is the Neumann one denoted by  $\eta$ . The mixed boundary value problems defined by (23) and (24) become:

$$\left\{ \begin{array}{lll} \nabla \cdot k(x)\nabla w_1^o = 0 & \text{in } \Omega \\ w_1^o = 0 & \text{on } \Gamma_m \\ k(x)\nabla w_1^o \cdot n = \bar{\eta} & \text{on } \Gamma_u \end{array} \right. \text{ and } \left\{ \begin{array}{lll} \nabla \cdot k(x)\nabla w_1^* = 0 & \text{in } \Omega \\ w_1^* = T & \text{on } \Gamma_m \\ k(x)\nabla w_1^* \cdot n = -\frac{1}{|\Gamma_u|} \int_{\Gamma_m} \Phi & \text{on } \Gamma_u \end{array} \right. \quad (46)$$

$$\left\{ \begin{array}{ll} \nabla.k(x)\nabla w_2^o = 0 & \text{in } \Omega \\ k(x)\nabla w_2^o.n = 0 & \text{on } \Gamma_m \\ k(x)\nabla w_2^o.n = \bar{\eta} & \text{on } \Gamma_u \end{array} \right. \text{ and } \left\{ \begin{array}{ll} \nabla.k(x)\nabla w_2^* = 0 & \text{in } \Omega, \\ k(x)\nabla w_2^*.n = \Phi & \text{on } \Gamma_m \\ k(x)\nabla w_2^*.n = -\frac{1}{|\Gamma_u|} \int_{\Gamma_m} \Phi & \text{on } \Gamma_u \end{array} \right. \quad (47)$$

with

$$\bar{\eta} = \eta - \frac{1}{|\Gamma_u|} \int_{\Gamma_u} \eta$$

Here the fields  $\mathcal{H}_i$  for  $i = 1, 2$  are function of  $\bar{\eta}$  only. The following supplementary condition is necessary in this case:

$$\int_{\Gamma_m} \Phi + \int_{\Gamma_u} \eta = 0$$

The Cauchy problem (1) is solved, if and only if  $u_1 = u_2 + K$  on  $\Gamma_u$  which is the first optimality condition defined in 2.2, the second one is satisfied by the definition of  $\eta$ .

**Lemma 3.6.** *The first optimality condition of the functional  $E_N$  reads:*

$$S\eta = (\tilde{S}_2 - \tilde{S}_1)\eta = \mathcal{X}_1 = \mathcal{R}_1(T) - \mathcal{R}_2(\Phi) - K$$

*Proof.* In this case the function depends only on the variable  $\eta$

$$E_N(\eta) = \frac{1}{2} \int_{\Omega} k(x) (\nabla u_1(\eta) - \nabla u_2(\eta)) (\nabla u_1(\eta) - \nabla u_2(\eta))$$

The optimality condition is then:

$$\begin{aligned} \frac{\partial E_N(\eta)}{\partial \eta} . \delta \eta &= \int_{\Omega} k(x) (\nabla u_1(\eta) - \nabla u_2(\eta)) (\nabla \mathcal{H}_1(\delta \eta) - \nabla \mathcal{H}_2(\delta \eta)) \\ &= \int_{\Omega} k(x) (\nabla u_1(\eta) - \nabla u_2(\eta)) . \nabla \mathcal{H}_1(\delta \eta) - \int_{\Omega} k(x) (\nabla u_1(\eta) - \nabla u_2(\eta)) . \nabla \mathcal{H}_2(\delta \eta) \\ &= \int_{\partial \Omega} k(x) (\nabla u_1(\eta) - \nabla u_2(\eta)) . n . \mathcal{H}_1(\delta \eta) - \int_{\partial \Omega} (u_1(\eta) - u_2(\eta)) . k(x) \nabla \mathcal{H}_2(\delta \eta) . n \end{aligned} \quad (48)$$

Since  $\mathcal{H}_1(\delta \eta) = 0$  on  $\Gamma_m$ ,  $\nabla u_1(\eta) . n = \nabla u_2(\eta) . n$  on  $\Gamma_u$  and  $k(x) \nabla \mathcal{H}_2(\delta \eta) . n = 0$  on  $\Gamma_m$ :

$$\frac{\partial E_N(\eta)}{\partial \eta} . \delta \eta = - \int_{\Gamma_u} (u_1(\eta) - u_2(\eta)) k(x) \nabla \mathcal{H}_2(\delta \eta) . n = - \int_{\Gamma_u} (u_1(\eta) - u_2(\eta)) . \delta \bar{\eta}$$

$$\frac{\partial E_N(\eta)}{\partial \eta} . \delta \eta = 0 \forall \delta \bar{\eta} \implies \int_{\Gamma_u} (u_1(\eta) - u_2(\eta)) . \delta \bar{\eta} = 0 \quad (49)$$

$$\implies u_1(\eta) - u_2(\eta) = K \text{ because } \int_{\Gamma_u} \delta \bar{\eta} = 0$$

$$\implies (\tilde{S}_2 - \tilde{S}_1)\eta = \mathcal{X}_1 = \mathcal{R}_1(T) - \mathcal{R}_2(\Phi) - K \quad (50)$$

□

Therefore the optimality condition is equivalent to the interfacial equation defined in the case 2 of the Lemma 3.1:

$$(\tilde{S}_2 - \tilde{S}_1)\eta = \mathcal{X}_1$$

Remark that the classical Neumann to Neumann preconditioner is recovered.

3.5. The case:  $0 < \alpha = \beta < \infty$ 

This case correspond to the Dirichlet to Robin operator used in the domain decomposition field. Let  $v$  be the unknown data on  $\Gamma_u$ . The mixed boundary value problems defined by (23) and (24) become:

$$\left\{ \begin{array}{ll} \nabla.k(x)\nabla w_1^o = 0 & \text{in } \Omega \\ w_1^o = 0 & \text{on } \Gamma_m \\ k(x)\nabla w_1^o.n + \alpha w_1^o = v & \text{on } \Gamma_u \end{array} \right. \text{ and } \left\{ \begin{array}{ll} \nabla.k(x)\nabla w_1^* = 0 & \text{in } \Omega \\ w_1^* = T & \text{on } \Gamma_m \\ k(x)\nabla w_1^*.n + \alpha w_1^* = 0 & \text{on } \Gamma_u \end{array} \right.$$

$$\left\{ \begin{array}{ll} \nabla.k(x)\nabla w_2^o = 0 & \text{in } \Omega \\ k(x)\nabla w_2^o.n = 0 & \text{on } \Gamma_m \\ k(x)\nabla w_2^o.n + \alpha w_2^o = v & \text{on } \Gamma_u \end{array} \right. \text{ and } \left\{ \begin{array}{ll} \nabla.k(x)\nabla w_2^* = 0 & \text{in } \Omega \\ k(x)\nabla w_2^*.n = \Phi & \text{on } \Gamma_m \\ k(x)\nabla w_2^*.n + \alpha w_2^* = 0 & \text{on } \Gamma_u \end{array} \right.$$

The Cauchy problem (1) is solved, if and only if  $u_1 = u_2 + K$  on  $\Gamma_u$  which is the first optimality condition defined in 2.2, the second one is satisfied by the definition of  $v$ . Here  $K = 0$ .

**Lemma 3.7.** *The first optimality condition of the functional  $E_\alpha$  reads::*

$$\left( \tilde{S}_2 - \tilde{S}_1 \right) v = \mathcal{X}_1$$

*Proof.* The proof follows the same steps as that of above Lemma. In this case the function depends on  $v = \eta + \alpha\tau$ :

$$E_\alpha(v) = \frac{1}{2} \int_{\Omega} k(x) (\nabla u_1(v) - \nabla u_2(v)) (\nabla u_1(v) - \nabla u_2(v))$$

$$\begin{aligned} \frac{\partial E_\alpha(v)}{\partial v} . \delta v &= \int_{\Omega} k(x) (\nabla u_1(v) - \nabla u_2(v)) (\nabla \mathcal{H}_1(\delta v) - \nabla \mathcal{H}_2(\delta v)) \\ &= \int_{\partial\Omega} k(x) (\nabla u_1(v) - \nabla u_2(v)) .n . \mathcal{H}_1(\delta v) - (u_1(v) - u_2(v)) k(x) \nabla \mathcal{H}_2(\delta v) .n \end{aligned}$$

Since  $\mathcal{H}_1(\delta v)$  and  $\nabla \mathcal{H}_2(\delta v) .n = 0$  on  $\Gamma_m$  then:

$$\int_{\Gamma_m} k(x) (\nabla u_1(v) - \nabla u_2(v)) .n . \mathcal{H}_1(\delta v) - (u_1(v) - u_2(v)) \nabla \mathcal{H}_2(\delta v) .n = 0$$

Then:

$$\frac{\partial E_\alpha(v)}{\partial v} . \delta v = \int_{\Gamma_u} k(x) (\nabla u_1(v) - \nabla u_2(v)) .n . \mathcal{H}_1(\delta v) - (u_1(v) - u_2(v)) k(x) \nabla \mathcal{H}_2(\delta v) .n$$

Using the Robin boundary condition:

$$k(x) (\nabla u_1(v) - \nabla u_2(v)) .n = -\alpha(u_1(v) - u_2(v))$$

the above optimality condition becomes:

$$\frac{\partial E_\alpha(v)}{\partial v} . \delta v = - \int_{\Gamma_u} (u_1(v) - u_2(v)) (\alpha \mathcal{H}_1(\delta v) + \nabla \mathcal{H}_2(\delta v) .n) \forall \delta v$$

Using the Lemma 3.3 for  $\alpha = \beta$ , we denote by  $\psi$  and  $\xi$  respectively  $\mathcal{H}_1(\delta v)$  and  $\nabla \mathcal{H}_2(\delta v).n$ , then we can state that:

$$\begin{aligned} \frac{\partial E_\alpha(v)}{\partial v} . \delta v = 0 \quad \forall \delta v &\implies u_1(v) - u_2(v) = 0 \text{ on } \Gamma_u \quad \forall (\psi, \xi) \in H^{-1/2}(\Gamma_u) \times H^{1/2}(\Gamma_u) \\ &\implies \left( \tilde{S}_2 - \tilde{S}_1 \right) v = \mathcal{R}_1(T) - \mathcal{R}_2(\Phi) \end{aligned}$$

□

Therefore the optimality condition is equivalent to the interfacial equation defined in the case 4 of the Lemma 3.1:

$$\left( \tilde{S}_2 - \tilde{S}_1 \right) v = \mathcal{X}_1$$

Remark that the case  $\alpha = \beta = 0$  is found by setting in the above optimality condition  $\alpha = 0$ .

### 3.6. Alternating Direction Iterative method

We consider the Alternating Direction Iterative (ADI) method, which generates two sequences of traces  $v_1^k$  and  $v_2^k$  build with the traces of  $u_{1|\Gamma_u}^k$  and  $u_{2|\Gamma_u}^k$  respectively.

Consider an initial guess  $v_2^0$ ; then, for  $k \geq 0$  we look for  $u_1^{k+1}$  and then  $u_2^{k+1}$  such that:

$$\begin{cases} \nabla . k(x) \nabla u_1^{k+1} = 0 & \text{in } \Omega \\ u_1^{k+1} = T & \text{on } \Gamma_m \\ k(x) \nabla u_1^{k+1} . n + \alpha u_1^{k+1} = v_1^{k+1} & \text{on } \Gamma_u \end{cases} \quad (51)$$

with  $v_1^{k+1} = k(x) \nabla u_2^k . n + \alpha u_2^k$ .

$$\begin{cases} \nabla . k(x) \nabla u_2^{k+1} = 0 & \text{in } \Omega \\ k(x) \nabla u_2^{k+1} . n = \Phi & \text{on } \Gamma_m \\ k(x) \nabla u_2^{k+1} . n + \beta u_2^{k+1} = v_2^{k+1} & \text{on } \Gamma_u \end{cases} \quad (52)$$

with  $v_2^{k+1} = k(x) \nabla u_1^{k+1} . n + \beta u_1^{k+1}$ .

Using the operators  $S_1, \tilde{S}_1, S_2$  and  $\tilde{S}_2$  defined above, it is easy to show that it is a fixed-point iteration:  $v_2^{k+1} = (S_1 + \beta \tilde{S}_1) v_1^{k+1}$ , then using the expression of  $v_1^{k+1}$  we obtain  $v_2^{k+1} = (S_1 + \beta \tilde{S}_1)(S_2 + \alpha \tilde{S}_2) v_2^k$ . Then:

$$v_2^{k+1} = \hat{S}_{\alpha\beta} v_2^k, \quad k \geq 0$$

where the fixed point map is given as follows:

$$\begin{aligned} \hat{S}_{\alpha\beta} : H^{1/2}(\Gamma_u) &\longrightarrow H^{1/2}(\Gamma_u) \\ v_2^k &\longmapsto \hat{S}_{\alpha\beta} v_2^k = v_2^{k+1} \end{aligned}$$

with:  $\hat{S}_{\alpha\beta} = (S_1 + \beta \tilde{S}_1)(S_2 + \alpha \tilde{S}_2)$ . The KMF iterative method outlined above happens when  $\alpha = 0$  and  $\beta = \infty$  and then  $\hat{S}_{ND} = \tilde{S}_1 S_2$ . We can then conclude that the KMF method can be interpreted as fixed point resolution of an interface problem  $v = \hat{S}_{\alpha\beta} v$ .

In this section, we showed that all the above methods are equivalent, from the continuous point of view. However, we expect that their numerical behavior will be different, which will be addressed in the next sections.

#### 4. Hadamard example

As pointed out previously, at the discrete level the interfacial operators outlined in section 3 are expected to be differently conditioned. The aim of this section is to give an analytical *taste* to what goes on for the condition numbers.

Let us consider an annular domain with an outer  $r_m = 1$  and an inner  $r_u \equiv r < 1$  radii,  $k(x) = 1$  and the polar coordinates system. The overspecified data are available on the external boundary  $\Gamma_m$ , whereas the lacking data are on the inner boundary  $\Gamma_u$ . The analytical solution of the problems (4) and (5) take the general form of separate variables functions:

$$\mathcal{H}_1(v) = \sum_{n=1}^{\infty} (r^n - r^{-n}) g_1(n\theta) \text{ and } \mathcal{H}_2(v) = \sum_{n=1}^{\infty} (r^n + r^{-n}) g_2(n\theta)$$

respectively, which will be used to calculate the eigenvalues. The eigenvalues of  $S_1^n$ ,  $S_2^n$ ,  $\tilde{S}_1^n$  and  $\tilde{S}_2^n$  are then given by the following sequence:

$$\lambda_1^n = \frac{n(r^{2n} + 1)}{n(r^{2n} + 1) - \alpha r(r^{2n} - 1)} \quad (53)$$

$$\lambda_2^n = \frac{n(r^{2n} - 1)}{n(r^{2n} - 1) - \beta r(r^{2n} + 1)} \quad (54)$$

$$\tilde{\lambda}_1^n = \frac{-r(r^{2n} - 1)}{n(r^{2n} + 1) - \alpha r(r^{2n} - 1)} \quad (55)$$

$$\tilde{\lambda}_2^n = \frac{-r(r^{2n} + 1)}{n(r^{2n} - 1) - \beta r(r^{2n} + 1)} \quad (56)$$

corresponding to the eigenvectors  $g_1^n = \cos(n\theta)$  and  $g_2^n = \sin(n\theta)$ , respectively. The interfacial operator  $\mathcal{S}_{\alpha\beta}^n$  defined in the Lemma 3.1 can be expressed as follows for the  $n$ -rank:

$$\mathcal{S}_{\alpha\beta}^n = \begin{bmatrix} \tilde{\lambda}_2^n - \tilde{\lambda}_1^n & \beta \tilde{\lambda}_2^n - \alpha \tilde{\lambda}_1^n \\ \lambda_1^n - \lambda_2^n & \alpha \lambda_1^n - \beta \lambda_2^n \end{bmatrix} \quad (57)$$

For the special values of parameters  $\alpha$  and  $\beta$  the above operator becomes:

$$\mathcal{S}_{ND}^n = \begin{bmatrix} \frac{r}{n} \left( \frac{r^{2n}-1}{r^{2n}+1} \right) & 1 \\ 1 & \frac{n}{r} \left( \frac{r^{2n}-1}{r^{2n}+1} \right) \end{bmatrix} \quad (58)$$

$$\mathcal{S}_{N\beta}^n = \begin{bmatrix} \frac{r}{n} \frac{4nr^{2n} + \beta r^{4n+1} - \beta r}{(-nr^{4n} + \beta r^{4n+1} + 2\beta r^{2n+1} + n + \beta r)} & \frac{\beta(r^{2n+1} + r)}{-r^{2n}n + n + \beta r^{2n+1} + \beta r} \\ \frac{\beta(r^{2n+1} + r)}{-r^{2n}n + n + \beta r^{2n+1} + \beta r} & \frac{\beta n(r^{2n} - 1)}{-r^{2n}n + n + \beta r^{2n+1} + \beta r} \end{bmatrix} \quad (59)$$

$$\mathcal{S}_{D\beta}^n = \begin{bmatrix} \frac{r^{2n+1} + r}{-r^{2n}n + n + \beta r^{2n+1} + \beta r} & \frac{n(r^{2n} - 1)}{-r^{2n}n + n + \beta r^{2n+1} + \beta r} \\ \frac{n(r^{2n} - 1)}{-r^{2n}n + n + \beta r^{2n+1} + \beta r} & \frac{n}{r} \frac{(-r^{4n}n + 4\beta r^{2n+1} + n)}{(r^{4n}n - 2r^{2n}n - r^{4n+1}\beta + n + \beta r)} \end{bmatrix} \quad (60)$$

$$\mathcal{S}_D^n = -\frac{n}{r} \left( \frac{4r^{2n}}{r^{4n} - 1} \right) \quad (61)$$

$$\mathcal{S}_N^n = -\frac{r}{n} \left( \frac{4r^{2n}}{r^{4n} - 1} \right) \quad (62)$$

$$\mathcal{S}_\alpha^n = 4 \frac{r^{2n+1}n}{-n^2r^{4n} + 2r^{4n+1}\alpha n + n^2 + 2n\alpha r - \alpha^2r^{4n+2} + \alpha^2r^2} \quad (63)$$

Consider the one field operators:  $S_N^n$ ,  $S_D^n$  and  $S_\alpha^n$ , their asymptotic development when  $n \rightarrow \infty$  shows that  $S_N^n \approx -\frac{4}{n}r^{2n+1}$ ,  $S_D^n \approx -4nr^{2n-1}$  and  $S_\alpha^n \approx \frac{4}{n}r^{2n+1}$ . Then, one can deduces that  $S_D^n$  and  $S_\alpha^n$ , which have the same behavior, decrease faster than  $S_N^n$ . The operators which depend on two unknown fields, tend toward the following expressions when  $n \rightarrow \infty$ :

$$\mathcal{S}_{ND}^n \approx \begin{bmatrix} -\frac{r}{n} & 1 \\ 1 & -\frac{n}{r} \end{bmatrix}; \quad \mathcal{S}_{D\beta}^n \approx \begin{bmatrix} \frac{r}{n} & -1 \\ -1 & \frac{n}{r} \end{bmatrix}; \quad \mathcal{S}_{N\beta}^n \approx \begin{bmatrix} \frac{r^2}{n^2} & \frac{r}{n} \\ \frac{r}{n} & -1 \end{bmatrix} \quad (64)$$

From these expressions one can deduces that the first eigenvalue of each operator vanishes quickly. The second eigenvalues of  $S_{ND}^n$  and  $S_{D\beta}^n$  tend toward  $\frac{n}{r}$  and  $\frac{-n}{r}$ , respectively. However, the second eigenvalues of  $S_{N\beta}^n$  tend toward  $-\beta$ . the operators  $S_{ND}^n$  and  $S_{D\beta}^n$  have the same behavior when  $n \rightarrow \infty$ . Then the operator  $S_{N\beta}^n$  has the best behavior. The same results are obtained for the Hadamard example on the square. The figures 1 and 2 show the evolution of the condition number of the single field operators and the two fields operators, respectively. They show the same behavior for high frequencies for all these methods. These analysis give an idea on the behavior of each operator, but it is not sufficient to decide which method is better in an absolute way. In fact, others parameters such geometric and Cauchy data singularities, or the unknown data etc... control the behavior of each operator.

## 5. The constitutive law gap functionals: adjoint fields and derivatives evaluation

The aim of this section is the evaluation of the derivatives of the functional with respect to  $(\eta, \tau)$ . Let us consider  $u_1$  and  $u_2$  as defined in the subsection 3.3. Without forgetting their dependence on the fields  $(\eta, \tau) \in H^{-1/2}(\Gamma_u) \times H^{1/2}(\Gamma_u)$ , the energy error can be simply expressed as follows:

$$E_{\alpha,\beta}(\eta, \tau) = \frac{1}{2} \int_{\Omega} k(x) (\nabla u_1 - \nabla u_2) \cdot (\nabla u_1 - \nabla u_2) \quad (65)$$

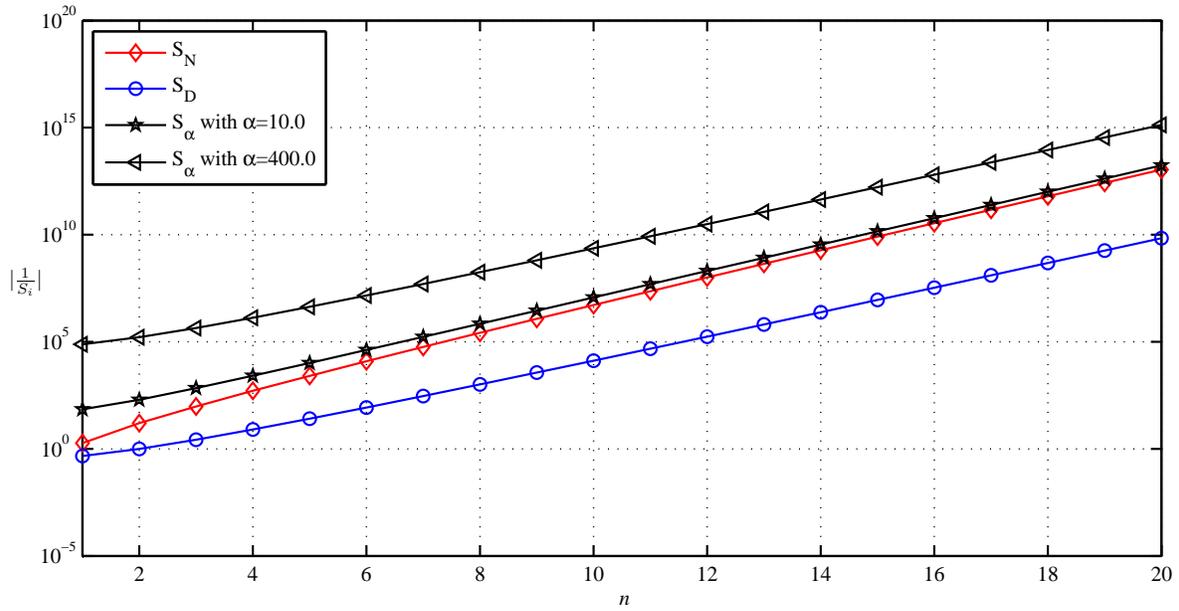
We consider the following spaces and fields:

$$V_1 = \{v \in H^1(\Omega) / v|_{\Gamma_m} = T\}$$

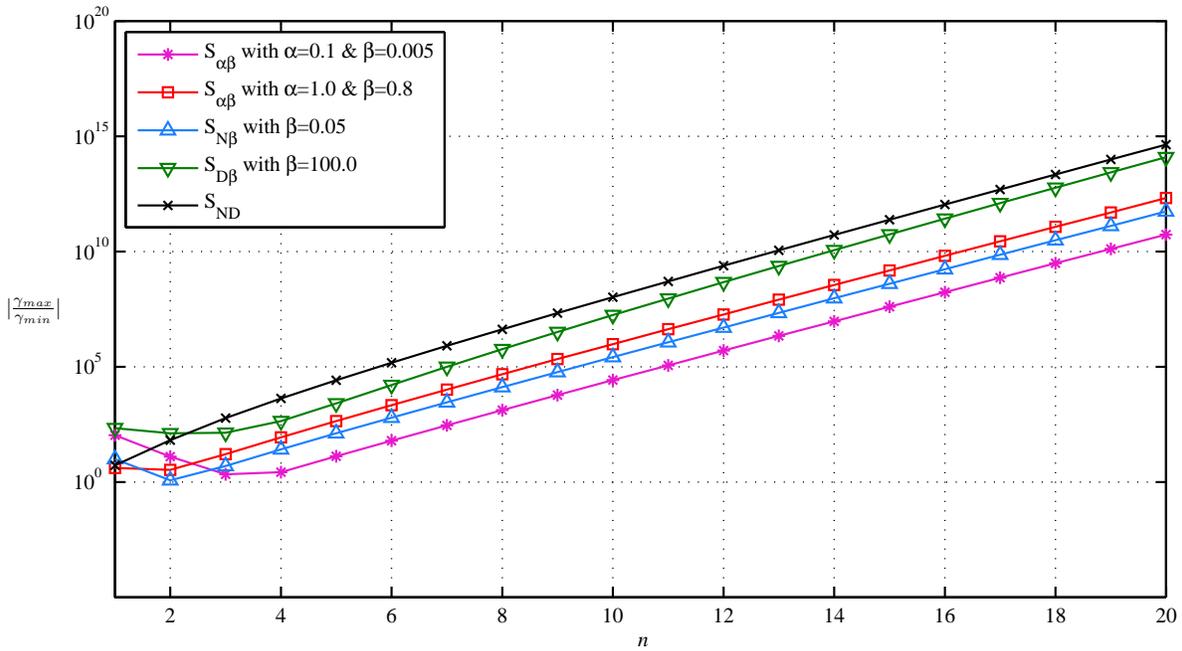
$$V_1^0 = \{v \in H^1(\Omega) / v|_{\Gamma_m} = 0\}$$

with  $(u_1, u_2, v_1, v_2) \in V_1 \times H^1(\Omega) \times V_1^0(\Omega) \times H^1(\Omega)$ . Then, we denote by  $\mathcal{J}_1$  and  $\mathcal{J}_2$  the weak formulations of the problems defined by (4) and (5).

$$\mathcal{J}_1(\eta, \tau) = \int_{\Omega} k(x) \nabla u_1 \nabla v_1 - \int_{\Gamma_u} v_1 k(x) \nabla u_1 \cdot n \quad (66)$$



**Figure 1.** Condition number of single field methods



**Figure 2.** Condition number of two field methods

$$\mathcal{J}_2(\eta, \tau) = \int_{\Omega} k(x) \nabla u_2 \nabla v_2 - \int_{\Gamma_m} \Phi v_2 - \int_{\Gamma_u} v_2 k(x) \nabla u_2 \cdot n \quad (67)$$

To evaluate the derivative we consider the following Lagrangian:

$$L_{\alpha, \beta}(u_1, u_2, v_1, v_2; \eta, \tau) = E_{\alpha, \beta}(\eta, \tau) - \mathcal{J}_1(\eta, \tau) - \mathcal{J}_2(\eta, \tau) \quad (68)$$

For any  $(\eta, \tau) \in H^{-1/2}(\Gamma_u) \times H^{1/2}(\Gamma_u)$  and the above defined field  $v_1$  and  $v_2$  it follows:

$$E_{\alpha,\beta}(\eta, \tau) = L_{\alpha,\beta}(u_1, u_2, v_1, v_2; \eta, \tau) \quad (69)$$

The gradient of  $E_{\alpha,\beta}$  can be obtained from the partial derivative of  $L_{\alpha,\beta}$  with respect to  $\eta$  and  $\tau$ .

$$\frac{\partial E_{\alpha,\beta}(\eta, \tau)}{\partial \eta} \cdot \delta\eta = - \int_{\Gamma_u} (v_1 + v_2) \cdot \delta\eta \quad (70)$$

$$\frac{\partial E_{\alpha,\beta}(\eta, \tau)}{\partial \tau} \cdot \delta\tau = - \int_{\Gamma_u} (\alpha v_1 + \beta v_2) \cdot \delta\tau \quad (71)$$

where  $v_1$  and  $v_2$  are solution of :

$$\begin{cases} \nabla k(x) \nabla v_1 & = 0 & \text{in } \Omega \\ v_1 & = 0 & \text{on } \Gamma_m \\ k(x) \nabla v_1 \cdot n + \alpha v_1 & = -k(x) (\nabla u_1 - \nabla u_2) \cdot n & \text{on } \Gamma_u \end{cases} \quad (72)$$

$$\begin{cases} \nabla k(x) \nabla v_2 & = 0 & \text{in } \Omega \\ k(x) \nabla v_2 \cdot n & = k(x) \nabla u_1 \cdot n - \Phi & \text{on } \Gamma_m \\ k(x) \nabla v_2 \cdot n + \beta v_2 & = k(x) (\nabla u_1 - \nabla u_2) \cdot n & \text{on } \Gamma_u \end{cases} \quad (73)$$

Remark that, as shown for the direct problem (4) and (5), the boundary conditions defined on  $\Gamma_u$  of the adjoint problems degenerate when the parameters  $\alpha$  and  $\beta$  tend toward to 0 and  $\infty$  as shown in the following table:

**Table 2.** Adjoint field boundary conditions on  $\Gamma_u$  as function of the parameters  $\alpha$  and  $\beta$ 

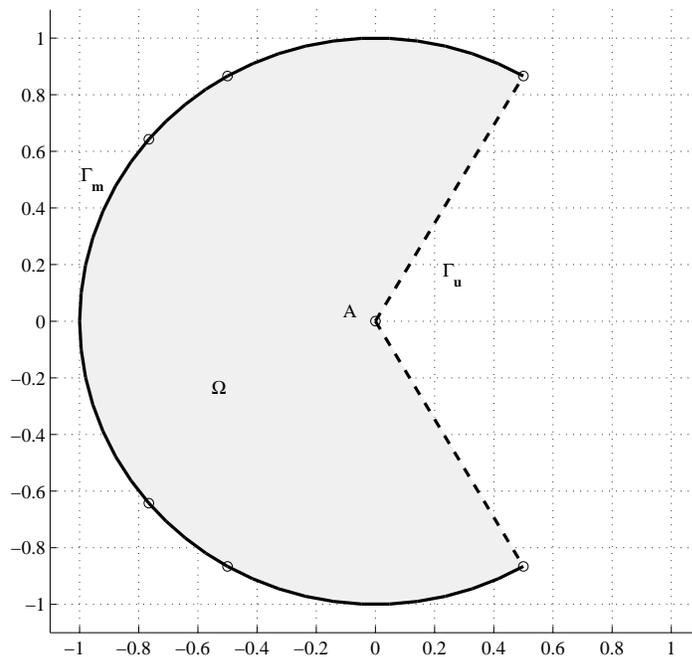
$\alpha$ and $\beta$	Variables	BC on $\Gamma_u$	Derivatives
Single field approaches			
$\alpha = 0$ $\beta = 0$	$\eta$	$k(x)\nabla v_1.n = 0$ $k(x)\nabla v_2.n = 0$	$\nabla_\eta E_{\alpha,\beta}.\delta\eta = -\int_{\Gamma_u} v_2\delta\eta$
$\alpha = \infty$ $\beta = \infty$	$\tau$	$v_1 = 0$ $v_2 = 0$	$\nabla_\tau E_{\alpha,\beta}.\delta\tau = \int_{\Gamma_u} k(x)\nabla v_2.n\delta\tau$
$0 < \alpha = \beta < \infty$	$v = \eta + \alpha\tau$	$k(x)\nabla v_1.n + \alpha v_1 = -k(x)\nabla(u_1 - u_2).n$ $k(x)\nabla v_2.n + \alpha v_2 = k(x)\nabla(u_1 - u_2).n$	$\nabla_v E_{\alpha,\beta}.\delta v = -\int_{\Gamma_u} (v_1 + v_2)\delta v$
Two fields approaches			
$\alpha = 0$ $\beta = \infty$	$\eta$ $\tau$	$k(x)\nabla v_1.n = -k(x)\nabla(u_1 - u_2).n$ $v_2 = 0$	$\nabla_\eta E_{\alpha,\beta}.\delta\eta = -\int_{\Gamma_u} v_1\delta\eta$ $\nabla_\tau E_{\alpha,\beta}.\delta\tau = \int_{\Gamma_u} k(x)\nabla(v_1 + v_2).n\delta\tau$
$\alpha = 0$ $0 < \beta < \infty$	$\eta$ $\tau$	$k(x)\nabla v_1.n = -k(x)\nabla(u_1 - u_2).n$ $k(x)\nabla v_2.n + \beta v_2 = k(x)\nabla(u_1 - u_2).n$	$\nabla_\eta E_{\alpha,\beta}.\delta\eta = -\int_{\Gamma_u} (v_1 + v_2)\delta\eta$ $\nabla_\tau E_{\alpha,\beta}.\delta\tau = \int_{\Gamma_u} k(x)\nabla(v_1 + v_2).n\delta\tau$
$\alpha = \infty$ $0 < \beta < \infty$	$\eta$ $\tau$	$v_1 = 0$ $k(x)\nabla v_2.n + \beta v_2 = k(x)\nabla(u_1 - u_2).n$	$\nabla_\eta E_{\alpha,\beta}.\delta\eta = -\int_{\Gamma_u} v_2\delta\eta$ $\nabla_\tau E_{\alpha,\beta}.\delta\tau = \int_{\Gamma_u} k(x)\nabla(v_2 - u_1 + u_2).n\delta\tau$
$0 < \alpha \neq \beta < \infty$	$\eta$ $\tau$	$k(x)\nabla v_1.n + \alpha v_1 = -k(x)\nabla(u_1 - u_2).n$ $k(x)\nabla v_2.n + \beta v_2 = k(x)\nabla(u_1 - u_2).n$	$\nabla_\eta E_{\alpha,\beta}.\delta\eta = -\int_{\Gamma_u} (v_1 + v_2)\delta\eta$ $\nabla_\tau E_{\alpha,\beta}.\delta\tau = \int_{\Gamma_u} k(x)(\nabla v_1 + \nabla v_2).n\delta\tau$

## 6. Numerical examples

The implementation of the above methods was carried out using the finite element method (FEM). Hence, the derivation of the adjoint state is preferably established on the basis of the FEM-discretized problem. The advantage of this fully discrete approach is that the exact gradient of the discrete objective function is obtained; moreover, it is easily implemented in existing FEM-sofware. In references [6] and [7], the FEM-discretized formulation is detailed for the case where  $\alpha = 0$  and  $\beta = \infty$ .

In order to show the performance of the aforementioned methods, we choose two examples for which the methods behave differently.

### 6.1. First example.



**Figure 3.** Geometry and boundaries of the studied domain

We consider a two-dimensional domain  $\Omega$  and assume that the boundary  $\Gamma$  of this domain is divided into two complementary parts  $\Gamma_m$  and  $\Gamma_u$  as shown on the figure 3. Note that because of the corner in the  $\Gamma_u$  part of the boundary, the problem addressed here is quite stiff as singularities appear near the point  $A$ . Furthermore at this point, there is a jump of the outer normal to the domain and then a jump of the normal derivatives of the solution fields.

Figures 4 and 5 show the identified Dirichlet and Neumann data. They are compared with the exact data. These results are obtained by using the same stopping criteria for the energy-like functional. All the methods converge with approximately

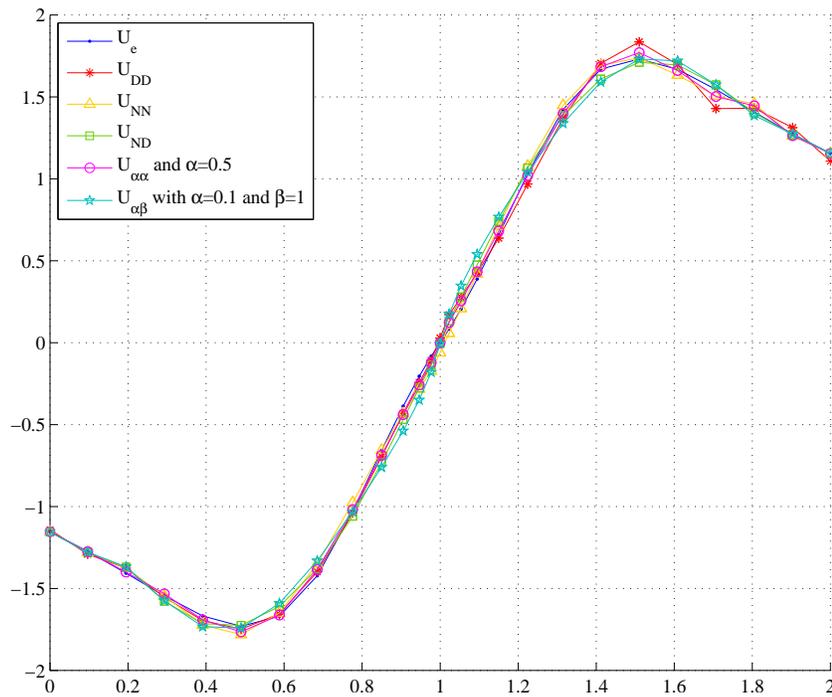


Figure 4. Exact and identified Dirichlet boundary condition on  $\Gamma_u$

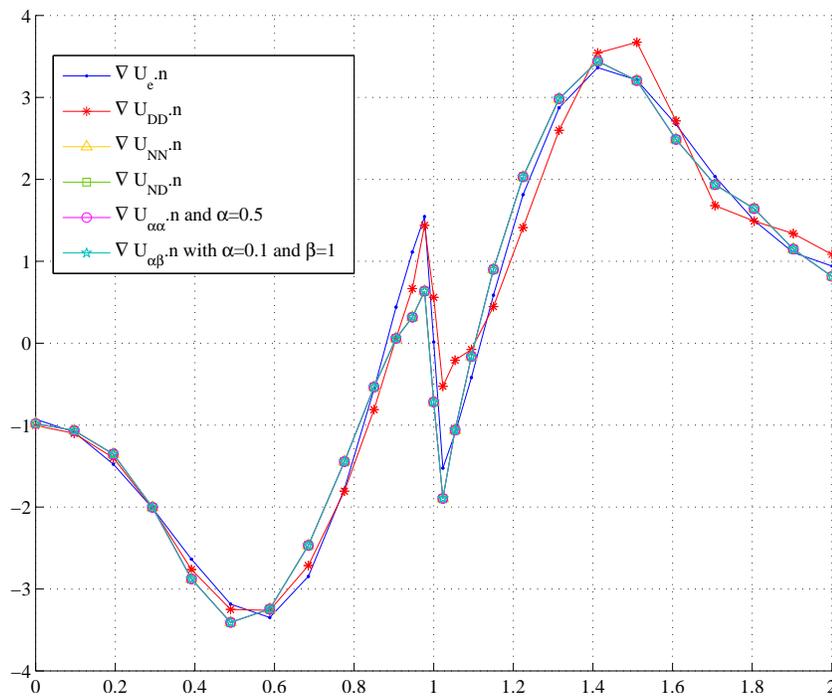
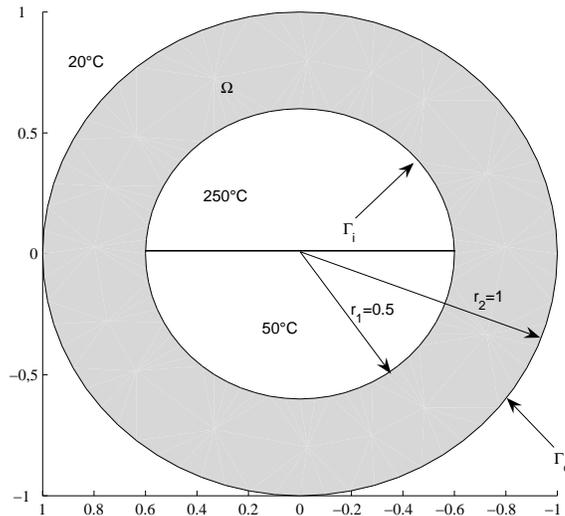


Figure 5. Exact and identified Neumann boundary condition on  $\Gamma_u$

the same number of iterations during the minimization process. Unlike the example of Hadamard, we observe here that the single field method with  $\alpha = \beta = \infty$  gives the worst result, whereas other methods give approximately the same (satisfactory) result.

## 6.2. Second example.

This example has been already addressed in [2], it deals with a practical case corresponding to stratified inner fluid. We consider therefore the reconstruction of temperature and flux in a pipeline of infinite length. We assume that the temperature does not depend on the longitudinal coordinate. We deal, therefore with a two dimensional problem as shown on figure 6. The overspecified boundary conditions used



**Figure 6.** Geometric data of the stratified inner fluid problem

in this example are generated by the finite element computation of a Robin problem with the following data:

- a constant thermal conductivity  $k = 17 \text{ W/m/}^\circ\text{C}$ ,
- on  $\Gamma_m$ ,  $T_{ext} = 20^\circ\text{C}$  with the Robin's coefficient  $\alpha_c = 12 \text{ W/m}^2/\text{}^\circ\text{C}$ ,
- on  $\Gamma_u$ ,  $T_{int} = 50^\circ\text{C}$  on the lower half circle of  $\Gamma_u$  and  $T_{int} = 250^\circ\text{C}$  on the upper half one with the Robin's coefficient  $\alpha_u = 1000 \text{ W/m}^2/\text{}^\circ\text{C}$ .

The cross section  $\Omega$  is an annular thick domain with radii  $r_1 = 1$  and  $r_2 = 0.5$ . To recover the temperature and the flux with accuracy, a mesh with 64 nodes on  $\Gamma_u$  is used. Trust Region Method of Matlab Optimization Toolbox [20] is used here to solve the optimization problem associated to each method. The same stopping criteria, termination tolerance on the function value set to  $10^{-6} \text{ W}^\circ\text{C/m}$  is imposed for all methods. Each one converges when the function value variation is less than the stopping criteria. Figures 7 and 8 show the reconstructed temperature and flux, and even in this case where the data is singular (discontinuity), they are in agreement with the actual ones, whatever the method. However, there are differences in the convergence process of these methods, the number of iterations needed to reach the stopping criteria is very

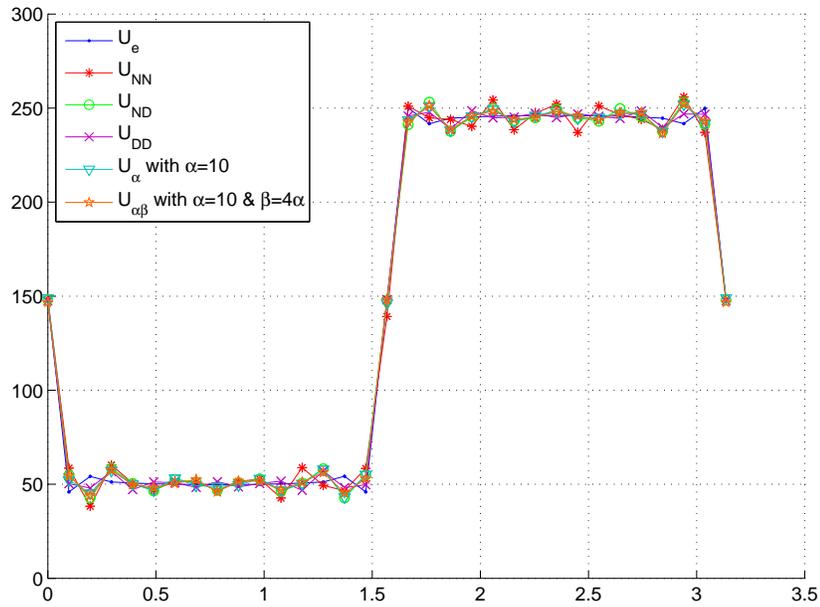


Figure 7. Exact and identified Neumann boundary condition on  $\Gamma_u$

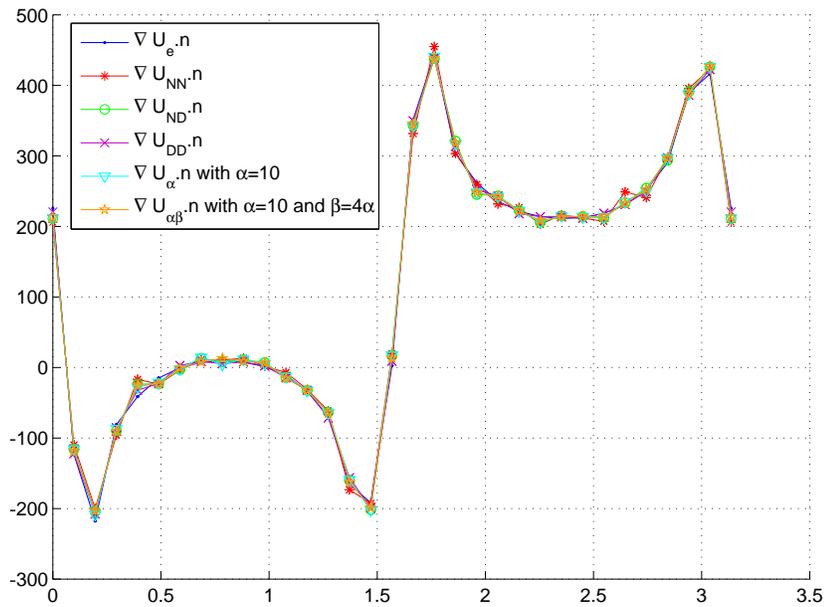


Figure 8. Exact and identified Neumann boundary condition on  $\Gamma_u$

**Table 3.** Number of iterations of the optimization process for each method

Functional	$E_{NN}$	$E_{\alpha}$	$E_{ND}$	$E_{DD}$	$E_{\alpha\beta}$
Functional value	$18.24 \cdot 10^{-3}$	$5.62 \cdot 10^{-3}$	$8.81 \cdot 10^{-3}$	$5.20 \cdot 10^{-3}$	$5.76 \cdot 10^{-3}$
Number of Iterations	233	215	375	1214	1460
Number of solved BVP	699	860	1500	3642	5840

different from one method to another, see table 3. Hence, computational cost, which depends on the number of BVP to solve, may also be a criterion for choosing a method

## 7. Conclusion

In this paper we proposed a general method based on minimizing constitutive law gap functional in order to solve the Cauchy problem for a linear elliptic PDE. This functional measures the gap between the solutions of two well-posed problems. Each of these problems has one of the Cauchy data as known boundary condition: Dirichlet or Neumann, and on the boundary where the data are lacking, an unknown Robin boundary conditions  $\eta + \alpha\tau$  and  $\eta + \beta\tau$  are imposed, respectively. The data  $\eta$  and  $\tau$  have to be identified and  $(\alpha, \beta)$  are positives scalars parameters controlling the functional behavior.

This approach generalizes that presented in Andrieux *et al* [2] and encompasses various methods proposed in the literature. According to the values of  $\alpha$  and  $\beta$  when they tend toward 0 or  $\infty$ , there are two groups of methods: the first group gathers those which depends on only one unknown data ( $\eta$ ,  $\tau$  or  $\eta + \alpha\tau$ ). The second group gathers those which depend on two unknown data  $\eta$  and  $\tau$ . Then, the equivalence between Euler-Lagrange conditions for the constitutive law functionals and interfacial operators usually used in the Domain Decomposition field is shown. Using the Hadamard example we analyze analytically the behavior of these operators as functions of the parameters  $(\alpha, \beta)$ . Then, the derivatives of the functional are given using adjoint fields which are parametrized by the same parameters.

Finally, numerical examples are given to illustrate the behavior of these methods which are not the only function of the parameters  $(\alpha, \beta)$  but also of the regularity of the Cauchy data and the overall geometry of the domain. Although, they are all equivalent from continuous point of view, we can not conclude definitely that one method, which corresponds on a choice of the parameters pair  $(\alpha, \beta)$ , is numerically more efficient than another. The choice between them depends mainly on the complexity of the problem from a geometrical point of view and singularity data. The amount of data to identify in comparison to that known, the number of variables in the minimization problem, are also important criteria for selecting the method.

## 8. Reference

- [1] Andrieux S. and Baranger T. N., Energy methods for Cauchy problems for evolution equations, 6th International Conference on Inverse Problems in Engineering: Theory and Practice, Journal of Physics: Conference Series 135 (2008) 012007, doi:10.1088/1742-6596/135/1/012007.
- [2] Andrieux S., Baranger T. N. and Ben Abda A. 2006, Solving Cauchy problems by minimizing an energy-like functional *Inverse Problems* **22** 115-133.
- [3] Andrieux S., Ben Abda A. and Baranger T. N. 2005, Data completion via an energy error functional *C.R. Mecanique* **333** 171-177.
- [4] Azaiez M., Ben Belgacem F., El Fekih H., (2006), On Cauchy's Problem. II. Completion, Regularization and Approximation, *Inverse Problems*, **22**, 1307-1336.
- [5] Baranger T.N. and Andrieux S., (2009), Data completion for linear symmetric operators as a Cauchy problem: an efficient method via energy like error minimization, *Vietnam Journal of Mechanics*, VAST, **31**, pp 247-261.
- [6] Baranger T. N. and Andrieux S., 2008, An Optimization Approach to solve Cauchy Problem in linear elasticity, *Journal of Structural and Multidisciplinary Optimization*, **35** 141-152.
- [7] Baranger T. N. and Andrieux S., 2008, An energy error-based method for the resolution of the Cauchy problem in 3D linear elasticity , *Computer Methods in Applied Mechanics and Engineering* **197** (9-12), 902-920.
- [8] Baumeister J. and Leitao A. 2002, On iterative methods for solving ill-posed problems modeled by partial differential equation *J. of Inverse and Ill-posed Problems*, **9** 13-30.
- [9] Ben Belgacem F., El Fekih H., 2006, On Cauchy's Problem. I. A Variational Steklov-Poincare, Theory, *Inverse problems*, **21**, 1915-1936.
- [10] Ben Belgacem, F., 2008, Why is the Cauchy's Problem Severely Ill-posed? *Inverse Problems*, **23**, 823-836.
- [11] Chakib, A. and Nachaoui, A. ,2006, Convergence analysis for finite element approximation to an inverse Cauchy problem, *Inverse Problems* **22** 1191-1206.
- [12] CimetiÅšre A., Delvare F., Jaoua M. and Pons F. 2001, Solution of the Cauchy problem using iterated Tikhonov regularization *Inverse Problems* **17** 553-570.
- [13] Escriva X., Baranger T. N. and Hariga-Tlatli N., 2007, Leaks identification in porous media by solving Cauchy problem, *CRAS Mecanique*, **335** (7), 401-406.
- [14] Escriva X. and Baranger T. N., 2008, Leaks identification on a Darcy model by solving Cauchy problem, 6th International Conference on Inverse Problems in Engineering: Theory and Practice, Dourdan (Paris), France - June 15-19, 2008.
- [15] Fursikov A. V. 2000 Optimal control of distributor systems: Theory and applications *Translations of mathematical Monograph* **187**.
- [16] Hadamard J. 1953 *Lectures on Cauchy's Problem in Linear Partial Differential Equation* Dover New York USA.
- [17] Hariga-Tlatli N., Baranger T., Erhel J., 2010, Misfit functional for recovering data in 2D ElectroCardioGraphy problems, *Engineering Analysis with Boundary Elements* **34** 492-500.
- [18] Kozlov V. A. , Maz'ya V. G. and Fomin A.V. 1991 An iterative method for solving the Cauchy problem for elliptic equations *Comput. Meth. Math. Phys.*, Vol. 31, N1, 45-52.
- [19] P.L. Lions, 1990, On the Swartz alternating method III: a variant for non-overlapping subdomains; Third International Symposium on Domain Decomposition Methods for Partial Differential Equations, Pages 202-231. SIAM, Philadelphia.
- [20] Matlab Software Copyright 1984-2000 The MathWorks, Inc.
- [21] Quarteroni A. and Valli A. 1999, *Domain Decomposition Methods for Partial Differential Equations*, Oxford University Press.
- [22] Tikhonov A. N. and Arsenin V. Y. 1977 *Solution to Ill-posed Problems* Winston-Wiley, New-York.

## 9. Annexe

**Lemma 9.1.** Consider  $(\psi, \xi) \in H^{1/2}(\Gamma_u) \times H^{-1/2}(\Gamma_u)$ , then there exists a pair  $(\varpi_\psi, \varpi_\xi) \in H^{-1/2}(\Gamma_u)^2$  such that:

$$\left\{ \begin{array}{ll} \nabla \cdot k(x) \nabla w_1^0 = 0 & \text{in } \Omega \\ w_1^0 = 0 & \text{on } \Gamma_m \\ k(x) \nabla w_1^0 \cdot n + \alpha w_1^0 = \varpi_\psi & \text{on } \Gamma_u \end{array} \right. \text{ and } \left\{ \begin{array}{ll} \nabla \cdot k(x) \nabla w_2^0 = 0 & \text{in } \Omega \\ k(x) \nabla w_2^0 = 0 & \text{on } \Gamma_m \\ k(x) \nabla w_2^0 \cdot n + \beta w_2^0 = \varpi_\xi & \text{on } \Gamma_u \end{array} \right.$$

with  $w_1^0 = \psi$  and  $k(x) \nabla w_2^0 \cdot n = \xi$  on  $\Gamma_u$ .

*Proof.* The fields  $w_i^0$  for  $i = 1, 2$  are characterized by the following variational properties:

$$\int_{\Omega} k(x) \nabla w_1^0 \cdot \nabla w + \int_{\Gamma_u} \alpha w_1^0 w = \int_{\Gamma_u} \varpi_\psi w, \quad \forall w \in H^1(\Omega), w = 0 \text{ on } \Gamma_m$$

$$\int_{\Omega} k(x) \nabla w_2^0 \cdot \nabla w + \int_{\Gamma_u} \beta w_2^0 w = \int_{\Gamma_u} \varpi_\xi w, \quad \forall w \in H^1(\Omega)$$

We consider the fields  $W_1^0$  and  $W_2^0$ , which are solution of the following well-posed problems:

$$\int_{\Omega} k(x) \nabla W_1^0 \cdot \nabla w = \int_{\Gamma_u} k(x) \nabla W_1^0 \cdot n w, \quad \forall w \in H^1(\Omega) \text{ and } w = 0 \text{ on } \Gamma_m$$

with  $W_1^0 = \psi$  on  $\Gamma_u$  and  $W_1^0 = 0$  on  $\Gamma_m$ .

$$\int_{\Omega} k(x) \nabla W_2^0 \cdot \nabla w = \int_{\Gamma_u} \xi w, \quad \forall w \in H^1(\Omega)$$

with  $k(x) \nabla W_2^0 \cdot n = \xi$  on  $\Gamma_u$ ,  $k(x) \nabla W_2^0 \cdot n = 0$  on  $\Gamma_m$  and  $\int_{\Gamma_u} \xi = 0$ . Assume that  $\varpi_\psi = (k(x) \nabla W_1^0 \cdot n + \alpha W_1^0)$  and  $\varpi_\xi = k(x) \nabla W_2^0 \cdot n + \beta W_2^0$ , we obtain:

$$\int_{\Omega} k(x) \nabla W_1^0 \cdot \nabla w = - \int_{\Gamma_u} \alpha W_1^0 w + \int_{\Gamma_u} \varpi_\psi w, \quad \forall w \in H^1(\Omega), w = 0 \text{ on } \Gamma_u.$$

$$\int_{\Omega} k(x) \nabla W_2^0 \cdot \nabla w = - \int_{\Gamma_u} \beta W_2^0 w + \int_{\Gamma_u} \varpi_\xi w, \quad \forall w \in H^1(\Omega).$$

This shows that the fields  $W_i^0$  and  $w_i^0$  for  $i = 1, 2$  are equal. Then, we have determined the values of the pair  $(\varpi_\psi, \varpi_\xi)$ , which ensure that solutions to Robin problems take the fixed values  $\psi$  and  $\xi$  of the field and the flux respectively on the boundary  $\Gamma_u$ .  $\square$