# Global-in-time solutions for the isothermal Matovich-Pearson equations 

Eduard Feireisl, Philippe Laurencot, Andro Mikelic

## To cite this version:

Eduard Feireisl, Philippe Laurencot, Andro Mikelic. Global-in-time solutions for the isothermal Matovich-Pearson equations. Nonlinearity, 2011, 24, pp.277-292. 10.1088/0951-7715/24/1/014 . hal00488835

## HAL Id: hal-00488835

## https://hal.science/hal-00488835

Submitted on 3 Jun 2010

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Global-in-time solutions for the isothermal Matovich-Pearson equations * 

Eduard Feireisl ${ }^{\dagger}$<br>Institute of Mathematics of the Academy of Sciences of the Czech Republic, Žitná 25, 11567 Praha 1, Prague, CZECH REPUBLIC<br>E-mail: feireisl@math.cas.cz<br>Philippe Laurençot<br>Institut de Mathématiques de Toulouse, CNRS UMR 5219<br>Université de Toulouse, F-31062 Toulouse Cedex 9, FRANCE<br>E-mail: laurenco@math.univ-toulouse.fr<br>Andro Mikelić ${ }^{\ddagger}$<br>Université de Lyon, Lyon, F-69003, FRANCE ;<br>Université Lyon 1, Institut Camille Jordan, CNRS UMR 5208, Département de Mathématiques, 43, Bd du 11 novembre 1918, 69622 Villeurbanne Cedex, FRANCE<br>E-mail: Andro.Mikelic@univ-lyon1.fr

June 3, 2010


#### Abstract

In this paper we study the Matovich-Pearson equations describing the process of glass fiber drawing. These equations may be viewed as a 1D-reduction of the incompressible Navier-Stokes equations including free boundary, valid for the drawing of a long and thin glass fiber. We


[^0]concentrate on the isothermal case without surface tension. Then the Matovich-Pearson equations represent a nonlinearly coupled system of an elliptic equation for the axial velocity and a hyperbolic transport equation for the fluid cross-sectional area. We first prove existence of a local solution, and, after constructing appropriate barrier functions, we deduce that the fluid radius is always strictly positive and that the local solution remains in the same regularity class. To the best of our knowledge, this is the first global existence and uniqueness result for this important system of equations.

Keywords: Fiber drawing; Matovich-Pearson equations; incompressible free boundary Navier-Stokes equations; non-local transport equation; iterated comparison; barrier function.

Mathematics Subject Classification MSC2000 (AMS classcode): 35Q30; 35Q35; 35R35; 35L80; 76D05; 76D27

## 1 Introduction

The drawing of continuous glass fibers is a widely used procedure. Industrial glass fibers are manufactured by a bushing with more than thousand nozzles. Bushings are supplied with a molten glass from a melting furnace. Its temperature ranges from $1300 K$ to $1800 K$. In order to understand the glass fiber forming process, it is important to study the drawing of a single glass fiber. This is, of course, a significant simplification because we disregard interaction between fibers and between fibers and the surrounding air. For a single glass fiber, the hot glass melt is forced by gravity to flow through a die into air. After leaving the die, the molten glass forms a free liquid jet. It is cooled and attenuated as it proceeds through the air. Finally, the cold fiber is collected on a rotating drum.

The molten glass can be considered as a Newtonian fluid and the process is described by the non-isothermal Navier-Stokes equations for a thermally dilatable but isochoric fluid. Since we deal with a free liquid jet, the problem is posed as a free boundary problem for the Navier-Stokes equations, coupled with the energy equation. We refer to [5] for detailed modeling and analysis of the equations describing the stationary flow inside the die. There are several models proposed to describe the various stages of the flow of a molten glass from the furnace to the winding spool: the slow flow in the die (the "first phase" of the drawing), the jet formation under rapid cooling (the "second phase"), and the terminal fiber profile (the "third phase") (see [6]).

Since we consider long (their length is approximately 10 m ) and thin (their diameter varies from 1 mm to $10 \mu \mathrm{~m}$ ) fibers, it is reasonable to apply
the lubrication approximation to the model equation. This approach yields good results, at least for flows far from the die exit and in the so-called "third phase" of the fiber drawing.

A standard engineering model for the isothermal glass fiber drawing in the "third phase" is represented by the Matovich-Pearson equations. For an axially symmetric fiber with a straight central line, they read

$$
\begin{equation*}
\partial_{t} \mathbf{A}+\partial_{x}(v \mathbf{A})=0 ; \quad \partial_{x}\left(3 \mu(T) \mathbf{A} \partial_{x} v\right)+\partial_{x}(\sigma(T) \sqrt{\mathbf{A}})=0 \tag{1}
\end{equation*}
$$

where $\mathbf{A}=\mathbf{A}(t, x)$ is the cross-area of the fiber section, $v=v(t, x)$ is the effective axial velocity, $3 \mu$ is Trouton's viscosity, and $\sigma$ denotes the surface tension. As the coefficients $\mu$ and $\sigma$ depend on the temperature, it is necessary to take into account an equation for the temperature $T=T(t, x)$.

The original derivation of the system (1) is purely heuristic and obtained under the assumptions that: (i) the viscous forces dominate the inertial ones; (ii) the effect of the surface tension is balanced with the normal stress at the free boundary; (iii) the heat conduction is small compared with the heat convection in the fiber; (iv) the fiber is almost straight, and all quantities are axially symmetric. We refer to the classical papers by Kase \& Matsuo [16, 17], and Matovich \& Pearson (19) for more details concerning the model.

Another derivation of the model based on a lubrication type asymptotic expansion can be found in the work by Schultz et al. [9, 23], Dewynne et al. [3, 4], and Hagen [12], with more emphasis on the mathematical aspects of the problem. The (formal) asymptotic expansion is developed with respect to a small parameter $\varepsilon$, proportional to the ratio of the characteristic thickness $R_{E}$ in the radial direction and the characteristic axial length of the fiber $L$. The fact that the viscosity changes over several orders of magnitude is surprisingly ignored in these studies. As a matter of fact, the viscosity coefficient depends effectively on the temperature, with values varying from 10 to $10^{12} \mathrm{~Pa}$ sec, while in the above mentioned asymptotic expansions it is considered simply of order one. A correct formal derivation was given in [1], and it is in full agreement with the model announced in [12]. Finally, a full non-stationary model of a thermally dilatable molten glass, with density depending on the temperature, was derived in [6].

A mathematical analysis of generalized stationary Matovich-Pearson equations is performed in [1] (see also [2]). The non-stationary case, without surface tension and with advection equation for the temperature, is studied by Hagen \& Renardy [10]. They prove a local-in-time existence result in the class of smooth solutions. Their approach is based on a precise analysis of the dependence of the solution of the mass conservation equation on the
velocity. This method requires controlling higher order Sobolev norms in the construction of solutions by means of an iterative procedure and works only for short time intervals. Hagen et al. 11, 13, 14] have also undertaken a detailed study of the linearized equations of forced elongation. Despite this considerable effort, global-in-time solvability of the Matovich-Pearson equations was left open.

The ratio $\sigma / \mu$ is small, and, furthermore, the inertia and gravity effects are negligible in most applications. Accordingly, we consider the MatovichPearson equations (罒) with $\sigma=0$, meaning, the isothermal drawing with constant positive viscosity and in the absence of surface tension. For a prescribed velocity at the fiber end points, the important parameter is the draw ratio, being the ratio between the outlet and inlet fluid velocities. It is well known that the instability known as a draw resonance occurs at draw ratios in excess of about 20.2. Linear stability analysis was rigorously undertaken by Renardy [22]. Moreover, in [26], it was established that the cross section, given by the Matovich-Pearson equations with $\sigma=0$, may vary chaotically at a draw ratio higher than 30, under the condition of periodic variations of the input cross section. There are also numerous articles devoted to numerical simulations confirming such a conclusion. Fairly complete simulations can be found in the papers by Gregory Forest \& Zhou [8, 25]. Their simulations predict various aspects of the physical process, like a linearized stability principle, bounds on the domain of convergence for linearly stable solutions, and transition to instability. Their analysis completes that of [7].

The above mentioned simplification of system (1) is briefly discussed in [15], however, without any rigorous proofs. Our idea is to use the particular structure of the system with $\sigma=0$, and to prove short-time existence of smooth solutions satisfying good uniform estimates. Then, performing a qualitative analysis of the solutions and constructing appropriate barrier functions, we show that the cross-section area remains bounded below away from zero. This observation allows us to deduce existence as well as uniqueness of global-in-time solutions.

## 2 Isothermal fiber drawing without surface tension

We study the system of equations

$$
\begin{gather*}
\partial_{t} A+\partial_{x}(v A)=0 \quad \text { in } \quad Q_{T}=(0, T) \times(0, L)  \tag{2}\\
\partial_{x}\left(3 \mu A \partial_{x} v\right)=0 \quad \text { in } \quad Q_{T}=(0, T) \times(0, L), \tag{3}
\end{gather*}
$$

supplemented with the boundary and initial conditions

$$
\begin{align*}
& A(t, 0)=S_{0}(t) \text { in }(0, T),  \tag{4}\\
& v(t, 0)=v_{i n}(t)\text { in }(0, T), x)=S_{1}(x)  \tag{5}\\
&(0, t, L)=v_{L}(t) \text { in }(0, L), \\
& \text { in }(0, T) .
\end{align*}
$$

Here $v$ is the axial velocity and $A$ denotes the cross section, $L, T$ are given positive numbers, and $3 \mu>0$ denotes Trouton's viscosity assumed to be constant.

The data satisfy

$$
\left\{\begin{array}{l}
0<v_{m} \leq v_{i n}(t)<v_{L}(t) \leq V_{M} \text { for any } t \in(0, T)  \tag{6}\\
0<S_{m} \leq S_{0}(t), S_{1}(x) \leq S_{M} \text { for all }(t, x) \in Q_{T}, S_{0}(0)=S_{1}(0)
\end{array}\right.
$$

Moreover, the functions $v_{i n}, v_{L}, S_{0}, S_{1}$ belong to certain regularity classes specified below.

### 2.1 A priori bounds

Our construction of global-in-time solutions is based on certain a priori estimates that hold, formally, for any smooth solution of problem (2) - (5), with the cross-section area $A>0$. The crucial observation is that, as a direct consequence of (3),

$$
\begin{equation*}
A(t, x) \partial_{x} v(t, x)=\chi(t) \text { for any } t \in(0, T), \tag{7}
\end{equation*}
$$

where $\chi$ is a function of the time variable only. Moreover, as $A$ is positive and the axial velocity satisfies the boundary conditions (6) , we deduce that

$$
\begin{equation*}
\chi(t)>0 \text { for any } t \in(0, T), \tag{8}
\end{equation*}
$$

which in turn implies

$$
\begin{equation*}
\partial_{x} v(t, x)>0 \text { for all }(t, x) \in Q_{T} . \tag{9}
\end{equation*}
$$

Accordingly,

$$
v_{i n}(t)<v(t, x)<v_{L}(t) \text { for all }(t, x) \in Q_{T}
$$

Next, we rewrite equation (2) in the form

$$
\partial_{t} A+v \partial_{x} A=-\chi \leq 0
$$

yielding

$$
\begin{equation*}
A(t, x) \leq S_{M} \text { for all }(t, x) \in Q_{T} \tag{10}
\end{equation*}
$$

Integrating $(\sqrt{7})$ over $(0, L)$ and using (5) and (10) give rise to the uniform bound

$$
\begin{equation*}
0<\chi(t)<\frac{v_{L}(t)-v_{i n}(t)}{L} S_{M} \text { for all } t \in(0, T) \tag{11}
\end{equation*}
$$

In order to deduce a lower bound for the cross section area $A$, we first observer that $A$ and $\partial_{x} A$ satisfy the same transport equation, namely,

$$
\begin{gather*}
\partial_{t} A+\partial_{x}(v A)=0  \tag{12}\\
\partial_{t}\left(\partial_{x} A\right)+\partial_{x}\left(v\left(\partial_{x} A\right)\right)=0 \tag{13}
\end{gather*}
$$

In particular,

$$
\begin{equation*}
\partial_{t}\left(\partial_{x} \log (A)\right)+v \partial_{x}\left(\partial_{x} \log (A)\right)=0 \tag{14}
\end{equation*}
$$

Evaluating the boundary values of $\partial_{x} \log (A)$ with (4), namely,

$$
\partial_{x} \log (A)(t, 0)=\frac{\partial_{x} A(t, 0)}{S_{0}(t)}, \partial_{x} \log (A)(0, x)=\frac{\partial_{x} S_{1}(x)}{S_{1}(x)}
$$

where, in accordance with (2), (4), (5), (7), and (11)

$$
\begin{aligned}
\partial_{x} A(t, 0) & =-\frac{1}{v_{i n}(t)}\left(\chi(t)+\frac{d S_{0}}{d t}(t)\right) \\
& \geq-\frac{1}{v_{i n}(t)}\left(\frac{v_{L}(t)-v_{i n}(t)}{L} S_{M}+\frac{d S_{0}}{d t}(t)\right) .
\end{aligned}
$$

We deduce easily the desired lower bound on $A$

$$
\begin{equation*}
A(t, x) \geq A_{m}>0 \text { for all }(t, x) \in Q_{T} \tag{15}
\end{equation*}
$$

where the constant $A_{m}$ is determined solely in terms of $v_{m}, V_{M}, S_{m}, S_{M}$, and the first derivatives of $S_{0}, S_{1}$.

The a priori bounds derived in (7) - (15) form a suitable platform for the existence theory developed in the remaining part of this paper.

## 3 Short time existence of regularized strong solutions

In addition to (6), we shall assume that

$$
\begin{equation*}
S_{0} \in W^{2, \infty}(0, T), \quad S_{1} \in H^{2}(0, L), \quad v_{i n}, v_{L} \in W^{1, \infty}(0, T) \tag{16}
\end{equation*}
$$

where the symbol $W^{k, p}$ denotes the standard Sobolev space of functions having $k$－derivatives $L^{p}$－integrable，and $H^{2} \equiv W^{2,2}$ ．For further use，we introduce the notation

$$
\begin{equation*}
Q^{0,0}=-\frac{d S_{0}}{d t}(0)-v_{i n}(0) \frac{d S_{1}}{d x}(0) . \tag{17}
\end{equation*}
$$

Let us begin with a list of definitions：
Definition 1．Let $t_{0}$ be a positive number．A pair $(A, v)$ ，defined on $Q_{t_{0}}=$ $\left(0, t_{0}\right) \times(0, L)$ ，is a strong solution of（园）－（因）if

$$
\begin{gather*}
A \in W^{1, \infty}\left(\left(0, t_{0}\right) \times(0, L)\right),  \tag{18}\\
v, \partial_{t} v, \partial_{x} v, \partial_{t x}^{2} v, \partial_{x}^{2} v \in L^{\infty}\left(\left(0, t_{0}\right) \times(0, L)\right), \tag{19}
\end{gather*}
$$

$(A, v)$ satisfy equations（2）－（3）a．e．in $\left(0, t_{0}\right) \times(0, L)$ ，

$$
\begin{gather*}
A>0 \text { on } Q_{t_{0}} \text { and } A \text { satisfies (4) pointwise, }  \tag{20}\\
v \text { satisfies (國) pointwise. } \tag{21}
\end{gather*}
$$

Definition 2．Let $t_{0}$ be a constant， $0<t_{0} \leq T$ ．For $h \in L^{\infty}\left(0, t_{0} ; H^{1}(0, L)\right)$ with $\partial_{x} h \in L^{\infty}\left(0, L ; L^{2}\left(0, t_{0}\right)\right)$ ，we define the energy functional $\mathcal{E}$ as

$$
\begin{equation*}
\mathcal{E}(h)^{2}=\operatorname{ess} \sup _{0<t<t_{0}}\|h(t, \cdot)\|_{H^{1}(0, L)}^{2}+v_{m} \operatorname{ess} \sup _{0<x<L}\left\|\partial_{x} h(\cdot, x)\right\|_{L^{2}\left(0, t_{0}\right)}^{2} . \tag{23}
\end{equation*}
$$

Similarly，the radius $R$ is defined by

$$
\begin{equation*}
\frac{1}{8} R^{2}=\int_{0}^{L}\left\{S_{1}^{2}+\left(\frac{d S_{1}}{d x}\right)^{2}\right\}(x) d x+\int_{0}^{T} v_{i n}(t)\left\{S_{0}^{2}+\frac{2}{v_{i n}^{2}}\left|\frac{d S_{0}}{d t}\right|^{2}\right\}(t) d t \tag{24}
\end{equation*}
$$

Definition 3．For $R$ given by（24）and $0<t_{0} \leq T$ ，we denote by $\mathcal{S}\left(t_{0}, R\right)$ the convex set of nonnegative functions $h$ defined on $Q_{t_{0}}$ such that

$$
\begin{gather*}
h \in W^{1, \infty}\left(0, t_{0} ; L^{2}(0, L)\right) \cap L^{\infty}\left(0, t_{0} ; H^{1}(0, L)\right) \cap \\
\cap W^{1, \infty}\left(0, L ; L^{2}\left(0, t_{0}\right)\right) \cap L^{\infty}\left(0, L ; H^{1}\left(0, t_{0}\right)\right),  \tag{25}\\
\mathcal{E}(h) \leq R, \quad \text { and } \\
\text { ess } \sup _{0<t<t_{0}}\left\|\partial_{t} h(t, \cdot)\right\|_{L^{2}(0, L)}+\sup _{x \in[0, L]}\left\|\partial_{t} h(\cdot, x)\right\|_{L^{2}\left(0, t_{0}\right)} \leq \\
R V_{M}\left(2+\frac{1}{L}+\frac{\sqrt{T}}{L^{3 / 2}}\right)+\left|Q^{0,0}\right|(\sqrt{T}+\sqrt{L})  \tag{26}\\
h(0, x)=S_{1}(x) \quad \text { and } \quad h(t, 0)=S_{0}(t) . \tag{27}
\end{gather*}
$$

Definition 4. For $R$ given by (24),

$$
\alpha=\frac{4}{v_{m}}\left|Q^{0,0}\right|^{2}+4 S_{M} L\left|Q^{0,0}\right|, \quad \beta=\frac{4 V_{M}^{2} R^{2}}{v_{m} L^{3}}+4 S_{M}^{2} V_{M}+\frac{4 S_{M} V_{M} R}{\sqrt{L}}
$$

we denote $t^{*} \in(0, T]$ a positive time satisfying

$$
\begin{equation*}
t^{*} \leq \frac{S_{m} L^{3 / 2}}{4 R V_{M}} \quad \text { and } \quad t^{*} \leq \frac{R^{2}}{8(\alpha+\beta)} \tag{28}
\end{equation*}
$$

In this section, for $\delta>0$ small enough, we construct a family of approximate solutions $\left(A^{\delta}, v^{\delta}\right)$ solving the following initial-boundary value problem:

$$
\begin{gather*}
\partial_{t} A^{\delta}+\partial_{x}\left(v^{\delta} A^{\delta}\right)=0 \quad \text { in } \quad Q_{t_{0}}^{\delta}=\left(\delta, t_{0}\right) \times(0, L),  \tag{29}\\
\partial_{t} A^{\delta}+v^{\delta} \partial_{x} A^{\delta}+Q^{0,0}\left(1-\frac{t}{\delta}\right)+ \\
\frac{v_{L}(\delta)-v_{i n}(\delta)}{\int_{0}^{L} \frac{d \xi}{A^{\delta}(\delta, \xi)}} \frac{t}{\delta}=0 \quad \text { in } \quad(0, \delta] \times(0, L),  \tag{30}\\
\partial_{x}\left(3 \mu A^{\delta} \partial_{x} v^{\delta}\right)=0 \quad \text { in } \quad Q_{t_{0}}=\left(0, t_{0}\right) \times(0, L),  \tag{31}\\
A^{\delta}(t, 0)=S_{0}(t) \text { in }(0, T), \quad A^{\delta}(0, x)=S_{1}(x) \quad \text { in }(0, L),  \tag{32}\\
v^{\delta}(t, 0)=v_{i n}(t) \text { in }(0, T), \quad v^{\delta}(t, L)=v_{L}(t) \quad \text { in }(0, T) \tag{33}
\end{gather*}
$$

Specifically, we prove the following result.
Theorem 1. Let $v_{L}, v_{i n}, S_{0}$ and $S_{1}$ satisfy (10). Consider $t^{*}>0$ given by (28), $t_{0} \in\left(0, t^{*}\right)$, and let $\delta \in\left(0, \min \left\{1, t_{0}\right\}\right)$ be a small number satisfying

$$
\begin{equation*}
\frac{S_{m}}{12}-\delta\left|Q^{0,0}\right|>0 \tag{34}
\end{equation*}
$$

Then the initial-boundary value problem (29) - (33) possesses a unique solution $\left(A^{\delta}, v^{\delta}\right)$ in the class

$$
\begin{align*}
A^{\delta} & \in C^{1}\left(\left[0, t_{0}\right] ; H^{1}(0, L)\right) \cap C\left(\left[0, t_{0}\right] ; H^{2}(0, L)\right)  \tag{35}\\
v^{\delta} & \in C^{1}\left(\left[0, t_{0}\right] ; H^{2}(0, L)\right) \cap C\left(\left[0, t_{0}\right] ; H^{3}(0, L)\right) \tag{36}
\end{align*}
$$

Note that for a strictly positive $h \in \mathcal{S}\left(t_{0}, R\right)$, the corresponding velocity
field $v$ solving $\partial_{x}\left(h \partial_{x} v\right)=0$ with boundary conditions (5) reads

$$
\begin{gather*}
v(t, x)=v_{i n}(t)+\frac{v_{L}(t)-v_{i n}(t)}{\int_{0}^{L} \frac{d \xi}{h(t, \xi)}} \int_{0}^{x} \frac{d \xi}{h(t, \xi)}  \tag{37}\\
\partial_{x} v(t, x)=\frac{v_{L}(t)-v_{i n}(t)}{\int_{0}^{L} \frac{d \xi}{h(t, \xi)}} \frac{1}{h(t, x)} \tag{38}
\end{gather*}
$$

Remark 1. Under the compatibility condition

$$
\begin{equation*}
Q^{0,0}=-\frac{d S_{0}}{d t}(0)-v_{i n}(0) \frac{d S_{1}}{d x}(0)=\frac{v_{L}(0)-v_{i n}(0)}{\int_{0}^{L} \frac{d x}{S_{1}(x)}} \tag{39}
\end{equation*}
$$

we could set $\delta=0$. However, imposing (3g) is not physically justified. Note that this condition is systematically used in 10] as well as [15].

The rest of this section is devoted to the proof of Theorem 1. The basic and rather standard idea is to construct a sequence approaching a fixed point of a suitable nonlinear mapping. The proof is carried over by means of several steps. We fix $t_{0} \in\left(0, t^{*}\right)$ and $\delta \in\left(0, \min \left\{1, t_{0}\right\}\right)$ such that (34) is satisfied.

## STEP 1

We take an arbitrary $A^{0} \in \mathcal{S}\left(t_{0}, R\right)$. Then we substitute $h=A^{0}$ into (37) and calculate $v^{0}=v$. We note that $A^{0} \in \mathcal{S}\left(t_{0}, R\right)$ implies

$$
\begin{gather*}
v^{0} \in W^{1, \infty}\left(\left[0, t_{0}\right] ; H^{1}(0, L)\right) \cap L^{\infty}\left(\left[0, t_{0}\right] ; H^{2}(0, L)\right),  \tag{40}\\
t \mapsto \partial_{x} v^{0}(t, 0)=\frac{v_{L}(t)-v_{i n}(t)}{\int_{0}^{L} \frac{d \xi}{A^{0}(t, \xi)}} \frac{1}{S_{0}(t)} \in H^{1}\left(0, t_{0}\right),  \tag{41}\\
\partial_{x} v^{0}(t, x)>0 \quad \text { and } \quad v_{i n}(t) \leq v^{0}(t, x) \leq v_{L}(t) \quad \text { in }\left[0, t_{0}\right] \times[0, L] . \tag{42}
\end{gather*}
$$

Next, we introduce functions $Q^{0}$ and $Q^{0, \delta}$ defined on $\left[0, t_{0}\right]$ by

$$
\begin{gather*}
Q^{0}(t)=\frac{v_{L}(t)-v_{i n}(t)}{\int_{0}^{L} \frac{d \xi}{A^{0}(t, \xi)}}  \tag{43}\\
Q^{0, \delta}(t)=\left\{\begin{array}{l}
Q^{0}(t), \quad \text { for } \delta \leq t \leq t_{0} \\
Q^{0,0}+\left(Q^{0}(\delta)-Q^{0,0}\right) \frac{t}{\delta}, \quad \text { for } 0 \leq t<\delta .
\end{array}\right. \tag{44}
\end{gather*}
$$

Obviously, $Q^{0, \delta} \in W^{1, \infty}\left(0, t_{0}\right)$, and, using Jensen's inequality and (6), we get

$$
\begin{align*}
0 \leq Q^{0}(t) & \leq \frac{v_{L}(t)}{\int_{0}^{L} \frac{d x}{A^{0}(t, x)} \leq \frac{v_{L}(t)}{L^{2}} \int_{0}^{L} A^{0}(t, x) d x} \\
& \leq \frac{V_{M}}{L^{3 / 2}}\left\|A^{0}(t, \cdot)\right\|_{L^{2}(0, L)} \leq \frac{V_{M} R}{L^{3 / 2}} \tag{45}
\end{align*}
$$

since $A^{0} \in \mathcal{S}\left(t_{0}, R\right)$, as well as

$$
\begin{equation*}
\left|Q^{0, \delta}(t)\right| \leq\left|Q^{0,0}\right|\left(1-\frac{t}{\delta}\right)_{+}+\frac{V_{M} R}{L^{3 / 2}} \tag{46}
\end{equation*}
$$

Now we are in a position to define the solution operator:
For $A^{0} \in \mathcal{S}\left(t_{0}, R\right)$ and $v^{0}$ given by (37) with $A^{0}$ instead of $h$, we solve the initial-boundary value problem

$$
\begin{gather*}
\partial_{t} u(t, x)=-v^{0}(t, x) \partial_{x} u(t, x)-Q^{0, \delta}(t), \quad(t, x) \in Q_{t_{0}}  \tag{47}\\
u(0, x)=S_{1}(x), x \in(0, L), \quad u(t, 0)=S_{0}(t), t \in\left(0, t_{0}\right) \tag{48}
\end{gather*}
$$

Because of (40)-(42) and regularity and compatibility of the data, we may apply a result of Hagen \& Renardy 10] (see Theorem 1 in Appendix) to problem (47)-(48) to obtain a unique solution $u \in C^{1}\left(\left[0, t_{0}\right] ; H^{1}(0, L)\right) \cap$ $C\left(\left[0, t_{0}\right] ; H^{2}(0, L)\right)$. We set

$$
\begin{equation*}
A^{1}(t, x)=u(t, x), \quad(t, x) \in Q_{t_{0}} \tag{49}
\end{equation*}
$$

Relation (49) defines a nonlinear operator, assigning to a given $A^{0} \in \mathcal{S}\left(t_{0}, R\right)$ the unique function $A^{1}$ in the class $C^{1}\left(\left[0, t_{0}\right] ; H^{1}(0, L)\right) \cap C\left(\left[0, t_{0}\right] ; H^{2}(0, L)\right)$. By (47), $\partial_{t}^{2} u+\partial_{t} Q^{0, \delta} \in L^{\infty}\left(Q_{t_{0}}\right)$. Using a compactness lemma of Aubin type (see e.g. [24]) we conclude that our nonlinear operator, defined on $\mathcal{S}\left(t_{0}, R\right)$, is compact.

At this stage we follow the ideas of 10] and 15] to show that this nonlinear operator has a fixed point for all $t_{0}<t^{*}$. They study the nonisothermal fiber spinning and their system of equations is different. Hence we are obliged to give an independent proof of short time existence, but the result remains close to their considerations.

The natural approach is to apply Schauder's fixed point theorem. To use it we have to prove that $\mathcal{S}\left(t_{0}, R\right)$ is a relatively compact convex set invariant under our nonlinear mapping. Finally, we should establish the continuity of the mapping (49).

## STEP 2

We establish uniform $L^{\infty}$ - bounds for the function $u$. We have
Lemma 1. Let $R$ be given by Definition 3 and let $t_{0} \leq t^{*}$. Then the solution $u$ of problem (47)-(48) satisfies the estimate

$$
\begin{equation*}
\frac{S_{m}}{4} \leq u(t, x) \leq S_{M}+\frac{2 S_{m}}{3} \leq 2 S_{M} \text { in }\left[0, t_{0}\right] \times[0, L] \tag{50}
\end{equation*}
$$

Proof. Let $s_{m} \in W^{2, \infty}\left(0, t_{0}\right)$ be the solution to the Cauchy problem

$$
\begin{equation*}
\frac{d s_{m}(t)}{d t}=-Q^{0, \delta}(t) \text { in }\left(0, t_{0}\right), s_{m}(0)=\frac{2 S_{m}}{3} \tag{51}
\end{equation*}
$$

Then $s_{m}$ clearly solves (47) and it follows from (6) and (48) that

$$
u(0, x)=S_{1}(x) \geq S_{m} \geq s_{m}(0), \quad x \in(0, L)
$$

while (6), (34), the nonnegativity of $Q^{0}$, (48), and (51) ensure that

$$
\begin{aligned}
s_{m}(t) & =\frac{2 S_{m}}{3}-\int_{0}^{t} Q^{0, \delta}(s) d s \leq \frac{2 S_{m}}{3}-\int_{0}^{\min \{\delta, t\}} Q^{0, \delta}(s) d s \\
& \leq \frac{2 S_{m}}{3}-\int_{0}^{\min \{\delta, t\}} Q^{0,0}\left(1-\frac{s}{\delta}\right) d s \leq \frac{2 S_{m}}{3}+\left|Q^{0,0}\right| \frac{\delta}{2} \\
& \leq S_{m} \leq S_{0}(t)=u(t, 0)
\end{aligned}
$$

The comparison principle then entails that $u(t, x) \geq s_{m}(t)$ for $(t, x) \in Q_{t_{0}}$, whence, by (28), (34), and (46),

$$
\begin{aligned}
u(t, x) & \geq s_{m}(t) \geq \frac{2 S_{m}}{3}-\int_{0}^{t}\left|Q^{0, \delta}(s)\right| d s \\
& \geq \frac{2 S_{m}}{3}-\left|Q^{0,0}\right| \frac{\delta}{2}-\frac{V_{M} R}{L^{3 / 2}} t^{*} \geq \frac{S_{m}}{4}
\end{aligned}
$$

This proves the lower bound. Proving the upper bound is analogous. The comparison function is now given by the solution to the Cauchy problem

$$
\begin{equation*}
\frac{d s_{M}(t)}{d t}=-Q^{0, \delta}(t) \text { in }\left(0, t_{0}\right), s_{M}(0)=S_{M}+\frac{S_{m}}{2} \tag{52}
\end{equation*}
$$

Indeed, $s_{M}$ clearly solves (47) with $s_{M}(0) \geq S_{M} \geq S_{1}(x)=u(0, x)$ for $x \in(0, L)$ and it follows from (6), (28), (34), (46), (48), and (52) that

$$
s_{M}(t) \geq s_{M}(0)-\left|Q^{0,0}\right| \frac{\delta}{2}-\frac{V_{M} R}{L^{3 / 2}} t^{*} \geq S_{M} \geq u(t, 0)
$$

Applying again the comparison principle gives $u(t, x) \leq s_{M}(t)$ for $(t, x) \in$ $Q_{t_{0}}$, which completes the proof since (34) guarantees that

$$
s_{M}(t) \leq S_{M}+\frac{S_{m}}{2}+\left|Q^{0,0}\right| \frac{\delta}{2} \leq S_{M}+\frac{2 S_{m}}{3} .
$$

This proves the Lemma.
Now we use equation (47) to calculate $G(t)=\partial_{x} u(t, 0) \in C\left(\left[0, t_{0}\right]\right)$ getting:

$$
\begin{equation*}
G(t)=\partial_{x} u(t, 0)=-\frac{1}{v_{i n}(t)}\left(\frac{d S_{0}(t)}{d t}+Q^{0, \delta}(t)\right) . \tag{53}
\end{equation*}
$$

Due to the assumptions on the data, $G \in W^{1, \infty}\left(0, t_{0}\right)$ and we have

$$
\begin{equation*}
|G(t)| \leq \frac{1}{v_{i n}(t)}\left(\left|\frac{d S_{0}(t)}{d t}\right|+\left|Q^{0, \delta}(t)\right|\right) . \tag{54}
\end{equation*}
$$

Next, we take the derivative of equation (47) with respect to the $x$ variable. This yields that $S=\partial_{x} u$ solves

$$
\begin{gather*}
\partial_{t} S=-v^{0} \partial_{x} S-\partial_{x} v^{0} S, \quad(t, x) \in Q_{t_{0}}  \tag{55}\\
S(0, x)=\frac{d S_{1}}{d x}(x), x \in(0, L) ; \quad S(t, 0)=G(t), t \in\left(0, t_{0}\right), \tag{56}
\end{gather*}
$$

the function $v^{0}$ being still given by (37) with $A^{0}$ instead of $h$.
Multiplying equation (47) by $u$, equation (55) by $S$, integrating both equations on $(0, t) \times(0, x),(t, x) \in Q_{t_{0}}$, and adding the obtained identities, we deduce:

$$
\begin{align*}
& \quad \frac{1}{2} \int_{0}^{x}\left\{u^{2}+S^{2}\right\}(t, \xi) d \xi+\frac{1}{2} \int_{0}^{t} v^{0}(\tau, x)\left\{u^{2}+S^{2}\right\}(\tau, x) d \tau+ \\
& \frac{1}{2} \int_{0}^{t} \int_{0}^{x} \partial_{x} v^{0}(\tau, \xi)\left\{S^{2}-u^{2}\right\}(\tau, \xi) d \xi d \tau+\int_{0}^{t} \int_{0}^{x} Q^{0, \delta}(\tau) u(\tau, \xi) d \xi d \tau \\
& =\frac{1}{2} \int_{0}^{x}\left\{S_{1}^{2}+\left(\frac{d S_{1}}{d x}\right)^{2}\right\}(\xi) d \xi+\frac{1}{2} \int_{0}^{t} v_{i n}(\tau)\left\{S_{0}^{2}+G^{2}\right\}(\tau) d \tau \tag{57}
\end{align*}
$$

Now we use (6), (46) with (54) to get

$$
\begin{gather*}
\int_{0}^{t} v_{i n}(\tau) G^{2}(\tau) d \tau \leq \int_{0}^{t} \frac{2}{v_{i n}(\tau)}\left\{\left|\frac{d S_{0}}{d \tau}\right|^{2}+\left|Q^{0, \delta}(\tau)\right|^{2}\right\} d \tau \leq \\
\int_{0}^{t} \frac{2}{v_{i n}(\tau)}\left|\frac{d S_{0}}{d \tau}\right|^{2} d \tau+\frac{4 \min \{t, \delta\}}{v_{m}}\left|Q^{0,0}\right|^{2}+\frac{4 V_{M}^{2} R^{2}}{v_{m} L^{3}} t . \tag{58}
\end{gather*}
$$

The third term on the left hand side of (57) is estimated with the help of (42) and (50) as

$$
\begin{gather*}
\int_{0}^{t} \int_{0}^{x} \partial_{x} v^{0}(\tau, \xi)\left(u^{2}-S^{2}\right)(\tau, \xi) d \xi d \tau \leq \int_{0}^{t} \int_{0}^{x} \frac{Q^{0}(\tau)}{A^{0}(\tau, \xi)} u^{2}(\tau, \xi) d \xi d \tau \\
\leq 4 S_{M}^{2} \int_{0}^{t} Q^{0}(\tau) \int_{0}^{x} \frac{d \xi}{A^{0}(\tau, \xi)} d \tau \leq 4 S_{M}^{2} \int_{0}^{t}\left(v^{0}(\tau, x)-v_{i n}(\tau)\right) d \tau \\
\leq 4 S_{M}^{2} V_{M} t \tag{59}
\end{gather*}
$$

while the fourth term obeys

$$
\begin{gather*}
\left|\int_{0}^{t} \int_{0}^{x} Q^{0, \delta}(\tau) u(\tau, \xi) d \xi d \tau\right| \leq 2 S_{M} L \int_{0}^{t}\left|Q^{0, \delta}(\tau)\right| d \tau \\
\leq 2 S_{M} L\left(\left|Q^{0,0}\right|+\frac{V_{M} R}{L^{3 / 2}}\right) t \tag{60}
\end{gather*}
$$

thanks to (46) and (50).
Now, setting

$$
y(t)=\sup _{x \in[0, L]}\left\{\int_{0}^{x}\left(u^{2}+S^{2}\right)(t, \xi) d \xi+\int_{0}^{t} v^{0}(\tau, x)\left\{u^{2}+S^{2}\right\}(\tau, x) d \tau\right\}
$$

we may insert (58)-(60) into (57) and use (24) and (28) to get the following estimate:

$$
\begin{aligned}
y(t) \leq & \int_{0}^{L}\left\{S_{1}^{2}+\left(\frac{d S_{1}}{d \xi}\right)^{2}\right\}(\xi) d \xi \\
& +\int_{0}^{t} v_{i n}(\tau)\left\{S_{0}^{2}+\frac{2}{v_{i n}^{2}(\tau)}\left|\frac{d S_{0}}{d \tau}\right|^{2}\right\}(\tau) d \tau+(\alpha+\beta) t \\
\leq & \frac{R^{2}}{8}+\frac{R^{2}}{8} \frac{t}{t^{*}} \leq \frac{R^{2}}{4}
\end{aligned}
$$

Recalling (6) and (42), we thus conclude that

$$
\begin{equation*}
\mathcal{E}(u)<R . \tag{61}
\end{equation*}
$$

Next, we use equation (47) and estimate (61) to control $\partial_{t} u$, obtaining

$$
\begin{gather*}
\sup _{t \in\left[0, t_{0}\right]}\left\|\partial_{t} u(t)\right\|_{L^{2}(0, L)}+\sup _{x \in[0, L]}\left\|\partial_{t} u(\cdot, x)\right\|_{L^{2}\left(0, t_{0}\right)}< \\
\quad R V_{M}\left(2+\frac{1}{L}+\frac{\sqrt{T}}{L^{3 / 2}}\right)+\left|Q^{0,0}\right|(\sqrt{T}+\sqrt{L}) \tag{62}
\end{gather*}
$$

Therefore the image of a nonnegative function from $\mathcal{S}\left(t_{0}, R\right)$ remains in $\mathcal{S}\left(t_{0}, R\right)$ and satisfies the $L^{\infty}$－bound（50）．Therefore our nonlinear map （49）maps the convex set

$$
\begin{equation*}
\mathcal{S}_{0}\left(t_{0}, R\right)=\left\{f \in \mathcal{S}\left(t_{0}, R\right) \mid f \text { satisfies (50) a.e. }\right\} \tag{63}
\end{equation*}
$$

into itself．

## STEP 3

Let $X$ be the intersection of the Banach spaces $W^{1, \infty}\left(\left[0, t_{0}\right] ; L^{2}(0, L)\right)$ ， $L^{\infty}\left(\left[0, t_{0}\right] ; H^{1}(0, L)\right), W^{1, \infty}\left(0, L ; L^{2}\left(0, t_{0}\right)\right)$ and $L^{\infty}\left(0, L ; H^{1}\left(0, t_{0}\right)\right)$ ．Clearly， $\mathcal{S}_{0}\left(t_{0}, R\right)$ is a convex，bounded and closed subset of the Banach space $X$ ． We apply the Schauder fixed point theorem to prove that the mapping（49） admits a fixed point in $\mathcal{S}_{0}\left(t_{0}, R\right)$ ．After Step 2 ，it remains only to prove the sequential continuity of the map（49）．Hence let $\left\{A^{k}\right\}_{k \in \mathbb{N}}$ ，be a sequence converging in $\mathcal{S}_{0}\left(t_{0}, R\right)$ to $A$ ．Then we have

$$
\begin{gather*}
A^{k} \rightarrow A \quad \text { strongly in } C\left(\left[0, t_{0}\right] ; L^{2}(0, L)\right) \cap L^{2}\left(0, t_{0} ; C([0, L])\right),  \tag{64}\\
Q_{k}^{0}=\frac{v_{L}-v_{i n}}{\int_{0}^{L} \frac{d x}{A^{k}(t, x)}} \rightarrow Q^{0}=\frac{v_{L}-v_{i n}}{\int_{0}^{L} \frac{d x}{A(t, x)}} \text { uniformly in } C\left(\left[0, t_{0}\right]\right),  \tag{65}\\
v^{k}=v_{i n}+Q_{k}^{0} \int_{0}^{x} \frac{d \xi}{A^{k}(t, \xi)} \rightarrow v=v_{i n}+Q^{0} \int_{0}^{x} \frac{d \xi}{A(t, \xi)} \\
\text { strongly in } C\left(\left[0, t_{0}\right] ; H^{1}(0, L)\right) . \tag{66}
\end{gather*}
$$

Let $u^{k}$ be the solution to（47）－（48），corresponding to $A^{k}$ and $u$ the so－ lution corresponding to $A$ ．Then by analogous calculations to those per－ formed in Step 2，we get $u^{k} \rightarrow u$ in $C\left(\left[0, t_{0}\right] ; H^{1}(0, L)\right) \cap L^{2}\left(0, t_{0} ; C^{1}([0, L])\right)$ ． Using the equation（47），we find out that one also has $\partial_{t} u^{k} \rightarrow \partial_{t} u$ in $C\left(\left[0, t_{0}\right] ; L^{2}(0, L)\right) \cap L^{2}\left(0, t_{0} ; C([0, L])\right)$ ．Therefore the mapping（49）is con－ tinuous and compact and by Schauder＇s fixed point theorem there is at least one fixed point $A^{\delta} \in \mathcal{S}_{0}\left(t_{0}, R\right)$ ．

Denoting the corresponding velocity field by $v^{\delta}$ ，we have

$$
v^{\delta} \in W^{1, \infty}\left(0, t_{0} ; H^{1}(0, L)\right) \cap L^{\infty}\left(0, t_{0} ; H^{2}(0, L)\right)
$$

and since $t \mapsto \partial_{x} v^{\delta}(t, 0) \in W^{1, \infty}\left(0, t_{0}\right)$ ，we may apply Theorem $⿴ 囗 十$ in Ap－ pendix to conclude that，in fact，

$$
A^{\delta} \in C^{1}\left(\left[0, t_{0}\right] ; H^{1}(0, L)\right) \cap C\left(\left[0, t_{0}\right] ; H^{2}(0, L)\right) .
$$

It remains to prove uniqueness.

## STEP 4

With the obtained smoothness, uniqueness is easy to establish. It suffices to notice that $L^{2}\left(0, t_{0} ; H^{1}(0, L)\right)$ perturbation of $v^{\delta}$ is controlled by the $L^{2}$ perturbation of $A^{\delta}$ in $x$ and $t$. Then we use this observation, regularity of $A^{\delta}$, and Gronwall's lemma to obtain uniqueness.

The proof of Theorem 1 is now complete.

## 4 Global existence of regularized strong solutions

Now we suppose that the regularity of the solution or/and the strict positivity of $A^{\delta}$, stated in (50), breaks down at the time $t_{p}$. Our goal is to prove that $t_{p}=T$.
Theorem 2. Under the hypotheses of Theorem 亿, the initial-boundary value problem (29)-(33) has a unique solution

$$
\begin{aligned}
& A^{\delta} \in C^{1}\left([0, T] ; H^{1}(0, L)\right) \cap C\left([0, T] ; H^{2}(0, L)\right), \\
& v^{\delta} \in C^{1}\left([0, T] ; H^{2}(0, L)\right) \cap C\left([0, T] ; H^{3}(0, L)\right) .
\end{aligned}
$$

The remaining part of this section is devoted to the proof of Theorem 2 .

## STEP 1

First we recall that, owing to (31) and (33), $v^{\delta}$ is given by

$$
\begin{equation*}
v^{\delta}(t, x)=v_{i n}(t)+\frac{v_{L}(t)-v_{i n}(t)}{\int_{0}^{L} \frac{d \xi}{A^{\delta}(t, \xi)}} \int_{0}^{x} \frac{d \xi}{A^{\delta}(t, \xi)} \tag{67}
\end{equation*}
$$

Next, by (6)

$$
\begin{equation*}
Q^{\delta}(t)=\frac{v_{L}(t)-v_{i n}(t)}{\int_{0}^{L} \frac{d x}{A^{\delta}(t, x)}}>0 \quad \text { in } \overline{Q_{t_{0}}} . \tag{68}
\end{equation*}
$$

and we put

$$
Q^{c u t, \delta}(t)=\left\{\begin{array}{l}
Q^{\delta}(t), \quad \text { for } \delta \leq t \leq t_{0}  \tag{69}\\
Q^{0,0}+\left(Q^{\delta}(\delta)-Q^{0,0}\right) \frac{t}{\delta}, \quad \text { for } 0 \leq t<\delta
\end{array}\right.
$$

We point out that the upper bound in ( $\overline{\boxed{0}}$ ) is independent of $t^{*}$. Indeed, we prove now that it is valid regardless the length of the time interval. To this end, we introduce the solution $S_{M}^{\delta} \in W^{2, \infty}\left(0, t_{0}\right)$ to the Cauchy problem

$$
\begin{equation*}
\frac{d S_{M}^{\delta}(t)}{d t}=\left|Q^{0,0}\right| \chi_{[0, \delta]}(t), \quad t \in\left(0, t_{0}\right), \quad S_{M}^{\delta}(0)=S_{M} \tag{70}
\end{equation*}
$$

Owing to the positivity (68) of $Q^{\delta}, S_{M}^{\delta}$ is a supersolution to (29)-(30) with $S_{M}^{\delta}(0)=S_{M} \geq S_{1}(x)=A^{\delta}(0, x)$ for $x \in[0, L]$ and $S_{M}^{\delta}(t) \geq S_{M}^{\delta}(0) \geq$ $S_{0}(t)=A^{\delta}(t, 0)$ for $t \in\left(0, t_{0}\right)$. The comparison principle then implies

$$
\begin{equation*}
A^{\delta}(t, x) \leq S_{M}^{\delta}(t) \leq S_{M}+\delta\left|Q^{0,0}\right| \tag{71}
\end{equation*}
$$

This proves the upper bound and, in addition, the estimate is independent of the length of the time interval.

Therefore, by Jensen's inequality, we may infer that

$$
\begin{align*}
& 0<Q^{\delta}(t) \leq \frac{V_{M}}{\int_{0}^{L} \frac{d \xi}{A^{\delta}(t, \xi)}} \leq \frac{V_{M}}{L^{2}} \int_{0}^{L} A^{\delta}(t, \xi) d \xi \\
& \leq \frac{V_{M}}{L}\left(S_{M}+\delta\left|Q^{0,0}\right|\right),  \tag{72}\\
& \left|G^{\delta}(t)\right|=\left|\partial_{x} A^{\delta}(t, 0)\right|=\left|-\frac{1}{v_{i n}(t)}\left(\frac{d S_{0}(t)}{d t}+Q^{c u t, \delta}(t)\right)\right| \\
& \leq \frac{1}{v_{m}}\left(\left|\frac{d S_{0}(t)}{d t}\right|+\left|Q^{0,0}\right|+\frac{V_{M}}{L}\left(S_{M}+\delta\left|Q^{0,0}\right|\right)\right), \tag{73}
\end{align*}
$$

for any $t<t_{p}$, where $t_{p}$ is the critical time at which $A^{\delta}(t, x)$ attains zero for some $x$.

## STEP 2

Having established that $\partial_{x} A^{\delta}(t, 0)$ is bounded in $L^{\infty}(0, t)$ independently of $t$, we now look for a strictly positive lower bound for $A^{\delta}$.

Proposition 1. There are constants $C_{1}$ and $C_{2}$, independent of the length of the time interval and of $\delta$, such that we have

$$
\begin{equation*}
C_{1} \leq \partial_{x} \log A^{\delta}(t, x) \leq C_{2} \quad \text { on } \quad \overline{Q_{t_{0}}} . \tag{74}
\end{equation*}
$$

Proof. We notice that $y=\partial_{x} \log A^{\delta}$ satisfies the equation

$$
\begin{equation*}
\partial_{t} y+v^{\delta} \partial_{x} y+\frac{y}{A^{\delta}}\left(Q^{\delta}-Q^{c u t, \delta}\right)=0 \text { in } Q_{t_{p}} \tag{75}
\end{equation*}
$$

and

$$
\begin{equation*}
y(0, x)=\frac{d \log S_{1}}{d x}(x), \quad x \in(0, L), \quad y(t, 0)=\frac{G^{\delta}(t)}{S_{0}(t)}, \quad t \in\left(0, t_{p}\right) . \tag{76}
\end{equation*}
$$

The function $F(t, x)=\left(Q^{\delta}-Q^{c u t, \delta}\right)(t) / A^{\delta}(t, x)$ vanishes for $t \geq \delta$ and, due to (50), satisfies the following bound

$$
\begin{align*}
\int_{0}^{t}\|F(\tau)\|_{L^{\infty}(0, L)} d \tau & =\int_{0}^{\min \{t, \delta\}}\|F(\tau)\|_{L^{\infty}(0, L)} d \tau \\
& \leq \frac{8 \delta}{S_{m}}\left(\left|Q^{0,0}\right|+\left\|Q^{\delta}\right\|_{L^{\infty}(0, \delta)}\right) \tag{77}
\end{align*}
$$

for every $t$ since $\delta \leq t_{0}$.
We next introduce the solution $y_{m}$ to the ordinary differential equation

$$
\begin{equation*}
\frac{d y_{m}}{d t}(t)=\|F(t)\|_{L^{\infty}(0, L)} y_{m}(t) \quad \text { in } \quad(0, T), \tag{78}
\end{equation*}
$$

with initial condition

$$
\begin{equation*}
y_{m}(0)=\min \left\{\min _{[0, L]}\left\{\frac{d \log S_{1}}{d x}\right\},-G_{M}\right\} \leq 0 \tag{79}
\end{equation*}
$$

with

$$
\begin{equation*}
G_{M}=\frac{1}{v_{m}}\left[\left\|\frac{d S_{0}}{d t}\right\|_{L^{\infty}(0, T)}+\left|Q^{0,0}\right|+\frac{V_{M}}{L}\left(S_{M}+\delta\left|Q^{0,0}\right|\right)\right] . \tag{80}
\end{equation*}
$$

Owing to (73), (76), (78), and (79), $y_{m}$ is nonpositive and thus a subsolution to (75), and satisfies $y_{m}(t) \leq y_{m}(0) \leq y(t, 0)$ for $t \in\left(0, t_{p}\right)$ and $y_{m}(0) \leq$ $y(0, x)$ for $x \in(0, L)$. The comparison principle then entails that

$$
y(t, x) \geq y_{m}(t)=y_{m}(0) \exp \int_{0}^{t}\|F(\tau)\|_{L^{\infty}(0, L)} d \tau
$$

Since $y_{m}(0) \leq 0$, we deduce from (77) that $y(t, x) \geq C_{1}$ for some positive constant $C_{1}$, independent of $\delta$ and $t_{0}$.

Similarly, let $Y_{M}$ be the solution to the ordinary differential equation

$$
\begin{equation*}
\frac{d Y_{M}}{d t}(t)=\|F(t)\|_{L^{\infty}(0, L)} Y_{M}(t) \quad \text { in } \quad(0, T), \tag{81}
\end{equation*}
$$

with initial condition

$$
\begin{equation*}
Y_{M}(0)=\max \left\{\max _{[0, L]}\left\{\frac{d \log S_{1}}{d x}\right\}, G_{M}\right\} \geq 0 \tag{82}
\end{equation*}
$$

the constant $G_{M}$ being defined in (80). It follows from (73), (76), (81), and (82) that $Y_{M}$ is a supersolution to (75) which satisfies $Y_{M}(t) \geq Y_{M}(0) \geq$ $y(t, 0)$ for $t \in\left(0, t_{p}\right)$ and $Y_{M}(0) \geq y(0, x)$ for $x \in(0, L)$. Using once more the comparison principle and (77), we conclude that

$$
y(t, x) \leq Y_{M}(t)=Y_{M}(0) \exp \int_{0}^{t}\|F(\tau)\|_{L^{\infty}(0, L)} d \tau \leq C_{2}
$$

the constant $C_{2}$ being independent on $\delta$ and $t$.
Since $\log A^{\delta}(t, 0)=\log S_{0}(t) \geq \log S_{m}>-\infty$ by (6) and

$$
\log A^{\delta}(t, x)=\log S_{0}(t)+\int_{0}^{x} \partial_{\xi} \log A^{\delta}(t, \xi) d \xi
$$

we infer from (74) that

$$
\begin{equation*}
\left\|\log A^{\delta}\right\|_{L^{\infty}\left(0, t_{0} ; W^{1, \infty}(0, L)\right)} \leq C \tag{83}
\end{equation*}
$$

Therefore $A^{\delta}$ is strictly positive on $\overline{Q_{t}}$ for all $t \leq t_{p}$, and bounded from below and from above by a constant which is independent of both $t$ and $\delta$.

## STEP 3

Having established (83) we easily obtain the following estimates:

$$
\begin{gather*}
\left\|\partial_{t} A^{\delta}\right\|_{L^{\infty}\left(Q_{t}\right)}+\left\|\partial_{x} A^{\delta}\right\|_{L^{\infty}\left(Q_{t}\right)} \leq C,  \tag{84}\\
\left\|v^{\delta}\right\|_{L^{\infty}\left(Q_{t}\right)}+\left\|\partial_{x} v^{\delta}\right\|_{L^{\infty}\left(Q_{t}\right)}+\left\|\partial_{x}^{2} v^{\delta}\right\|_{L^{\infty}\left(Q_{t}\right)}+\left\|\partial_{x t}^{2} v^{\delta}\right\|_{L^{\infty}\left(Q_{t}\right)} \leq C, \tag{85}
\end{gather*}
$$

where $C$ is again independent of $t$ and $\delta$.
The estimate (85) guarantees that the coefficients in equations (29)-(30) remain regular, whence Theorem is applicable. Consequently, $A^{\delta}$ remains bounded in $C^{1}\left([0, t] ; H^{1}(0, L)\right) \cap C\left([0, t] ; H^{2}(0, L)\right)$.

We conclude that, for all $t, A^{\delta}$ is bounded from below by a positive constant, independent of $t$, and the norm of $A^{\delta}$ in $C^{1}\left([0, t] ; H^{1}(0, L)\right) \cap$ $C\left([0, t] ; H^{2}(0, L)\right)$ remains bounded by a constant, also independent of $t$, that may, however, depend on $\delta$. The maximal solution therefore extends to $[0, T]$, and, in fact, we have established the existence of a unique strictly positive solution $A^{\delta}$ on $(0, T) \times(0, L)$. The corresponding velocity $v^{\delta}$ is given by (37), with $h=A^{\delta}$.

This completes the proof of Theorem .

## 5 Existence of a unique strong solution

At this stage, we are ready to establish the main result of the paper.
Theorem 3. Under the hypotheses (6), (16), the initial-boundary value problem (2)-(5) possesses a unique (strong) solution $A$, v on $\bar{Q}_{T}$, belonging to the class

$$
\begin{gathered}
A, \partial_{t} A, \partial_{x} A \in L^{\infty}((0, L) \times(0, T)), \\
v, \partial_{t} v, \partial_{t, x}^{2} v, \partial_{x, x}^{2} v \in L^{\infty}((0, L) \times(0, T)) .
\end{gathered}
$$

Proof. We just recall the estimates obtained in the proof of Theorem 2, valid independently of $\delta$ :

$$
\begin{align*}
& 0<C_{1} \leq A^{\delta}(t, x) \leq C \quad \text { in } \quad \overline{Q_{T}},  \tag{86}\\
& \left\|\partial_{x} A^{\delta}\right\|_{L^{\infty}\left(Q_{T}\right)}+\left\|\partial_{t} A^{\delta}\right\|_{L^{\infty}\left(Q_{T}\right)} \leq C . \tag{87}
\end{align*}
$$

Since $L^{\infty}\left(Q_{T}\right)$ is the dual space of the separable Banach space $L^{1}\left(Q_{T}\right)$, Alaoglu's weak* compactness theorem gives weak ${ }^{*}$ sequential compactness. Therefore there exist $A$ and $v$ such that

$$
\begin{gather*}
A^{\delta} \rightarrow A \quad \text { uniformly in } \overline{Q_{T}}, \text { as } \delta \rightarrow 0,  \tag{88}\\
\partial_{x} A^{\delta} \rightharpoonup \partial_{x} A \quad \text { weakly* in } \quad L^{\infty}\left(Q_{T}\right), \text { as } \delta \rightarrow 0,  \tag{89}\\
\partial_{t} A^{\delta} \rightharpoonup \partial_{t} A \quad \text { weakly* in } L^{\infty}\left(Q_{T}\right), \text { as } \delta \rightarrow 0,  \tag{90}\\
v^{\delta} \rightarrow v=  \tag{91}\\
v_{i n}(t)+\frac{v_{L}(t)-v_{i n}(t)}{\int_{0}^{L} \frac{d \xi}{A(t, \xi)}} \int_{0}^{x} \frac{d \xi}{A(t, \xi)} \text { uniformly in } \overline{Q_{T}}, \text { as } \delta \rightarrow 0 .  \tag{92}\\
\partial_{x} v^{\delta} \rightarrow \partial_{x} v=\frac{v_{L}(t)-v_{i n}(t)}{\int_{0}^{L} \frac{d \xi}{A(t, \xi)}} \frac{1}{A(t, x)} \text { uniformly in } \overline{Q_{T}}, \text { as } \delta \rightarrow 0,
\end{gather*}
$$

at least for suitable subsequences. Obviously, $(A, v)$ solves the system (2)(55). Moreover, by virtue of the interior regularity, the equations (2) -(3) are satisfied pointwise. Finally, according to the smoothness of $A$ and $v$, the proof of uniqueness is straightforward.

Remark 2. The standard way of proving uniqueness relies on Gronwall's inequality. For any bounded time interval, small $L^{2}$-perturbations of the data $v_{L}, v_{i n}, S_{0}$ in the $L^{2}$ - norm result in the corresponding variation
of the solution in the same norm, that may depend exponentially on the length of the time interval. Better estimates would require refined analytical arguments.

## 6 Appendix

Here we recall the result from [10], which is used in this paper:
Theorem 4. Let $f$ and $p$ be given continuous functions defined on $\left[0, t_{0}\right] \times$ $[0, b], b>0$. Let $u^{0} \in C([0, b])$ and $u^{b} \in C\left(\left[0, t_{0}\right]\right)$ and let us suppose $]$ that

$$
\begin{gather*}
p, f \in W^{1, \infty}\left(\left[0, t_{0}\right] ; H^{1}(0, b)\right) \cap L^{\infty}\left(\left[0, t_{0}\right] ; H^{2}(0, b)\right), \\
\text { are such that } \quad \partial_{x} p(\cdot, 0) \text { and } \partial_{x} f(\cdot, 0) \in H^{1}\left(0, t_{0}\right) ;  \tag{93}\\
p<0 \quad \text { on } \quad\left[0, t_{0}\right] \times[0, b] ;  \tag{94}\\
u^{0} \in H^{2}(0, b), \quad u^{b} \in H^{2}\left(0, t_{0}\right) ;  \tag{95}\\
u^{0}(0)=u^{b}(0) ; \quad \partial_{t} u^{b}(0)=p(0,0) \partial_{x} u^{0}(0)+f(0,0) \tag{96}
\end{gather*}
$$

Then the boundary-initial value problem

$$
\begin{align*}
\partial_{t} u & =p(t, x) \partial_{x} u+f(t, x), & & (x, t) \in\left(0, t_{0}\right) \times(0, b),  \tag{97}\\
u(0, x) & =u^{0}(x), x \in(0, b) ; & & u(t, 0)=u^{b}(t), t \in\left(0, t_{0}\right) . \tag{98}
\end{align*}
$$

has a solution

$$
\begin{equation*}
u \in C^{1}\left(\left[0, t_{0}\right] ; H^{1}(0, b)\right) \cap C\left(\left[0, t_{0}\right] ; H^{2}(0, b)\right) \tag{99}
\end{equation*}
$$

which is unique in $W^{1, \infty}\left(0, t_{0} ; L^{2}(0, b)\right) \cap L^{\infty}\left(0, t_{0} ; H^{1}(0, b)\right)$.

## References

[1] T. Clopeau, A. Farina, A. Fasano, A. Mikelić, Asymptotic equations for the terminal phase of glass fiber drawing and their analysis, accepted for publication in Nonlinear Analysis TMA: Real World Applications, 2009.http://dx.doi.org/10.1016/j.nonrwa.2008.09.017
[2] J.N. Dewynne, J. R. Ockendon, P. Wilmott, On a mathematical model for fibre tapering. SIAM J. Appl. Math. 49 (1989), 983-990.

[^1][3] J.N. Dewynne, J. R. Ockendon, P. Wilmott, A systematic derivation of the leading-order equations for extensional flows in slender geometries. J. Fluid Mech. 244 (1992), 323-338.
[4] J.N. Dewynne,P.D. Howell, J. R. Ockendon, P. Wilmott, Slender viscous fibres with inertia and gravity. Quart. J. Mech. Appl. Math. 47 (1994), no. 4, 541-555.
[5] A. Farina, A. Fasano, A. Mikelić , On the equations governing the flow of mechanically incompressible, but thermally expansible, viscous fluids, $\mathrm{M}^{3}$ AS : Math. Models Methods Appl. Sci., Vol. 18 (2008), no. 6, p. 813-858.
[6] A. Fasano, A. Mikelić , A. Farina, Non-isothermal flow of molten glass: mathematical challenges and industrial questions, chapter in Lectures given at the Centro Internazionale Matematico Estivo (C.I.M.E.)- European Mathematical Society (EMS) Summer School in applied mathematics held in Montecatini, Italia, September 8 - September 19, 2008, to appear in Lecture Notes Centro Internazionale Matematico Estivo (C.I.M.E.) Series, Lecture Notes in Mathematics, Springer, 2009.
[7] F. T. Geyling, G. M. Homsy, Extensional instabilities of the glass fiber drawing process, Glass Tech. 21, 2, 95-102, 1980.
[8] M. Gregory Forest, H. Zhou, Unsteady analyses of thermal glass fibre drawing processes. Euro. Jnl of Applied Mathematics, Vol. 12 (2001), p. 479-496.
[9] G. Gupta, W.W. Schultz, Non-isothermal flows of Newtonian slender glass fibers. Int. J. Non-Linear Mech., Vol. 33 (1998), p. 151-163.
[10] T. Hagen, M. Renardy, On the equations of fiber spinning in nonisothermal viscous flow. Topics in nonlinear analysis, 321-348, Progr. Nonlin. Diff. Eqs. Appls., 35, Birkhäuser, Basel, 1999.
[11] T. Hagen, M. Renardy, Studies on the linear equations of melt-spinning of viscous fluids, Diff. Int. Eqs., 14 (2001), 19-36.
[12] T. Hagen, On the effects of spinline cooling and surface tension in fiber spinnning, ZAMM - Z. Angew. Math. Mech., Vol. 82 (2002), no. 8, p. 545-558.
[13] T. Hagen, On the semigroup of linearized forced elongation. Appl. Math. Lett. 18 (2005), no. 6, 667-672.
[14] T. Hagen, Linear theory of nonisothermal forced elongation. J. Evol. Equ. 5 (2005), no. 3, 417-440.
[15] T. Hagen, Advances in fiber and film flow, Nonlin. Anal., 63 (2005), e1119-e1130.
[16] S. Kase, T. Matsuo, Studies of melt spinning I. Fundamental Equations on the Dynamics of Melt Spinning, J. Polym. Sci. A, 3 (1965), pp. 25412554.
[17] S. Kase, T. Matsuo, Studies of melt spinning. II, Steady state and transient solutions of fundamental equations compared with experimental results, J. Polym. Sci 11, 251-287 (1967).
[18] J.L. Lions, Quelques méthodes de rsolution des problèmes aux limites non linéaires, Dunod, Paris, 1969.
[19] M. A. Matovich, J. R. A. Pearson, Spinning a molten threadline-steady state isothermal viscous flows, Ind. Engrg. Chem. Fundam., 8 (1969), pp. 512-520.
[20] R. von der Ohe, Simulation of glass fiber forming processes, Thesis PhD (2005) - AUC Imprint: ISBN: 8791200253. - Aalborg : Department of Production, Aalborg University, 2005. pp. 192.
[21] J. R. A. Pearson, Mechanics of Polymer Processing. Elsevier, 1985.
[22] M. Renardy, Draw resonance revisited. SIAM J. Appl. Math. 66 (2006), no. 4, 1261-1269.
[23] W.W. Schultz, S.H. Davis: One-dimensional liquid fibers, J. Rheol., Vol. 26 (1982), p. 331-345.
[24] J. Simon, Compact sets in the space $L^{p}(0, T ; B)$. Ann. Mat. Pura Appl. (4) 146 (1987), 65-96.
[25] H. Zhou, M. Gregory Forest, A numerical study of unsteady, thermal, glass fiber drawing processes, Commun. Math. Sci. 3 (2005), no. 1, 27-45
[26] A.L. Yarin, P. Gospodinov, O. Gottlieb, M.D. Graham, Newtonian glass fiber drawing: Chaotic variation of the cross-sectional radius, Physics of Fluids, Vol. 11 (1999), no. 11, p. 3201-3208.


[^0]:    *This paper was partly written during the visit of Ph.L. and A.M. at Nečas Center for Mathematical Modelling in March/April 2008 and they wish to express their thanks for the hospitality during the stay.
    ${ }^{\dagger}$ The work of E.F. was supported by Grant 201/09/0917 of GA ČR as a part of the general research programme of the Academy of Sciences of the Czech Republic, Institutional Research Plan AV0Z10190503.
    ${ }^{\ddagger}$ Research of A.M. on the mathematical modeling of glass fiber drawing was supported in part by the EZUS LYON 1 INGENIERIE during the period $2004-2007$.

[^1]:    ${ }^{1}$ Hagen and Renardy suppose $\partial_{x} p(\cdot, b), \partial_{x} f(\cdot, b) \in H^{1}\left(0, t_{0}\right)$ as well. This does not seem to be necessary.

