On the compared expressiveness of arc, place and transition time Petri nets

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Fundamenta Informaticae 88 (2008) 1–25
IOS Press

On the compared expressiveness of Arc, Place and Transition Time Petri Nets

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Abstract. In this paper, we consider safe Time Petri Nets where time intervals (strict and large) are associated with places (*P-TPN*), arcs (*A-TPN*) or transitions (*T-TPN*). We give the formal strong and weak semantics of these models in terms of Timed Transition Systems. We compare the expressiveness of the six models w.r.t. (weak) timed bisimilarity (behavioral semantics). The main results of the paper are : (i) with strong semantics, *A-TPN* is strictly more expressive than *P-TPN* and *T-TPN* ; (ii) with strong semantics *P-TPN* and *T-TPN* are incomparable ; (iii) *T-TPN* with strong semantics and *T-TPN* with weak semantics are incomparable. Moreover, we give a complete classification by a set of 9 relations explained in Fig. 19 (p. 23).

1. Introduction

The two main extensions of Petri nets with time are Time Petri Nets (TPNs) [20] and Timed Petri Nets [22]. For TPNs a transition can fire within a time interval whereas for Timed Petri Nets it has a duration and fires as soon as possible or with respect to a scheduling policy, depending on the authors. Among Timed Petri Nets, time can be considered relative to places (P-Timed Petri Nets), arcs (A-Timed Petri Nets) or transitions (T-Timed Petri Nets) [23, 21]. The same classes are defined for TPNs i.e.*T-TPN* [20, 4], *A-TPN* [16, 1, 15] and *P-TPN* [18, 19]. It is known that P-Timed Petri Nets and T-Timed Petri

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Nets are expressively equivalent [23, 21] and these two classes of Timed Petri Nets are included in the two corresponding classes *T-TPN* and *P-TPN* [21]

Depending on the authors, two semantics are considered for $\{T,A,P\}$-*TPN*: a weak one, where no transition is never forced to be fired, and a strong one, where each transition must be fired when the upper bound of its time condition is reached. Moreover there are a single-server and several multi-server semantics [6, 3]. The number of *clocks* to be considered is finite with single-server semantics (one clock per transition, one per place or one per arc) whereas it is not with multi-server semantics.

A-TPN have mainly been studied with weak (lazy) multi-server semantics [16, 1, 15]: this means that the number of clocks is not finite but the firing of transitions may be delayed, even if this implies that some transitions are disabled because their input tokens become too old. The reachability problem is undecidable for this class of *A-TPN* but thanks to this weak semantics, it enjoys *monotonic* properties and falls into a class of models for which coverability and boundedness problems are decidable.

Conversely *T-TPN* [20, 4] and *P-TPN* [18, 19] have been studied with strong single-server semantics. They do not have monotonic features of weak semantics although the number of *clocks* is finite. The marking reachability problem is known undecidable [17] but marking coverability, k-boundedness, state reachability and liveness are decidable for bounded *T-TPN* and *P-TPN* with strong semantics.

Related work : Expressiveness of models extended with time *Time Petri Nets versus Timed Automata*. Some works compare the expressiveness of Time Petri Nets and Timed Automata. In [24], the author exposes mutual isomorphic translations between 1-safe Time-Arc Petri Nets (*A-TPN*) and networks of Timed Automata.

In [13, 11] it was proved that bounded *T-TPN* with strong semantics form a strict subclass of the class of timed automata w.r.t. timed bisimilarity. Authors give in [12] a characterisation of the subclass of timed automata which admit a weakly timed bisimilar *T-TPN*. Moreover it was proved in [11] that bounded *T-TPN* and timed automata are equally expressive w.r.t. timed language acceptance.

Arc, Place and Transition Time Petri Nets. The comparison of the expressiveness between *A-TPN*, *P-TPN* and *T-TPN* models with strong and weak semantics w.r.t. timed language acceptance and timed bisimulation have been very little studied[1].

In [14] authors compared these models w.r.t. language acceptance. With strong semantics, they established *P-TPN* $\subseteq_{\mathcal{L}}$ *T-TPN* $\subseteq_{\mathcal{L}}$ *A-TPN*[2] and with weak semantics the result is *P-TPN* $=_{\mathcal{L}}$ *T-TPN* $=_{\mathcal{L}}$ *A-TPN*.

In [9] authors study only the strong semantics and obtain the following results: *T-TPN* $\subset_{\mathcal{L}}$ *A-TPN* and *P-TPN* $\not\subset_{\mathcal{L}}$ *T-TPN*.

These results of [14] and [9] are inconsistent.

Concerning bisimulation, in [9] (with strong semantics) we have *T-TPN* $\subset_{\approx}$ *A-TPN*, *P-TPN* $\subseteq_{\approx}$ *A-TPN* and *P-TPN* $\not\subseteq_{\approx}$ *T-TPN*. But the counter-example given in this paper to show *P-TPN* $\not\subseteq_{\approx}$ *T-TPN* uses the fact that the *T-TPN* 'à la Merlin' cannot model strict timed constraint[3]. This counter example fails if we extend these models to strict constraints.

[1]Moreover, all studies consider only closed interval constraints, and from results in [9], offering strict constraints makes a difference on expressiveness.

[2]We note $\sim_{\mathcal{L}}$ and $\sim_{\approx}$ with $\sim \in \{\subset, \subseteq, =\}$ respectively for the expressiveness relation w.r.t. timed language acceptance and timed bisimilarity.

[3]The intervals are of the form $[a, b]$ and they can not handle a behavior like "if $x < 1$".

In [19] *P-TPN* and *T-TPN* are declared incomparable but no proof is given.
Many problems remain open concerning the relationships between these models.

Our Contribution. In this paper, we consider safe Arc, Place and Transition Time Petri Nets with strict and large timed constraints and with single-server semantics. We give the formal strong and weak semantics of these models in terms of Timed Transition Systems. We compare each model with the two others in the weak and the strong semantics, and also the relationships between the weak and the strong semantics for each model (see Fig. 19, p. 23). The comparison criterion is the weak timed bisimulation. In [8], a previous version of this work, only 7 of the 9 relations where covered. Here, the 2 missing ones are also presented, in Theorems 4.10 and 4.12.

The paper is organised as follows: Section 2 gives some "framework" definitions. Section 3 presents the three timed Petri nets models, with strong and weak semantics. Section 4 is the core of our contribution: it lists all the new results we propose. Section 5 concludes.

2. Framework definition

We denote A^X the set of mappings from X to A. If X is finite and $|X| = n$, an element of A^X is also a vector in A^n. The usual operators $+, -, <$ and $=$ are used on vectors of A^n with $A = \mathbb{N}, \mathbb{Q}, \mathbb{R}$ and are the point-wise extensions of their counterparts in A. For a *valuation* $\nu \in A^X, d \in A, \nu + d$ denotes the vector $(\nu + d)(x) = \nu(x) + d$. The set of boolean is denoted by $\mathbb{B}$. The set of non negative intervals in $\mathbb{Q}$ is denoted by $\mathcal{I}(\mathbb{Q}_{\geq 0})$. An element of $\mathcal{I}(\mathbb{Q}_{\geq 0})$ is a constraint φ of the form $\alpha \prec_1 x \prec_2 \beta$ with $\alpha \in \mathbb{Q}_{\geq 0}, \beta \in \mathbb{Q}_{\geq 0} \cup \{\infty\}$ and $\prec_1, \prec_2 \in \{<, \leq\}$, such that $I = [\![\varphi]\!]$. We let $I^{\downarrow} = [\![0 \leq x \prec_2 \beta]\!]$ be the *downward closure* of I and $I^{\uparrow} = [\![\alpha \prec_1 x]\!]$ be the *upward closure* of I. Let Σ be a fixed finite alphabet s.t. $\varepsilon \notin \Sigma$ and $\Sigma_\varepsilon = \Sigma \cup \{\varepsilon\}$, with ε the neutral element of sequence ($\forall a \in \Sigma_\varepsilon : \varepsilon a = a\varepsilon = a$).

Definition 2.1. (Timed Transition Systems)
A *timed transition system (TTS)* over the set of actions Σ_ϵ is a tuple $S = (Q, Q_0, \Sigma_\epsilon, \longrightarrow)$ where Q is a set of states, $Q_0 \subseteq Q$ is the set of initial states, Σ_ϵ is a finite set of actions disjoint from $\mathbb{R}_{\geq 0}$, $\longrightarrow \subseteq Q \times (\Sigma_\epsilon \cup \mathbb{R}_{\geq 0}) \times Q$ is a set of edges. If $(q, e, q') \in \longrightarrow$, we also write $q \xrightarrow{e} q'$. Moreover, it should verify some time-related conditions: time determinism (td), time-additivity (ta), nul delay (nd) and time continuity (tc). $\forall d, d' \in \mathbb{R}_{\geq 0}, \forall q, q', q'' \in Q$:

$$
\begin{array}{llll}
td \equiv q \xrightarrow{d} q' \wedge q \xrightarrow{d} q'' \Rightarrow q' = q'' & & ta \equiv q \xrightarrow{d} q' \wedge q' \xrightarrow{d'} q'' \Rightarrow q \xrightarrow{d+d'} q'' \\
nd \equiv q \xrightarrow{0} q & & tc \equiv q \xrightarrow{d} q' \Rightarrow \forall d' \leq d, \exists q_{d'}, q \xrightarrow{d'} q_{d'}
\end{array}
$$

In the case of $q \xrightarrow{d} q'$ with $d \in \mathbb{R}_{\geq 0}$, d denotes a delay and not an absolute time.

In a TTS $S = (Q, Q_0, \Sigma_\epsilon, \longrightarrow)$, a *run* ρ of length $n \geq 0$ is a finite ($n < \omega$) or infinite ($n = \omega$) sequence of alternating time and discrete transitions (starting from $q_0 \in Q_0$) of the form:

$$\rho = q_0 \xrightarrow{d_0} q_0' \xrightarrow{a_0} q_1 \xrightarrow{d_1} q_1' \xrightarrow{a_1} \cdots q_n \xrightarrow{d_n} q_n' \cdots$$

A *run* ρ from a state q_1 is a run starting from $q_1 \in Q$.

A trace of ρ is the timed word $w = (a_0, d_0)(a_1, d_1)\cdots(a_n, d_n)\cdots$ that consists of the sequence of letters of Σ.

We write *Untimed*(ρ) = *Untimed*$(w) = a_0 a_1 \cdots a_n \cdots$ for the untimed part of w, and *Duration*(ρ) = *Duration*$(w) = \sum d_k$ for the duration of the timed word w and then of the run ρ.

As a shorthand, we denote :

- $\rho = q \xrightarrow{abc} q'$ or $\rho = q \xrightarrow{a} q_a \xrightarrow{b} q_b \xrightarrow{c} q'$ for the sequence in null time of discrete steps a, b and c like $\rho = q \xrightarrow{0} q_{a_0} \xrightarrow{a} q_a \xrightarrow{0} q_{b_0} \xrightarrow{b} q_b \xrightarrow{0} q_{c_0} \xrightarrow{c} q'$
- $\rho = q \xrightarrow{\epsilon * a} q'$ for a sequence in null time of some epsilon transition followed by $a \in \Sigma$ like in the run $\rho = q \xrightarrow{\epsilon} \cdots \xrightarrow{\epsilon} q_{\varepsilon *} \xrightarrow{a} q'$
- $q \xrightarrow{(a,d)} q'$ for a sequence of time elapsing and discrete steps like $q \xrightarrow{d} q'' \xrightarrow{a} q'$.
- $\rho = q \xrightarrow{(\epsilon *, d)} q'$ for a run $\rho = q \xrightarrow{(\epsilon, d_1)} q_1 \xrightarrow{(\epsilon, d_2)} \cdots q_n \xrightarrow{(\epsilon, d_n)} q'$ such that $\sum_{1 \le i \le n} d_i = d$

Definition 2.2. (Strong Timed Bisimilarity)
Let $S_1 = (Q_1, Q_0^1, \Sigma, \longrightarrow_1)$ and $S_2 = (Q_2, Q_0^2, \Sigma, \longrightarrow_2)$ be two TTS[4] and $\approx_{\mathcal{S}}$ be a binary relation over $Q_1 \times Q_2$. We write $q \approx_{\mathcal{S}} q'$ for $(q, q') \in \approx_{\mathcal{S}}$. $\approx_{\mathcal{S}}$ is a *timed bisimulation relation* between S_1 and S_2 if:

- $q_1 \approx_{\mathcal{S}} q_2$, for all $(q_1, q_2) \in Q_0^1 \times Q_0^2$;
- if $q_1 \xrightarrow{t}_1 q_1'$ with $t \in \mathbb{R}_{\geq 0}$ and $q_1 \approx_{\mathcal{S}} q_2$ then $q_2 \xrightarrow{t}_2 q_2'$ for some q_2', and $q_1' \approx_{\mathcal{S}} q_2'$; conversely if $q_2 \xrightarrow{t}_2 q_2'$ and $q_1 \approx_{\mathcal{S}} q_2$ then $q_1 \xrightarrow{t}_1 q_1'$ for some q_1' and $q_1' \approx_{\mathcal{S}} q_2'$;
- if $q_1 \xrightarrow{a}_1 q_1'$ with $a \in \Sigma$ and $q_1 \approx_{\mathcal{S}} q_2$ then $q_2 \xrightarrow{a}_2 q_2'$ and $q_1' \approx_{\mathcal{S}} q_2'$; conversely if $q_2 \xrightarrow{a}_2 q_2'$ and $q_1 \approx_{\mathcal{S}} q_2$ then $q_1 \xrightarrow{a}_1 q_1'$ and $q_1' \approx_{\mathcal{S}} q_2'$.

Two TTS S_1 and S_2 are *timed bisimilar* if there exists a timed bisimulation relation between S_1 and S_2. We write $S_1 \approx_{\mathcal{S}} S_2$ in this case.

Let $S = (Q, Q_0, \Sigma_\varepsilon, \longrightarrow)$ be a TTS. We define the ε-abstract TTS $S^\varepsilon = (Q, Q_0^\varepsilon, \Sigma, \longrightarrow_\varepsilon)$ (with no ε-transitions) by:

- $q \xrightarrow{d}_\varepsilon q'$ with $d \in \mathbb{R}_{\geq 0}$ iff there is a run $\rho = q \xrightarrow{*} q'$ with *Untimed*$(\rho) = \varepsilon$ and *Duration*$(\rho) = d$,
- $q \xrightarrow{a}_\varepsilon q'$ with $a \in \Sigma$ iff there is a run $\rho = q \xrightarrow{*} q'$ with *Untimed*$(\rho) = a$ and *Duration*$(\rho) = 0$,
- $Q_0^\varepsilon = \{q \mid \exists q' \in Q_0 \mid\ q' \xrightarrow{*} q \textit{ and } \textit{Duration}(\rho) = 0 \wedge \textit{Untimed}(\rho) = \varepsilon\}$.

Definition 2.3. (Weak Timed Bisimilarity)
Let $S_1 = (Q_1, Q_0^1, \Sigma_\varepsilon, \longrightarrow_1)$ and $S_2 = (Q_2, Q_0^2, \Sigma_\varepsilon, \longrightarrow_2)$ be two TTS and $\approx_{\mathcal{W}}$ be a binary relation over $Q_1 \times Q_2$. $\approx_{\mathcal{W}}$ is a *weak (timed) bisimulation relation* between S_1 and S_2 if it is a strong timed bisimulation relation between S_1^ε and S_2^ε.

[4]Note that they contain no ε-transitions.

Note that if $S_1 \approx_{\mathcal{S}} S_2$ then $S_1 \approx_{\mathcal{W}} S_2$ and if $S_1 \approx_{\mathcal{W}} S_2$ then S_1 and S_2 have the same timed language.

In this paper, we consider weak timed bisimilarity and we note $\approx$ for $\approx_{\mathcal{W}}$.

Definition 2.4. (Expressiveness w.r.t. (Weak) Timed Bisimilarity)
The class $\mathcal{C}$ is *more expressive* than $\mathcal{C}'$ w.r.t. timed bisimilarity if for all $B' \in \mathcal{C}'$ there is a $B \in \mathcal{C}$ s.t. $B \approx B'$. We write $\mathcal{C}' \subseteq_{\approx} \mathcal{C}$ in this case. If moreover there is a $B \in \mathcal{C}$ s.t. there is no $B' \in \mathcal{C}'$ with $B \approx B'$, then $\mathcal{C}$ is *strictly more expressive* than $\mathcal{C}'$, denoted by $\mathcal{C}' \subset_{\approx} \mathcal{C}$. If both $\mathcal{C}' \subseteq_{\approx} \mathcal{C}$ and $\mathcal{C} \subseteq_{\approx} \mathcal{C}'$ then $\mathcal{C}$ and $\mathcal{C}'$ are equally expressive w.r.t. timed bisimilarity, and we write $\mathcal{C} =_{\approx} \mathcal{C}'$.

3. $\{$*T,A,P*$\}$-*TPN*: definitions and semantics

The classical definition of *T-TPN* [20] is based on a single server semantics (see [6, 3] for other semantics). With this semantics, bounded-TPN and safe-TPN (i.e. one-bounded) are equally expressive w.r.t. timed-bisimilarity and then w.r.t. timed language acceptance [11]. For multi-sever semantics, it is easy to show for $\{$*T,A,P*$\}$-*TPN* that bounded-TPN and safe-TPN are equally expressive w.r.t. timed-bisimilarity and then w.r.t. timed language acceptance. A proof of this result can be found for $\{$*T,A,P*$\}$-*TPN* in appendix **??**. [7]. Thus, in the sequel, we will consider safe TPN. We now give definitions and semantics of safe $\{$*T,A,P*$\}$-*TPN*.

3.1. Common definitions

We assume the reader is aware of Petri net theory, and only recall a few definitions.

Definition 3.1. (Petri Net)
A *Petri Net* $\mathcal{N}$ is a tuple $(P, T, {}^\bullet(.), (.)^\bullet, M_0, \Lambda)$ where: $P = \{p_1, p_2, \cdots, p_m\}$ is a finite set of *places* and $T = \{t_1, t_2, \cdots, t_n\}$ is a finite set of *transitions*; ${}^\bullet(.) \in (\{0,1\}^P)^T$ is the *backward* incidence mapping; $(.)^\bullet \in (\{0,1\}^P)^T$ is the *forward* incidence mapping; $M_0 \in \{0,1\}^P$ is the *initial* marking, $\Lambda : T \to \Sigma \cup \{\varepsilon\}$ is the labeling function.

Notations for all Petri nets We use the following common shorthands: $p \in M \stackrel{\text{def}}{=} M(p) \geq 1$, $M \geq {}^\bullet t \stackrel{\text{def}}{=} \forall p : M(p) \geq {}^\bullet(t,p)$, ${}^\bullet t \stackrel{\text{def}}{=} \{p | {}^\bullet(t,p) \geq 1\}$, $t^\bullet \stackrel{\text{def}}{=} \{p | (t,p)^\bullet \geq 1\}$, ${}^\bullet p \stackrel{\text{def}}{=} \{t | (t,p)^\bullet \geq 1\}$, $p^\bullet \stackrel{\text{def}}{=} \{t | {}^\bullet(t,p) \geq 1\}$.

A *marking* M is an element $M \in \{0,1\}^P$. $M(p)$ is the number of tokens in place p. A transition t is said to be *enabled* by marking M iff $M \geq {}^\bullet t$, denoted $t \in$ *enabled*(M). The firing of t leads to a marking $M' = M - {}^\bullet t + t^\bullet$, denoted by $M \xrightarrow{t} M'$.

Often, the alphabet is the set of transitions and the labeling function the identity ($\Sigma = T, \Lambda(t) = t$). In these cases, the label of the transition will not be put in figures.

Notations for all timed Petri nets In timed extensions of Petri nets, a transition can be fired only if the enabling condition *and* some time related condition are satisfied. In the following, the expressions *enabled* and *enabling* refer only to the marking condition, and *firable* is the conjunction of enabling and the model-specific timed condition.

Then, $t \in$ *firable*(S) denotes that t is firable in timed state S, and $t \in$ *enabled*(M) that t is enabled by marking M.

Weak vs. strong semantics The basic strong semantics paradigm is expressed in different ways depending on the authors: one expression could be "time elapsing can not disable the firable property of a transition", or "whenever the upper bound of a firing interval is reached, the transition must be fired". Depending on the models and the authors, this principle is described by different equations. In this paper, the one we are going to use is: a delay d is admissible from state S ([5]) iff

$$t \notin \textit{firable}(S+d) \Rightarrow \forall d' \in [0,d] : t \notin \textit{firable}(S+d') \tag{1}$$

which means that from S, if a transition is not firable after a delay d, it never was between S and $S+d$, which is equivalent[6] to say that, if a transition is enabled now or in the future (without discrete transition firing), it remains firable with time elapsing.

3.2. Transition Time Petri Nets (*T-TPN*)

The model. Time Petri Nets were introduced in [20] and extend Petri nets with timing constraints on the firings of transitions.

Definition 3.2. (Transition Time Petri Net)
A *Time Petri Net* $\mathcal{N}$ is a tuple $(P, T, {}^\bullet(.), (.)^\bullet, M_0, \Lambda, I)$ where: $(P, T, {}^\bullet(.), (.)^\bullet, M_0, \Lambda)$ is a Petri net and $I : T \rightarrow \mathcal{I}(\mathbb{Q}_{\geq 0})$ associates with each transition a *firing interval*.

Semantics of Transition Time Petri Nets.

The state of *T-TPN* is a pair (M, ν), where M is a marking and $\nu \in \mathbb{R}_{\geq 0}^T$ is a *valuation* such that each value $\nu(t_i)$ is the elapsed time since the last time transition t_i was enabled. $\mathbf{0}$ is the initial valuation with $\forall i \in [1..n], \mathbf{0}(t_i) = 0$.

For Transition Time Petri Net, notations $enabled$ and $firable$ are defined as follows :

$$t \in \textit{enabled}(M) \textit{ iff } M \geq {}^\bullet t \qquad\qquad t \in \textit{firable}(M, \nu) \textit{ iff } \begin{cases} t \in \textit{enabled}(M) \\ \nu(t) \in I(t) \end{cases}$$

The *newly enabled* function $\uparrow\!enabled(t_k, M, t_i) \in \mathbb{B}$ is true if t_k is enabled by the firing of transition t_i from marking M, and false otherwise. This definition of enabledness is based on [4, 2] which is the most common one. In this framework, a transition t_k is *newly enabled* after firing t_i from marking M if "it is not enabled by $M - {}^\bullet t_i$ and is enabled by $M' = M - {}^\bullet t_i + t_i^\bullet$" [4]. A discussion on other semantics can be found in [10]. Formally this gives:

$$\uparrow\!enabled(t_k, M, t_i) = \left(M - {}^\bullet t_i + t_i^\bullet \geq {}^\bullet t_k\right) \wedge \left((M - {}^\bullet t_i < {}^\bullet t_k) \vee (t_k = t_i)\right) \tag{2}$$

Notice that the condition $t_k = t_i$ is useless in the safe context.

Lemma 3.1. (Equivalent definition of "newly enabled" in safe *T-TPN*)
If $\mathcal{N}$ is a safe *T-TPN*, then

$$\uparrow\!enabled(t_k, M, t_i) = \begin{cases} t_k \in \textit{enabled}(M - {}^\bullet t_i + t_i^\bullet) \\ t_k^\bullet \cup {}^\bullet t_i \neq \emptyset \end{cases}$$

[5]The encoding of the state depends on the model.
[6]Because $(A \Rightarrow B) \equiv (\neg B \Rightarrow \neg A)$, then $(eq\ 1) \equiv \exists d' \in [0,d] : t \in \textit{firable}(S+d') \Rightarrow t \in \textit{firable}(S+d)$.

Definition 3.3. (Strong Semantics of *T-TPN*)
The semantics of a *T-TPN* $\mathcal{N}$ is a timed transition system $S_{\mathcal{N}} = (Q, q_0, \rightarrow)$ where: $Q = \{0,1\}^P \times (\mathbb{R}_{\geq 0})^n$, $q_0 = (M_0, \mathbf{0})$, $\longrightarrow \in Q \times (\Sigma_\varepsilon \cup \mathbb{R}_{\geq 0}) \times Q$ consists of the discrete and continuous transition relations:

- the discrete transition relation is defined $\forall t \in T$:

$$(M,\nu) \xrightarrow{\Lambda(t)} (M',\nu') \text{ \emph{iff} } \begin{cases} t \in \textit{firable}(M,\nu) \\ M' = M - {}^\bullet t + t^\bullet \\ \forall t' \in T : \nu'(t') = \begin{cases} 0 & \textit{if } \uparrow\!enabled(t', M, t), \\ \nu(t') & \textit{otherwise}. \end{cases} \end{cases}$$

- the continuous transition relation is defined $\forall d \in \mathbb{R}_{\geq 0}$:

$$(M,\nu) \xrightarrow{d} (M,\nu') \text{ \emph{iff} } \begin{cases} \nu' = \nu + d \\ \forall t \in T : t \notin \textit{firable}(M, v+d) \Rightarrow \\ \qquad (\forall d' \in [0,d] : t \notin \textit{firable}(M, v+d')) \end{cases} \tag{3}$$

Notice that, for the sake of simplicity, a valuation is associated to each transition, even those who are not enabled. The same will apply for *P-TPN* and *A-TPN*.

Lemma 3.2. (Equivalent definition of the continuous transition relation)
An equivalent definition of the continuous transition relation is:

$$(M,\nu) \xrightarrow{d} (M,\nu') \text{ \emph{iff} } \begin{cases} \nu' = \nu + d \\ \forall t \in T, \left(M \geq {}^\bullet t \implies \nu'(t) \in I(t)^{\downarrow}\right) \end{cases} \tag{4}$$

Proof:
First of all, a little property is needed: for all reachable states, if a transition is enabled, its clock is in the downward closure of its timing interval.

$$M \geq {}^\bullet t \Rightarrow \nu(t) \in I(t)^{\downarrow} \tag{5}$$

This comes from the fact that, in strong semantics, the clock of a transition can never overtake the upper bound of its interval. It can be easily proved by induction.

(3) $\Rightarrow$ (4) Let be t a transition and $\nu' = \nu + d$.
Then (3) $\iff$ $t \in \textit{firable}(M, \nu+d) \vee (\forall d' \in [0,d], t \notin \textit{firable}(M, \nu+d'))$. If $t \in \textit{firable}(M, \nu+d)$, then $M \geq {}^\bullet t$ and $\nu+d \in I(t) \subset I(t)^{\downarrow}$. Otherwise, we have $\forall d' \in [0,d], t \notin \textit{firable}(M, \nu+d')$. If $M < {}^\bullet t$, (4) olds. If not, from (5), we know that $\nu(t) \in I(t)^{\downarrow}$, then, condition $\forall d' \in [0,d], \nu(t)+d' \notin I(t)$ means that $\nu(t)+d'$ does not reach the lower bound of $I(t)$ and remains in $I(t)^{\downarrow}$.

(4) $\Rightarrow$ (3) There are two cases: either $t \in \textit{firable}(M, \nu+d)$ (and (3) is satisfied) or it is not. In this case, it either comes from marking ($M < {}^\bullet t$) or time constraint ($\nu(t)+d \notin I(t)$)). If $M < {}^\bullet t$, (3) is obviously satisfied. Otherwise, $\nu(t)+d \in I(t)^{\downarrow}$ but $\nu(t)+d \notin I(t)$, that is to say, the clock did not reach the lower bound and $\forall d' \in [0,d], \nu(t)+d' \notin I(t)$. □

Definition 3.4. (Weak Semantics of *T-TPN*)
For safe *T-TPN*, the only difference of the weak semantics is on the continuous transition relation defined $\forall d \in \mathbb{R}_{\geq 0}$:

$$(M, \nu) \xrightarrow{d} (M, \nu') \textit{ iff } \nu' = \nu + d$$

Examples.

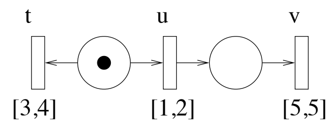

Figure 1. Priority in strong semantics

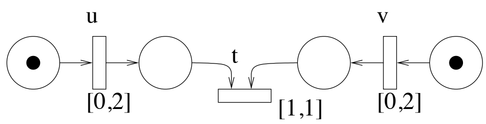

Figure 2. Synchronization

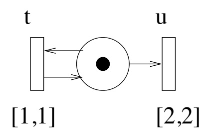

Figure 3. Continuous enabling

Figure 1 illustrates the difference between weak and strong semantics: in the initial marking, only t and u are enabled. After one time unit delay, u is firable, in both semantics. Then, the behaviours split:

- in the strong semantics, because u reaches its upper interval always before t becomes firable ($3 > 2$), t can never be fired. v is fired exactly five time units after the firing of u.
- in the weak semantics, u can overlap its upper bound, and t can be fired after being enabled 3 times units up to 4 time units. It also can not be fired. If u is fired, v can be fired 5 time units after firing of u, but it also may not.

Figure 2 illustrates the synchronization rule: u (resp. v) is fired at an absolute date $\theta_u \leq 2$ (resp. $\theta_v \leq 2$), and t can be fired at $max(\theta_u, \theta_v) + 1$. The difference between the weak and strong semantics is that, in the weak semantics, transitions may not be fired.

Figure 3 illustrates another important point: the continuous enabling. In this *T-TPN* with the strong semantics, transition u will *never* be fired, because, at each time unit, t is fired, removing the token and putting it back immediately. Then, u is at most 1 time unit continuously enabled, never 2 time units. With the weak semantics, u is fired iff t overlaps its upper bound.

3.3. Place Time Petri Nets (*P-TPN*)

The model. Place Time Petri Nets were introduced in [18], adding interval on places and considering a strong semantics.

Putting interval on places implies that clocks are handled by tokens: a token can be use to fire a transition iff its age in the place is in the interval of the place. A particularity of this model is the notion of *dead token*. A token whose age is greater than the upper bound of its place can never leave this place: it is a *dead token*.

Let $dead$ be a mapping in $\{0,1\}^P$. $dead(p)$ is the number of dead tokens in place p ($\forall p \in P$: $dead(p) \leq M(p)$). We use the following shorthands : $M \backslash dead$ for $M - dead$ and thus $p \in M \backslash dead$ for $M(p) - dead(p) \geq 1$.

Definition 3.5. (Place Time Petri Net)
A *Place Time Petri Net* $\mathcal{N}$ is a tuple $(P, T, {}^\bullet(.), (.)^\bullet, M_0, \Lambda, I)$ where: $(P, T, {}^\bullet(.), (.)^\bullet, M_0, \Lambda)$ is a Petri net and $I : P \to \mathcal{I}(\mathbb{Q}_{\geq 0})$ associates with each place a *residence time* interval.

Semantics of Place Time Petri Nets.

The state of *P-TPN* is a tuple $(M, dead, \nu)$ where M is a marking, $dead$ is the dead token mapping and $\nu \in \mathbb{R}_{\geq 0}^{M}$ the age of tokens in places. A transition can be fired iff all tokens involved in the firing respect the residence interval in their places. Tokens are dropped with age 0. In strong semantics, if a token reaches its upper bound, and if there exists a firable transition that can consume this tokens, it must be fired.

For Place Time Petri Net, notations $enabled$ and $firable$ are defined as follows :

$$t \in \textit{enabled}(M \backslash dead) \textit{ iff } M - dead \geq {}^{\bullet}t$$

$$t \in \textit{firable}(M, dead, \nu) \textit{ iff } \begin{cases} t \in \textit{enabled}(M \backslash dead) \\ \forall p \in {}^{\bullet}t, \nu(p) \in I(p) \end{cases}$$

Definition 3.6. (Strong Semantics of *P-TPN*)
The semantics of a *P-TPN* $\mathcal{N}$ is a timed transition system $S_{\mathcal{N}} = (Q, q_0, \rightarrow)$ where: $Q = \{0,1\}^P \times \{0,1\}^P \times (\mathbb{R}_{\geq 0})^P$, $q_0 = (M_0, \mathbf{0}, \mathbf{0})$, $\longrightarrow \in Q \times (\Sigma_\varepsilon \cup \mathbb{R}_{\geq 0}) \times Q$ consists of the discrete and continuous transition relations:

The discrete transition relation is defined $\forall t \in T$:

$$(M, dead, \nu) \xrightarrow{\Lambda(t)} (M', dead, \nu') \textit{ iff } \begin{cases} t \in \textit{firable}(M, dead, \nu) \\ M' = M - {}^{\bullet}t + t^{\bullet} \\ \nu'(p) = \begin{cases} 0 & \textit{if } (dead(p) = 0) \wedge (p \in t^{\bullet}) \\ \nu(p) & \textit{otherwise}. \end{cases} \end{cases}$$

The continuous transition relation is defined $\forall d \in \mathbb{R}_{\geq 0}$:

$$(M, dead, \nu) \xrightarrow{d} (M, dead', \nu') \textit{ iff}$$

$$\begin{cases} \nu' = \nu + d \\ \forall t \in T : t \notin \textit{firable}(M, dead, v + d) \Rightarrow (\forall d' \in [0, d] : t \notin \textit{firable}(M, dead, v + d')) \\ dead'(p) = \begin{cases} 1 & \text{if } (p \in M \backslash dead) \wedge (v'(p) \notin I(p)^{\downarrow}) \\ dead(p) & \text{otherwise} \end{cases} \end{cases}$$

Definition 3.7. (Weak Semantics of *P-TPN*)
The weak semantics is exactly the same as the strong one without the condition $\forall t \in T : t \notin \textit{firable}(M, dead, v + d) \Rightarrow (\forall d' \in [0, d] : t \notin \textit{firable}(M, dead, v + d')$ in the continuous transition relation.

3.4. Arc Time Petri Nets (*A-TPN*)

The model. Arc Time Petri Nets were introduced in [25], adding interval on arcs and considering a weak semantics.

Like in *P-TPN*, an age is associated with each token. A transition t can be fired iff the tokens in the input places p satisfy the constraint on the arc from the place to the transition.

As for *P-TPN*, there could exist *dead tokens*, that is to say, tokens whose age is greater than the upper bound of all output arcs.

Definition 3.8. (Arc Time Petri Net)
An *Arc Time Petri Net* $\mathcal{N}$ is a tuple $(P, T, {}^\bullet(.), (.)^\bullet, M_0, \Lambda, I)$ where: $(P, T, {}^\bullet(.), (.)^\bullet, M_0, \Lambda)$ is a Petri net and $I : P \times T \to \mathcal{I}(\mathbb{Q}_{\geq 0})$ associates with each arc from place to transition a *time interval*.

For Arc Time Petri Net, notations $enabled$ and $firable$ are defined as follows:

$$t \in \textit{enabled}(M \backslash dead) \text{ \textit{iff} } M - dead \geq {}^\bullet t$$

$$t \in \textit{firable}(M, dead, \nu) \text{ \textit{iff} } \begin{cases} t \in \textit{enabled}(M \backslash dead) \\ \forall p \in {}^\bullet t, \nu(p) \in I(p,t) \end{cases}$$

Semantics of Arc Time Petri Nets. Like for *P-TPN*, the state of *A-TPN* is a tuple $(M, dead, \nu)$ where M is a marking, $dead$ is the dead token mapping and $\nu \in \mathbb{R}_{\geq 0}^M$ the age of tokens in places. A transition t can be fired iff all tokens involved in the firing respect the constraint on arc from their place to the transition. Tokens are dropped with age 0. In strong semantics, if a token reaches one of its upper bound, and if there exists a transition that consumes this tokens, it must be fired.

Definition 3.9. (Strong Semantics of *A-TPN*)
The semantics of a *P-TPN* $\mathcal{N}$ is a timed transition system $S_\mathcal{N} = (Q, q_0, \to)$ where: $Q = \{0,1\}^P \times \{0,1\}^P \times (\mathbb{R}_{\geq 0})^P$, $q_0 = (M_0, \mathbf{0})$, $\longrightarrow \in Q \times (\Sigma_\varepsilon \cup \mathbb{R}_{\geq 0}) \times Q$ consists of the discrete and continuous transition relations: The discrete transition relation has the same definition that the one of *A-TPN* (with its specific definition of *firable*). The continuous transition relation is defined $\forall d \in \mathbb{R}_{\geq 0}$:

$$(M, dead, \nu) \xrightarrow{d} (M, dead', \nu') \text{ \textit{iff} }$$
$$\begin{cases} \nu' = \nu + d \\ \forall t \in T : t \notin \textit{firable}(M, dead, v + d) \Rightarrow (\forall d' \in [0,d] : t \notin \textit{firable}(M, dead, v + d')) \\ dead'(p) = \begin{cases} 1 & \text{if } \begin{cases} p \in M \backslash dead \\ \forall t \in p^\bullet, \nu'(p) \notin I(p,t)^\downarrow \end{cases} \\ dead(p) & \textit{otherwise} \end{cases} \end{cases} \tag{6}$$

The definition of semantics of *A-TPN* and *P-TPN* are very similar: the only difference is that, in the definition of *A-TPN*, the timing condition for firable is $\forall p \in {}^\bullet t : \nu(p) \in I(p,t)$ as in *P-TPN*, it's $\forall p \in {}^\bullet t : \nu(p) \in I(p)$, and the same for the condition associated with $dead$.

Definition 3.10. (Weak Semantics of *A-TPN*)
The weak semantics is exactly the same as the strong one without the condition $\forall t \in T : t \notin \textit{firable}(M, dead, v + d) \Rightarrow (\forall d' \in [0,d] : t \notin \textit{firable}(M, dead, v + d'))$ in the continuous transition relation.

4. Comparison of the expressiveness w.r.t. bisimulation

In the sequel we will compare various classes of safe TPN w.r.t. bisimulation. We note $\overline{\textit{T-TPN}}$ and $\underline{\textit{T-TPN}}$, for the classes of safe Transition Time Petri Nets respectively with strong and weak semantics. We note $\overline{\textit{A-TPN}}$ and $\underline{\textit{A-TPN}}$, for the classes of safe Arc Time Petri Nets respectively with strong and weak semantics. We note $\overline{\textit{P-TPN}}$ and $\underline{\textit{P-TPN}}$, for the classes of safe Place Time Petri Nets respectively with strong and weak semantics.

A *run* of a time Petri net $\mathcal{N}$ is a (finite or infinite) path in $S_\mathcal{N}$ starting in q_0. As a shorthand, we write that there is a *run* from a state q in $\mathcal{N}$ if there is a run $q_0 \xrightarrow{\rho_0} q \xrightarrow{\rho}$ in $S_\mathcal{N}$.

Moreover, we write $\mathcal{N}$ for $S_\mathcal{N}$ (i.e. we will use the shorthand : *a run* ρ *of* $\mathcal{N}$ or *a state* q *of* $\mathcal{N}$).

4.1. $\overline{\textit{X-TPN}} \not\subseteq_\approx \underline{\textit{X-TPN}}$ with $X \in \{T, A, P\}$

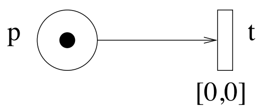

Figure 4. A “non-delay” $\overline{\textit{T-TPN}}$

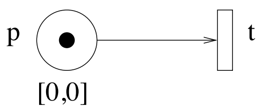

Figure 5. A “non-delay” $\overline{\textit{P-TPN}}$

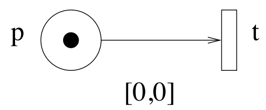

Figure 6. A “non-delay” $\overline{\textit{A-TPN}}$

Theorem 4.1. (Weak semantics can not emulate strong semantics)

$$\overline{\textit{P-TPN}} \not\subseteq_\approx \underline{\textit{P-TPN}} \qquad \overline{\textit{T-TPN}} \not\subseteq_\approx \underline{\textit{T-TPN}} \qquad \overline{\textit{A-TPN}} \not\subseteq_\approx \underline{\textit{A-TPN}}$$

Proof:

By contradiction: assume there exists a $\underline{\textit{T-TPN}}$ weakly timely bisimular to the $\overline{\textit{T-TPN}}$ of Figure 4. From its initial state, a delay of duration $d > 0$ is possible (in weak semantics, a delay is always possible). By bisimulation hypothesis, it should also be possible from the initial state of the strong *T-TPN* of Figure 4. Since t is a visible action ($\Lambda(t) = t \neq \epsilon$), this contradicts our assumption.

The same applies for *P-TPN* and *A-TPN*. □

4.2. $\underline{\textit{P-TPN}} \subset_\approx \overline{\textit{P-TPN}}$

Let be $\mathcal{N} \in \underline{\textit{P-TPN}}$. We construct a TPN $\overline{\mathcal{N}} \in \overline{\textit{P-TPN}}$ as follow :

- we start from $\overline{\mathcal{N}} = \mathcal{N}$ and $\overline{M_0} = M_0$,
- for each place p of $\mathcal{N}$,
  - we add to $\overline{\mathcal{N}}$, the net in the gray area of the Figure 7 with a token in place p_2^t.
  - for each transition t such that $p \in {}^\bullet t$, we add an arc from p_2^t to t and an arc from t to p_2^t.

Note that in the gray area, there is always a token either in place p_1^t or in the place p_2^t.

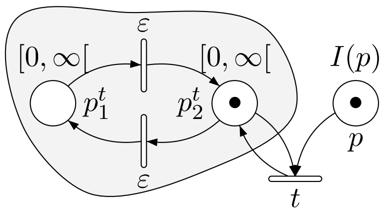

Figure 7. The translation from $\underline{\textit{P-TPN}}$ into $\overline{\textit{P-TPN}}$

Lemma 4.1. (Translating a $\underline{P\text{-}TPN}$ into a $\overline{P\text{-}TPN}$)
Let $\mathcal{N} \in \underline{P\text{-}TPN}$ and $\overline{\mathcal{N}} \in \overline{P\text{-}TPN}$ its translation into $\overline{P\text{-}TPN}$ as defined previously, $\mathcal{N}$ and $\overline{\mathcal{N}}$ are timed bisimilar.

Proof:
$\mathcal{N} = (P, T, {}^\bullet(.), (.)^\bullet, M_0, I)$ and $\overline{\mathcal{N}} = (\overline{P}, \overline{T}, \overline{{}^\bullet(.)}, \overline{(.)^\bullet}, \overline{M_0}, \overline{I})$. Note that $P \subset \overline{P}$ and $T \subset \overline{T}$.

Let $(M, dead, \nu)$ be a state of $\mathcal{N}$ and $(\overline{M}, \overline{dead}, \overline{\nu})$ be a state of $\overline{\mathcal{N}}$. We define the relation $\approx \subseteq ((\{0,1\} \times \mathbb{R}_{\geq 0})^P \times (\{0,1\} \times \mathbb{R}_{\geq 0})^{\overline{P}}$ by:

$$(M, dead, \nu) \approx (\overline{M}, \overline{dead}, \overline{\nu}) \iff \forall p \in P \begin{cases} (1)\, M(p) = \overline{M}(p) \\ (2)\, dead(p) = \overline{dead}(p) \\ (3)\, \nu(p) = \overline{\nu}(p) \end{cases} \tag{7}$$

Now we can prove that $\approx$ is a weak timed bisimulation relation between $\mathcal{N}$ and $\overline{\mathcal{N}}$.

Proof : First we have $(M_0, dead_0, \nu_0) \approx (\overline{M_0}, \overline{dead_0}, \overline{\nu_0})$.

Let us consider a state $q = (M, dead, \nu) \in \mathcal{N}$ and a state $\overline{q} = (\overline{M}, \overline{dead}, \overline{\nu}) \in \overline{\mathcal{N}}$ such that $(\overline{M}, \overline{dead}, \overline{\nu}) \approx (M, dead, \nu)$.

- *Discrete transitions* Let t be a firable transition from $q = (M, dead, \nu)$ in $\mathcal{N}$. There is a run $\rho_1 = (M, dead, \nu) \xrightarrow{t} (M_1, dead_1, \nu_1)$ (with $dead = dead_1$). It means that $\forall p \in {}^\bullet(t)\ \nu(p) \in I(p)^\downarrow$. Moreover, $M_1 = M - {}^\bullet t + t^\bullet$ and $\forall p \in M_1 \backslash dead_1$, $\nu_1(p) = 0$ if $p \in t^\bullet$.

  In $\overline{\mathcal{N}}$, as $(\overline{M}, \overline{dead}, \overline{\nu}) \approx (M, dead, \nu)$ we have $\forall p \in {}^\bullet(t)\ \overline{\nu}(p) \in \overline{I}(p)^\downarrow$. Moreover $\overline{\nu}(p_2^t) \in \overline{I}(p_2^t)^\downarrow$(with upper bound : ∞) and there is a token either in p_1^t or in p_2^t. Thus, there is a run $\overline{\rho}_1 = (\overline{M}, \overline{dead}, \overline{\nu}) \xrightarrow{\epsilon *} (\overline{M}_{\varepsilon_1}, \overline{dead}_{\varepsilon_1}, \overline{\nu}_{\varepsilon_1}) \xrightarrow{t} (\overline{M}_1, \overline{dead}_1, \overline{\nu}_1)$ with $(\overline{M}_{\varepsilon_1}, \overline{dead}_{\varepsilon_1}, \overline{\nu}_{\varepsilon_1}) \approx (M, dead, \nu)$ and $\overline{M}_{\varepsilon_1}(p_2^t) = 1$. We have $\overline{M_1} = \overline{M} - \overline{{}^\bullet t} + \overline{t^\bullet}$ that is to say $\overline{M}_1(p_2^t) = 1$, $\overline{M}_1(p_1^t) = 0$ and $\forall p \in P$, $\overline{M}_1(p) = M_1(p)$. Moreover $\overline{dead} = \overline{dead_1}$ and $\forall p \in \overline{M}_1 \backslash \overline{dead}_1$, $\overline{\nu}_1(p) = 0$ if $p \in \overline{t^\bullet}$ and then $\forall p \in P, \overline{\nu}_1(p) = \nu_1(p)$. Thus $(\overline{M}_1, \overline{dead}_1, \overline{\nu}_1) \approx (M_1, dead_1, \nu_1)$.

- *Continuous transitions* In $\mathcal{N}$, from $q = (M, dead, \nu)$, there is a run $\rho_2 = (M, dead, \nu) \xrightarrow{d} (M_2, dead_2, \nu_2)$ such that $\forall p \in M(p), \nu_2(p) = \nu(p) + d$ and $M = M_2$. Moreover, $\forall p \in M \backslash dead$, $M_2(p) = 1$ and $dead_2(p) = 0$ if $\nu_2(p) \in I(p)^\downarrow$ and $M_2(p) = dead_2(p) = 1$ if $\nu_2(p) \notin I(p)^\downarrow$.

  - if there is no firable transition t such that $\exists p_t \in {}^\bullet(t)$ with $\nu(p_t) \in I(p_t)^\downarrow$ and $\nu_2(p_t) \notin I(p_t)^\downarrow$. As $(M, dead, \nu) \approx (\overline{M}, \overline{dead}, \overline{\nu})$, we have $\forall p \in P$, $M(p) = \overline{M}(p)$, $dead(p) = \overline{dead}(p)$ and $\nu(p) = \overline{\nu}(p)$ and then in $\overline{\mathcal{N}}$, there is a run $\overline{\rho}_2 = (\overline{M}, \overline{dead}, \overline{\nu}) \xrightarrow{d} (\overline{M}_2, \overline{dead}_2, \overline{\nu}_2)$ such that $\overline{dead}_2 = \overline{dead}$ and $\forall p \in P$, $\overline{M}_2(p) = M_2(p)$ and $\overline{\nu}_2(p) = \overline{\nu}_2(p) + d = \nu_2(p)$. Thus $(\overline{M}_2, \overline{dead}_2, \overline{\nu}_2) \approx (M_2, dead_2, \nu_2)$
  - if there is a firable transition t such that $\exists p_t \in {}^\bullet(t)$ with $\nu(p_t) \in I(p_t)^\downarrow$ and $\nu_2(p_t) \notin I(p_t)^\downarrow$ (and then $dead(p_t) = 0$ and $dead_2(p_t) = 1$). As $(M, dead, \nu) \approx (\overline{M}, \overline{dead}, \overline{\nu})$, we have $\forall p \in P$, $M(p) = \overline{M}(p)$, $dead(p) = \overline{dead}(p)$ and $\nu(p) = \overline{\nu}(p)$. In $\overline{\mathcal{N}}$, there is a run $\overline{\rho}_2 = (\overline{M}, \overline{dead}, \overline{\nu}) \xrightarrow{\epsilon *} (\overline{M}_{\varepsilon_2}, \overline{dead}_{\varepsilon_2}, \overline{\nu}_{\varepsilon_2})$ such that $(\overline{M}_{\varepsilon_2}, \overline{dead}_{\varepsilon_2}, \overline{\nu}_{\varepsilon_2}) \approx (M, dead, \nu)$ and $\overline{M}_{\varepsilon_2}(p_2^t) = 0$. Thus, there is a run $(\overline{M}_{\varepsilon_2}, \overline{dead}_{\varepsilon_2}, \overline{\nu}_{\varepsilon_2}) \xrightarrow{d} (\overline{M}_2, \overline{dead}_2, \overline{\nu}_2)$ such that $\overline{M}_2(p_t) = \overline{dead}_2(p_t) = 1$ and $\forall p \in P$, $\overline{\nu_2}(p) = \overline{\nu}(p) + d = \nu_2(p)$ and then $(\overline{M}_2, \overline{dead}_2, \overline{\nu}_2) \approx (M_2, dead_2, \nu_2)$.

The converse is straightforward following the same steps as the previous ones. □

Theorem 4.2. (The strong semantics is strictly more expressive for *P-TPN*)

$$\underline{P\text{-}TPN} \subset_{\approx} \overline{P\text{-}TPN}$$

Proof:
As $\overline{P\text{-}TPN} \not\subseteq_{\approx} \underline{P\text{-}TPN}$ (Theorem 4.1) and thanks to Lemma 4.1. □

4.3. $\underline{A\text{-}TPN} \subset_{\approx} \overline{A\text{-}TPN}$

Theorem 4.3. (The strong semantics is strictly more expressive for *A-TPN*)

$$\underline{A\text{-}TPN} \subset_{\approx} \overline{A\text{-}TPN}$$

Proof:
As for Theorem 4.2 □

4.4. $\underline{T\text{-}TPN} \not\subseteq_{\approx} \overline{T\text{-}TPN}$

We first recall the following theorem :

Theorem 4.4. ([11])
There is no TPN $\in \overline{T\text{-}TPN}$ weakly timed bisimilar to $\mathcal{A}_0 \in \mathcal{TA}$ (Fig. 8).

Theorem 4.5. (The strong semantics does not generalise the weak one for *T-TPN*)

$$\underline{T\text{-}TPN} \not\subseteq_{\approx} \overline{T\text{-}TPN}$$

Proof:
We first prove that the TPN $\mathcal{N}_{T0} \in \underline{T\text{-}TPN}$ of Fig. 9 is weakly timed bisimilar to $\mathcal{A}_0 \in \mathcal{TA}$ (Fig. 8).
Let (ℓ, v) be a state of $\mathcal{A}_0 \in \mathcal{TA}$ where $\ell \in \{\ell_0, \ell_1\}$ and $v(x) \in \mathbb{R}_{\geq 0}$ is the valuation of the clock x. We define the relation $\approx \subseteq (\{\ell_0, \ell_1\} \times \mathbb{R}_{\geq 0}) \times (\{0,1\} \times \mathbb{R}_{\geq 0})$ by:

$$(\ell, v) \approx (M, \nu) \iff \begin{cases} (1)\, \ell = \ell_0 \iff M(P_1) = 1 \\ \quad\;\; \ell = \ell_1 \iff M(P_1) = 0 \\ (2)\, v(x) = \nu(a) \end{cases} \tag{8}$$

$\approx$ is a weak timed bisimulation (The proof is straightforward).
From Theorem 4.4, there is no TPN $\in \overline{T\text{-}TPN}$ weakly timed bisimilar to $\mathcal{A}_0 \in \mathcal{TA}$ (Fig. 8) and the TPN $\mathcal{N}_{T0} \in \underline{T\text{-}TPN}$ of Fig. 9 is weakly timed bisimilar to $\mathcal{A}_0$. □

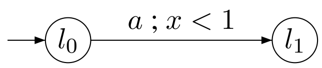

Figure 8. The Timed Automaton $\mathcal{A}_0$

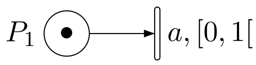

Figure 9. The TPN $\mathcal{N}_{T0} \in \underline{\textit{T-TPN}}$ bisimilar to $\mathcal{A}_0$

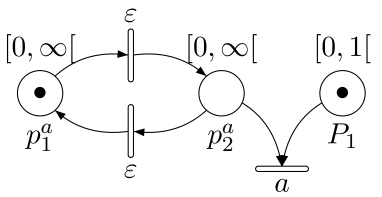

Figure 10. The TPN $\mathcal{N}_{P0} \in \overline{\textit{P-TPN}}$ bisimilar to $\mathcal{A}_0$

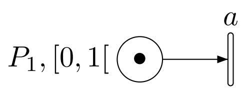

Figure 11. A TPN $\mathcal{N}_{P1} \in \underline{\textit{P-TPN}}$ bisimilar to $\mathcal{N}_{P0}$

4.5. $\overline{\textit{P-TPN}} \not\subseteq_{\approx} \overline{\textit{T-TPN}}$

Lemma 4.2. The TPN $\mathcal{N}_{P0} \in \overline{\textit{P-TPN}}$ (Fig.10) is weakly timed bisimilar to $\mathcal{A}_0 \in \mathcal{TA}$ (Fig. 8).

Proof:
From Lemma 4.1, $\mathcal{N}_{P0} \approx \mathcal{N}_{P1}$. Obviously, $\mathcal{N}_{P1} \approx \mathcal{N}_{T0}$. And, from proof of Theorem 4.5, $\mathcal{N}_{T0} \approx \mathcal{A}_0$. By transitivity, $\mathcal{N}_{P0} \approx \mathcal{A}_0$.

($\mathcal{N}_{P0}, \mathcal{N}_{P1}, \mathcal{N}_{T0}$ and $\mathcal{A}_0$ are respectivly presented in Figures 10, 11, 9, 8). □

Theorem 4.6. (In strong semantics, *T-TPN* does not generalise *P-TPN*)

$$\overline{\textit{P-TPN}} \not\subseteq_{\approx} \overline{\textit{T-TPN}}$$

Proof:
From Theorem 4.4, there is no TPN $\in \overline{\textit{T-TPN}}$ weakly timed bisimilar to $\mathcal{A}_0 \in \mathcal{TA}$ (Fig. 8) and from Lemma 4.2, the TPN $\mathcal{N}_{P0} \in \overline{\textit{P-TPN}}$ is weakly timed bisimilar to $\mathcal{A}_0$. □

4.6. $\overline{\textit{T-TPN}} \not\subseteq_{\approx} \overline{\textit{P-TPN}}$ and $\underline{\textit{T-TPN}} \not\subseteq_{\approx} \underline{\textit{P-TPN}}$

Definition 4.1. (Relevant clock of a *P-TPN*)
Let $\mathcal{N} = (P, T, {}^{\bullet}(.), (.)^{\bullet}, M_0, \Lambda, I)$ be a *P-TPN* ($\overline{\textit{P-TPN}}$ or $\underline{\textit{P-TPN}}$), and $q = (M, dead, \nu)$ be a state of $\mathcal{N}$. In q, a clock x associated to a place $p \in P$ is said to be *relevant* iff $M(p) = 1$.

We first give a lemma stating that "in *P-TPN* ($\overline{\textit{P-TPN}}$ or $\underline{\textit{P-TPN}}$) a relevant clock (associated to a token in a marked place p) can become irrelevant or can be reset only in its firing interval ($\nu(p) \in I(p)$)".

Lemma 4.3. (Reset of relevant clock in *P-TPN*)
In P-TPN, a relevant clock can become irrelevant or can be reset only in its firing interval. Let $\mathcal{N}$, be a *P-TPN* ($\overline{\textit{P-TPN}}$ or $\underline{\textit{P-TPN}}$). Let $(M, dead, \nu)$ be a state of $\mathcal{N}$ such that $M(p) > 0$ and $\nu(p) > 0$. If $(M, dead, \nu) \rightarrow (M', dead', \nu')$ (where $\rightarrow$ is a discrete or a continuous transition) and $\nu'(p) = 0$ or $M'(p) = 0$ then $\nu(p) \in I(p)$

Proof:
From the semantics of *P-TPN* ($\overline{\textit{P-TPN}}$ or $\underline{\textit{P-TPN}}$), a relevant clock associated to a place p $(M(p) = 1)$ can become irrelevant or can be reset only by a discrete transition $(M, dead, \nu) \xrightarrow{t} (M', dead, \nu')$ such that $p \in {}^{\bullet}t$ (if $p \in t^{\bullet}$ the relevant clock is reset, otherwise it become irrelevant). Then, as $t \in$ *firable*$(M, dead, \nu)$, we have $\nu(p) \in I(p)$. $\square$

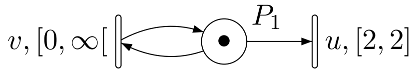

Figure 12. The TPN $\mathcal{N}_{T1} \in \overline{\textit{T-TPN}}$

Theorem 4.7. There is no TPN $\in \overline{\textit{P-TPN}}$ weakly timed bisimilar to $\mathcal{N}_{T1} \in \overline{\textit{T-TPN}}$ (Fig. 12).

Proof:
The idea of the proof is that in the *T-TPN* $\mathcal{N}_{T1}$ the clock associated to the transition u can be reset at any time (in particular before 2 time units). In the *P-TPN*, time measure is performed by a finite number of clock.

Corollary 4.1. (In strong semantics, *P-TPN* does not generalise *T-TPN*)

$$\overline{\textit{T-TPN}} \not\subseteq_{\approx} \overline{\textit{P-TPN}}$$

Proof:
Direct from Theorem 4.7. $\square$

Moreover, the Theorem 4.7 remains valid in weak semantics. Indeed, we can consider the net of the Fig. 12 with a weak semantics and the proof of Theorem 4.7 remains identical. It just needs to rewrite the no-death assumption as : there exists q_0'' such that behaviour from q_0'' in $\mathcal{N}'$ is bisimilar to $\mathcal{N}_{T1}$ without death of any token. The proof is then : if a state q_0' require the death of a token in a place p to be bisimilar to q_0, we can kill this token (by firing the corresponding run) and then fire v to go back to a state which must be bisimilar to q_0 and so on until q_0''.

We have then the following corollary.

Corollary 4.2. (In weak semantics, *P-TPN* does not generalise *T-TPN*)

$$\underline{\textit{T-TPN}} \not\subseteq_{\approx} \underline{\textit{P-TPN}}$$

4.7. $\overline{\textbf{\textit{T-TPN}}} \subset_{\approx} \overline{\textbf{\textit{A-TPN}}}$ and $\underline{\textbf{\textit{T-TPN}}} \subseteq_{\approx} \underline{\textbf{\textit{A-TPN}}}$

The proof of this strict inclusion is done in two steps: Lemma 4.4 (in Section 4.7.1) shows that $\overline{\textit{T-TPN}} \subseteq_{\approx} \overline{\textit{A-TPN}}$ (by construction: for each $\overline{\textit{T-TPN}}$, a weak-bisimilar $\overline{\textit{A-TPN}}$ is built), and Lemma 4.5 shows that there exists a $\overline{\textit{A-TPN}}$ bisimilar to $\mathcal{A}_0 \in \mathcal{TA}$ (Fig. 8) already used in Theorem 4.4. With these two lemmas, the strict inclusion is straightforward (Section 4.7.3).

4.7.1. Weak inclusion: : $\overline{\textit{T-TPN}} \subseteq_{\approx} \overline{\textit{A-TPN}}$ and $\underline{\textit{T-TPN}} \subseteq_{\approx} \underline{\textit{A-TPN}}$

Lemma 4.4. (From *T-TPN* to *A-TPN*)

$$\overline{\textit{T-TPN}} \subseteq_{\approx} \overline{\textit{A-TPN}} \qquad\qquad \underline{\textit{T-TPN}} \subseteq_{\approx} \underline{\textit{A-TPN}}$$

The proof is done by construction: for each *T-TPN* $\mathcal{N}$, a weak-bisimilar *A-TPN* $\mathcal{N}'$ is built. The main issue is to emulate the *T-TPN* "start clock when all input places are marked" rule with the *A-TPN* rule "start clock as soon as the token is in place".

The main idea is, for each transition t in a *T-TPN* $\mathcal{N}$, to build a *chain* of places ${}^\circ t^0, \ldots, {}^\circ t^n$ (with $n = |{}^\bullet t|$) in the translated *A-TPN* $\mathcal{N}'$, such that $\sum_{p \in {}^\bullet t} M_{\mathcal{N}}(p) = i \iff M_{\mathcal{N}'}({}^\circ t^i) = 1$ (with $i \in [1, n]$). Therefor, the time interval $I_{\mathcal{N}}(t)$ is set to arc from ${}^\circ t^{|{}^\bullet t|}$ to t. Then, the rule "start clock in $I(t)$ when all input places of t are marked" is emulated by the rule "start clock constraint in $I({}^\circ t^{|{}^\bullet t|}, t)$ when ${}^\circ t^{|{}^\bullet t|}$ is marked" which is equivalent because $I_{\mathcal{N}'}({}^\circ t^{|{}^\bullet t|}, t) = I_{\mathcal{N}}(t)$ and $\sum_{p \in {}^\bullet t} M_{\mathcal{N}}(p) = n \iff M_{\mathcal{N}'}({}^\circ t^n) = 1$.

Once this done, a little stuff has to be added to handle conflict and reversible nets[7].

It should be noticed that exactly the same translation applies for weak and strong semantics. Nevertheless, to improve readability, two proofs are given. One technical difficulty of the proof comes from the *dead* tokens: the are no dead tokens in *T-TPN* definitions, but there are in *P-TPN*. With the strong semantics, these dead tokens never appear, then, they can be neglected. But in weak semantics, they have to be handled.

Emulating the *T-TPN* firing rule The emulation pattern is presented with help of an example: the *T-TPN* of Figure 2 is translated into the *A-TPN* of Figure 13.

In *T-TPN*, a timed condition is activated when a transition t is enabled, that is to say, when there are enough tokens in the places ${}^\bullet t$. Conversely, in *A-TPN* a timed condition is activated when a token enter into a place. To emulate the first condition with the second one, a chain of places $\left\{{}^\circ t^0, \ldots, {}^\circ t^{|{}^\bullet t|}\right\}$ is introduced[8], like in Figure 13. Then, the firing condition is activated only when there is one token in place ${}^\circ t^{|{}^\bullet t|}$ (${}^\circ t^2$ in the example), that is to say, when there are enough tokens in the emulated places ${}^\bullet t$.

With this chain structure, the firing of the transition u (resp. v) must increase the marking of ${}^\bullet t$, *i.e.* put a token in ${}^\circ t^1$ *or* ${}^\circ t^2$ (depending on the previous marking).

Since bisimulation is based on the timed transition system where only labels of transitions are visible, the transition u can be replaced by two transitions, one putting a token in ${}^\circ t^1$ and the other in ${}^\circ t^2$, as long as they have the same label.

In Figure 13, these two transitions are called $u_{(t:0,1)}$ and $u_{(t:1,2)}$ and $\Lambda(u_{(t:0,1)}) = \Lambda(u_{(t:1,2)}) = \Lambda(u)$ ([9]).

[7] This translation pattern have been used in [5] to translate $\overline{\textit{T-TPN}}$ into $\overline{\textit{P-TPN}}$, but it was a mistake. The translation only apply in some specific cases: when transitions are conflict-free or when the lower bound of time intervals is 0 for example (see[7]).

[8] Be careful to this notation : ${}^\bullet t$ is the set of input places of a transition t, and ${}^\circ t^i$ is a place in the built *P-TPN*. This notation has been chosen to underline the fact that this chain of places $\left\{{}^\circ t^0, \ldots, {}^\circ t^{|{}^\bullet t|}\right\}$ in the *A-TPN* emulates the marking of the places ${}^\bullet t$ in the *T-TPN*.

[9] Notation $u_{(t:1,2)}$ is used to denotes that this firing of u makes the marking of ${}^\bullet t$ going form 1 to 2.

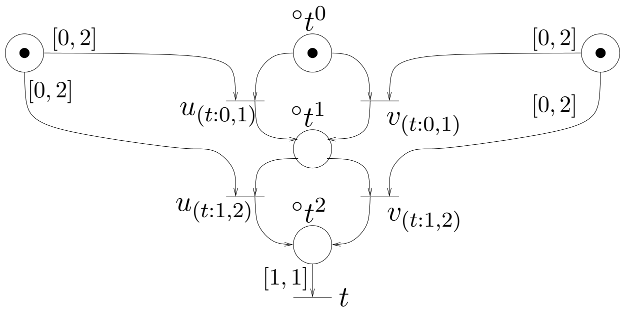

Figure 13. A translation of the $\overline{\textit{T-TPN}}$ of Figure 2 into $\overline{\textit{A-TPN}}$

Full formal translation With the chain structure introduced in the previous subsection, we are able to emulate the firing rule of *T-TPN* with *A-TPN* in simple examples: incrementing the marking of $^\bullet t$ up to enabling, and starting clock just when all input places are marked.

But the firing of a transition does not, in the general case, increment the input marking of just one transition: it can modify several input marking transition, adding or removing tokens.

The full translation should also be able to modify several chains, by increase or decrease.

For a given transition $t \in T$, its firing will remove tokens in the input places of some transitions and add in the input places of others. Then, for each of these transitions, for each possible marking of the input chain of these transition, the impact of the firing of t must be encoded in $\mathcal{N}'$.

Let $\mathcal{N} = (P, T, {}^\bullet(.), (.)^\bullet, M_0, \Lambda, I)$ be a *T-TPN*. For each t, let us define:

$$\textit{InfluencedBy}\,(t) = \{u \in T \,|\, {}^\bullet u \cap ({}^\bullet t \cup t^\bullet) \neq \emptyset\} \qquad \textit{InflOf}\,(t,u) = \sum_{p \in {}^\bullet u} (t,p)^\bullet - {}^\bullet(t,p)$$

Then, let $\mathcal{N}' = (P', T', Pre', Post', M_0', \Lambda', I')$ built as follow:

- the set of places is the one of $\mathcal{N}$ augmented with the chains $^{\circ}t^{0,\dots,|^\bullet t|}$

$$P' = P \cup \bigcup_{t \in T} \bigcup_{i \in \{0,\dots,|^\bullet t|\}} \{^{\circ}t^i\}$$

- each transition t is replaced by a set of transitions that emulates all it possible firing: for all transition u_k influenced[10] by the firing of t, and for each pair i_k, j_k of possible marking of $^\bullet u_k$ such that firing of t leads from i_k to j_k, a transition $t_{\{(u_1:i_1,j_1)\cdots(u_n:i_n,j_n)\}}$ is created. Let *Change* (t) be the set of tuples $\{(u_1 : i_1, j_1) \cdots (u_n : i_n, j_n)\}$.

$$\textit{Change}\,(t) = \bigcup_{u \in \textit{InfluencedBy}(t)} \bigcup_{\substack{i,j \in \{0,\dots,|^\bullet u|\} \\ j = i + \textit{InflOf}(t,u)}} \{(u : i, j)\}$$

$$T' = \bigcup_{t \in T} \bigcup_{\mathbf{u} \in \textit{Change}(t)} t_{\mathbf{u}}$$

[10] That is to say: $\{u_k\}_k = \textit{InfluencedBy}\,(t)$.

- the Pre and $Post$ set of transition $t_{\{(u_1:i_1,j_1)\cdots(u_n:i_n,j_n)\}}$ are:

$$Pre\left(t_{\{(u_1:i_1,j_1)\cdots(u_n:i_n,j_n)\}}\right) = {}^\bullet t \cup \left\{{}^\circ t^{|{}^\bullet t|}\right\} \cup \left\{{}^\circ u_0^{i_0}, \ldots, {}^\circ u_n^{i_n}\right\}$$
$$Post\left(t_{\{(u_1:i_1,j_1)\cdots(u_n:i_n,j_n)\}}\right) = t^\bullet \cup \left\{{}^\circ u_0^{j_0}, \ldots, {}^\circ u_n^{j_n}\right\}$$

- the new initial marking is $M_0' = M_0 \cup \left\{ {}^\circ t^n \;\middle|\; t \in T \wedge n = \sum_{p \in {}^\bullet t} M_0(p) \right\}$
- the labeling function is very simple: $\Lambda'(t_{\mathbf{u}}) = \Lambda(t)$ for all $\mathbf{u} \in$ *Change* (t)
- and the time interval function associates interval $I(t)$ to the arcs $({}^\circ t^{|{}^\bullet t|}, t_{\mathbf{u}})$ for all $\mathbf{u} \in$ *Change* (t) and $[0, \infty[$ otherwise.

Proof

4.7.2. A specific $\overline{\textit{A-TPN}}$

Lemma 4.5. The TPN $\mathcal{N}_{A0} \in \overline{\textit{A-TPN}}$ of Fig.15 is weakly timed bisimilar to $\mathcal{A}_0 \in \mathcal{TA}$ (Fig. 8).

The bisimulation relation and the proof are identical to those of Lemma 4.2.

4.7.3. Strict inclusion in strong semantics

Theorem 4.8. (Strict inclusion of *T-TPN* into *A-TPN* in strong semantics)

$$\overline{\textit{T-TPN}} \subset_\approx \overline{\textit{A-TPN}}$$

Proof:
Thanks to Lemma 4.4 we have $\overline{\textit{T-TPN}} \subseteq_\approx \overline{\textit{A-TPN}}$. Moreover from Theorem 4.4, there is no TPN $\in \overline{\textit{T-TPN}}$ weakly timed bisimilar to $\mathcal{A}_0 \in \mathcal{TA}$ (Fig. 8) and from Lemma 4.5, the TPN $\mathcal{N}_{A0} \in \overline{\textit{A-TPN}}$ is weakly timed bisimilar to $\mathcal{A}_0$. □

4.8. $\overline{\textit{P-TPN}} \subset_\approx \overline{\textit{A-TPN}}$ and $\underline{\textit{P-TPN}} \subset_\approx \underline{\textit{A-TPN}}$

Lemma 4.6. (*P-TPN* included in *A-TPN* (strong and weak semantics))

$$\underline{\textit{P-TPN}} \subseteq_\approx \underline{\textit{A-TPN}} \qquad\qquad \overline{\textit{P-TPN}} \subseteq_\approx \overline{\textit{A-TPN}}$$

Proof:
The translation is obvious: for a given *P-TPN* $\mathcal{N}$, a *A-TPN* $\mathcal{N}'$ is built, with the same untimed Petri net, and such that, $\forall p, \forall t \in p^\bullet : I'(p,t) = I(p)$. Then, considering their respective definitions for $enabled$, $firable$ and the discrete and continuous translation, the only difference is that, when the *P-TPN* condition is $\nu(p) \in I(p)$ or $\nu(p) \in I(p)^\downarrow$, the *A-TPN* condition is $\forall t \in p^\bullet : \nu(p) \in I(p,t)$ or $\nu(p) \in I(p,t)^\downarrow$. And in our translation, $I'(p,t) = I(p)$.

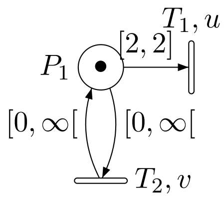

Figure 14. The TPN $\mathcal{N}_{A1} \in \overline{\textit{A-TPN}}$

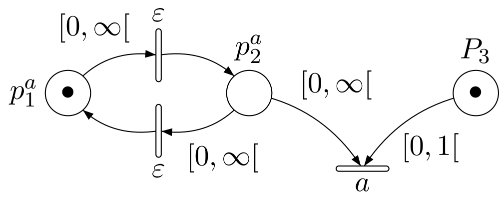

Figure 15. The TPN $\mathcal{N}_{A0} \in \overline{\textit{A-TPN}}$ bisimilar to $\mathcal{A}_0$

Then, all evolution rules are the same and both are strongly bisimilar.

□

Lemma 4.7. (No $\overline{\textit{P-TPN}}$ is bisimilar to a $\overline{\textit{A-TPN}}$)
There exists $\mathcal{N}_{A1} \in \overline{\textit{A-TPN}}$ such that there is no $\mathcal{N} \in \overline{\textit{P-TPN}}$ weakly timed bisimilar to $\mathcal{N}_{A1}$.

Proof:
The proof is based on Theorem 4.7. The *A-TPN* $\mathcal{N}_{A1}$ (cf. Fig. 14) is the same net than the *T-TPN* $\mathcal{N}_{T1}$ (cf. Fig. 12). Obviously, $\mathcal{N}_{A1}$ and $\mathcal{N}_{T1}$ are (strongly) bisimilar. Then, from Theorem 4.7 that states that there is no $\overline{\textit{P-TPN}}$ weakly bisimilar to $\mathcal{N}_{T1}$, there neither is any $\overline{\textit{P-TPN}}$ weakly bisimilar to $\mathcal{N}_{A1}$. □

Lemma 4.8. (No $\underline{\textit{P-TPN}}$ is bisimilar to a $\underline{\textit{A-TPN}}$)
There exists $\mathcal{N}_{A1} \in \underline{\textit{A-TPN}}$ such that there is no $\mathcal{N} \in \underline{\textit{P-TPN}}$ weakly timed bisimilar to $\mathcal{N}_{A1}$.

The proof is the same as for Lemma 4.7.

Theorem 4.9. (*A-TPN* are strictly more expressive than *P-TPN*)

$$\overline{\textit{P-TPN}} \subset_{\approx} \overline{\textit{A-TPN}} \qquad\qquad \underline{\textit{P-TPN}} \subset_{\approx} \underline{\textit{A-TPN}}$$

Proof:
Obvious from Lemma 4.6, 4.7 and 4.8. □

4.9. $\underline{\textit{P-TPN}} \not\subseteq_{\approx} \underline{\textit{T-TPN}}$

According to Corollary 4.2, we have $\underline{\textit{T-TPN}} \not\subseteq_{\approx} \underline{\textit{P-TPN}}$. To prove that $\underline{\textit{T-TPN}}$ is not more expressive than $\underline{\textit{P-TPN}}$, we prove that the TPN $\mathcal{N}_{\underline{P}}$ of Figure 16 can not be bisimulated by any net of $\underline{\textit{T-TPN}}$. The intuition of the proof is that, in weak semantics, the ϵ transition can never be forced to be fired. Then, the bisimulation relation should be achieved with some "direct" mapping, which is impossible due to the very different synchronisation rules, like in the strong semantics case.

The Lemma 4.9 proves this "direct" mapping property in a specific case, used in the proof of the Theorem 4.10.

A technical point of the proof could be highlighted: the proof is, like the one of Theorem 4.7, based on the smallest constant of the nets, but, to simplify the notations, the problem can be reduced to a problem on integer, by multiplying by the least common multiple of all denominators.

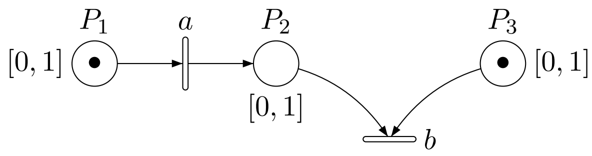

Figure 16. A TPN $\mathcal{N}_{\underline{P}} \in$ *P-TPN*

Lemma 4.9. Let us consider a TPN $\mathcal{N} \in$ *T-TPN* such that $\forall t \in T$, $\alpha(t) \in \mathbb{N}$ and $\beta(t) \in \mathbb{N}$. Let us define a state q such that for all run from q, an action b is continuously possible during 0.5 time unit and impossible after 0.5 time unit. Formally it gives :

$$\forall q' \text{ st } q \xrightarrow{(\epsilon*,d)} q' \text{ we have } \begin{cases} d \in [0, 0.5] \Rightarrow \exists q' \xrightarrow{\epsilon * b} \\ d > 0.5 \Rightarrow \nexists q' \xrightarrow{\epsilon * b} \end{cases}$$

If such state $q = (\nu, M)$ is a state of $\mathcal{N}$ then $\exists t_b$ with $\Lambda(t_b) = b$ such that $\begin{cases} \nu(t_b) + 0.5 = \beta(t_b) \\ I(t_b) \textit{ is closed on the right} \\ \alpha(t_b) \le \beta(t_b) - 1 \end{cases}$

The lemma could also states with every value in $]0, 1[$, other than 0.5. But for the proof, 0.5 is sufficient.

Proof:
When q is a state, and d a delay (a real number), let $q + d$ denotes the state reached from q by a delay of duration d (because of the time determinism property, it is unique). Formally, it gives : $q \xrightarrow{d} q + d$ with $q = (M, \nu)$ and $q + d = (M, \nu + d)$

Because of the weak semantics, the state $q + d$ can always be reached,

The proof is decomposed into several steps.

1. *Let m be the minimal delay such that no transition met its upper bound between $q+m$ and $q+0.5$.*

   As $q = (M, \nu)$, for each enabled transition t, $\beta(t) - \nu(t)$ is the remaining time before disabling of the transition.

   m is then formally defined by : $m = \max\{\beta(t) - \nu(t) \mid M \ge {}^\bullet t \text{ and } \beta(t) - \nu(t) < 0.5\}$ If the set is empty, $m = 0$.

   As the number of transitions is finite, m obviously exists.

2. *From the state $q + m$, action b is accessible, by a null duration path ρ.*

   From the hypothesis, there exists a sequence of transitions ρ such that $q + m \xrightarrow{\rho}\xrightarrow{t_b}$, with $\Lambda(t_b) = b$, $untimed(\rho) = \epsilon*$ and $duration(\rho) = 0$ (the sequence ρ may be emtpy).

3. *The same path ρ can be used a little later*

   Let d be such that $m + d < 0.5$. Because of the definition of m, the state $q + m + d$ can be reached without disabling any transition. Let us denote $q_1 = q + m + d$.

   By definition of m, every transition firable in $q + m$ is still firable in q_1: no upper bound $\beta(t)$ have been ovelapped by its clock value $\nu(t)$. Then, the same transition sequence can be used: $q_1 \xrightarrow{\rho}\xrightarrow{t_b}$.

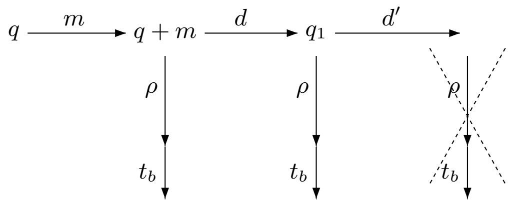

Figure 17. States and path illustrating steps 2, 3, 4 and 5 of the proof of lemma 4.9

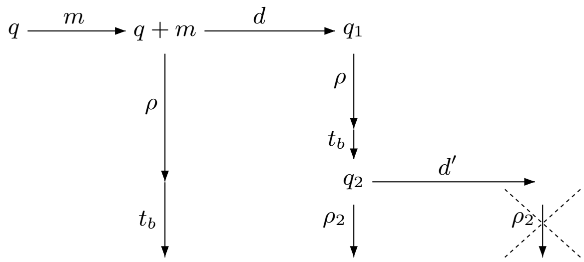

Figure 18. States and path illustrating step 6 of he proof of lemma 4.9

4. *But this path can no more be used once the 0.5 limits have been overlapped: it means a transition have overlapped its upper bound*

   Let now be d' such that $m+d+d' > 0.5$. From our hypothesis, the action b is no more reachable, and then, the path ρt_b can no more be used as shown in Fig. 17. It means that there exists a transition t_1 in ρt_b (i.e. $\rho t_b = \rho_1 t_1 \rho_2$) that was firable from q_1 and whose upper bound have been overlapped. That is to say a transition t_1 such that $\nu_{q_1}(t_1) \leq \beta(t_1) \leq \nu_{q_1+d'}(t_1)$.

5. *The transition t_1 was firable from $q+m$ and $\alpha(t_1) < \beta(t_1)$.*

   As ρ_1 is in null time, t_1 is not newly enabled by the firing of a transition in ρ_1(Indeed, obviously, $\beta(t_1)$ can not be overlapped without time elapsing). Then t_1 is enabled in state q_1 and then in q and $q+m$ since the marking of these states is the same. Moreover, as t_1 is firable from q_1 and from $q+m$ we have $\beta(t_1) \geq d > 0$ and $\alpha(t_1) \leq \beta(t_1) - d < \beta(t_1)$.

6. *The transition t_b was firable from $q+m$ and $\alpha(t_b) < \beta(t_b)$.*

   If $t_1 = t_b$, we can stop (cf previous step).

   If not, ρ_2 is not empty and $\rho_2 = \rho_2' t_b$. We can consider q_2 defined by $q_1 \xrightarrow{\rho_1 t_1} q_2$ (see Fig.18). Because q_2 is reachable from q with a path of duration $m+d < 0.5$, the same reasoning can be applied, with ρ_2' instead of ρ. Each t_i is enabled in $q+m$ (see previous item). Thus, since the number of enabled transitions in $q+m$ is finite, after a finite n number of steps, we get $t_n = t_b$ and transition t_b was firable from $q+m$ and $\alpha(t_b) < \beta(t_b)$.

   $\nu_{q+(m+d)}(t_b) \leq \beta(t_b) \leq \nu_{q+(m+d+d')}(t_b)$

7. $\beta(t_b) = \nu(t_b) + 0.5$ *and* $\alpha(t_b) \leq \nu(t_b) - 0.5$

We have proved $\nu_{q+m+d}(t_b) \leq \beta(t_b) \leq \nu_{q+m+d+d'}(t_b) \iff \nu_q(t_b) + m + d \leq \beta(t_b) \leq \nu_q(t_b) + m + d + d'$ for all values of d, d' such that $m + d < 0.5$ and $m + d + d' > 0.5$. When $m + d$ tends to 0.5 from the left, and $m + d + d'$ tends to 0.5 from the right, both values tends to $\nu(t_b) + 0.5$, then $\nu(t_b) + 0.5 = \beta(t_b)$.

Moreover, because $\alpha(t_b)$ and $\beta(t_b)$ are integers, and $\alpha(t_b) < \beta(t_b)$ we have $\alpha(t_b) \leq \beta(t_b) - 1$, and also $\alpha(t_b) \leq \nu(t_b) - 0.5$.

□

Let us consider the TPN $\mathcal{N}_{\underline{P}} \in$ *P-TPN* of the Figure 16.

Theorem 4.10. There is no TPN $\in$ *T-TPN* weakly timed bisimilar to $\mathcal{N}_{\underline{P}} \in$ *P-TPN* (Fig. 16).

Proof:
The proof is done by contradiction.

Let $\mathcal{N}_{\underline{P}} \in$ *P-TPN* the net of the Figure 16. Assume there exists $\mathcal{N}_{\underline{T}} \in$ *T-TPN* $= (P, T, ^\bullet(.), (.)^\bullet, M_0, \Lambda, I)$ that is timed bisimilar to $\mathcal{N}_{\underline{P}}$. We denote $\sim$ the bisimulation relation such that $\mathcal{N}_{\underline{T}} \sim \mathcal{N}_{\underline{P}}$.

Let $Const = \{\alpha(t) > 0, \beta(t) > 0\}$ be the set of constant of $\mathcal{N}_{\underline{T}}$ and k, be the least common denominator of $Const$.

Let $\mathcal{N}_{\underline{T}^k} \in$ *T-TPN* $= (P, T, ^\bullet(.), (.)^\bullet, M_0, \Lambda, k.I)$ be the TPN obtained by multiplying by k all bound α and β of the firing intervals I. The bounds of the firing intervals of $\mathcal{N}_{\underline{T}^k}$ are in $\mathbb{N}$.

Moreover $\mathcal{N}_{\underline{T}^k}$ is timed bisimilar to the net $\mathcal{N}_{\underline{P}^k}$ obtained by the same operation.

Let be the run $\rho_P^k = q_0 \xrightarrow{k-0.5} q_1 \xrightarrow{a} q_2$ in $\mathcal{N}_{\underline{P}^k}$. From q_2, every delay of duration $d \leq 0.5$ can be followed by a firing of b, and every delay of duration $d > 0.5$ can not.

Let q'_0 be the initial state of $\mathcal{N}_{\underline{T}^k}$. By bisimulation assumption, there exists a run $\rho_T^k = q'_0 \xrightarrow{(\epsilon*, k-0.5)} q'_1 \xrightarrow{\epsilon*a} q'_2$, with $q'_2 = (M'_2, \nu'_2)$. By bisimulation assumption, q'_2 respects the hypotheses of Lemma 4.9. It implies that there exists t_b with $\Lambda_{\underline{T}^k}(t_b) = b$ such that $\alpha_{\underline{T}^k}(t_b) \leq \beta_{\underline{T}^k}(t_b) - 1$.

In q'_2, t_b is enabled, and t_b is firable since 0.5 time unit, that is to say, t_b is firable *before* the firing of the transition of label a, which contradicts the bisimulation assumption. □

Theorem 4.11. (In weak semantics, *T-TPN* does not generalise *P-TPN*)

$$\underline{\textit{P-TPN}} \not\subseteq_{\approx} \underline{\textit{T-TPN}}$$

Proof:
This is a direct application of Theorem 4.10. □

4.10. *T-TPN* $\subset_{\approx}$ *A-TPN*

Theorem 4.12. (In weak semantics, *A-TPN* are strictly more expressive than *T-TPN*)

$$\underline{\textit{T-TPN}} \subset_{\approx} \underline{\textit{A-TPN}}$$

Proof:
We already know that $\underline{\textit{T-TPN}} \subseteq_{\approx} \underline{\textit{A-TPN}}$ (Lemma 4.4, p. 16).

From the previous Theorem 4.10, there exists $\mathcal{N}_{\underline{P}} \in \underline{\textit{P-TPN}}$ (Figure 16) that can not be bisimulated by any $\underline{\textit{T-TPN}}$.

From Lemma 4.6, there exists $\mathcal{N}_{\underline{A}} \in \underline{\textit{A-TPN}}$ bisimilar to $\mathcal{N}_{\underline{P}}$.

Then, $\mathcal{N}_{\underline{A}}$ can not be bisimulated by any $\underline{\textit{T-TPN}}$, that is to say $\underline{\textit{A-TPN}} \not\subseteq_{\approx} \underline{\textit{T-TPN}}$. □

4.11. Sum up

We are now going to sum-up all results in a single location, Figure 19.

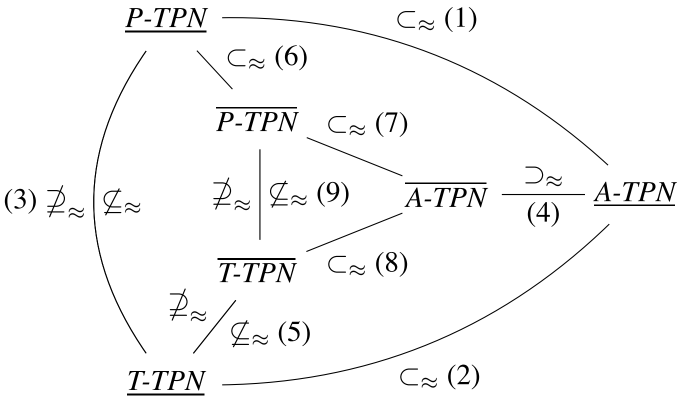

Figure 19. The classification explained

(1) and (7) A *P-TPN* can always be translated into a *A-TPN* and there exist some *A-TPN* that can not be simulated by any *P-TPN* (Theorem 4.9).

(2) A $\underline{\textit{T-TPN}}$ can be translated into a $\underline{\textit{A-TPN}}$ (Lemma 4.4) and there exist a $\underline{\textit{A-TPN}}$ that can not be simulated by any $\underline{\textit{P-TPN}}$ (Theorem 4.12).

(3) Corrolary 4.2 states that $\underline{\textit{T-TPN}} \not\subseteq_{\approx} \underline{\textit{P-TPN}}$, and Theorem 4.11 states the opposite. Both model are incomparable.

(4) The strong semantics of *A-TPN* strictly generalises the weak one (Theorem 4.3).

(5) Strong and weak *T-TPN* are incomparable: the weak semantics can not emulate the strong one (Theorem 4.1) but there also exist *T-TPN* with weak semantics that can not been emulated by any strong *T-TPN* (Theorem 4.5).

(6) Theorem 4.2 states that $\underline{\textit{P-TPN}} \subset_{\approx} \overline{\textit{P-TPN}}$: in *P-TPN*, the strong semantics can emulate the weak one (Lemma 4.1), but weak semantics can not do the opposite (Theorem 4.1).

(8) A $\overline{\textit{T-TPN}}$ can be translated into a $\overline{\textit{A-TPN}}$ (Lemma 4.4) and there exists a $\overline{\textit{A-TPN}}$ (Lemma 4.5) that can not be emulated by any $\overline{\textit{T-TPN}}$. Then strict inclusion follows (Theorem 4.8).

(9) *T-TPN* and *P-TPN* with strong semantics are incomparable: Theorem 4.6 states that there is a $\overline{\textit{P-TPN}}$ that can be simulated by no $\overline{\textit{T-TPN}}$ and Corollary 4.1 states the symmetric.

5. Conclusion

Several timed Petri nets models have been defined for years for different purposes. They have been individually studied, some analysis tools exists for some of them, and the users know that a given problem can be modelled with one or the other with more or less difficulty, but a clear map of their relationships was missing. This paper draws most of this map (cf. Fig. 19).

Behind the details of the results, a global view of the main results is the following:

- *P-TPN* and *A-TPN* are really close models, since their firing rule is the conjunction of some local clocks, whereas the *T-TPN* has another point of view, its firing rule taking into account only the last clock;
- the *A-TPN* model generalises all the other models, but emulating the *T-TPN* firing rule with *A-TPN* ones is not possible in practice for human modeller;
- the strong semantics generalises the weak one for *P-TPN* and *A-TPN*, but not for *T-TPN*.

The next step will be to study the language-based relationships.

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