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► **To cite this version:**

| Patrick Le Meur. Crossed-products of Calabi-Yau algebras by finite groups. 2010. hal-00488814v1

HAL Id: hal-00488814

<https://hal.science/hal-00488814v1>

Preprint submitted on 3 Jun 2010 (v1), last revised 20 Apr 2020 (v3)

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CROSSED PRODUCTS OF CALABI-YAU ALGEBRAS BY FINITE GROUPS

PATRICK LE MEUR

ABSTRACT. Let a finite group G act on a differential graded Calabi-Yau algebra A over a field whose characteristic does not divide the order of G . This note studies when the crossed-product of A by G is still Calabi-Yau. Under a compatibility condition between the action of G on A and the Calabi-Yau structure of A , it is proved that this is indeed the case. Similar results are proved for general constructions of Calabi-Yau algebras such as deformed Calabi-Yau completions and Ginzburg algebras.

INTRODUCTION

Ginzburg introduced Calabi-Yau algebras in [11] as non commutative versions of coordinate rings of Calabi-Yau varieties. They have been object of much interest recently. For instance, in the study of Donaldson-Thomas invariants, Calabi-Yau algebras are associated to brane tilings ([17]). The algebras considered there are Jacobian algebras, that is, the 0-th cohomology algebra of Ginzburg algebras $\mathcal{A}(Q, W)$ of quivers with potentials (Q, W) ([11, 3.6]). The Ginzburg algebras were proved to be Calabi-Yau in [15]. Moreover, in representation theory, Calabi-Yau algebras satisfying some finiteness condition are used to construct the generalised cluster categories [2] which serve to categorify cluster algebras [10]. Finally, it is proved in [16] that the mutation of quivers with potentials in the sense of [8] define equivalences between the derived categories of the associated Ginzburg algebras. Since the definition of Calabi-Yau algebras was given in [11], many known algebras have been proved to be Calabi-Yau: The algebra $\mathbb{C}[x_1, \dots, x_n]$ of polynomials in n variables is Calabi-Yau of dimension n ; if X is a smooth affine variety with trivial canonical sheaf, then $\mathbb{C}[X]$ is Calabi-Yau of dimension $\dim X$ (see [11, 3.2]); also the Weyl algebra $A_n = \mathbb{C}[x_1, \dots, x_n, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}]$ is Calabi-Yau of dimension $2n$ (see [3]); and, finally, the Yang-Mills algebras ([6]) are Calabi-Yau of dimension 3 ([12]).

Let k be a field and A be a differential graded algebra over k (dg algebra, for short). The enveloping algebra of A is the algebra $A^e = A \otimes A^{op}$. The definition of the Calabi-Yau property is expressed in terms of the duality

$$\mathrm{RHom}_{A^e}(-, A \otimes A): \mathrm{per}(A^e) \rightarrow \mathrm{per}((A^e)^{op})$$

denoted by $M \mapsto M^\vee$, for simplicity. Here, $\mathrm{RHom}_{A^e}(-, A \otimes A)$ is the total derived functor, and $\mathrm{per}(A^e)$ is the perfect derived category of A^e , that is, the full triangulated subcategory of the derived category of right dg A^e -modules containing A^e and stable under shifts in both directions and under taking direct summands. Following [11], the dg algebra A is called *Calabi-Yau of dimension d* if:

- (1) A is *homologically smooth*, that is, $A \in \mathrm{per}(A^e)$, and
- (2) there exists an isomorphism $f: A^\vee[d] \rightarrow A$ in the derived category of left dg A^e -modules such that $f^\vee[d] = f$.

In the last equality, right dg $(A^e)^{op}$ -modules (equivalently, left dg A^e -modules) are identified with right dg A^e -modules using the canonical anti-involution of A^e . Recall that if A is an algebra concentrated in degree 0 (with zero differential) then condition (1) above means that the A -bimodule ${}_A A_A$ has a resolution of finite length by finitely generated projective A -bimodules, and condition (2) gives the following isomorphisms of A -bimodules:

$$\mathrm{H}^i(A; A \otimes A) \stackrel{\mathrm{def}}{=} \mathrm{Ext}_{A^e}^i(A, A \otimes A) \simeq \begin{cases} 0 & \text{if } i \neq d, \\ A & \text{if } i = d \end{cases}$$

where $\mathrm{H}^i(A; -)$ denotes the Hochschild cohomology of A . For a Calabi-Yau algebra A , the complex A^\vee plays an important rôle. If $\theta_A \rightarrow A^\vee$ is a cofibrant resolution (of left dg A^e -bimodule), then θ_A is called the *inverse dualizing complex* of A ([21]). It is unique up to quasi-isomorphism.

This text studies the actions of finite groups on dg Calabi-Yau algebras. The motivation for this work is the following. Let X be a smooth affine Calabi-Yau variety and G be a finite group acting on X . Then, the quotient variety X/G has singularities (if $G \neq 1$), G acts on $\mathbb{C}[X]$ and $\mathbb{C}[X/G]$ is the algebra of invariants $\mathbb{C}[X]^G$. A crepant resolution of X/G is a resolution of singularities $f: Y \rightarrow X/G$ such that $f^*\omega_{X/G} \simeq \omega_Y$ (which implies that Y is itself Calabi-Yau). Such resolutions do not always exist and are not always unique. But it is conjectured by Bondal and Orlov that given crepant resolutions $Y_1 \rightarrow X/G$ and $Y_2 \rightarrow X/G$, there exists an equivalence $\mathcal{D}^b(\mathrm{Coh} Y_1) \simeq \mathcal{D}^b(\mathrm{Coh} Y_2)$. In some cases, these resolutions, are related to the crossed-product

Date: June 3, 2010.

2000 *Mathematics Subject Classification*. Primary 18G20, Secondary 16W22.

Key words and phrases. Calabi-Yau algebra; crossed product; quiver with potential.

algebra (or, skew-group algebra) $\mathbb{C}[X] \rtimes G$. For example, if $X = V$ is a vector space and $G \stackrel{\text{finite}}{<} \text{SL}(V)$, then $\mathbb{C}[X]$ is the symmetric algebra $S(V)$ and $S(V) \rtimes G$ is a non commutative crepant resolution of V/G in the sense of [22]; also, if $V = \mathbb{C}^3$, then $S(V) \rtimes G$ is Morita equivalent to the 0-th cohomology algebra $H^0(\mathcal{A}(Q, W))$ for some quiver with potential (Q, W) , and, for any crepant resolution $Y \rightarrow \mathbb{C}^3/G$, there is an equivalence $\mathcal{D}^b(S(V) \rtimes G) \simeq \mathcal{D}^b(\text{Coh } Y)$ ([11, 4.4] and [5]).

This motivates the following question:

(Q): *Let A be a dg Calabi-Yau algebra over a field k and G be a finite group acting on A by dg automorphisms such that $\text{car}(k)$ does not divide $\sharp G$. Is the crossed-product $A \rtimes G$ a Calabi-Yau algebra?*

Recall the following result closely related to (Q): Given an algebra A (concentrated in degree 0), if A has the Van den Bergh duality (that is, $\text{Ext}_{A^e}^i(A, A \otimes A)$ is 0 if $i \neq d$ and is an invertible A -bimodule if $i = d$), then so does $A \rtimes G$ ([1, Prop. 7.1], see also [9] for a generalisation to crossed-products of algebras by Hopf algebras). The main result of this text is the following. Recall that the definition of the Calabi-Yau property involves a quasi-isomorphism $f: \theta_A[d] \simeq A^\vee[d] \rightarrow A$.

Theorem 1. *Let k be a field and A be a dg algebra over k . Let G be a finite group whose order is not divisible by $\text{car } k$ and assume that G acts on A by dg automorphisms.*

(1) *Up to an adequate choice of the inverse dualizing complex θ_A of A , the group G acts on θ_A by automorphisms of dg modules and the inverse dualizing complex of $A \rtimes G$ is:*

$$\theta_{A \rtimes G} = \theta_A \rtimes G.$$

(2) *If A is Calabi-Yau of dimension d and if the quasi-isomorphism $f: \theta_A[d] \simeq A^\vee[d] \rightarrow A$ is G -equivariant, then $A \rtimes G$ is Calabi-Yau of dimension d .*

When A is concentrated in degree 0, part (1) was proved in [9, Thm. 17, Ex. 21] and it is easy to check that part (2) follows from the same result (see also [1]). Unfortunately, the proofs in [1] and [9] are based on Stefan's spectral sequence on Hochschild cohomology for Hopf-Galois extensions ([18]) so their proof cannot apply directly to dg algebras not concentrated in degree 0 (for instance, all Calabi-Yau algebras used to construct generalised cluster categories). Therefore, in order to prove the above theorem, different techniques are needed. Recently, Witherspoon introduced methods to study the Hochschild cohomology ring of the crossed-products of algebras by finite groups ([23]). It appears that the techniques introduced there can be adapted to dg algebras and applied to prove Theorem 1.

The text is organised as follows. Section 1 fixes notation and all the definitions. Section 2 proves Theorem 1. Section 3 studies (Q) for dg Calabi-Yau algebras arising from a general construction by Keller and called deformed Calabi-Yau completion ([15]): Given a homologically smooth dg algebra A , an integer $d \geq 3$ and $c \in \text{HH}_{d-2}(A)$ (the Hochschild homology of A with coefficients in A), the construction defines a dg Calabi-Yau algebra of dimension d and denoted by $\Pi_d(A, c)$. This section proves that if G acts on A and if $c \in \text{HH}_{d-2}(A)^G$, then G acts naturally on $\Pi_d(A, c)$ and $\Pi_d(A, c) \rtimes G$ is itself a deformed Calabi-Yau completion and, therefore, is Calabi-Yau. Section 4 applies the results of Section 3 to crossed products of Ginzburg algebras by finite groups. In particular, sufficient conditions are given for the crossed-product of a Ginzburg algebra by a finite group to have a derived category equivalent to that of a Ginzburg algebra. Finally, Section 5 comments the equivariance hypothesis of Theorem 1, it appears that this hypothesis is of geometric nature.

1. DEFINITIONS AND NOTATION

Throughout the text, k will denote a field.

1.1. Differential graded algebras and their modules. Tensor product over k is denoted by \otimes . The reader is referred to [14] for a background on differential graded algebras (dg algebras, for short). All dg algebras are over k and with differential of degree 1. Let A be a dg algebra. Its *opposite algebra* has A as underlying complex of vector spaces and its product map is defined by $A^{op} \otimes A^{op} \rightarrow A^{op}$, $a \otimes b \mapsto (-1)^{\text{deg}(a)\text{deg}(b)}ba$, for homogeneous $a, b \in A$ (and ba is the product in A). If B is another dg algebra, then the graded vector space $A \otimes B$ is a dg algebra with product defined by $(a \otimes b).(a' \otimes b') = (-1)^{\text{deg}(a')\text{deg}(b)}aa' \otimes bb'$ and differential defined by $d(a \otimes b) = d(a) \otimes b + (-1)^{\text{deg}(a)}a \otimes d(b)$, for homogeneous $a, a' \in A$ and $b, b' \in B$. In particular, the *enveloping algebra* of A is the dg algebra $A^e = A \otimes A^{op}$. Note that the map $A^e \rightarrow (A^e)^{op}$ defined by $a \otimes b \mapsto (-1)^{\text{deg}(a)\text{deg}(b)}b \otimes a$, for homogeneous $a, b \in A$, is an isomorphism of dg algebras.

The category of (right) *differential graded (dg) A -modules* is denoted by $\mathcal{C}(A)$: Its objects are the right dg A -modules (with differential of degree 1) and its morphisms are the homogeneous of degree 0 morphisms of graded A -modules commuting with the differentials. The suspension in $\mathcal{C}(A)$ is denoted by $M \mapsto M[1]$. For convenience, $\mathcal{C}(A^{op})$ will be considered as the category of left A -modules: If M is a right dg A^{op} -module and $a \in A$, $m \in M$ are homogeneous, then $a.m$ is defined as $(-1)^{\text{deg}(a)\text{deg}(m)}m.a$. Similarly, $\mathcal{C}(A \otimes B^{op})$ will be considered as the category of dg $B - A$ -bimodules (or, simply, A -bimodules, if $A = B$): If M is a right dg $A \otimes B^{op}$ -module and

$a \in A$, $b \in B$, $m \in M$ are homogeneous, then $b.m.a$ is defined as $(-1)^{\deg(b) \cdot (\deg(m) + \deg(a))} m.(a \otimes b)$. Note that the canonical isomorphism $A^e \rightarrow (A^e)^{op}$ induces an isomorphism between $\mathcal{C}(A^e)$ and $\mathcal{C}((A^e)^{op})$.

For example, A is a dg A^e -module for the action defined by $x.(a \otimes b) = (-1)^{\deg(b)(\deg(a) + \deg(x))} bxa$ for homogeneous $x, a, b \in A$. Therefore, it is also a left dg A^e -module, for the action defined by $(a \otimes b).x = (-1)^{\deg(b)\deg(x)} axb$. Also, the product in A^e endows $A \otimes A$ with a structure of $A^e - A^e$ -bimodule.

Let $L \in \mathcal{C}(A)$ and $M \in \mathcal{C}(A \otimes B^{op})$. Then $\mathcal{H}om_A(L, M)$ will denote the left dg B -module defined as follows: Its component of degree $n \in \mathbb{Z}$ is the vector space of homogeneous of degree n morphisms $f: L \rightarrow M$ of graded A -modules; the differential of f is $d_M \circ f - (-1)^n f \circ d_L$ (if d_L and d_M denote the differentials of L and M , respectively); and the action of $b \in B$ on f is $b.f: x \mapsto b.f(x).1$. If $M \in \mathcal{C}(A \otimes B^{op})$ is fixed, this defines the functor $\mathcal{H}om_A(-, M): \mathcal{C}(A) \rightarrow \mathcal{C}(B^{op})$. The associated *right total derived functor* is denoted by $\mathbf{R}Hom_A(-, M): \mathcal{D}(A) \rightarrow \mathcal{D}(B^{op})$. Recall that it may be computed on $L \in \mathcal{D}(A)$ as follows: Take a cofibrant resolution $X \rightarrow L$ in $\mathcal{C}(A)$, then $\mathbf{R}Hom_A(L, M) = \mathcal{H}om_A(X, M)$.

1.2. Duality for bimodules, inverse dualizing complex and the Calabi-Yau property. Let A be dg algebra. The total right derived functor $\mathbf{R}Hom_{A^e}(-, A \otimes A): \mathcal{D}(A^e) \rightarrow \mathcal{D}((A^e)^{op})$ induces, by restriction, a duality (see [11, 3.2])

$$\mathbf{R}Hom_{A^e}(-, A \otimes A): \text{per}(A^e) \rightarrow \text{per}((A^e)^{op}).$$

For convenience, this duality will sometimes be denoted by $M \mapsto M^\vee$.

Let $X \rightarrow A$ be a cofibrant resolution in $\mathcal{C}(A^e)$. Then $\mathbf{R}Hom_{A^e}(A, A \otimes A) = \mathcal{H}om_{A^e}(X, A \otimes A)$. Let $\theta_A \rightarrow \mathbf{R}Hom_{A^e}(A, A \otimes A)$ be a cofibrant resolution in $\mathcal{C}((A^e)^{op})$. Then θ_A is unique up to quasi-isomorphism and is called the *inverse dualizing complex* of A ([20]).

Let d be an integer. Following [11, 3.2.3], the dg algebra A is called *Calabi-Yau of dimension d* if the two following conditions hold:

- (1) A is *homologically smooth*, that is, $A \in \text{per}(A^e)$,
- (2) there is an isomorphism $f: \mathbf{R}Hom_{A^e}(A, A \otimes A)[d] \rightarrow A$ in $\mathcal{D}((A^e)^{op})$ such that $\mathbf{R}Hom_{(A^e)^{op}}(f, A \otimes A)[d] = f$.

In the last equality, left and right dg A^e -modules are identified using the canonical isomorphism $A^e \rightarrow (A^e)^{op}$, and $A^{\vee\vee}$ is identified with A (because $A \in \text{per}(A^e)$), so that $\mathbf{R}Hom_{(A^e)^{op}}(f, A \otimes A)$ is indeed a morphism $A^\vee[d] \rightarrow A$ in $\mathcal{D}((A^e)^{op})$.

1.3. Crossed-products. Let G be a finite group and A be a dg algebra. As usual, conjugation of an element $g \in G$ by an element $h \in G$ is denoted by ${}^h g = hgh^{-1}$. An action of G on A (by dg automorphisms) is an action $(g, a) \mapsto {}^g a$ of G on A by automorphisms of algebras such that:

- ${}^g a$ is homogeneous and $\deg({}^g a) = \deg(a)$ if $a \in A$ is homogeneous,
- the differential d of A verifies $d({}^g a) = {}^g d(a)$,

for $a \in A$ and $g \in G$. This defines the *crossed-product* (or *skew-group*) dg algebra $A \rtimes G$ as follows: As a complex of vector spaces it equals $A \otimes kG$ where kG is concentrated in degree 0; the product is defined by $(a \otimes g) \cdot (b \otimes h) = a {}^g b \otimes gh$; and the differential is defined by $d(a \otimes g) = d(a) \otimes g$, for $a, b \in A$ and $g, h \in G$. As usual, A and kG are considered as subalgebras of $A \rtimes G$ and $a \otimes g$ is written ag , for short. Hence, $ga = {}^g ag$, for $a \in A$ and $g \in G$.

2. THE MAIN THEOREM

This section proves Theorem 1. The first paragraphs introduce the needed material for the proof, it is inspired from the work of Witherspoon in [23].

2.1. The dg algebra Δ . Let $g \in G$. Denote by $Ag = gA$ the subcomplex of vector spaces of Λ equal to $\{ag \mid g \in G\} = \{ga \mid g \in G\}$. Then $Ag = gA$ is a dg A^e -submodule of Λ and $A \simeq Ag$ both in $\mathcal{C}(A)$ and $\mathcal{C}(A^{op})$ (but not in $\mathcal{C}(A^e)$). Note that $\Lambda = \bigoplus_{g \in G} Ag$ in $\mathcal{C}((A^e)^{op} \otimes A^e)$.

Let $g, h \in G$. Then, $Ag \otimes hA$ is a dg A^e -sub-bimodule of $\Lambda \otimes \Lambda$. There are obvious isomorphisms $A \otimes A \simeq Ag \otimes hA$ in $\mathcal{C}(A^e)$ and $\mathcal{C}((A^e)^{op})$. Note that $\Lambda \otimes \Lambda = \bigoplus_{g, h \in G} Ag \otimes hA$ both in $\mathcal{C}(A^e)$ and in $\mathcal{C}((A^e)^{op})$. Besides, the product in Λ^e induces a map $(Ag \otimes hA) \otimes (Ag' \otimes h'A) \rightarrow Agg' \otimes h'hA$, for all $g, g', h, h' \in G$. In particular, $\bigoplus_{g \in G} Ag \otimes g^{-1}A$ is a dg subalgebra of Λ^e . In the sequel, it will be denoted by Δ :

$$\Delta = \bigoplus_{g \in G} Ag \otimes Ag^{-1}.$$

Note that A^e is a dg subalgebra of Δ .

The dg algebra Δ has the following simple interpretation: The group G acts diagonally on the dg algebra A^e : ${}^g(a \otimes b) = {}^g a \otimes {}^g b$, for $a, b \in A$ and $g \in G$. For this action, the map $\Delta \rightarrow A^e \rtimes G$, $ag \otimes g^{-1}b \mapsto (a \otimes b).g$ is an isomorphism of dg algebras. The dg Δ -modules also have a similar interpretation. By definition, *an action*

of G on $M \in \mathcal{C}(A^e)$ compatible with the action of A^e is an action $G \times M \rightarrow M$, $(g, m) \mapsto {}^g m$ of G on M by automorphism of complexes of vector spaces such that ${}^g(m.(a \otimes b)) = {}^g m.({}^g a \otimes {}^g b)$, for $a, b \in A$, $m \in M$ and $g \in G$. Then, the dg Δ -modules are precisely the dg A^e -modules M endowed with a compatible action of G (with $m.(g \otimes g^{-1}) = {}^{g^{-1}} m$, for $m \in M$ and $g \in G$). Similarly, one defines the notion of *compatible action of G on left dg A^e -modules*, and left dg Δ -modules are precisely left dg A^e -modules endowed with a compatible action of G (with $(g \otimes g^{-1}).m = {}^g m$, for $m \in M$ and $g \in G$).

In the sequel, the following collection $\{M_g\}_{g \in G}$ of dg $A^e - \Delta$ -bimodules will be useful. Let $g \in G$. The complex of vector spaces $\bigoplus_{h \in G} Ah \otimes h^{-1}gA$ is a dg $A^e - \Delta$ -sub-bimodule of Λ^e . It will be denoted by M_g :

$$M_g = \bigoplus_{h \in G} Ah \otimes h^{-1}gA.$$

The following lemma collects some information on these bimodules. *Here and after, products are computed in Λ^e , unless otherwise specified.*

Lemma. *Let $t \in G$.*

(a) *There is an isomorphism in $\mathcal{C}(\Delta)$:*

$$\begin{aligned} \Delta &\longrightarrow M_t \\ u &\longmapsto (1 \otimes t).u \quad . \end{aligned}$$

(b) $\Lambda^e = \bigoplus_{g \in G} M_g$ in $\mathcal{C}((A^e)^{op} \otimes \Delta)$.

(c) *Let $g \in G$. Multiplication by $g \otimes g^{-1}$ induces an isomorphism in $\mathcal{C}(\Delta)$:*

$$\begin{aligned} M_t &\longrightarrow M_{gtg^{-1}} \\ u &\longmapsto (g \otimes g^{-1}).u \quad . \end{aligned}$$

(d) *There is an isomorphism in $\mathcal{C}((A^e)^{op} \otimes \Delta)$:*

$$\begin{aligned} \mathcal{H}om_{A^e}(\Delta, A \otimes tA) &\longrightarrow M_t \\ \varphi &\longmapsto \sum_{g \in G} \varphi(g^{-1} \otimes g).(g \otimes g^{-1}) \quad . \end{aligned}$$

Proof. Only (d) needs details. The given map is a morphism in $\mathcal{C}((A^e)^{op} \otimes \Delta)$. One easily checks that the following map is its inverse:

$$\begin{aligned} M_t &\longrightarrow \mathcal{H}om_{A^e}(\Delta, A \otimes tA) \\ \bigoplus_{g \in G} Ag \otimes g^{-1}tA \ni (m_g)_g &\longmapsto (ag^{-1} \otimes gb \mapsto m_g.(ag^{-1} \otimes gb)) \quad , \end{aligned}$$

where, on the right-hand side, $a, b \in A$ and $g \in G$. □

2.2. Extensions of scalars between $\mathcal{C}(A^e)$, $\mathcal{C}(\Delta)$ and $\mathcal{C}(\Lambda^e)$. Recall ([14, 3.8]) the following adjunctions:

- $-\otimes_{A^e} \Lambda^e: \mathcal{C}(A^e) \rightarrow \mathcal{C}(\Lambda^e)$ is left adjoint to the restriction-of-scalars functor,
- $-\otimes_{A^e} \Delta: \mathcal{C}(A^e) \rightarrow \mathcal{C}(\Delta)$ is left adjoint to the restriction-scalars-functor,
- $-\otimes_{\Delta} \Lambda^e: \mathcal{C}(\Delta) \rightarrow \mathcal{C}(\Lambda^e)$ is left adjoint to the restriction-of-scalars functor,
- $\mathcal{H}om_{A^e}(\Delta, -): \mathcal{C}(A^e) \rightarrow \mathcal{C}(\Delta)$ is right adjoint to the restriction-of-scalars functor.

Of course, similar adjunctions hold true for left dg modules. In particular, $\Lambda^e \otimes_{\Delta} -: \mathcal{C}(\Delta^{op}) \rightarrow \mathcal{C}((\Lambda^e)^{op})$ has the following well-known interpretation that will be used later. Let M be a left dg A^e -module with a compatible action of G . Equivalently (2.1), it is naturally a left dg Δ -module. Then, the *crossed-product $M \rtimes G$ of M by G* is the left dg Λ^e -module constructed as follows: As a complex of vector spaces, it is equal to $M \otimes kG$ (with $\deg(m \otimes g) = \deg(m)$, for $g \in G$ and homogeneous m); and the action of Λ^e on $M \rtimes G$ is defined by $(a \otimes b).(m \otimes g) = ((a \otimes {}^g b).m) \otimes g$ and $(h \otimes k).(m \otimes g) = {}^h m \otimes h g k$, for $a, b \in A$, $m \in M$ and $g, h, k \in G$. Then, $M \rtimes G \simeq \Lambda^e \otimes_{\Delta} M$ in $\mathcal{C}((\Lambda^e)^{op})$. Actually, the two following maps are isomorphisms in $\mathcal{C}((\Lambda^e)^{op})$ and inverse to each other:

$$\begin{aligned} M \rtimes G &\longrightarrow \Lambda^e \otimes_{\Delta} M \\ m \otimes g &\longmapsto (1 \otimes g) \otimes m \quad , \text{ and} \\ \Lambda^e \otimes_{\Delta} M &\longrightarrow M \rtimes G \\ (ag \otimes hb) \otimes m &\longmapsto ((a \otimes {}^{gh} b).{}^g m) \otimes gh \quad , \end{aligned}$$

where $a, b \in A$, $m \in M$ and $g, h \in G$.

The adjunctions recalled at the beginning of the paragraph satisfy the following properties that will be useful later. Notice that (d) was first proved in [4, Lem. 3.3] for group algebras and in [23, Lem. 3.5] for crossed-products of algebras. The proof for is recalled for convenience.

Lemma. *Denote by N the order of G .*

(a) *There are isomorphisms*

$$\begin{cases} \Delta \simeq (A \otimes A)^N & \text{in } \mathcal{C}(A^e), \text{ and} \\ \Lambda \otimes \Lambda \simeq \Delta^N & \text{in } \mathcal{C}(\Delta). \end{cases}$$

(b) *The following functors preserve quasi-isomorphisms and componentwise surjective morphisms:*

- $\mathcal{H}om_{A^e}(\Delta, -): \mathcal{C}(A^e) \rightarrow \mathcal{C}(\Delta)$,
- $- \otimes_{A^e} \Lambda^e: \mathcal{C}(A^e) \rightarrow \mathcal{C}(\Lambda^e)$,
- $- \otimes_{\Delta} \Lambda^e: \mathcal{C}(\Delta) \rightarrow \mathcal{C}(\Lambda^e)$.

(c) *The following functors preserve cofibrant objects and cofibrant resolutions:*

- $- \otimes_{A^e} \Delta: \mathcal{C}(A^e) \rightarrow \mathcal{C}(\Delta)$,
- $- \otimes_{\Delta} \Lambda^e: \mathcal{C}(\Delta) \rightarrow \mathcal{C}(\Lambda^e)$,
- *the restriction-of-scalars functor* $\mathcal{C}(\Delta) \rightarrow \mathcal{C}(A^e)$.

(d) *The following map is an isomorphism in* $\mathcal{C}(\Lambda^e)$:

$$\begin{aligned} A \otimes_{\Delta} \Lambda^e &\longrightarrow \Lambda \\ a \otimes (x \otimes y) &\longmapsto a.(x \otimes y) = (-1)^{\deg(y) \cdot (\deg(x) + \deg(a))} y a x \end{aligned}$$

(defined on homogeneous $a \in A$ and $x, y \in \Lambda$).

(e) *Let* $M \in \mathcal{C}(\Delta)$, *then:*

- M *is a direct summand of* $M \otimes_{\Delta} \Lambda^e$ *in* $\mathcal{C}(\Delta)$,
- M *is a direct summand of* $M \otimes_{A^e} \Delta$ *in* $\mathcal{C}(\Delta)$ *if, moreover,* $N \in \mathbf{k}^\times$,
- *if* M *is cofibrant in* $\mathcal{C}(\Delta)$, *then so is it in* $\mathcal{C}(A^e)$. *The converse holds true if* $N \in \mathbf{k}^\times$.

Proof. (a) follows directly from the discussion made in 2.1.

(b) follows from (a). Notice that all restriction-of-scalars functors preserve quasi-isomorphisms and componentwise surjective morphisms.

(c) Recall the following basic fact on adjunctions: If B and C are dg algebras and $F_1: \mathcal{C}(B) \rightarrow \mathcal{C}(C)$ and $F_2: \mathcal{C}(C) \rightarrow \mathcal{C}(B)$ are additive functors such that (F_1, F_2) is adjoint and F_2 preserves morphisms which are both quasi-isomorphisms and componentwise surjective, then F_1 preserves cofibrant objects. This and (b) show that (c) holds true.

(d) The given map is a morphism in $\mathcal{C}(\Lambda^e)$. It is straightforward to check that the following map is its inverse:

$$\begin{aligned} \Lambda &\longrightarrow A \otimes_{\Delta} \Lambda^e \\ ag &\longmapsto a \otimes (g \otimes 1) \quad . \end{aligned}$$

(e) Recall that $M_t = \bigoplus_{g \in G} Ag \otimes g^{-1}tA \in \mathcal{C}((A^e)^{op} \otimes \Delta)$ was defined in 2.1. Then, both M_1 and $M_{\neq 1} := \bigoplus_{t \neq 1} M_t$ are dg Δ -sub-bimodules of $\Lambda \otimes \Lambda$. Therefore, $\Lambda \otimes \Lambda = M_1 \oplus M_{\neq 1}$ and $M_1 = \Delta$ in $\mathcal{C}(\Delta)$. Thus, $M = M \otimes_{\Delta} M_1$ is a direct summand of $M \otimes_{\Delta} \Lambda^e$ in $\mathcal{C}(\Delta)$.

Next, assume that $N \in \mathbf{k}^\times$. The functor $- \otimes_{A^e} \Delta: \mathcal{C}(A^e) \rightarrow \mathcal{C}(\Delta)$ is left adjoint to the restriction-of-scalars functor and the counit of this adjunction evaluates at M as:

$$\begin{aligned} \eta_M: M \otimes_{A^e} \Delta &\longrightarrow M \\ m \otimes u &\longmapsto mu \end{aligned} \quad \text{in } \mathcal{C}(\Delta).$$

On the other hand, there is a well-defined morphism in $\mathcal{C}(\Delta)$:

$$\begin{aligned} M &\longrightarrow M \otimes_{A^e} \Delta \\ m &\longmapsto \frac{1}{\#G} \sum_{g \in G} (m.(g \otimes g^{-1})) \otimes (g^{-1} \otimes g) \quad . \end{aligned}$$

A simple verification shows that it is right inverse to η_M . Thus, M is a direct summand of $M \otimes_{A^e} \Delta$ in $\mathcal{C}(\Delta)$.

Finally, if M is cofibrant in $\mathcal{C}(\Delta)$, then so is it in $\mathcal{C}(A^e)$, because of (c). Conversely, if M is cofibrant in $\mathcal{C}(A^e)$ and $\#G \in \mathbf{k}^\times$, then $M \otimes_{A^e} \Delta$ is cofibrant in $\mathcal{C}(\Delta)$, because of (c), and it admits M as a direct summand in $\mathcal{C}(\Delta)$, as proved before. Thus, M is cofibrant in $\mathcal{C}(\Delta)$. \square

The last assertion in (e) has the following practical consequence, where A is a dg algebra acted on by a finite group G whose order is invertible in \mathbf{k} : Cofibrant objects in $\mathcal{C}(\Delta)$ are precisely cofibrant objects in $\mathcal{C}(A^e)$ endowed with a compatible action of G ; in particular, given $M \in \mathcal{C}(\Delta)$, the cofibrant resolutions of M in $\mathcal{C}(\Delta)$ are precisely the cofibrant resolutions $P \rightarrow M$ in $\mathcal{C}(A^e)$ which are G -equivariant, where P is endowed with a compatible action of G . Of course, if $\#G = 0$ in \mathbf{k} , then a cofibrant object in $\mathcal{C}(A^e)$ endowed with a compatible action of G needs not be cofibrant in $\mathcal{C}(\Delta)$.

2.3. Interaction between duality and extension of scalars. This paragraph is a key-step towards a comparison of the inverse dualizing complexes of A and Λ . For this purpose, it is convenient to introduce a functor $\mathcal{C}(\Delta) \rightarrow \mathcal{C}(\Delta^{op})$ induced by $\mathcal{H}om_{A^e}(-, A \otimes A): \mathcal{C}(A^e) \rightarrow \mathcal{C}((A^e)^{op})$.

Let $M \in \mathcal{C}(\Delta)$. Then, the left dg A^e -module structure on $\mathcal{H}om_{A^e}(M, A \otimes A)$ arises from the action of A^e on $A \otimes A$ on the left. Let $g \in G$ and $f \in \mathcal{H}om_{A^e}(M, A \otimes A)$. Let ${}^g f: M \rightarrow A \otimes A$ be the map defined by:

$$(\forall m \in M) \quad ({}^g f)(m) = (g \otimes g^{-1}).f(m.(g \otimes g^{-1})).(g^{-1} \otimes g).$$

This defines an action of G on $\mathcal{H}om_{A^e}(M, A \otimes A)$ compatible with the structure of left dg A^e -module. In other words, the structure of left dg A^e -module on $\mathcal{H}om_{A^e}(M, A \otimes A)$ extends to a structure of left dg Δ -module by setting $(g \otimes g^{-1}).f = {}^g f$ for $g \in G$ and $f \in \mathcal{H}om_{A^e}(M, A \otimes A)$. This defines an additive functor $\mathcal{H}om_{A^e}(-, A \otimes A): \mathcal{C}(\Delta) \rightarrow \mathcal{C}(\Delta^{op})$ such that the following diagram commutes:

$$(D_1) \quad \begin{array}{ccc} \mathcal{C}(\Delta) & \xrightarrow{\mathcal{H}om_{A^e}(-, A \otimes A)} & \mathcal{C}(\Delta^{op}) \\ \downarrow & & \downarrow \\ \mathcal{C}(A^e) & \xrightarrow{\mathcal{H}om_{A^e}(-, A \otimes A)} & \mathcal{C}((A^e)^{op}) \end{array}$$

where the vertical arrows are the restriction-of-scalars functors, they preserve quasi-isomorphisms and cofibrant resolutions (2.2). Therefore, the preceding diagram yields a commutative diagram (up to isomorphism of functors), by taking right total derived functors:

$$(D_2) \quad \begin{array}{ccc} \mathcal{D}(\Delta) & \xrightarrow{\mathbf{R}\mathcal{H}om_{A^e}(-, A \otimes A)} & \mathcal{D}(\Delta^{op}) \\ \downarrow & & \downarrow \\ \mathcal{D}(A^e) & \xrightarrow{\mathbf{R}\mathcal{H}om_{A^e}(-, A \otimes A)} & \mathcal{D}((A^e)^{op}) \end{array}$$

where the notation $\mathbf{R}\mathcal{H}om_{A^e}(-, A \otimes A): \mathcal{D}(\Delta) \rightarrow \mathcal{D}(\Delta^{op})$ is adopted for the right total derived functor of $\mathcal{H}om_{A^e}(-, A \otimes A): \mathcal{C}(\Delta) \rightarrow \mathcal{C}(\Delta^{op})$. This notation is justified by (D_1) .

The following result makes a comparison between $\mathbf{R}\mathcal{H}om_{A^e}(-, A \otimes A): \mathcal{D}(\Delta) \rightarrow \mathcal{D}(\Delta^{op})$ and $\mathbf{R}\mathcal{H}om_{\Lambda^e}(-, \Lambda \otimes \Lambda): \mathcal{D}(\Lambda^e) \rightarrow \mathcal{D}((\Lambda^e)^{op})$, using extensions of scalars.

Lemma. *The following diagram commutes up to isomorphism of functors:*

$$(D_3) \quad \begin{array}{ccc} \mathcal{D}(\Delta) & \xrightarrow{\mathbf{R}\mathcal{H}om_{A^e}(-, A \otimes A)} & \mathcal{D}(\Delta^{op}) \\ \downarrow -\otimes_{\Delta} \Lambda^e & & \downarrow \Lambda^e \otimes_{\Delta} - \\ \mathcal{D}(\Lambda^e) & \xrightarrow{\mathbf{R}\mathcal{H}om_{\Lambda^e}(-, \Lambda \otimes \Lambda)} & \mathcal{D}((\Lambda^e)^{op}) \end{array} .$$

Proof. Recall (2.2) that $-\otimes_{\Delta} \Lambda^e: \mathcal{C}(\Delta) \rightarrow \mathcal{C}(\Lambda^e)$ and $\Lambda^e \otimes_{\Delta} -: \mathcal{C}(\Delta^{op}) \rightarrow \mathcal{C}((\Lambda^e)^{op})$ preserve quasi-isomorphisms and cofibrant resolutions. Therefore, the vertical arrows of the diagram make sense and, in order to prove the lemma, it suffices to prove that the following diagram commutes up to an isomorphism of functors:

$$(D_4) \quad \begin{array}{ccc} \mathcal{C}(\Delta) & \xrightarrow{\mathcal{H}om_{A^e}(-, A \otimes A)} & \mathcal{C}(\Delta^{op}) \\ \downarrow -\otimes_{\Delta} \Lambda^e & & \downarrow \Lambda^e \otimes_{\Delta} - \\ \mathcal{C}(\Lambda^e) & \xrightarrow{\mathcal{H}om_{\Lambda^e}(-, \Lambda \otimes \Lambda)} & \mathcal{C}((\Lambda^e)^{op}) \end{array} .$$

Let $M \in \mathcal{C}(\Delta)$. The construction of a functorial isomorphism $\mathcal{H}om_{\Lambda^e}(M \otimes_{\Delta} \Lambda^e, \Lambda \otimes \Lambda) \rightarrow \Lambda^e \otimes_{\Delta} \mathcal{H}om_{A^e}(M, A \otimes A)$ will be made in three steps:

- (a) First, construct an isomorphism $\mathcal{H}om_{\Lambda^e}(M \otimes_{\Delta} \Lambda^e, \Lambda \otimes \Lambda) \rightarrow \bigoplus_{g \in G} \mathcal{H}om_{A^e}(M, A \otimes gA)$ in $\mathcal{C}((A^e)^{op})$, using the tools introduced in 2.1 and 2.2.
- (b) Then, endow $\bigoplus_{g \in G} \mathcal{H}om_{A^e}(M, A \otimes gA)$ with a structure of left dg Λ^e -module such that the isomorphism of the first step actually lies in $\mathcal{C}((\Lambda^e)^{op})$.
- (c) Finally, construct an isomorphism $\bigoplus_{g \in G} \mathcal{H}om_{A^e}(M, A \otimes gA) \rightarrow \Lambda^e \otimes_{\Delta} \mathcal{H}om_{A^e}(M, A \otimes A)$ in $\mathcal{C}((\Lambda^e)^{op})$.

(a) By adjunction, the following map is an isomorphism in $\mathcal{C}((\Lambda^e)^{op})$ and functorial in M :

$$(1) \quad \begin{array}{ccc} \mathcal{H}om_{\Lambda^e}(M \otimes_{\Delta} \Lambda^e, \Lambda \otimes \Lambda) & \longrightarrow & \mathcal{H}om_{\Delta}(M, \Lambda \otimes \Lambda) \\ f & \longmapsto & (m \mapsto f(m \otimes (1 \otimes 1))) \end{array} .$$

Moreover, $\Lambda \otimes \Lambda = \bigoplus_{g \in G} M_g$ in $\mathcal{C}((A^e)^{op} \otimes \Delta)$ (2.1, (b)), therefore:

$$(2) \quad \mathcal{H}om_{\Delta}(M, \Lambda \otimes \Lambda) = \bigoplus_{g \in G} \mathcal{H}om_{\Delta}(M, M_g) \quad \text{in } \mathcal{C}((A^e)^{op}).$$

On the other hand, 2.1, (d), yields an isomorphism in $\mathcal{C}((A^e)^{op})$, for every $g \in G$:

$$(3) \quad \begin{array}{ccc} \mathcal{H}om_{\Delta}(M, M_g) & \longrightarrow & \mathcal{H}om_{\Delta}(M, \mathcal{H}om_{A^e}(\Delta, A \otimes gA)) \\ f & \longmapsto & (m \mapsto (at^{-1} \otimes tb \mapsto (\text{projection of } f(m) \text{ on } At \otimes t^{-1}gA).at^{-1} \otimes tb)) \end{array} ,$$

where, on the right-hand side, $a, b \in A$ and $t \in G$. Finally, by adjunction, the following map is an isomorphism in $\mathcal{C}((A^e)^{op})$ and functorial in M , for every $g \in G$:

$$(4) \quad \begin{array}{ccc} \mathcal{H}om_{\Delta}(M, \mathcal{H}om_{A^e}(\Delta, A \otimes gA)) & \longrightarrow & \mathcal{H}om_{A^e}(M, A \otimes gA) \\ f & \longmapsto & (m \mapsto (f(m))(1 \otimes 1)) \end{array} .$$

Note that (2) and (3) are functorial in M . Putting together (1), (2), (3) and (4) yields the following functorial isomorphism in $\mathcal{C}((A^e)^{op})$:

$$(5) \quad \begin{array}{ccc} \mathcal{H}om_{\Lambda^e}(M \otimes_{\Delta} \Lambda^e, \Lambda \otimes \Lambda) & \longrightarrow & \bigoplus_{g \in G} \mathcal{H}om_{A^e}(M, A \otimes gA) \\ f & \longmapsto & \sum_{g \in G} (m \mapsto \text{projection of } f(m \otimes (1 \otimes 1)) \text{ on } A \otimes gA) \end{array} .$$

This achieves the first step.

(b) The second step consists in extending the action of A^e on $\bigoplus_{g \in G} \mathcal{H}om_{A^e}(M, A \otimes gA)$ to an action of

Λ^e , such that (4) is an isomorphism in $\mathcal{C}((\Lambda^e)^{op})$. For this purpose, note that the compatible action of G on $\mathcal{H}om_{A^e}(M, A \otimes A) \in \mathcal{C}((A^e)^{op})$ may be extended to $\mathcal{H}om_{A^e}(M, \Lambda \otimes \Lambda)$ in a similar way: If $g \in G$ and $\varphi \in \mathcal{H}om_{A^e}(M, \Lambda \otimes \Lambda)$, then let ${}^g\varphi$ be the map $M \rightarrow \Lambda \otimes \Lambda$, $m \mapsto (g \otimes g^{-1}).\varphi(m.(g \otimes g^{-1})).(g^{-1} \otimes g)$. It is easy to check that this defines an action $(g, \varphi) \mapsto {}^g\varphi$ on $\mathcal{H}om_{A^e}(M, \Lambda \otimes \Lambda)$ compatible with the action of A^e (but not with that of Λ^e) on the left. Therefore, the subspace $\mathcal{H}om_{A^e}(M, A \otimes A)$ of $\mathcal{H}om_{A^e}(M, \Lambda \otimes \Lambda)$ is stabilized by this action; and the action of $g \in G$ maps the subspace $\mathcal{H}om_{A^e}(M, Ah \otimes kA)$ to the subspace $\mathcal{H}om_{A^e}(M, A{}^g h \otimes {}^g kA)$, for $h, k \in G$. In order to extend the left action of A^e on $\bigoplus_{g \in G} \mathcal{H}om_{A^e}(M, A \otimes gA)$ to

a left action of Λ^e , proceed as follows. Let $t, g \in G$ and $\varphi \in \mathcal{H}om_{A^e}(M, A \otimes gA)$. Denote by $(1 \otimes t) * \varphi$ the following map:

$$(1 \otimes t) * \varphi: \quad \begin{array}{ccc} M & \longrightarrow & A \otimes gtA \\ m & \longmapsto & (1 \otimes t).\varphi(m) \end{array} .$$

It obviously lies in $\mathcal{H}om_{A^e}(M, A \otimes gtA)$. Denote by $(t \otimes 1) * \varphi$ the following map:

$$(t \otimes 1) * \varphi: \quad \begin{array}{ccc} M & \longrightarrow & A \otimes gtA \\ m & \longmapsto & (1 \otimes t).({}^t\varphi)(m) \end{array} .$$

It obviously lies in $\mathcal{H}om_{A^e}(M, A \otimes tgA)$. Finally, let $(a \otimes b) * \varphi$ be $(a \otimes b).\varphi$, for $a, b \in A$. It is then easy to check that the assignment:

$$(at \otimes sb) * \varphi := (a \otimes b) * [(t \otimes 1) * ((1 \otimes s) * \varphi)], \quad a, b \in A, s, t \in G$$

extends the action of A^e on $\bigoplus_{g \in G} \mathcal{H}om_{A^e}(M, A \otimes gA)$ on the left to an action of Λ^e . Also easy is the verification that (4) then becomes an isomorphism in $\mathcal{C}((\Lambda^e)^{op})$. This finishes the second step.

(c) Now, proceed the third step. The following map is certainly an isomorphism in $\mathcal{C}((\Lambda^e)^{op})$ and functorial in M :

$$(6) \quad \begin{array}{ccc} \bigoplus_{g \in G} \mathcal{H}om_{A^e}(M, A \otimes gA) & \longrightarrow & \Lambda^e \otimes_{\Delta} \mathcal{H}om_{A^e}(M, A \otimes A) \\ \sum_{g \in G} \varphi_g & \longmapsto & \sum_{g \in G} (1 \otimes g) \otimes ((1 \otimes g^{-1}) * \varphi_g) \end{array} .$$

Putting together (4) and (6) shows that (D_4) commutes up to an isomorphism of functors. This proves the lemma. \square

2.4. Main theorem. All the necessary tools have been introduced in order to prove the main result of the text. Recall that crossed-products of left dg A^e -modules acted on compatibly by G were defined in 2.2 and that this operation is precisely the extension of scalars from Δ to Λ^e . The following implies Theorem 1

Theorem 2. *Let A be a dg algebra over the field k . Assume that a finite group G acts on A by dg automorphisms. Let $\Lambda = A \rtimes G$ be the crossed-product and $\Delta = \bigoplus_{g \in G} A \otimes g^{-1}A$.*

- (a) *If $\Lambda \in \text{per}(\Lambda^e)$, then $A \in \text{per}(A^e)$. If $\sharp G \in k^{\times}$, then the converse holds true.*
- (b) *The dg algebra A admits an inverse dualizing complex $\theta_A \in \mathcal{C}((A^e)^{op})$ endowed with a compatible action of G . For this action, $\theta_A \rtimes G$ is the inverse dualizing complex of Λ .*

(c) Assume that A is Calabi-Yau of dimension d and $\Lambda \in \text{per}(\Lambda^e)$ (see (a)). If an isomorphism $f: \text{RHom}_{A^e}(A, A \otimes A)[d] \rightarrow A$ exists in $\mathcal{D}(\Delta^{op})$ and is such that $\text{RHom}_{A^e}(f, A \otimes A)[d] = f$ in $\mathcal{D}(\Delta)$, then Λ is Calabi-Yau of dimension d .

Proof. Recall that the functor $\text{RHom}_{A^e}(-, A \otimes A): \mathcal{D}(\Delta) \rightarrow \mathcal{D}(\Delta^{op})$ was defined in 2.3, (D_2) .

Fix a cofibrant resolution $p: X \rightarrow A$ in $\mathcal{C}(\Delta)$. Following 2.2, (c), it is also a cofibrant resolution in $\mathcal{C}(A^e)$ and so is $p \otimes_{\Delta} \Lambda^e: X \otimes_{\Delta} \Lambda^e \rightarrow A \otimes_{\Delta} \Lambda^e$. Composing with the isomorphism $A \otimes_{\Delta} \Lambda^e \rightarrow \Lambda$ in $\mathcal{C}(\Lambda^e)$ (2.2, (d)) therefore gives a cofibrant resolution $X \otimes_{\Delta} \Lambda^e \rightarrow \Lambda$ in $\mathcal{C}(\Lambda^e)$.

(a) Assume first that $\Lambda \in \text{per}(\Lambda^e)$. Therefore, $A \otimes_{\Delta} \Lambda^e \simeq \Lambda \in \text{per}(\Lambda^e)$. The restriction-of-scalars functor $\mathcal{C}(\Lambda^e) \rightarrow \mathcal{C}(\Delta)$ preserves quasi-isomorphisms and maps Λ^e to $\Lambda^e \simeq \Delta^{\sharp G}$ in $\mathcal{C}(\Delta)$ (2.2, (a)). Therefore, it induces a functor $\mathcal{D}(\Lambda^e) \rightarrow \mathcal{D}(\Delta)$ which maps $\text{per}(\Lambda^e)$ into $\text{per}(\Delta)$. In particular, $A \otimes_{\Delta} \Lambda^e \in \text{per}(\Delta)$. Since A is a direct summand of $A \otimes_{\Delta} \Lambda^e$ in $\mathcal{C}(\Delta)$ (2.2, (e)), and therefore in $\mathcal{D}(\Delta)$, this implies that $A \in \text{per}(\Delta)$. On the other hand the restriction-of-scalars functor $\mathcal{C}(\Delta) \rightarrow \mathcal{C}(A^e)$ preserves quasi-isomorphisms and maps Δ to $\Delta \simeq (A^e)^{\sharp G}$ in $\mathcal{C}(A^e)$ (2.2, (a)). Therefore, it induces a functor $\mathcal{D}(\Delta) \rightarrow \mathcal{D}(A^e)$ which maps $\text{per}(\Delta)$ into $\text{per}(A^e)$. It also maps A to A . Thus, $A \in \text{per}(A^e)$.

Now, assume that $A \in \text{per}(A^e)$ and $\sharp G \in k^{\times}$. The two functors $- \otimes_{A^e} \Delta: \mathcal{C}(A^e) \rightarrow \mathcal{C}(\Delta)$ and $- \otimes_{\Delta} \Lambda^e: \mathcal{C}(\Delta) \rightarrow \mathcal{C}(\Lambda^e)$ preserve quasi-isomorphisms (2.2, (b)), the former maps A^e to $A^e \otimes_{A^e} \Delta = \Delta$ and the latter maps Δ to $\Delta \otimes_{\Delta} \Lambda^e = \Lambda^e$. Therefore, these functors induce functors $- \otimes_{A^e} \Delta: \mathcal{D}(A^e) \rightarrow \mathcal{D}(\Delta)$ and $- \otimes_{\Delta} \Lambda^e: \mathcal{D}(\Delta) \rightarrow \mathcal{D}(\Lambda^e)$, the former maps $\text{per}(A^e)$ into $\text{per}(\Delta)$, the latter maps $\text{per}(\Delta)$ into $\text{per}(\Lambda^e)$. The former maps A to $A \otimes_{A^e} \Delta$ and A is a direct summand of $A \otimes_{A^e} \Delta$ in $\mathcal{C}(\Delta)$, thus $A \in \text{per}(\Delta)$. The latter maps A to $A \otimes_{\Delta} \Lambda^e \simeq \Lambda$, thus $\Lambda \in \text{per}(\Lambda^e)$.

(b) By construction, $\mathcal{H}om_{A^e}(X, A \otimes A) \in \mathcal{C}(\Delta^{op})$. Let $Y \rightarrow \mathcal{H}om_{A^e}(X, A \otimes A)$ be a cofibrant resolution in $\mathcal{C}(\Delta^{op})$. Therefore, it is also a cofibrant resolution in $\mathcal{C}((A^e)^{op})$ (2.2, (c)). Thus, θ_A may be chosen equal to Y and, as a left dg Δ -module, it has an action of G compatible with the left action of A^e . Of course, not any cofibrant resolution of $\mathcal{H}om_{A^e}(X, A \otimes A)$ has this property.

By definition of the inverse dualizing complex of Λ , there is a quasi-isomorphism $\theta_{\Lambda}: \mathcal{H}om_{A^e}(X \otimes_{\Delta} \Lambda^e, \Lambda \otimes \Lambda)$ in $\mathcal{C}((\Lambda^e)^{op})$. On the other hand, the proof in 2.3 gave an isomorphism $\mathcal{H}om_{A^e}(X \otimes_{\Delta} \Lambda^e, \Lambda \otimes \Lambda) \rightarrow \Lambda^e \otimes_{\Delta} \mathcal{H}om_{A^e}(X, A \otimes A)$ in $\mathcal{C}((\Lambda^e)^{op})$ and $\Lambda^e \otimes_{\Delta} \mathcal{H}om_{A^e}(M, A \otimes A) \simeq \mathcal{H}om_{A^e}(X, A \otimes A) \rtimes G$ in $\mathcal{C}((\Lambda^e)^{op})$. Thus, there is a quasi-isomorphism $\theta_{\Lambda} \rightarrow \theta_A \rtimes G$ in $\mathcal{C}((\Lambda^e)^{op})$. For this reason, θ_{Λ} may be chosen equal to $\theta_A \rtimes G$.

(c) Assume that $A \in \text{per}(A^e)$, $\Lambda \in \text{per}(\Lambda^e)$ and that there is an isomorphism $f: \text{RHom}_{A^e}(A, A \otimes A)[d] \rightarrow A$ in $\mathcal{D}(\Delta^{op})$ such that $\text{RHom}_{A^e}(f, A \otimes A)[d] = f$ in $\mathcal{D}(\Delta)$. Applying the functor $\Lambda^e \otimes_{\Delta} -: \mathcal{D}(\Delta^{op}) \rightarrow \mathcal{D}((\Lambda^e)^{op})$ yields an isomorphism in $\mathcal{D}((\Lambda^e)^{op})$:

$$\Lambda^e \otimes_{\Delta} f: \Lambda^e \otimes_{\Delta} \text{RHom}_{A^e}(A, A \otimes A)[d] \xrightarrow{\sim} \Lambda^e \otimes_{\Delta} A \simeq \Lambda.$$

As explained above, $\Lambda^e \otimes_{\Delta} \text{Rhom}_{A^e}(A, A \otimes A)$ identifies with $\text{RHom}_{\Lambda^e}(\Lambda, \Lambda \otimes \Lambda)$. Thus, in order to conclude, it only remains to prove that $\text{RHom}_{(\Lambda^e)^{op}}(\Lambda^e \otimes_{\Delta} f, \Lambda \otimes \Lambda)[d] = \Lambda^e \otimes_{\Delta} f$. But this follows from the equality $\text{RHom}_{A^e}(f, A \otimes A)[d] = f$ in $\mathcal{D}(\Delta)$ and from the commutativity of the following diagram obtained by applying 2.3 (and interchanging the rôles played by left and right modules):

$$\begin{array}{ccc} \mathcal{D}(\Delta^{op}) & \xrightarrow{\text{RHom}_{(A^e)^{op}}(-, A \otimes A)} & \mathcal{D}(\Delta) \\ \Lambda^e \otimes_{\Delta} - \downarrow & & \downarrow - \otimes_{\Delta} \Lambda^e \\ \mathcal{D}((\Lambda^e)^{op}) & \xrightarrow{\text{RHom}_{(\Lambda^e)^{op}}(-, \Lambda \otimes \Lambda)} & \mathcal{D}(\Lambda^e) \end{array} \quad .$$

Thus, Λ is Calabi-Yau of dimension d . □

Here are some technical comments on the hypotheses made in part (c) of the above theorem. Let $X \rightarrow A$ be a cofibrant resolution in $\mathcal{C}(\Delta)$ and $\theta_A \rightarrow \mathcal{H}om_{A^e}(X, A \otimes A)$ be a cofibrant resolution in $\mathcal{C}(\Delta^{op})$. The data of an isomorphism $\text{RHom}_{A^e}(A, A \otimes A)[d] \rightarrow A$ in $\mathcal{D}(\Delta^{op})$ is equivalent to that of a quasi-isomorphism $\theta_A[d] \rightarrow A$ in $\mathcal{C}(\Delta^{op})$ that is, a quasi-isomorphism in $\mathcal{C}((A^e)^{op})$ which is compatible with the actions of G on θ_A and on A . Given an isomorphism $f: \text{RHom}_{A^e}(A, A \otimes A)[d] \rightarrow A$ in $\mathcal{D}(\Delta^{op})$, the equality $\text{RHom}_{(A^e)^{op}}(f, A \otimes A)[d] = f$ holds true in $\mathcal{D}(A^e)$ if it does so in $\mathcal{D}(\Delta)$, because of (D_2) (after interchanging the rôles of left and right modules). The converse needs not hold true because the restriction-of-scalars functor $\mathcal{D}(\Delta) \rightarrow \mathcal{D}(A^e)$ is not necessarily faithful. However, $\mathcal{D}(\Delta) \rightarrow \mathcal{D}(A^e)$ is faithful if $\sharp G \in k^{\times}$: Indeed, it is right adjoint to $- \otimes_{A^e} \Delta: \mathcal{D}(A^e) \rightarrow \mathcal{D}(\Delta)$ and the counit of this adjunction splits if $\sharp G \in k^{\times}$ (see the proof of 2.1, (e)).

These considerations give Theorem 1 as a simplification of Theorem 2 in case $\sharp G \in \mathbf{k}^\times$:

Theorem 1. *Let A be a dg algebra acted on by a finite group G such that $\sharp G \in \mathbf{k}^\times$. Fix the following data:*

- $X \rightarrow A$ is a cofibrant resolution in $\mathcal{C}(A^e)$ such that X is endowed with a compatible action of G and the resolution is G -equivariant,
- $\theta_A \rightarrow \mathcal{H}om_{A^e}(X, A \otimes A) = \mathrm{RHom}_{A^e}(A, A \otimes A)$ is a cofibrant resolution in $\mathcal{C}((A^e)^{op})$ such that θ_A is endowed with a compatible action of G and the resolution is G -equivariant.

Then:

- (1) $\theta_A \rtimes G$ is the inverse dualizing complex of $A \rtimes G$.
- (2) Assume that A is homologically smooth and there exists a G -equivariant quasi-isomorphism $f: \theta_A[d] \rightarrow A$ in $\mathcal{C}((A^e)^{op})$ such that $\mathrm{RHom}_{(A^e)^{op}}(f, A \otimes A)[d] = f$ in $\mathcal{D}(A^e)$ (in particular, A is Calabi-Yau of dimension d). Then $A \rtimes G$ is Calabi-Yau of dimension d .

Deleting the equivariance condition on f in Theorem 1 gives exactly the definition of the Calabi-Yau property for A . Later, this equivariance condition will be discussed on and referred to as the *compatibility condition*, for short.

3. CROSSED-PRODUCTS OF CALABI-YAU COMPLETIONS

This section investigates when the crossed-product of a dg Calabi-Yau algebra A by a finite group G is still Calabi-Yau when A arises from a particular construction introduced in [15] and called *deformed Calabi-Yau completion*.

3.1. A reminder on deformed Calabi-Yau completions. The reader is referred to [15] for a detailed account on deformed Calabi-Yau completions. This paragraph recalls this construction briefly. Let A be a dg algebra, d be an integer and $\theta_A \in \mathcal{C}((A^e)^{op})$ be the inverse dualizing complex of A . The *d -Calabi-Yau completion* of A is defined as the dg tensor algebra

$$\Pi_d(A) = T_A(\theta_A[d-1]) = A \oplus \theta_A[d-1] \oplus (\theta_A[d-1] \otimes_A \theta_A[d-1]) \oplus \cdots$$

It does not depend on the choice of θ_A up to quasi-isomorphism of dg algebras. If A is homologically smooth, then it is Calabi-Yau of dimension d ([15, Thm. 4.8]).

The definition of the deformed Calabi-Yau completion uses the following canonical isomorphisms where A is assumed to be homologically smooth:

$$(7) \quad \begin{aligned} \mathrm{HH}_{d-2}(A) = \mathrm{Tor}_{d-2}^{A^e}(A, A) &= \mathrm{H}^{d-2}(A \overset{\mathbb{L}}{\otimes}_{A^e} A) \simeq \mathrm{H}^{d-2}(A \overset{\mathbb{L}}{\otimes}_{A^e} A^{\vee\vee}) \\ &\simeq \mathrm{Hom}_{\mathcal{D}((A^e)^{op})}(A^\vee, A[d-2]) \\ &\simeq \mathrm{Hom}_{\mathcal{D}((A^e)^{op})}(\theta_A[d-1], A[1]) \end{aligned}$$

where, for short, $-\vee$ stands for $\mathrm{RHom}_{A^e}(-, A \otimes A)$ (hence θ_A is any cofibrant resolution of A^\vee in $\mathcal{C}((A^e)^{op})$, and $A \simeq A^{\vee\vee}$ in $\mathcal{D}(A^e)$ because A is homologically smooth). Also HH denotes the Hochschild homology. If $X \rightarrow A$ and $p_A: \theta_A \rightarrow \mathcal{H}om_{A^e}(X, A \otimes A)$ are cofibrant resolutions in $\mathcal{C}(A^e)$ and $\mathcal{C}((A^e)^{op})$, respectively, then $\mathrm{HH}_{d-2}(A) = \mathrm{H}^{d-2}(X \overset{\mathbb{L}}{\otimes}_{A^e} A)$. Also $\mathrm{Hom}_{\mathcal{D}((A^e)^{op})}(\theta_A[d-1], A[1]) = \mathrm{H}^0(\mathcal{H}om_{(A^e)^{op}}(\theta_A[d-1], A[1]))$, and the isomorphism in (7) is induced by the following morphism of complexes (which is therefore a quasi-isomorphism):

$$(8) \quad \begin{array}{ccc} X \overset{\mathbb{L}}{\otimes}_{A^e} A[2-d] & \longrightarrow & \mathcal{H}om_{(A^e)^{op}}(\theta_A[d-1], A[1]) \\ x \otimes a & \longmapsto & (\alpha \mapsto (p_A(\alpha))(x).a) \end{array}$$

Let $c \in \mathrm{HH}_{d-2}(A)$. The associated morphism in $\mathrm{Hom}_{\mathcal{D}((A^e)^{op})}(\theta_A[d-1], A[1])$ (see (7)) admits, say, $D: \theta_A[d-1] \rightarrow A[1]$ as a representative. Then D is a morphism of graded left A^e -modules. Therefore, it induces a unique A^e -linear (skew-)derivation of degree 1:

$$\overline{D}: T_A(\theta_A[d-1]) \rightarrow T_A(\theta_A[d-1]).$$

Writing $\delta: T_A(\theta_A[d-1]) \rightarrow T_A(\theta_A[d-1])$ for the differential of the dg algebra $\Pi_d(A)$, the *deformed Calabi-Yau completion* $\Pi_d(A, c)$ is defined as the dg algebra

$$\Pi_d(A, c) = (T_A(\theta_A), \delta + \overline{D}).$$

Here, $\delta + \overline{D}$ is indeed a differential because $c \in \mathrm{HH}_{d-2}(A)$. It is proved in [15, Thm. 5.2] that $\Pi_d(A, c)$ is Calabi-Yau of dimension d , for every c (under the assumption that A is homologically smooth).

Throughout this section, A will denote a dg algebra, acted on by a finite group G . The crossed-product $A \rtimes G$ will be denoted by Λ and the dg subalgebra $\bigoplus_{g \in G} Ag \otimes g^{-1}A$ of Λ will be denoted by Δ (see 2.1). Also, a cofibrant resolution $X \rightarrow A$ in $\mathcal{C}(\Delta)$ will be fixed, recall that this is also a cofibrant resolution in $\mathcal{C}(A^e)$ and it induces a cofibrant resolution $X \overset{\mathbb{L}}{\otimes}_{\Delta} \Lambda^e \rightarrow A \overset{\mathbb{L}}{\otimes}_{\Delta} \Lambda^e \xrightarrow{\sim} \Lambda$ in $\mathcal{C}(\Lambda^e)$ (2.2, (c) and (d)). Finally, a cofibrant resolution $p_A: \theta_A \rightarrow \mathcal{H}om_{A^e}(X, A \otimes A)$ in $\mathcal{C}(\Delta^{op})$ is fixed (so it is a cofibrant resolution in $\mathcal{C}((A^e)^{op})$).

Following 2.3, left dg Δ -modules will be considered as left dg A^e -modules endowed with a compatible action of G . In particular, given $M \in \mathcal{C}(\Delta)$, the compatible action of G on $\mathcal{H}om_{A^e}(M, A \otimes A) \in \mathcal{C}(\Delta^{op})$ is such that ${}^g f = (g \otimes g^{-1}).f$ maps $m \in M$ to $(g \otimes g^{-1}).f(m.(g \otimes g^{-1})).(g^{-1} \otimes g)$, for $g \in G$ and $f \in \mathcal{H}om_{A^e}(M, A \otimes A)$.

3.2. Group actions on deformed Calabi-Yau completions. Since $\theta_A \in \mathcal{C}(\Delta^{op})$, there is an action of G on θ_A which is compatible with the action of A^e on the left. This induces an action of G on $\Pi_d(A) = T_A(\theta_A[d-1])$ by dg automorphisms ($g \in G$ acts diagonally on $(\theta_A[d-1])_{A^e}^{\otimes n}$ for all $n \geq 1$).

On the other hand, the actions of G on $X \in \mathcal{C}(\Delta)$ and on A induce a diagonal action of G on $X \otimes_{A^e} A$ and, therefore, on $\mathbb{H}_*(A)$. To be more precise: ${}^g(x \otimes a) = {}^g x \otimes {}^g a = (x.(g^{-1} \otimes g)) \otimes ((g \otimes g^{-1}).a)$, for $x \in X$, $a \in A$ and $g \in G$ (see 2.1). As usual, $\mathbb{H}_{d-2}(A)^G$ denotes the set of fixed points in $\mathbb{H}_{d-2}(A)$ under this action. The following lemma gives a necessary and sufficient condition on $c \in \mathbb{H}_{d-2}(A)$ for the action of G on $\Pi_d(A)$ to be an action on $\Pi_d(A, c)$ by dg automorphisms (of course, G already acts on $\Pi_d(A, c)$ by graded automorphisms, since $\Pi_d(A, c) = \Pi_d(A)$ as graded algebras).

Lemma. *Let A be a homologically smooth dg algebra acted on by a finite group G such that $\sharp G \in \mathbf{k}^\times$. The following conditions are equivalent:*

- (i) $c \in \mathbb{H}_{d-2}(A)^G$,
- (ii) *there exists a cocycle $D \in \mathcal{H}om_{(A^e)^{op}}(\theta_A[n-1], A[1])$ which corresponds to c under (7) (see also (8)) and such that the action of G on $\Pi_d(A)$ is also an action on $\Pi_d(A, c) = (T_A(\theta_A[d-1]), \delta + \overline{D})$ by dg automorphisms.*

Proof. Let $u \in X \otimes_{A^e} A$ be any representative of $c \in \mathbb{H}_{d-2}(A) = \mathbb{H}^0(X \otimes_{A^e} A)$. Let $D_u \in \mathcal{H}om_{(A^e)^{op}}(\theta_A[d-1], A[1])$ be the image of u under the isomorphism (8). Note that (8) is G -equivariant. On the other hand, c lies in $\mathbb{H}_{d-2}(A)^G$ if and only if there is a choice of the representative u such that ${}^g u = u$ for all $g \in G$, because $\sharp G \in \mathbf{k}^\times$. Therefore, c lies in $\mathbb{H}_{d-2}(A)^G$ if and only if there is a representative u of c such that ${}^g D_u = D_u$ for all $g \in G$.

Next, let $D \in \mathcal{H}om_{(A^e)^{op}}(\theta_A[d-1], A[1])$ be a 0-cocycle. Then, the action of G on $\Pi_d(A)$ is an action by dg automorphisms on $(T_A(\theta_A[d-1]), \delta + \overline{D})$ if and only if $\delta + \overline{D}$ is a G -equivariant differential. Since δ is itself G -equivariant, this is equivalent to say that \overline{D} is G -equivariant. Clearly, this is equivalent to D being G -equivariant. Note that if this is the case and if the cohomology class of D in $\text{Hom}_{\mathcal{D}((A^e)^{op})}(\theta_A[d-1], A[1])$ corresponds to c under (7), then c lies in $\mathbb{H}_{d-2}(A)^G$, because (7) is G -equivariant.

These considerations show the equivalence between (i) and (ii). \square

Note that not all the actions of G on $\Pi_d(A, c)$ arise from actions on A such that $c \in \mathbb{H}_{d-2}(A)^G$.

3.3. Crossed-products of Calabi-Yau completions. The following lemma shows that taking crossed-products commutes with Calabi-Yau completion.

Lemma. *Let A be a dg algebra acted on by a finite group G . Let d be an integer. Then, the dg algebras $\Pi_d(A \rtimes G)$ and $\Pi_d(A) \rtimes G$ are quasi-isomorphic.*

Proof. Recall that $\theta_A \rtimes G$ is the inverse dualizing complex of $\Lambda = A \rtimes G$ (2.4, (b)). Therefore, $\Pi_d(A \rtimes G) = T_{A \rtimes G}(\theta_A \rtimes G[d-1])$. The following map is therefore a morphism of dg algebras:

$$\begin{aligned} \Pi_d(A) \rtimes G &\longrightarrow \Pi_d(A \rtimes G) \\ a \otimes g &\longmapsto a \otimes g \\ (f_1 \otimes \cdots \otimes f_n) \otimes g &\longmapsto f_1 \otimes \cdots \otimes f_{n-1} \otimes (f_n \otimes g) \end{aligned} ,$$

where $a \in A$, $f_1, \dots, f_n \in \theta_A[d-1]$ and $g \in G$. It is straightforward to check that the following map is its inverse:

$$\begin{aligned} \Pi_d(A \rtimes G) &\longrightarrow \Pi_d(A) \rtimes G \\ a \otimes g &\longmapsto a \otimes g \\ (f_1 \otimes g_1) \otimes \cdots \otimes (f_n \otimes g_n) &\longmapsto (f_1 \otimes g_1 f_2 \otimes \cdots \otimes g_1 \cdots g_l f_{l+1} \otimes \cdots \otimes g_1 \cdots g_{n-2} f_{n-1} \otimes g_1 \cdots g_{n-1} f_n) \otimes g_1 \cdots g_n \end{aligned} ,$$

where $a \in A$, $g, g_1, \dots, g_n \in G$ and $f_1, \dots, f_n \in \theta_A[d-1]$. \square

3.4. Compatibility between crossed-products and deformed Calabi-Yau completions. The following proposition is the main result of the section. It shows that under compatibility conditions, the crossed-product of a deformed Calabi-Yau completion is still a deformed Calabi-Yau completion and, therefore, is Calabi-Yau.

Proposition. *Let A be a homologically smooth algebra. Let G be a finite group acting on A by dg automorphisms and such that $\sharp G \in \mathbf{k}^\times$. Finally let $c \in \mathbb{H}_{d-2}(A)^G$. Then, up to a choice of the inverse dualizing complex of A , the action of G on A induces an action on $\Pi_d(A, c)$ by dg automorphisms. For this action, there is an isomorphism of dg algebras:*

$$\Pi_d(A, c) \rtimes G \simeq \Pi_d(A \rtimes G, \overline{c}) ,$$

where \bar{c} is the image of c under the canonical map $\mathrm{HH}_{d-2}(A) \rightarrow \mathrm{HH}_{d-2}(A \rtimes G)$. In particular, $\Pi_d(A, c) \rtimes G$ is Calabi-Yau of dimension d .

Proof. For the purpose of the proposition, it is necessary to first introduce an adequate cofibrant resolution $p_\Lambda: \theta_\Lambda \rightarrow \mathrm{RHom}_{\Lambda^e}(\Lambda, \Lambda \otimes \Lambda)$ in $\mathcal{C}((\Lambda^e)^{op})$. Following 2.4, the inverse dualizing complex of Λ is $\theta_\Lambda := \Lambda^e \otimes_{\Delta} \theta_A$. Moreover, $X \otimes_{\Delta} \Lambda^e$ is a cofibrant resolution of Λ in $\mathcal{C}(\Lambda^e)$, because X is a cofibrant resolution of A in $\mathcal{C}(\Delta)$ (see 2.2, (c) and (d)). Therefore, an explicit quasi-isomorphism $p_\Lambda: \theta_\Lambda \rightarrow \mathcal{H}om_{\Lambda^e}(X \otimes_{\Delta} \Lambda^e, \Lambda \otimes \Lambda) = \mathrm{RHom}_{\Lambda^e}(\Lambda, \Lambda \otimes \Lambda)$ in $\mathcal{C}((\Lambda^e)^{op})$ is obtained as follows. On the one hand, $\Lambda^e \otimes_{\Delta} p_A: \theta_\Lambda \rightarrow \Lambda^e \otimes_{\Delta} \mathcal{H}om_{A^e}(X, A \otimes A)$ is a quasi-isomorphism in $\mathcal{C}((\Lambda^e)^{op})$, because $p_A: \theta_A \rightarrow \mathcal{H}om_{A^e}(X, A \otimes A)$ is a quasi-isomorphism in $\mathcal{C}(\Delta^{op})$ (see 2.2, (c)). On the other hand, the following map is an isomorphism in $\mathcal{C}(\Delta^{op})$, inverse to the composition of (5) with (6):

$$\begin{aligned} \Lambda^e \otimes_{\Delta} \mathcal{H}om_{A^e}(X, A \otimes A) &\longrightarrow \mathcal{H}om_{\Lambda^e}(X \otimes_{\Delta} \Lambda^e, \Lambda \otimes \Lambda) \\ (ag \otimes hb) \otimes f &\longmapsto (x \otimes u \mapsto (a \otimes ghb).(({}^g f)(x)).u) \end{aligned} ,$$

where $a, b \in A$, $g, h \in G$, $f \in \mathcal{H}om_{A^e}(X, A \otimes A)$ and $u \in \Lambda^e$. Thus, the following map is an isomorphism in $\mathcal{C}((\Lambda^e)^{op})$:

$$\begin{aligned} p_\Lambda: \theta_\Lambda &\longrightarrow \mathcal{H}om_{\Lambda^e}(X \otimes_{\Delta} \Lambda^e, \Lambda \otimes \Lambda) \\ (ag \otimes hb) \otimes f &\longmapsto (x \otimes u \mapsto (a \otimes ghb).({}^g(p_A(f)))(x).u) \end{aligned} .$$

Following 3.3, there exists $D \in \mathcal{H}om_{(A^e)^{op}}(\theta_A[d-1], A[1])$ which is G -invariant (that is, ${}^g D = D$ for every $g \in G$) and such that $\Pi_d(A, c)$ is the graded algebra $T_A(\theta_A[d-1])$ endowed with the differential $\delta_A + \overline{D}$, where δ_A denotes the differential of $\Pi_d(A)$. In particular, the action of G on A induces an action on $\Pi_d(A, c)$. Therefore, $\Pi_d(A, c) \rtimes G$ is the graded algebra $T_A(\theta_A[d-1]) \rtimes G$ endowed with the differential whose restriction to $\theta_A[d-1]$ equals to $\delta_A + \overline{D}$ and which vanishes on G . It then follows from 3.2 that $\Pi_d(A, c) \rtimes G$ is isomorphic to the graded algebra $T_{A \rtimes G}(\theta_\Lambda[d-1])$ endowed with the differential $\delta_\Lambda + \overline{\Lambda^e \otimes_{\Delta} D}$, where $\Lambda^e \otimes_{\Delta} D \in \mathcal{H}om_{(\Lambda^e)^{op}}(\theta_\Lambda[d-1], \Lambda[1])$ (recall that $\theta_\Lambda = \Lambda^e \otimes_{\Delta} \theta_A$ and $\Lambda^e \otimes_{\Delta} A$ is identified to Λ in $\mathcal{C}((\Lambda^e)^{op})$, via $u \otimes a \mapsto u.a$). In other words, $\Pi_d(A, c) \rtimes G$ is the deformed Calabi-Yau completion $\Pi_d(\Lambda, c')$ where $c' \in \mathrm{HH}_{d-2}(\Lambda)$ corresponds to $\Lambda^e \otimes_{\Delta} D \in \mathcal{H}om_{(\Lambda^e)^{op}}(\theta_\Lambda[d-1], \Lambda[1])$ under the map (8) corresponding to Λ .

Therefore, in order to prove the proposition, it suffices to prove that the following diagram commutes:

$$(D_5) \quad \begin{array}{ccc} \mathrm{HH}_{d-2}(A)^G \hookrightarrow \mathrm{HH}_{d-2}(A) & \xrightarrow{(7)} & \mathrm{H}^0(\mathcal{H}om_{(A^e)^{op}}(\theta_A[d-1], A[1])) \\ \downarrow & & \downarrow \mathrm{H}^0(\Lambda^e \otimes_{\Delta} -) \\ \mathrm{HH}_{d-2}(\Lambda) & \xrightarrow{\quad} & \mathrm{H}^0(\mathcal{H}om_{(\Lambda^e)^{op}}(\theta_\Lambda[d-1], \Lambda[1])) \end{array}$$

where:

- the vertical arrow on the left is the composition $\mathrm{HH}_{d-2}(A)^G \hookrightarrow \mathrm{HH}_{d-2}(A) \rightarrow \mathrm{HH}_{d-2}(\Lambda)$,
- the bottom horizontal arrow is the map (7) corresponding to Λ ,

For this purpose, it suffices to prove that the following diagram commutes (recall that $\mathrm{HH}_{d-2}(A)^G = \mathrm{H}^{d-2}((X \otimes_{A^e} A)^G)$ because $\sharp G \in \mathbf{k}^\times$):

$$(D_6) \quad \begin{array}{ccc} (X \otimes_{A^e} A)^G \hookrightarrow X \otimes_{A^e} A & \xrightarrow{(8)} & \mathcal{H}om_{(A^e)^{op}}(\theta_A[d-1], A[1]) \\ \downarrow & & \downarrow \Lambda^e \otimes_{\Delta} - \\ (X \otimes_{\Delta} \Lambda^e) \otimes_{\Lambda^e} \Lambda = X \otimes_{\delta} \Lambda & \xrightarrow{\quad} & \mathcal{H}om_{(\Lambda^e)^{op}}(\theta_\Lambda[d-1], \Lambda[1]) \end{array}$$

where:

- the vertical arrow on the left is the composition $(X \otimes_{A^e} A)^G \hookrightarrow X \otimes_{A^e} A \rightarrow X \otimes_{\Delta} \Lambda$,
- the bottom horizontal arrow is the map (8) corresponding to Λ .

Let $\sum_i x_i \otimes a_i \in (X \otimes_{A^e} A)^G$, that is, $\sum_i {}^g x_i \otimes {}^g a_i = \sum_i x_i \otimes a_i$ for every $g \in G$. Then, the image of $\sum_i x_i \otimes a_i$ under the composition of the top horizontal arrows in (D_6) is the map (see (8)):

$$\begin{aligned} \mathcal{H}om_{A^e}(X, A \otimes A)[d-1] &\longrightarrow A[1] \\ f &\longmapsto \sum_i (p_A(f))(x_i).a_i \end{aligned} .$$

Applying the right-hand side vertical arrow in (D_6) yields the map:

$$(9) \quad \begin{array}{ccc} \Lambda^e \otimes_{\Delta} \mathcal{H}om_{A^e}(X, A \otimes A)[d-1] & \longrightarrow & \Lambda[1] \\ u \otimes f & \longmapsto & \sum_i u.(p_A(f))(x_i).a_i \end{array} .$$

On the other hand, the image of $\sum_i x_i \otimes a_i$ under the left-hand side vertical arrow of (D_6) is equal to $\sum_i x_i \otimes a_i \in X \otimes_{\Delta} \Lambda$. Applying the bottom horizontal arrow of (D_6) to $\sum_i x_i \otimes a_i$ yields the following map (see (8)):

$$(10) \quad \begin{array}{ccc} \Lambda^e \otimes_{\Delta} \mathcal{H}om_{A^e}(X, A \otimes A)[d-1] & \longrightarrow & \Lambda[1] \\ (ag \otimes hb) \otimes f & \longmapsto & \sum_i (p_{\Lambda}((ag \otimes hb) \otimes g))(x_i).a_i \end{array} .$$

The proof of the commutativity of (D_6) therefore reduces to proving that the two maps (9) and (10) are equal. Let $a, b \in A$, $g \in G$ and $f \in \mathcal{H}om_{A^e}(X, A \otimes A)$. Using the expression of p_{Λ} that was computed at the beginning of the proof, gives:

$$\sum_i (p_{\Lambda}((ag \otimes hb) \otimes f))(x_i).a_i = \sum_i (a \otimes ghb).({}^g(p_A(f)))(x_i).a_i .$$

Recall that ${}^g(p_A(f))(x_i) = (g \otimes g^{-1}).((p_A(f))(x_i).(g \otimes g^{-1})).(g^{-1} \otimes g)$, therefore:

$$\sum_i p_{\Lambda}((ag \otimes hb) \otimes f)(x_i).a_i = \sum_i (a \otimes ghb).(g \otimes g^{-1}).(p_A(f)(x_i).(g \otimes g^{-1})).(g^{-1} \otimes g).a_i .$$

Note that $(x_i.(g \otimes g^{-1})) \otimes ((g^{-1} \otimes g).a_i) = g^{-1} a_i \otimes g^{-1} a_i$, by definition of the action of G on $X \otimes_{A^e} A$. Therefore:

$$\begin{aligned} \sum_i p_{\Lambda}((ag \otimes hb) \otimes g)(x_i).a_i &= \sum_i (ag \otimes hb).p_A(f)(g^{-1} x_i).g^{-1} a_i \\ &= \sum_i (ag \otimes hb)p_A(f)(x_i).a_i \end{aligned}$$

where the last equality holds true because $\sum_i x_i \otimes a_i \in (X \otimes_{A^e} A)^G$. This proves that the maps (9) and (10) are equal. Thus, (D_6) and (D_5) commute, which proves that $\Pi_d(A, c) \times G \simeq \Pi_d(A \rtimes G, \bar{c})$ where \bar{c} is the image of c under the map $\mathrm{HH}_{d-2}(A) \rightarrow \mathrm{HH}_{d-2}(A \rtimes G)$. The last assertion of the proposition holds true because $\Pi_d(A \rtimes G, \bar{c})$ itself is Calabi-Yau of dimension d . \square

4. APPLICATION TO GINZBURG ALGEBRAS

This section applies the previous study on crossed-products of deformed Calabi-Yau completions to the particular case of Ginzburg algebras.

4.1. A reminder on Ginzburg algebras. Ginzburg algebras were introduced by Ginzburg in [11, 4.2]. The definition used here is the more general one of [15, Sect. 6.3].

Let Q be a finite *graded k-quiver*, that is, the set Q_0 of vertices is finite and the kQ_0 -bimodule kQ_1 spanned by the (finite) set Q_1 of arrows has a \mathbb{Z} -grading. Thus, the path algebra kQ is a graded algebra, such that the idempotents associated to trivial paths have degree 0. As usual, the idempotent associated to a vertex x is denoted by e_x . A *potential* on Q is an element W of the quotient vector space $\left(kQ \otimes_{(kQ_0)^e} kQ_0 \right) / \langle uv \otimes e_x - (-1)^{\deg(u).\deg(v)} vu \otimes e_y \mid u, v \rangle$ of $kQ \otimes_{(kQ_0)^e} kQ_0$ by the subspace generated by all elements of the form $uv \otimes e_x - (-1)^{\deg(u).\deg(v)} vu \otimes e_y$ for homogeneous $u \in e_x kQ$ and $v \in e_y kQ$. Hence, W is a linear combination of oriented cycles in Q , each of which is considered up to cyclic permutation with Koszul-type signs. The grading on kQ defines a grading on the space of potentials.

Let $d \geq 3$ be an integer and W a homogeneous of degree $3-d$ potential on Q . The Ginzburg algebra $\mathcal{A}(Q, W)$ associated to (Q, W) is defined as follows. As a graded algebra, it is freely generated over kQ_0 by the direct sum of the following graded kQ_0 -bimodules:

- kQ_1 , hence kQ is a graded subalgebra of $\mathcal{A}(Q, W)$,
- $\mathrm{Hom}_{(kQ_0)^e}(kQ_1, kQ_0 \otimes kQ_0)[d-2]$, hence, every arrow $a: x \rightarrow y$ in Q defines a new generator $a^*: y \rightarrow x$ and the linear combination $\sum_t \lambda_t a_t^*: y \rightarrow x$ has degree $2-d-n$ if $\sum_t \lambda_t a_t: x \rightarrow y$ has degree n (with $\lambda_t \in k$ and $a_t \in Q_1$),
- $kQ_0[d-1]$, hence, every vertex $i \in Q_0$ defines a new generator $c_i: i \rightarrow i$ of degree $1-d$.

The differential on $\mathcal{A}(Q, W)$ (of degree $+1$) is uniquely determined by the following rules:

- it vanishes on kQ ,
- it maps $a^*: y \rightarrow x$ to the partial derivative $\partial_a W$,

- it maps $c_i: i \rightarrow i$ to $\sum_{a: i \rightarrow \cdot} aa^* - \sum_{a: \cdot \rightarrow i} a^*a$, where the first sum runs over all the arrows in Q starting in i and the second sum runs over all the arrows in Q arriving in i .

Recall that the partial derivation ∂_a is the linear map which takes a path p to:

$$\partial_a p = \sum_{p=p_1 a p_2} p_2 p_1$$

where the sum runs over all decompositions $p = p_1 a p_2$ with paths p_1 and p_2 . Of course, this extends linearly to define the partial derivation ∂_u with respect to any $u \in kQ_1$.

As said above, this definition is taken from [15, Sect. 6.2]. The original definition of Ginzburg corresponds to the case where kQ has the trivial grading and $d = 3$. It is proved in [15, Thm. 6.3] (see also [15, Thm. A.12]) that $\mathcal{A}(Q, W)$ is quasi-isomorphic to the deformed Calabi-Yau completion $\Pi_d(kQ, c)$, where the graded algebra kQ is endowed with the zero differential and $c \in \text{HH}_{d-2}(kQ)$ is as follows (see [15, 6.1 and 6.2] for more details): As a homogeneous of degree $3-d$ potential, W may be considered as lying in the cyclic homology space $\text{HC}_{d-3}(kQ)$ (and any element of $\text{HC}_{d-3}(kQ)$ arises in this way if $\text{car}(k) = 0$ or if $n = 3$), then $c \in \text{HH}_{d-2}(kQ)$ is defined as the image of W under Connes' map $\text{HC}_{d-3}(kQ) \xrightarrow{B} \text{HH}_{d-2}(kQ)$.

4.2. The Morita reduced version of $kQ \rtimes G$. Let Q be a graded k -quiver and G a finite group such that $\text{car}(k)$ does not divide $\sharp G$. Assume that G acts on the set Q_0 (with action denoted by $(g, i) \mapsto g.i$) and also on the kQ_0 -bimodule kQ_1 (with action denoted by $(g, u) \mapsto {}^g u$), the two actions are supposed to be compatible to each other, that is, ${}^g a \in e_{g.i} kQ_1 e_{g.j}$ if $a \in e_i kQ_1 e_j$ and $g \in G$. This defines an action of G on kQ by graded automorphisms and any action of G on kQ by graded automorphisms stabilising Q_0 and kQ_1 arises in this way.

It is proved in [7] that if kQ has the trivial grading, then there exists a quiver Q' such that $\mathcal{D}(kQ')$ and $\mathcal{D}(kQ \rtimes G)$ are equivalent. It is not difficult to check that the proof given there works for the graded situation and actually shows the existence of a graded k -quiver Q' and of an injective homomorphism of graded algebras $kQ' \rightarrow kQ \rtimes G$ which, by restriction, induces an equivalence $\mathcal{D}(kQ \rtimes G) \rightarrow \mathcal{D}(kQ')$. Here is a description of Q' , the reader is referred to [7] for a detailed proof.

If a group H acts on a set E on the left, $[H \backslash E]$ will denote an arbitrarily fixed set of representatives of the orbit set $H \backslash E$. For every $i \in Q_0$, let $\kappa_i \in G$ be such that $\kappa_i.i \in [G \backslash Q_0]$. The stabilizer of i is denoted by G_i .

The construction of kQ' is done in two steps (see [7, 3.1] for more details). Let S be the semi-simple algebra

$$S = \prod_{i \in [G \backslash Q_0]} kG_i.$$

For $i, j \in [G \backslash Q_0]$, consider the following graded $kG_i - kG_j$ -sub-bimodule of $kQ \rtimes G$:

$$\bigoplus_{(i', j') \in [G \backslash G.i \times G.j]} G_i \kappa_{i'}^{-1} e_{i'} kQ_1 e_{j'} \kappa_{j'} G_j \subseteq kQ \rtimes G.$$

Here, the action of G on $G.i \times G.j$ is diagonal. Taking the direct sum of all these modules for $(i, j) \in [G \backslash Q_0] \times [G \backslash Q_0]$ yields a graded S -subbimodule of $kQ \rtimes G$. Denote it by M . Then, the inclusion $M \hookrightarrow kQ \rtimes G$ induces a homomorphism of graded algebras $T_S(M) \rightarrow kQ \rtimes G$. It is injective and the associated restriction-of-scalars functor $\mathcal{D}(kQ \rtimes G) \rightarrow \mathcal{D}(T_S(M))$ is an equivalence. The second step is as follows. For every $i \in [G \backslash Q_0]$, let $e_1^i, \dots, e_{n_i}^i$ be primitive (orthogonal) idempotents in kG_i such that $e_1^i kG_i, \dots, e_{n_i}^i kG_i$ form a complete set of representatives of the irreducible representations of G_i (recall that $\sharp G \in k^\times$). Let $e = \sum_{i,j} e_i^j \in S$. This is an idempotent and eSe is the algebra:

$$eSe = \prod_{i,j} ke_j^i.$$

Moreover, eMe is a graded eSe -sub-bimodule of M . Then, any choice of a basis of eMe yields a quiver Q' and an isomorphism of graded algebras $kQ' \simeq T_{eSe}(eMe)$. Besides, the inclusion $eMe \hookrightarrow M$ induces a homomorphism of graded algebras $kQ' \simeq T_{eSe}(eMe) \rightarrow T_S(M)$. It is injective and the associated restriction-of-scalars functor $\mathcal{D}(T_S(M)) \rightarrow \mathcal{D}(T_{eSe}(eMe)) \simeq \mathcal{D}(kQ')$ is an equivalence. The composition $kQ' \simeq T_{eSe}(eMe) \rightarrow T_S(M) \rightarrow kQ \rtimes G$ is the desired homomorphism of graded algebras whose associated restriction-of-scalars functor $\mathcal{D}(kQ \rtimes G) \rightarrow \mathcal{D}(kQ')$ is an equivalence. By construction, if kQ has the trivial grading, then so does kQ' .

4.3. Crossed-products of Ginzburg algebras by finite groups. Let $d \geq 3$ be an integer, Q a graded k -quiver and W a homogeneous of degree $3-d$ potential on Q . Let G be a finite group such that $\sharp G \in k^\times$ and assume that G acts on the set Q_0 and on the kQ_0 -bimodule kQ_1 (with the two actions compatible to each other), thus defining an action on kQ by graded automorphisms. This action extends to an action on $\mathcal{A}(Q, W)$ by graded automorphisms as follows:

- given $g \in G$ and an arrow $a: x \rightarrow y$, if ${}^g a = \sum_t \lambda_t a_t$ as linear combination of arrows in Q , then ${}^g(a^*)$ is defined

$$\text{as } {}^g a = \sum_t \lambda_t a_t^*,$$

- given $g \in G$ and $i \in Q_0$, then ${}^g c_i = c_{g.i}$.

Recall (4.2) that there exists a graded k -quiver Q and a morphism of graded algebras $kQ' \rightarrow kQ \rtimes G$ whose associated restriction-of-scalars functor $\mathcal{D}(kQ \rtimes G) \rightarrow \mathcal{D}(kQ')$ is an equivalence.

Proposition. *Keep the above setting. If $W \in \mathrm{HC}_{d-3}(kQ)^G$, then the action of G on $\mathcal{A}(Q, W)$ is an action by dg automorphisms. Moreover, there is a quasi-isomorphism of dg algebras*

$$\mathcal{A}(Q, W) \rtimes G \underset{\mathrm{qis}}{\simeq} \Pi_d(kQ \rtimes G, c_1)$$

where $c_1 \in \mathrm{HH}_{d-2}(kQ \rtimes G)$ is the image of $W \in \mathrm{HH}_{d-3}(kQ)$ under the composite map $\mathrm{HC}_{d-3}(kQ) \rightarrow \mathrm{HC}_{d-3}(kQ \rtimes G) \xrightarrow{B} \mathrm{HH}_{d-2}(kQ \rtimes G)$. In particular, $\mathcal{A}(Q, W) \rtimes G$ is Calabi-Yau of dimension d .

If, moreover, $\mathrm{car}(k) = 0$ or $d = 3$, then the image of W under the composite map $\mathrm{HC}_{d-3}(kQ) \rightarrow \mathrm{HC}_{d-3}(kQ \rtimes G) \simeq \mathrm{HC}_{d-3}(kQ')$ is a homogeneous of degree $d-3$ potential W' on Q' and there exists a homomorphism of dg algebras

$$\mathcal{A}(Q', W') \rightarrow \mathcal{A}(Q, W) \rtimes G$$

whose associated restriction-of-scalars functor $\mathcal{D}(\mathcal{A}(Q, W) \rtimes G) \rightarrow \mathcal{D}(\mathcal{A}(Q', W'))$ is an equivalence.

Proof. Since the restriction-of-scalars functor $\mathcal{D}(kQ \rtimes G) \rightarrow \mathcal{D}(kQ')$ is an equivalence, the canonical maps $\mathrm{HC}_{d-3}(kQ') \rightarrow \mathrm{HC}_{d-3}(kQ \rtimes G)$ and $\mathrm{HH}_{d-2}(kQ') \rightarrow \mathrm{HH}_{d-2}(kQ \rtimes G)$ are isomorphisms ([13]). These embed into the diagram:

$$\begin{array}{ccccccc} \mathrm{HC}_{d-3}(kQ)^G & \xrightarrow{c} & \mathrm{HC}_{d-3}(kQ) & \longrightarrow & \mathrm{HC}_{d-3}(kQ \rtimes G) & \xleftarrow{\sim} & \mathrm{HC}_{d-3}(kQ') \\ \downarrow & & \downarrow B & & \downarrow B & & \downarrow B \\ \mathrm{HH}_{d-2}(kQ)^G & \xrightarrow{c} & \mathrm{HH}_{d-2}(kQ) & \longrightarrow & \mathrm{HH}_{d-2}(kQ \rtimes G) & \xleftarrow{\sim} & \mathrm{HH}_{d-2}(kQ') \end{array}$$

where B stands for the Connes' map, the leftmost vertical arrow is induced by B and the horizontal arrows are the canonical ones. Denote by $c \in \mathrm{HH}_{d-2}(kQ)$, $c_1 \in \mathrm{HH}_{d-2}(kQ \rtimes G)$, $c' \in \mathrm{HH}_{d-2}(kQ')$ and $W' \in \mathrm{HC}_{d-3}(kQ')$ the corresponding images of $W \in \mathrm{HC}_{d-3}(kQ)$. Note that, if $\mathrm{car} k = 0$ or if $d = 3$, then W' is a homogeneous of degree $3-d$ potential on Q' .

Note that $\partial_{s_a}(p) = {}^g \left(\partial_a({}^{g^{-1}} p) \right)$, for $g \in G$, $p \in kQ$ and $a \in kQ_1$. Therefore, if $W \in \mathrm{HC}_{d-3}(kQ)^G$, then:

- the action of G on $\mathcal{A}(Q, W)$ is an action by dg automorphisms, and
- $c \in \mathrm{HH}_{d-2}(kQ)^G$, so that G acts on $\Pi_d(kQ, c)$ by dg automorphisms, following 3.4.

From now on, suppose that $W \in \mathrm{HC}_{d-3}(kQ)^G$. In [15, Thm. 6.3] is constructed a quasi-isomorphism of dg algebras $\mathcal{A}(Q, W) \simeq \Pi_d(kQ, c)$ which is easily checked to be G -equivariant for the corresponding actions of G on $\mathcal{A}(Q, W)$ and $\Pi_d(kQ, c)$, respectively. Whence a quasi-isomorphism of dg algebras

$$(11) \quad \mathcal{A}(Q, W) \rtimes G \underset{\mathrm{qis}}{\simeq} \Pi_d(kQ, c) \rtimes G.$$

In particular, $\mathcal{A}(Q, W) \rtimes G$ is Calabi-Yau of dimension d because $c \in \mathrm{HH}_{d-2}(kQ)^G$ and because of 3.4. Next, according to 3.4, there is a quasi-isomorphism of dg algebras

$$(12) \quad \Pi_d(kQ, c) \rtimes G \underset{\mathrm{qis}}{\simeq} \Pi_d(kQ \rtimes G, c_1).$$

Combining (11) and (12) shows that $\mathcal{A}(Q, W) \rtimes G$ and $\Pi_d(kQ \rtimes G, c_1)$ are quasi-isomorphic.

Now assume that $\mathrm{car}(k) = 0$ or $d = 3$. Then W' is a homogeneous of degree $3-d$ potential on W , as recalled earlier. Applying [15, Thm. 5.5] to $kQ' \rightarrow kQ \rtimes G$ gives a morphism of dg algebras

$$(13) \quad \Pi_d(kQ', c') \rightarrow \Pi_d(kQ \rtimes G, c_1)$$

whose associated restriction-of-scalars functor $\mathcal{D}(\Pi_d(kQ \rtimes G, c_1)) \rightarrow \mathcal{D}(\Pi_d(kQ', c'))$ is an equivalence. Finally, [15, Thm. 6.3] gives a quasi-isomorphism of dg algebras

$$(14) \quad \Pi_d(kQ', c') \underset{\mathrm{qis}}{\simeq} \mathcal{A}(Q', W') \text{ if } \mathrm{car}(k) = 0 \text{ or } d = 3.$$

Combining (11), (12), (13) and (14) shows that $\mathcal{A}(Q, W) \rtimes G$ and $\mathcal{A}(Q', W')$ have equivalent derived categories. \square

5. GEOMETRIC INTERPRETATION OF THE COMPATIBILITY CONDITION

Recall the following examples of Calabi-Yau algebras: The algebra $\mathbb{C}[x_1, \dots, x_n]$ of polynomials in n variables; the coordinate ring $\mathbb{C}[X]$ of a smooth affine variety with trivial canonical sheaf; and the Weyl algebra. The crossed-product of any of these algebras by a finite group G has the Van den Bergh duality ([1, Prop. 7.1]). Actually, it can be checked that the compatibility condition of Theorem 1 is satisfied (and, therefore, the crossed-product is Calabi-Yau, in each of these three examples) under the following geometric conditions:

- $G < \mathrm{SL}_n(\mathbb{C})$, in the case of the algebra $\mathbb{C}[x_1, \dots, x_n]$,

- the action of G on $\mathbb{C}[X]$ arises from an action of G on X such that the canonical sheaf is G -invariant,
- $G < \mathrm{Sp}_{2n}(\mathbb{C})$, in the case of the Weyl algebra.

Here are some details on this fact in the case of the Weyl algebra A_n (the three cases are very similar).

Denote by V the vector space $\mathbb{C}^{2n} = \bigoplus_{i=1}^n \mathbb{C} \cdot x_i \oplus \bigoplus_{i=1}^n \mathbb{C} \cdot \frac{\partial}{\partial x_i}$ and let G be a subgroup of $\mathrm{Sp}(V) = \mathrm{Sp}_{2n}(\mathbb{C})$. Then G acts on V and this action extends naturally to an action on A_n by algebra automorphisms. Recall that A_n admits the following projective resolution (that is, a cofibrant resolution) in $\mathcal{C}(A_n^e)$ ([19, 3.2]):

$$(15) \quad 0 \rightarrow \Lambda^{2n}V \otimes A_n^e \rightarrow \cdots \rightarrow \Lambda^dV \otimes A_n^e \rightarrow \cdots \rightarrow V \otimes A_n^e \rightarrow A_n^e \rightarrow A_n \rightarrow 0$$

where the differential $\Lambda^{d+1}V \otimes A_n^e \rightarrow \Lambda^dV \otimes A_n^e$ is defined by

$$(v_1 \wedge \cdots \wedge v_{d+1}) \otimes u \mapsto \sum_{i=1}^{d+1} (-1)^{d-i+1} (v_1 \wedge \cdots \wedge \widehat{v}_i \wedge \cdots \wedge v_{d+1}) \otimes (v_i \otimes 1 - 1 \otimes v_i) \cdot u.$$

This differential is G -equivariant for the diagonal action of G on $\Lambda^\bullet V \otimes A_n^e$. As earlier, let $\Delta = \bigoplus_{g \in G} A_n g \otimes g^{-1} A_n$.

Therefore, (15) is a cofibrant resolution of A_n in $\mathcal{C}(\Delta)$ (2.2, (e)). This implies that $\mathrm{RHom}_{A_n^e}(A_n, A_n \otimes A_n) = \mathrm{Hom}_{A_n^e}(\Lambda^\bullet V \otimes A_n^e, A_n \otimes A_n)$ in $\mathcal{C}(\Delta^{op})$, which is isomorphic in $\mathcal{C}(\Delta^{op})$ to:

$$(16) \quad 0 \rightarrow A_n^e \otimes \Lambda^0 V^* \rightarrow \cdots \rightarrow A_n^e \otimes \Lambda^d V^* \rightarrow A_n^e \otimes \Lambda^{d+1} V^* \rightarrow \cdots \rightarrow A_n^e \otimes \Lambda^{2n} V^* \rightarrow 0$$

where $A_n^e \otimes \Lambda^d V^*$ is in degree d ; the action of G on $A_n^e \otimes \Lambda^d V^*$ is diagonal (and given by the adjoint action of G on V^*); and the differential is defined by:

$$\begin{aligned} A_n^e \otimes \Lambda^d V^* &\longrightarrow A_n^e \otimes \Lambda^{d+1} V^* \\ u \otimes e_{i_1}^* \wedge \cdots \wedge e_{i_d}^* &\longmapsto \sum_{j=1}^{2n} u \cdot (e_j \otimes 1 - 1 \otimes e_j) \otimes e_{i_1}^* \wedge \cdots \wedge e_{i_d}^* \wedge e_j^* \quad , \end{aligned}$$

where (e_1^*, \dots, e_{2n}^*) is the basis of V^* dual to $(e_1, \dots, e_{2n}) = (p_1, \dots, p_n, q_1, \dots, q_n)$. Since $A_n^e \otimes \Lambda^d V^*$ is a projective left A_n^e -module for every d , the complex (16) is cofibrant in $\mathcal{C}(\Delta^{op})$ and is therefore equal to θ_{A_n} . Consider the morphism of complexes $\theta_{A_n}[2n] \rightarrow A_n$ given by the multiplication $A_n^e \otimes \Lambda^{2n} V^* = A_n^e \rightarrow A_n$ in degree 0 (and the zero map in non zero degrees). Since A_n is Calabi-Yau of dimension $2n$, the cohomology of $\theta_{A_n}[2n]$ vanishes in non-zero degrees. Besides, $\theta_{A_n}[2n] \rightarrow A_n$ is clearly a quasi-isomorphism in degree 0. Finally, $\theta_{A_n}[2n] \rightarrow A_n$ is obviously a morphism in $\mathcal{C}(\Delta^{op})$, that is, it is G -equivariant. This shows that the compatibility condition in Theorem 1 holds true if $G < \mathrm{Sp}_{2n}(\mathbb{C})$ acts on A_n . As a consequence, the crossed-product $A_n \rtimes G$ is Calabi-Yau of dimension $2n$.

ACKNOWLEDGEMENTS

The author acknowledges Jacques Alev for stimulating comments and questions

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