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Example of supersonic solutions to a steady state Euler-Poisson system

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Abstract. We give an example of supersonic solutions to a one-dimensional steady state Euler-Poisson system arising in the modeling of plasmas and semiconductors. The existence of the supersonic solutions which correspond to large current density is proved by Schauder's fixed point theorem. We show also the uniqueness of solutions in the supersonic region.

Keywords. Euler-Poisson equations, steady state flow, supersonic solutions, existence, uniqueness.

2000 Mathematics Subject Classification. 35J25, 35Q35, 76J20.

1 Introduction

The Euler-Poisson system plays an important role in the mathematical modeling and numerical simulation for plasmas and semiconductors [2, 7, 8]. In the steady state isentropic case the existence and uniqueness of smooth solutions are obtained in the subsonic region for a one-dimensional flow [3] or potential flows [4]. See also [1] for the subsonic solutions to a one-dimensional non-isentropic model. In [5, 6], the stationary transonic solutions are studied by an artificial viscosity approximation. The existence of the transonic solutions is proved by passing to the limit in the approximate Euler-Poisson system as the viscosity coefficient goes to zero. However, the existence of the purely supersonic solutions has not been discussed yet.

In this paper, we give an example of the supersonic solutions in a one-dimensional steady state Euler-Poisson system :

$$\partial_x j = 0, \tag{1.1}$$

$$\partial_x \left(\frac{j^2}{n} + p(n) \right) = n \partial_x \phi - j / \tau, \tag{1.2}$$

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$$-\partial_{xx}\phi = b - n. \quad (1.3)$$

Equation (1.1) implies that j is a constant. Here, n , j and ϕ are the electron density, the current density and the electric potential, respectively. The parameter $\tau > 0$ stands for the momentum relaxation time depending on n and j in general. For simplicity, we assume that τ is a constant. The given function $b = b(x)$ is the doping profile for the semiconductors. The pressure function $p = p(n)$ is assumed to be smooth and strictly increasing for $n > 0$. As in [3], we consider equations (1.1)-(1.3) in the interval $(0, 1)$ subject to the following Dirichlet boundary conditions :

$$n(0) = n_0, \quad n(1) = n_1, \quad \phi(0) = \phi_0, \quad \phi(1) = \phi_1, \quad (1.4)$$

where $n_0 > 0$, $n_1 > 0$ and $\phi_0, \phi_1 \in \mathbb{R}$ are given data. If $n > 0$ is a smooth function, after eliminating ϕ in (1.2)-(1.3), we obtain a Dirichlet problem for n :

$$-\partial_{xx}F_j(n) - \frac{1}{j}\partial_x\left(\frac{1}{\tau n}\right) + \frac{1}{j^2}(n - b) = 0 \quad \text{in } (0, 1), \quad (1.5)$$

$$n(0) = n_0, \quad n(1) = n_1, \quad (1.6)$$

where

$$F_j(n) = \frac{1}{2n^2} + \frac{h(n)}{j^2} \quad \text{with } h(n) = \int_1^n \frac{p'(y)}{y} dy.$$

Once n is solved, from (1.2) ϕ is given explicitly by :

$$\phi(x) = \phi_0 + j^2(F_j(n(x)) - F_j(n_0)) + \int_0^x \frac{j}{\tau n(y)} dy. \quad (1.7)$$

Then ϕ_1 is linked with j by the following relation

$$\phi_1 = \phi_0 + j^2(F_j(n_1) - F_j(n_0)) + \int_0^1 \frac{j}{\tau n(y)} dy. \quad (1.8)$$

It is easy to see that (n, ϕ) with $n > 0$ is a smooth solution of (1.2)-(1.4) if and only if (n, ϕ) is a smooth solution of (1.5)-(1.7). Therefore, we may first solve n to the Dirichlet problem (1.5)-(1.6) and then determine ϕ by (1.7).

Now the equation (1.5) is elliptic if and only if $F'_j(n) \neq 0$. Since p is strictly increasing, there is a unique $n_c(j)$ such that $F'_j(n_c(j)) = 0$, or equivalently

$$\sqrt{p'(n_c(j))} = \frac{|j|}{n_c(j)}.$$

Here the quantities $c = \sqrt{p'(n)}$ and j/n stand for the speed of sound and the electron velocity, respectively. If $n \rightarrow n^2 p'(n)$ is strictly increasing, we obtain the following alternative :

$$\text{subsonic flow} \iff F'(n) > 0 \iff n > n_c(j) \implies (1.5) \text{ is elliptic}, \quad (1.9)$$

$$\text{supersonic flow} \iff F'(n) < 0 \iff n < n_c(j) \implies (1.5) \text{ is elliptic.} \quad (1.10)$$

Note that the linear term n/j^2 in (1.5) has not a good sign. Nevertheless, it is small as j is large and then can be controlled by the $L^2(0, 1)$ norm of $\partial_x n$ by Poincaré's inequality. Similar argument holds for the term $\partial_x(1/j\tau n)$. This is the main feature of the problem to yield the existence and uniqueness of solutions.

2 Existence of solutions

Assume $b \in L^\infty(0, 1)$. In view of (1.9), the subsonic solutions to (1.2)-(1.4) correspond to the small value of j . They have been considered in [3]. We study here the supersonic solutions which correspond to the case (1.10). To this end, let M_1 and M_2 be any two constants satisfying

$$0 < M_1 < \min(n_0, n_1), \quad \max(n_0, n_1) < M_2. \quad (2.1)$$

Choosing j such that $n_c(j) > M_2$, then (1.10) and (2.1) imply that the boundary data n_0 and n_1 are in the supersonic region. Since the maximum principle can not be applied to (1.5) in the supersonic region, the solutions of (1.5)-(1.6) may not be supersonic flow. To seek for a supersonic solution, we define a smooth and strictly decreasing function \tilde{F}_j on \mathbb{R}^+ such that

$$\tilde{F}_j(+\infty) = 0, \quad \tilde{F}_j(n) = F_j(n) \text{ for all } n \leq M_2.$$

Then we study the following problem instead of (1.5)-(1.6) :

$$-\partial_{xx}\tilde{F}_j(n) - \frac{1}{j}\partial_x\left(\frac{1}{\tau n}\right) + \frac{1}{j^2}(n - b) = 0 \quad \text{in } (0, 1), \quad (2.2)$$

$$n(0) = n_0, \quad n(1) = n_1. \quad (2.3)$$

Our strategy is to prove the existence of a smooth solution n to (2.2)-(2.3) such that $0 < n \leq M_2$. Then n is a supersonic solution of (1.5)-(1.6) by the definition of \tilde{F}_j .

Since \tilde{F}_j is smooth and strictly decreasing from \mathbb{R}^+ to \mathbb{R}^+ , we may make a change of variable $v = \tilde{F}_j(n)$ for $n > 0$. Let G_j be the inverse of \tilde{F}_j , which is also smooth and strictly decreasing from \mathbb{R}^+ to \mathbb{R}^+ . Then the problem (2.2)-(2.3) is equivalent to

$$-\partial_{xx}v - \frac{1}{j}\partial_x\left(\frac{1}{\tau G_j(v)}\right) + \frac{1}{j^2}(G_j(v) - b) = 0 \quad \text{in } (0, 1), \quad (2.4)$$

$$v(0) = v_{0j} = F_j(n_0), \quad v(1) = v_{1j} = F_j(n_1). \quad (2.5)$$

To study the problem (2.4)-(2.5), we will apply Schauder's fixed point theorem. For this purpose, let's define a closed convex set

$$S = \{v \in C([0, 1]); F_j(M_2) \leq v \leq F_j(M_1)\},$$

and a map T by $v = T(\sigma)$ for $\sigma \in S$, where v solves the linear problem :

$$-\partial_{xx}v + \frac{1}{j^\tau}\alpha_j(\sigma)\partial_xv + \frac{1}{j^2}\beta_j(x, \sigma) = 0 \quad \text{in } (0, 1), \quad (2.6)$$

$$v(0) = v_{0j}, \quad v(1) = v_{1j}, \quad (2.7)$$

with

$$\alpha_j(\sigma) = \frac{G'_j(\sigma)}{G_j^2(\sigma)} = \frac{1}{G_j^2(\sigma)\tilde{F}'_j(G_j(\sigma))}, \quad \beta_j(x, \sigma) = G_j(\sigma) - b(x).$$

We observe that $\sigma \in S$ implies that

$$F_j(M_2) \leq \sigma \leq F_j(M_1).$$

From $\tilde{F}_j(\sigma) = F_j(\sigma)$ for $\sigma \leq M_2$, we have

$$M_1 \leq G_j(\sigma) \leq M_2.$$

Therefore, from the definition of F_j , there is a $j_1 > 0$ depending only on M_1 and M_2 such that α_j and β_j are two bounded functions with bounds depending on M_1 and M_2 but independent of j and σ for any $j \in \mathbb{R}$ satisfying $|j| \geq j_1$.

For $v \in H^1(0, 1)$ and $z \in H_0^1(0, 1)$, let

$$a(v, z) = \int_0^1 \left(\partial_x v \partial_x z + \frac{1}{j^\tau} \alpha_j(\sigma) z \partial_x v \right) dx, \quad l(z) = -\frac{1}{j^2} \int_0^1 \beta_j(x, \sigma) z dx.$$

It is clear that $l(\cdot)$ is linear and continuous on $H_0^1(0, 1)$, and $a(\cdot, \cdot)$ is bilinear and continuous on $H_0^1(0, 1) \times H_0^1(0, 1)$. Moreover, by Poincaré's inequality,

$$\begin{aligned} a(z, z) &= \int_0^1 \left((\partial_x z)^2 + \frac{1}{j^\tau} \alpha_j(\sigma) z \partial_x z \right) dx \\ &\geq \|\partial_x z\|_{L^2(0,1)}^2 - \frac{1}{|j|^\tau} \|\alpha_j\|_{L^\infty(0,1)} \|z\|_{L^2(0,1)} \|\partial_x z\|_{L^2(0,1)} \\ &\geq \left(1 - \frac{C_1}{|j|^\tau} \|\alpha_j\|_{L^\infty(0,1)} \right) \|\partial_x z\|_{L^2(0,1)}^2, \quad \forall z \in H_0^1(0, 1), \end{aligned}$$

where $C_1 > 0$ is the constant in Poincaré's inequality. Then there exists a $j_2 \geq \frac{2C_1}{\tau} \|\alpha_j\|_{L^\infty(0,1)}$ depending only on M_1 and M_2 such that

$$a(z, z) \geq \frac{1}{2} \|\partial_x z\|_{L^2(0,1)}^2, \quad \forall |j| \geq j_2, \quad \forall z \in H_0^1(0, 1). \quad (2.8)$$

Therefore, $a(\cdot, \cdot)$ is coercive. By Lax-Milgram's theorem, there exists a unique solution $v \in H^1(0, 1)$ to the variational problem $a(v, z) = l(z)$, $\forall z \in H_0^1(0, 1)$ and (2.7). This shows that the map T is well defined.

We prove now that $T(S)$ is a compact set of $C([0, 1])$. Indeed, let $\bar{v}_j = (1-x)v_{0j} + xv_{1j}$. Then $v - \bar{v}_j \in H_0^1(0, 1)$. From the continuity of $l(\cdot)$ and $a(\cdot, \cdot)$, the coercivity estimate (2.8) and

$$a(v - \bar{v}_j, v - \bar{v}_j) = l(v - \bar{v}_j) - a(\bar{v}_j, v - \bar{v}_j),$$

it is easy to obtain

$$\|\partial_x(v - \bar{v}_j)\|_{L^2(0,1)} \leq \frac{2C_1}{j^2} \|\beta_j\|_{L^\infty(0,1)} + \frac{2C_1}{|j|^\tau} \|\alpha_j\|_{L^\infty(0,1)} \|\partial_x \bar{v}_j\|_{L^2(0,1)}. \quad (2.9)$$

Recall that α_j and β_j are bounded independent of σ . We conclude from Poincaré's inequality and the compact imbedding from $H^1(0, 1)$ into $C([0, 1])$ that $T(S)$ is a compact set of $C([0, 1])$. Moreover, there are constants $C_2 > 0$ and $j_3 \geq j_2$ which depend only on M_1 and M_2 such that

$$|v(x) - \bar{v}_j(x)| \leq \frac{C_2}{|j|}, \quad \forall |j| \geq j_3, \quad \forall x \in [0, 1].$$

Since

$$F_j(\max(n_0, n_1)) \leq \bar{v}_j(x) \leq F_j(\min(n_0, n_1)), \quad \forall x \in [0, 1],$$

it follows that

$$F_j(\max(n_0, n_1)) - \frac{C_2}{|j|} \leq v(x) \leq F_j(\min(n_0, n_1)) + \frac{C_2}{|j|}, \quad \forall |j| \geq j_3, \quad \forall x \in [0, 1].$$

The function $n \rightarrow F_j(n)$ being strictly decreasing for $n \leq M_2$, from (2.1) there is a $j_4 \geq j_3$ depending only on M_1 and M_2 such that

$$F_j(M_2) \leq v(x) \leq F_j(M_1), \quad \forall |j| \geq j_4, \quad \forall x \in [0, 1]. \quad (2.10)$$

Hence, $v \in S$ and then T is a self map from S to S . Finally, the continuity of T follows from a standard argument. More precisely, for $\sigma_1, \sigma_2 \in S$, we can prove that there is a constant $C_3 > 0$ depending only on M_1 and M_2 such that

$$\left(1 - \frac{C_3}{|j|^\tau}\right) \|T(\sigma_1) - T(\sigma_2)\|_{C([0,1])} \leq \frac{C_3}{|j|^\tau} \|\sigma_1 - \sigma_2\|_{C([0,1])}.$$

Thus, T is continuous for $|j| > j_5 = \max(j_4, C_3/\tau)$. We conclude from Schauder's fixed point theorem the existence of a solution $v \in H^1(0, 1) \cap S$ of $v = T(v)$.

This shows the existence of a solution $v \in H^1(0, 1) \cap S$ to the problem (2.4)-(2.5), and then the existence of a solution $n = G_j(v) \in H^1(0, 1)$ to the problem (2.2)-(2.3). Since $v = \tilde{F}_j(n) = F_j(n)$ for $n \leq M_2$, from (2.10) we obtain

$$M_1 \leq n(x) \leq M_2, \quad \forall |j| \geq j_5, \quad \forall x \in [0, 1]. \quad (2.11)$$

Therefore, $n \in H^1(0, 1)$ is a supersonic solution to the problem (1.5)-(1.6). Thus, we have proved

Theorem 1 *Let $n_0 > 0$ and $n_1 > 0$. Let M_1, M_2 be two constants satisfying (2.1) and $b \in L^\infty(0, 1)$. Then there exists a $j_e > 0$ depending only on M_1 and M_2 such that for any current density j satisfying $|j| \geq j_e$, the problem (1.2)-(1.4) admits a solution $(n, \phi) \in H^1(0, 1) \times H^1(0, 1)$. This solution is located in the supersonic region and satisfies (2.11).*

3 Uniqueness of solutions

There doesn't exist a general result on the uniqueness of solutions when the boundary data are located in the supersonic region. Indeed, for large j the formation of shocks cannot be avoided and the transonic solutions should be investigated. We refer to [5, 6] for the analysis of the transonic solutions. Here we give a uniqueness result in the supersonic region for large j . This result can be stated as follows.

Theorem 2 *Let M_1 and M_2 be two constants with $0 < M_1 < M_2$. Let $(n^{(1)}, \phi^{(1)})$ and $(n^{(2)}, \phi^{(2)})$ be two supersonic solutions of (1.2)-(1.3) in $H^1(0, 1) \times H^1(0, 1)$ with $M_1 \leq n^{(1)}, n^{(2)} \leq M_2$. Then there exists a $j_u > 0$ depending only on M_1 and M_2 such that for any current density $j \in \mathbb{R}$ satisfying $|j| \geq j_u$, we have $(n^{(1)}, \phi^{(1)}) = (n^{(2)}, \phi^{(2)})$.*

Proof. In view of (1.7), it suffices to show that $n^{(1)} = n^{(2)}$. Let $w = n^{(2)} - n^{(1)}$. By subtracting the equation (1.5) satisfied by $n^{(1)}$ and $n^{(2)}$ we obtain :

$$\partial_{xx}(A_j(x)w) + \frac{1}{j\tau}\partial_x(B(x)w) + \frac{1}{j^2}w = 0 \quad \text{in } (0, 1), \quad (3.1)$$

where

$$A_j(x) = - \int_0^1 \frac{\partial F_j}{\partial n} \left(n^{(1)}(x) + s \left(n^{(2)}(x) - n^{(1)}(x) \right) \right) ds,$$

$$\frac{1}{M_2^2} \leq B(x) = \frac{1}{n^{(1)}n^{(2)}} \leq \frac{1}{M_1^2} \quad \text{in } (0, 1).$$

From

$$F_j'(n) = -\frac{1}{n^3} + \frac{h'(n)}{j^2},$$

it is easy to check that there are constants $C_4 > 0$ and $j_6 > 0$ which depend only on M_1 and M_2 such that

$$A_j(x) \geq C_4, \quad \forall |j| \geq j_6, \quad \forall x \in [0, 1].$$

Multiplying (3.1) by $A_j w \in H_0^1(0, 1)$ and integrating over $(0, 1)$ give :

$$\int_0^1 [\partial_x(A_j(x)w)]^2 dx = \int_0^1 \left(-\frac{1}{j\tau}B(x)w\partial_x(A_j(x)w) + \frac{1}{j^2}A_j(x)w^2 \right) dx.$$

It follows from Poincaré's inequality that :

$$\|\partial_x(A_j w)\|_{L^2(0,1)}^2 \leq \frac{1}{C_4} \left(\frac{C_0}{M_1^2 |j| \tau} + \frac{C_0^2}{j^2} \right) \|\partial_x(A_j w)\|_{L^2(0,1)}^2.$$

This shows that $A_j w = 0$ and then $w = 0$ provided that $|j| \geq j_7$ for some large $j_7 > 0$ depending only on M_1 and M_2 .

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References

- [1] P. Amster, M.P. Beccar Varela, A. Jüngel and M.C. Mariani, Subsonic solutions to a one-dimensional non-isentropic hydrodynamic model for semiconductors, *J. Math. Anal. Appl.* 258, 52–62 (2001).
- [2] F. Chen, Introduction to Plasma Physics and Controlled Fusion, Vol. 1, *Plenum Press, New-York*, 1984.
- [3] P. Degond and P. Markowich, On a one-dimensional steady-state hydrodynamic model for semiconductors, *Appl. Math. Letters*, 3, 25–29 (1990).
- [4] P. Degond and P. Markowich, A steady state potential flow model for semiconductors, *Ann. Math. Pura Appl.* IV, 87–98 (1993).
- [5] I.M. Gamba, Stationary transonic solutions of a one-dimensional hydrodynamic model for semiconductors, *Comm. Part. Diff. Eqs.* 17, 553–577 (1992).
- [6] I.M. Gamba and C.S. Morawetz, A viscous approximation for a 2-D steady semiconductor or transonic gas dynamic flow : existence theorem for potential flow, *Comm. Pure Appl. Math.* XLIX, 999–1049 (1996).
- [7] C. Gardner, J. Jerome and D. Rose, Numerical methods for the hydrodynamic device model : subsonic flow, *IEEE Trans. CAD*, 8, 501–507 (1989).
- [8] P. Markowich, The Stationary Semiconductor Device Equations, Springer, New York, 1986.