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CROSSED PRODUCTS OF CALABI-YAU ALGEBRAS BY FINITE GROUPS

PATRICK LE MEUR

Abstract. Let a finite group G act on a differential graded algebra A. This article presents necessary conditions and sufficient conditions for the skew group algebra $A * G$ to be Calabi-Yau. In particular, when A is the Ginzburg dg algebra of a quiver with an invariant potential, then $A * G$ is Calabi-Yau and Morita equivalent to a Ginzburg dg algebra. Some applications of these results are derived to compare the generalised cluster categories of A and $A * G$ when they are defined and to compare the higher Auslander-Reiten theories of A and $A * G$ when A is a finite dimensional algebra.

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Introduction

Calabi-Yau algebras occur in many parts of mathematics and play a special role in several areas related to representation theory. In the non commutative algebraic geometry initiated by Artin and Schelter ([4]), they appear naturally among the regular algebras used as non commutative analogues of polynomial algebras. They also serve to define generalised cluster categories ([2]), which are used to categorify the cluster algebras of Fomin and Zelevinsky. When a cluster algebra is given by a marked surface, quivers with potentials are associated to the triangulations of the surface ([5, 26]); recall that quivers with potentials are the base ingredients in the definition of Ginzburg differential graded (dg) algebras, which are Calabi-Yau. Finally, in the higher Auslander-Reiten theory developed by Iyama et al., the higher representation infinite algebras ([19]) are defined using higher derived preprojective algebras, which are Calabi-Yau completions of finite dimensional algebras of finite global dimension ([24]).

Recall that a dg algebra A over a field $\Bbbk$ is called d-Calabi-Yau, where d is an integer, when

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- A is homologically smooth, that is $A \in \mathrm{per}(A^e)$, and
- $\Sigma^d \mathrm{RHom}_{A^e}(A, A^e) \simeq A$ in the derived category $\mathcal{D}(A^e)$ of A^e.

Here, Σ denotes the suspension for all triangulated categories, A^e denotes the enveloping dg algebra $A \otimes_{\Bbbk} A^{\mathrm{op}}$ and $\mathrm{per}(A^e)$ denotes the perfect derived category of A^e.

This article investigates the Calabi-Yau duality of skew group algebras $A * G$ of dg algebras A under the action of a finite group G with a particular interest on the dg algebras which are involved in the above mentioned theories. This is motivated by several developments some of which being recent and others being older.

On one hand, some of the early developments of non commutative algebraic geometry focused to the regularity properties of invariant subrings under actions of finite groups. And this has led to the study of skew group algebras from the viewpoint of regularity. Recently, many articles have studied when the skew group algebra of an ordinary (not dg) algebra is Calabi-Yau and fairly complete answers are known when the original algebra is Koszul ([36]) connected graded ([35]), or is the universal enveloping algebra of a finite dimensional Lie algebra ([16]).

On the other hand, the recent works [3] and [33] use skew group algebras of Ginzburg dg algebras and the associated generalised cluster categories for marked surfaces. While the former uses actions of $\mathbb{Z}/2\mathbb{Z}$ to relate punctured surfaces to unpunctured surfaces, the latter uses free actions of finite groups for orbifolds. In both cases, the considered skew group algebra happens to be a Ginzburg dg algebra up to a Morita equivalence. Recall that, in the context of a generalised cluster category $\mathcal{C}$, important information is provided by the cluster tilting objects and the cluster tilting subcategories. A functorially finite subcategory $\mathcal{T} \subseteq \mathcal{C}$ is called *cluster tilting* if

$$\mathcal{T} \quad = \quad \{X \in \mathcal{C} \mid (\forall M \in \mathcal{T})\ \mathcal{C}(X, \Sigma M) = 0\},$$

and an object $T \in \mathcal{C}$ is called cluster tilting if its closure $\mathrm{add}(T)$ under direct sums and direct summands is cluster tilting.

The present article therefore aims at giving general answers to the following questions for a given dg algebra A acted upon by a finite group G by dg automorphisms.[1]

- Under which conditions (necessary and/or sufficient) is $A * G$ Calabi-Yau?
- To what extent are constructions of Calabi-Yau algebras such as Ginzburg dg algebras or deformed Calabi-Yau completions compatible with taking skew group algebras?
- Assuming that A and $A * G$ are Calabi-Yau and such that their generalised cluster categories are defined, what kind of relation is there between these categories, and what kind of relation is there between their respective cluster tilting objects?
- When A is a finite dimensional algebra, is there a connection between A being d-representation infinite and $A * G$ being n-representation infinite?

Note that the dg algebras considered in this article do not satisfy particular conditions such as being Koszul, connected graded or Artin-Schelter.

1. Main results and structure of the article

The main results assume the following setting

Setting 1.1. Let d be an integer. Let $\Bbbk$ be a field. Let A be a dg algebra over $\Bbbk$. Let G be a finite group acting on A by dg automorphisms $(g, a) \mapsto {}^g a$. Denote by Λ the skew group dg algebra $A * G$.

Recall that Λ is defined as follows. As a complex it is equal to $A \otimes \Bbbk G$, where $\Bbbk G$ is a complex concentrated in degree 0, and the product is such that $(a * g) \cdot (b * h) = a\, {}^g b * gh$ for all $a, b \in A$ and $g, h \in G$, where $a * g$ stands for $a \otimes g$.

[1] In the original version of this article, [29], only the first two of these questions are treated.

The investigation of the Calabi-Yau duality for Λ makes use of an auxiliary dg subalgebra of Λ^e denoted by Δ,

$$\Delta = \bigoplus_{g\in G}(A * g) \otimes (A * g^{-1})\,. \tag{1.1.1}$$

It features the following properties.

- A is a dg Δ-module and $A \otimes^{\mathrm{L}}_{\Delta} \Lambda^e \simeq \Lambda$ in $\mathcal{D}(\Lambda^e)$.
- The functor $\mathrm{Hom}_{A^e}(-, A^e)\colon \mathrm{Mod}(A^e) \to \mathrm{Mod}(A^e)$ induces a functor $\mathrm{Mod}(\Delta) \to \mathrm{Mod}(\Delta)$ also denoted by $\mathrm{Hom}_{A^e}(-, A^e)$ and whose total derived functor $\mathrm{RHom}_{A^e}(-, A^e)$ is such that $\mathrm{RHom}_{A^e}(-, A^e) \otimes^{\mathrm{L}}_{\Delta} \Lambda^e \simeq \mathrm{RHom}_{\Lambda^e}(- \otimes^{\mathrm{L}}_{\Delta} \Lambda^e, \Lambda^e)$.

These are used to prove the first main result of this article.

Theorem 1.2 (Theorem 3.4.1)**.** *Assume setting 1.1. Assume that* $\mathrm{char}(\Bbbk) \nmid \mathrm{Card}(G)$*, that A is homologically smooth and that* $\Sigma^d \mathrm{RHom}_{A^e}(A, A^e) \simeq A$ *in* $\mathcal{D}(\Delta)$*, then A and Λ are d-Calabi-Yau.*

See Proposition 3.4.2 for a necessary condition for Λ to be d-Calabi-Yau expressed in terms of $\mathrm{RHom}_{A^e}(A, A^e)$ when the Hochschild cohomology group $\mathrm{HH}^0(A)$ is a local algebra.

Several constructions of Calabi-Yau dg algebras behave nicely with respect to taking skew group algebras. This article concentrates on two constructions, namely, the Calabi-Yau completions (and their deformations) of [24] and the Ginzburg dg algebras of [15, 24]. When A is homologically smooth, its d-Calabi-Yau completion $\mathbf{\Pi}_d(A)$ is defined as $T_A(\Sigma^{d-1}\Theta_A)$ where Θ_A is any cofibrant replacement of $\mathrm{RHom}_{A^e}(A, A^e)$ also called an *inverse dualising complex* of A. It is d-Calabi-Yau. Any Hochschild homology class $c \in \mathrm{HH}_{d-2}(A)$ yields a deformation of the differential of $T_A(\Sigma^{d-1}\Theta_A)$, thus giving rise to the deformed Calabi-Yau completion $\mathbf{\Pi}_d(A, c)$. If c lies in the image of the natural mapping $\mathrm{HN}_{d-2}(A) \to \mathrm{HH}_{n-2}(A)$, where HN is the negative cyclic homology, then $\mathbf{\Pi}_d(A, c)$ is d-Calabi-Yau ([37]). Both constructions depend on the choice of Θ_A and on the choice of a representative of c, but they are independent of these choices up to a quasi-isomorphism of dg algebras. Now, here is how these constructions are compatible with taking skew group algebras.

Theorem 1.3 (Theorem 3.5.4)**.** *Assume setting 1.1. Assume that A is homologically smooth and that* $\mathrm{char}(\Bbbk) \nmid \mathrm{Card}(G)$*.*

(1) $A * G$ is homologically smooth and, up to appropriate choices of inverse dualising complexes of A and $A*G$, the group G acts on $\mathbf{\Pi}_d(A)$ by dg automorphisms and $\mathbf{\Pi}_d(A)*G \simeq \mathbf{\Pi}_d(A * G)$ as dg algebras. In particular, $\mathbf{\Pi}_d(A) * G$ is d-Calabi-Yau.

(2) Let $c \in \mathrm{HH}_{d-2}(A)^G$. Let $\overline{c}$ be the image of c under the mapping $\mathrm{HH}_{d-2}(A) \to \mathrm{HH}_{d-2}(A*G)$ induced by $A \to A * G$, $a \mapsto a * e$. Up to appropriate choices as in (1) and up to appropriate choices of representatives of c and $\overline{c}$, the group G acts on $\mathbf{\Pi}_d(A, c)$ by dg automorphisms and $\mathbf{\Pi}_d(A, c) * G \simeq \mathbf{\Pi}_d(A * G, \overline{c})$ as dg algebras. In particular, if c lifts to $\mathrm{HN}_{d-2}(A)$ (so that $\mathbf{\Pi}_d(A, c)$ is d-Calabi-Yau), then $\mathbf{\Pi}_d(A, c) * G$ is d-Calabi-Yau.

Note that, if A is a finite dimensional algebra of global dimension d, then $H^0(\mathbf{\Pi}_{d+1}(A))$ is isomorphic to the $(d+1)$-preprojective algebra of A in the sense of [21, Definition 2.11]. Hence, part (1) of Theorem 1.3 entails that, in this case, G acts on the $(d+1)$-preprojective algebra of A and that the resulting skew group algebra is isomorphic to the $(d+1)$-preprojective algebra of $A * G$ (see Proposition 6.2.1). In the more particular case where $d = 1$, these are classical preprojective algebras of hereditary algebras and this result is proved in [9].

The Ginzburg dg algebras are particular instances of deformed Calabi-Yau completions. They are Calabi-Yau. Here, they are considered using the following setting.

Setting 1.4. Let Q, G, W be as follows.

- Q is a finite graded $\Bbbk$-quiver.
- G is a finite group with order not divisible by $\mathrm{char}(\Bbbk)$ and acting on the path algebra $\Bbbk Q$ by degree preserving automorphisms in such a way that both the set of (idempotents associated with) vertices and the vector space generated by the arrows of Q are stabilised by the action.
- W is a homogeneous potential of degree $d-3$ on Q.

The Ginzburg dg algebra $\mathcal{A}(Q,W)$ is quasi-isomorphic to the deformed Calabi-Yau completion $\mathbf{\Pi}_d(\Bbbk Q, c)$, where the graded path algebra $\Bbbk Q$ has zero differential and c is the image of W under Connes' boundary $\mathrm{HH}_{d-3}(\Bbbk Q) \to \mathrm{HH}_{d-2}(\Bbbk Q)$. Theorem 1.3 applies as follows.

Corollary 1.5 (Corollary 4.4.3)**.** *Assume setting 1.4. Assume that W is G-invariant up to cyclic permutation.*

(1) The action of G on $\Bbbk Q$ extends to an action of G on $\mathcal{A}(Q,W)$ by dg automorphisms and $\mathcal{A}(Q,W) * G$ is d-Calabi-Yau.

(2) For all graded $\Bbbk$-quivers Q' and for all (non unital) graded algebra homomorphisms $\Bbbk Q' \to \Bbbk Q * G$ whose restriction-of-scalars functor is an equivalence from $\mathrm{Mod}(\Bbbk Q * G)$ to $\mathrm{Mod}(\Bbbk Q')$ (see [9]), there exists a homogeneous of degree $d-3$ potential W' on Q' such that $\Bbbk Q' \to \Bbbk Q * G$ extends to a (non unital) dg algebra homomorphism

$$\mathcal{A}(Q',W') \to \mathcal{A}(Q,W) * G$$

whose restriction-of-scalars functor induces an equivalence

$$\mathcal{D}(\mathcal{A}(Q,W) * G) \xrightarrow{\sim} \mathcal{D}(\mathcal{A}(Q',W'))\,.$$

More precisely, $\Bbbk Q' \to \Bbbk Q * G$ induces an isomorphism $\mathrm{HH}_{d-3}(\Bbbk Q') \to \mathrm{HH}_{d-3}(\Bbbk Q * G)$. Using it as an identification, W' may be taken equal to the image of W under the mapping $\mathrm{HH}_{d-3}(\Bbbk Q) \to \mathrm{HH}_{d-3}(\Bbbk Q * G)$ induced by the natural embedding $\Bbbk Q \to \Bbbk Q * G$, $a \mapsto a * e$. When $d=3$, Corollary 1.5 is proved in [3, Theorem 2.6, Corollary 2.7] assuming that $G = \mathbb{Z}/2\mathbb{Z}$ and the action of G on $\Bbbk Q$ is induced by an action on Q by quiver automorphisms.

As mentioned in the introduction, 3-Calabi-Yau algebras are base ingredients in the construction of generalised cluster categories. Theorem 1.2 and Corollary 1.5 raise the question of the comparison of generalised cluster categories of A and $A * G$ when they are defined. The setting in which this article addresses this question is the following.

Setting 1.6. Let $\Bbbk$ be a field. Let G be a finite group. Let A be dg $\Bbbk$-algebra acted upon by G by dg automorphisms. Assume that

- A is concentrated in non positive degrees and $H^0(A)$ is finite dimensional,
- $\mathrm{char}(\Bbbk)$ does not divide $\mathrm{Card}(G)$,
- and both A and $A * G$ are 3-Calabi-Yau.

This ensures the existence of the generalised cluster categories $\mathcal{C}_A$ and $\mathcal{C}_{A*G}$ of A and $A * G$, respectively. In particular, they have distinguished cluster tilting objects $\overline{A}$ and $\overline{A * G}$, respectively. In this setting, the group G acts (strictly) on $\mathcal{C}_A$ by (strict) automorphisms of triangulated categories. Also, the extension-of-scalars functor $\mathcal{D}(A) \to \mathcal{D}(A * G)$ and the restriction-of-scalars functor $\mathcal{D}(A * G) \to \mathcal{D}(A)$ induce triangle functors

$$F_\lambda \colon \mathcal{C}_A \rightleftarrows \mathcal{C}_{A*G} \colon F_\cdot$$

which form a biadjoint pair with split units (see Proposition 5.1.2). These functors are the basement of the comparison between the cluster tilting objects of $\mathcal{C}_A$ and those of $\mathcal{C}_{A*G}$.

Theorem 1.7 (Corollary 5.2.2 and Proposition 5.2.3)**.** *Assume setting 1.6.*

(1) *The assignments* $\mathcal{T} \mapsto \mathrm{add}(F_\lambda \mathcal{T})$ *and* $\mathcal{T}' \mapsto \mathrm{add}(F.\mathcal{T}')$ *induce mutually inverse bijections between the set of* G*-stable cluster tilting subcategories of* $\mathcal{C}_A$ *and the set of cluster tilting subcategories of* $\mathcal{C}_{A*G}$ *which are stable under* $F_\lambda F.$.
(2) *For all* G*-stable cluster tilting subcategories* $\mathcal{T}$ *of* $\mathcal{C}_A$*, there exists a cluster tilting object* $T \in \mathcal{C}_A$ *such that* $\mathcal{T} = \mathrm{add}(T)$ *and* $T^g = T$ *for all* $g \in G$*. Then, (*$\mathrm{add}(F_\lambda \mathcal{T}) = \mathrm{add}(F_\lambda T)$*), the group* G *acts on* $\mathrm{End}(T)$ *by algebra automorphisms and* $\mathrm{End}(T) * G$ *is Morita equivalent to* $\mathrm{End}(F_\lambda T)$.

Part (1) of Theorem 1.7 is similar to some of the results of [10, Section 3] where a correspondence is established between the G-stable cluster tilting objects of a given Hom-finite Krull-Schmidt and stably 2-Calabi-Yau Frobenius category endowed with a suitable action of G and the $\Bbbk[G]$-stable cluster tilting objects of the associated G-equivariant category.

With additional assumptions, part (1) of Theorem 1.7 has a more precise formulation. Indeed, assume that the action of G on the isomorphism classes of indecomposable projective $H^0(A)$-modules is free. Then, Proposition 5.3.5 asserts that, for all cluster tilting objects $T \in \mathcal{C}_{A*G}$ in the mutation class of $\overline{A * G}$, every indecomposable direct summand of T lies in the essential image of F_λ; moreover, T is isomorphic to $F_\lambda \widetilde{T}$ for some cluster tilting object $\widetilde{T} \in \mathcal{C}_A$ such that $\widetilde{T}^g = T$ for all $g \in G$. When A is given by a quiver with potential like in Corollary 1.5, the above condition relative to $H^0(A)$ is equivalent to the action of G on the set of vertices of Q being free.

These results have particular implications in the acyclic case. A Hom-finite, Krull-Schmidt and 2-Calabi-Yau category is called *acyclic* when it is equivalent as a triangulated category to the cluster category, in the sense of [6], of a hereditary algebra. Following [25], this is the case if and only if the category is algebraic and contains a cluster tilting object whose endomorphism algebra is hereditary.

Corollary 1.8 (Corollary 5.4.2)**.** *Assume setting 1.6.*

(1) *The following assertions are equivalent.*
  (i) $\mathcal{C}_A$ *is equivalent to the cluster category of a Dynkin quiver.*
  (ii) $\mathcal{C}_{A*G}$ *is equivalent to the cluster category of a Dynkin quiver.*
(2) *If* $\mathcal{C}_A$ *is acyclic and has infinitely many isomorphism classes of indecomposable objects, then*
  (a) *there exists a cluster tilting object* $T \in \mathcal{C}_A$ *such that* $\mathrm{End}(T)$ *is hereditary and* $T^g = T$ *for all* $g \in G$ *and*
  (b) *for any such* T*, there exists a cluster tilting object* $T' \in \mathcal{C}_{A*G}$ *such that the algebras* $\mathrm{End}(T) * G$ *and* $\mathrm{End}(T')$ *are Morita equivalent.*
  Consequently, $\mathcal{C}_{A*G}$ *is acyclic.*
(3) *Assume that* G *acts freely on the set of isomorphism classes of indecomposable projective* $H^0(A)$*-modules. If* $\mathcal{C}_{A*G}$ *is acyclic, then so is* $\mathcal{C}_A$.

Note that part (1) is proved in [33, Proposition 7.12] when A is given by a quiver with potential (Q, W) and the action of G on A is induced by a free action on Q by quiver automorphisms in such a way that W is G-invariant up to cyclic permutation.

The last main result of this article is an application of Theorem 1.3 to higher Auslander-Reiten theory. Indeed, a finite dimensional algebra of global dimension d is d-representation infinite, in the sense of [19], if and only if the cohomology of its derived $(d+1)$-preprojective algebra is concentrated in degree 0. Since the derived $(d+1)$-preprojective algebra is equal to the $(d+1)$-Calabi-Yau completion, Theorem 1.3 applies to prove the following.

Corollary 1.9 (Propositions 6.1.2 and 6.2.1)**.** *Let* A *be a finite dimensional algebra of global dimension* d*. Let* G *be a finite group acting on* A *by algebra automorphisms. Assume that* $\mathrm{char}(\Bbbk)$ *does not divide* $\mathrm{Card}(G)$.

(1) A is d-representation finite if and only if so is $A * G$.
(2) A is d-representation infinite if and only if so is $A * G$.

The article is organised as follows. Basic material is recalled in section 2. Section 3 develops the needed properties of the dg algebra Δ and proves Theorems 1.2 and 1.3. Section 4 is devoted to the application to Ginzburg dg algebras, it proves Corollary 1.5. The comparison between the generalised cluster categories of A and $A * G$, when these are defined, is made in section 4; this includes proofs of Theorem 1.7 and Corollary 1.8. Finally, Corollary 1.9 on higher representation (in)finite algebras is proved in section 6.

2. Definitions and notation

Throughout the text, $\Bbbk$ denotes a field. The tensor product over $\Bbbk$ is denoted by $\otimes$. Let A be a dg algebra over $\Bbbk$. All dg algebras are over $\Bbbk$ and with differential of degree 1. Denote by A^{op} the *opposite dg algebra* of A. Denote by A^e the *enveloping dg algebra* $A \otimes A^{\mathrm{op}}$ of A, with product given by $(a \otimes b) \cdot (c \otimes d) = (-1)^{\deg(b)\cdot(\deg(c)+\deg(d))} ac \otimes db$ for all (homogeneous) $a, b, c, d \in A$.

The dg category of dg (right) A-modules is denoted by $\mathrm{Mod}(A)$. See [23] for a background. In all places, the suspension functor is denoted by Σ. Given dg A-modules M, N the complex of morphisms $M \to N$ in $\mathrm{Mod}(A)$ is denoted by $\mathrm{Hom}_A(M, N)$; in particular $Z^0\mathrm{Hom}_A(M, N)$ is the vector space of morphisms of dg A-modules, that is, with degree 0 and commuting with the differentials, and $H^0\mathrm{Hom}_A(M, N)$ is the vector space of morphisms of dg A-modules up to homotopy. The *derived category* of A is denoted by $\mathcal{D}(A)$, this is the localisation of the graded category $H^0\mathrm{Mod}(A)$ at quasi-isomorphisms. It contains two full subcategories used here,

- the *perfect* derived category denoted by $\mathrm{per}(A)$, this is the thick triangulated subcategory of $\mathcal{D}(A)$ generated by A and
- the full subcategory $\mathcal{D}_{\mathrm{fd}}(A)$ of objects $X \in \mathcal{D}(A)$ such that $\dim(\oplus_{i\in\mathbb{Z}} H^i(X)) < \infty$.

In particular, A is a dg A^e-module for the following structure map

$$\begin{array}{rcl} A \otimes A^e & \to & A^e \\ a \otimes (b \otimes c) & \mapsto & (-1)^{\deg(c)\cdot(\deg(b)+\deg(a))} cba \end{array}$$

and A is called *homologically smooth* if $A \in \mathrm{per}(A^e)$.

2.1. **Calabi-Yau algebras.** For all $X \in \mathrm{Mod}(A)$, the complex $\mathrm{Hom}_{A^e}(X, A^e)$ is viewed as a dg A^e-module such that $(f \cdot (a \otimes b))(x) = (-1)^{\deg(f)\cdot(\deg(a)+\deg(b))+\deg(b)\cdot\deg(a)}(b \otimes a) \cdot f(x)$ for all $f \in \mathrm{Hom}_{A^e}(X, A^e)$, $x \in X$ and $a, b \in A$. When A is homologically smooth, $\mathrm{RHom}_{A^e}(-, A^e)$ induces a duality $\mathrm{RHom}_{A^e}(-, A^e)\colon \mathrm{per}(A^e)^{\mathrm{op}} \to \mathrm{per}(A^e)$. Let d be an integer. Following [15, 3.2.3], the dg algebra A is called d-Calabi-Yau if the two following assertions hold true:

(1) A is homologically smooth,
(2) $\Sigma^d\mathrm{RHom}_{A^e}(A, A^e) \simeq A$ in $\mathcal{D}(A^e)$.

2.2. **Skew group algebras.** Let G be a finite group and A be a dg algebra. In all places, e denotes the neutral element of G. By an *action of G on A by dg automorphism* is meant a morphism of complexes $\Bbbk G \otimes A \to A$, where $\Bbbk G$ is a complex in degree 0 and with zero differential, such that, denoting by ${}^g a$ the image of $g \otimes a$ for all $g \in G$ and $a \in A$, then

- ${}^g(ab) = {}^g a\, {}^g b$ for all $a, b \in A$ and $g \in G$ and
- ${}^e a = a$ and ${}^{gh} a = {}^g({}^h a)$ for all $a \in A$ and $g, h \in G$.

This defines the *skew group* dg algebra $A * G$ with underlying complex of vector spaces equal to $A \otimes \Bbbk G$, where any tensor $a \otimes g$ with $a \in A$ and $g \in G$ is denoted by $a * g$ and with product such that $(a * g) \cdot (b * h) = a\, {}^g b * gh$ for all $a, b \in A$ and $g, h \in G$.

For all $g \in G$, denote by $A * g$ the dg A^e-module $A * g = \{a * g \mid a \in A\}$. It is isomorphic to A as a dg A-module. Besides, $A * G = \oplus_{g\in G} A * g$ as a dg A^e-module.

2.3. **Adjunctions.** Let G be a finite group acting on a dg algebra A by dg automorphism. For all $g \in G$ denote the restriction of scalars along the dg automorphism $A \to A\,, a \mapsto {}^g a$ as follows

$$\begin{array}{rcl} \mathrm{Mod}(A) & \to & \mathrm{Mod}(A) \\ M & \mapsto & M^g \\ M \xrightarrow{f} M' & \mapsto & M^g \xrightarrow{f^g} M'^g\,. \end{array} \tag{2.3.1}$$

This defines a *strict* action of G on $\mathrm{Mod}(A)$ on the right by automorphisms of dg categories. It preserves both projective and injective model structures of $\mathrm{Mod}(A)$, and hence defines a strict action of G on $\mathcal{D}(A)$ by strict automorphisms of triangulated category.

Since $A * g \simeq A$ as dg A-modules for all $g \in G$, the extension-of-scalars functor $- \otimes_A (A * G)\colon \mathrm{Mod}(A) \to \mathrm{Mod}(A * G)$ preserves quasi-isomorphisms and so does the restriction-of-scalars functor $\mathrm{res}\colon \mathrm{Mod}(A * G) \to \mathrm{Mod}(A)$. Consider the adjoint pair of triangle functors

$$- \otimes^{\mathrm{L}}_A (A * G)\colon \mathcal{D}(A) \rightleftarrows \mathcal{D}(A * G)\colon \mathrm{Rres}$$

Lemma 2.3.1. *The functors* $- \otimes^{\mathrm{L}}_A A * G$ *and* Rres *have the following properties.*

(1) The pairs $(\mathrm{Rres}\,, - \otimes^{\mathrm{L}}_A (A * G))$ *and* $(- \otimes^{\mathrm{L}}_A (A * G), \mathrm{Rres}\,)$ *are adjoint.*

(2) The unit of the adjoint pair $(- \otimes^{\mathrm{L}}_A (A * G), \mathrm{Rres}\,)$ *splits.*

(3) If $\mathrm{char}(\Bbbk) \nmid \mathrm{Card}(G)$*, then the unit morphism* $N \to (\mathrm{Rres}\, N) \otimes^{\mathrm{L}}_A (A * G)$ *of the adjoint pair* $(\mathrm{Rres}\,, - \otimes^{\mathrm{L}}_A (A * G))$ *splits functorially for all* $N \in \mathcal{D}(A * G)$*.*

(4) For all $M \in \mathcal{D}(A)$ *there is a functorial isomorphism in* $\mathcal{D}(A)$

$$\mathrm{Rres}\,(M \otimes^{\mathrm{L}}_A (A * G)) \simeq \bigoplus_{g \in G} M^g\,. \tag{2.3.2}$$

(5) For all $M, M' \in \mathcal{D}(A)$ *there is a bifunctorial isomorphism*

$$\bigoplus_{g \in G} \mathcal{D}(A)(M, M'^g) \xrightarrow{\sim} \mathcal{D}(A * G)(M \otimes^{\mathrm{L}}_A (A * G), M' \otimes^{\mathrm{L}}_A (A * G))\,.$$

(6) There exists an isomorphism of functors $\lambda_g\colon -^g \otimes^{\mathrm{L}}_A (A * G) \xrightarrow{\sim} - \otimes^{\mathrm{L}}_A (A * G)$*, for all* $g \in G$*, such that the following equalities hold true, for all* $M \in \mathcal{D}(A)$ *and* $g, h \in G$*,*

$$\left\{ \begin{array}{rcl} (\lambda_e)_M & = & \mathbb{1}_M \\ (\lambda_g)_M \circ (\lambda_h)_{M^g} & = & (\lambda_{gh})_M\,. \end{array} \right.$$

(7) $\mathrm{Rres} \simeq (\mathrm{Rres})^g$ *for all* $g \in G$*.*

Proof. This is known for finitely generated modules over Artin algebras. Here are some details.

(1) See [34, Theorem 1.4]. Given $X \in \mathrm{Mod}(A * G)$ and $Y \in \mathrm{Mod}(A)$, the following is an isomorphism of complexes

$$\Phi\colon \mathrm{Hom}_A(\mathrm{res}\, X, Y) \to \mathrm{Hom}_{A*G}(X, Y \otimes_A (A * G))\,,$$

where $\Phi(f)$ is defined as follows, for all $f \in \mathrm{Hom}_A(\mathrm{res}\, X, Y)$,

$$\begin{array}{rrcl} \Phi(f)\colon & X & \to & Y \otimes_A (A * G) \\ & x & \mapsto & \sum_{g \in G} f(x \cdot (1 * g^{-1})) \otimes (1 * g)\,. \end{array}$$

The inverse of Φ maps any $f' \in \mathrm{Hom}_{A*G}(X, Y \otimes_A (A * G))$ to its composition with the mapping $Y \otimes_A A * G \to Y$ induced by the counit $\Bbbk G \to \Bbbk$. Hence, the pair $(\mathrm{res}, - \otimes_A (A * G))$ of functors between $\mathrm{Mod}(A)$ and $\mathrm{Mod}(A * G)$ is adjoint. This is a Quillen adjunction, which yields the adjunction at the level of derived categories (note that both res and $- \otimes_A (A * G)$ preserve quasi-isomorphisms).

(2) See [34, Theorem 1.4]. The unit morphism $M \to \operatorname{Rres}(M \otimes^{\mathrm{L}}_A (A * G))$ is induced by

$$\begin{array}{rcl} M & \to & M \otimes_A (A * G) \\ m & \mapsto & m \otimes (1 * e)\,. \end{array}$$

A splitting in $\mathcal{D}(A)$ is induced by the counit $\Bbbk G \to \Bbbk$.

(3) The unit morphism $N \to (\operatorname{Rres} N) \otimes^{\mathrm{L}}_A (A * G)$ is induced by

$$\begin{array}{rcl} N & \to & N \otimes_A (A * G) \\ n & \mapsto & \sum_{g \in G} (n \cdot (1 * g^{-1})) \otimes (1 * g)\,. \end{array}$$

A splitting in $\mathcal{D}(A * G)$ is induced by $n \otimes (a * g) \mapsto \frac{1}{\operatorname{Card}(G)} \cdot n \cdot (a * g)$.

(4) See [34, Proposition 1.8]. The isomorphism is induced by

$$\begin{array}{rcl} \bigoplus_{g \in G} M^g & \to & M \otimes_A (A * G) \\ \sum_{g \in G} m_g & \mapsto & \sum_{g \in G} m_g \otimes (1 * g)\,. \end{array}$$

(5) follows from (1) and (4).

(6) There exists an isomorphism of dg $A * G$-modules, for all $M \in \operatorname{Mod}(A)$ and $g \in G$,

$$\begin{array}{rrcl} \ell_{g,M}\colon & M^g \otimes_A (A * G) & \to & M \otimes_A (A * G) \\ & m \otimes (a * h) & \mapsto & m \otimes ({}^g a \otimes h)\,. \end{array}$$

And $\ell_{e,M} = \mathbb{1}_M$ and $\ell_{h,M^g} \circ \ell_{g,M} = \ell_{gh,M}$ for all $M \in \operatorname{Mod}(A)$ and $g, h \in G$. This proves (6).

(7) $N \to N^g$, $n \mapsto n \cdot (1 * g^{-1})$ is an isomorphism of dg A-modules, for all $N \in \operatorname{Mod}(A * G)$. □

3. The main theorems

This section assumes setting 1.1. This setting is restated below for convenience.

Setting 3.1. Let d be an integer. Let $\Bbbk$ be a field. Let A be a dg algebra over $\Bbbk$. Let G be a finite group acting on A by dg automorphisms $(g, a) \mapsto {}^g a$. Denote by Λ the dg algebra $A * G$.

The section aims at proving Theorem 1.2 which gives sufficient conditions for Λ to be d-Calabi-Yau and Theorem 1.3 which explains when G acts on the (deformed) Calabi-Yau-completions of A so that the resulting skew group algebras are Calabi-Yau.

For this purpose, an auxiliary dg subalgebra Δ of Λ^e such that $A \otimes_\Delta \Lambda^e \simeq \Lambda$ is introduced in subsection 3.1. This allows to relate A being homologically smooth to Λ being homologically smooth in subsection 3.2. The proofs of these results are based on a relationship between the inverse dualising complexes of A and Λ established in subsection 3.3. Theorem 1.2 is proved in subsection 3.4 and Theorem 1.3 is proved in subsection 3.5.

3.1. The dg algebra Δ and its dg modules.

Recall that Δ is defined in (1.1.1). The group G acts on A^e by dg automorphisms as follows

$$\begin{array}{rcl} G \times A^e & \to & A^e \\ (g, a \otimes b) & \mapsto & {}^g(a \otimes b) = {}^g a \otimes {}^g b\,, \end{array}$$

and the resulting skew group algebra is isomorphic to Δ *via* the following isomorphism

$$\begin{array}{rcl} \Delta & \to & A^e * G \\ (a * g) \otimes (b * g^{-1}) & \mapsto & (a \otimes {}^{g^{-1}} b) * g\,. \end{array} \tag{3.1.1}$$

To be given a dg Δ-module is equivalent to be given a dg A^e-module M together with a *compatible* action of G on M by automorphisms of complexes of vector spaces $G \times M \to M$, $(g, m) \mapsto {}^g m$. Here, by "compatible", is meant that, for all $g \in G$, $a, b \in A$ and $m \in M$,

$$ {}^g(m \cdot (a \otimes b)) = {}^g m \cdot {}^g(a \otimes b)\,, \tag{3.1.2}$$

the structure of dg Δ-module is then such that $m \cdot ((a * g) \otimes (b * g^{-1})) = ({}^{g^{-1}} m) \cdot ({}^{g^{-1}} a \otimes b)$.

To any dg Δ-module M is associated a dg Λ^e-module $M*G$ with underlying complex of vector spaces $M\otimes \Bbbk G$ and with action of Λ^e such that $(m*g)\cdot(a*h\otimes b*k) = ({}^k m\cdot({}^{kg}a\otimes b))*kgh$ for all $m\in M$, $a,b\in A$, and $g,h,k\in G$. Note that $A\in \mathrm{Mod}(\Delta)$ by means of the following structure mapping

$$\begin{array}{rcl} A\otimes\Delta & \to & A \\ a\otimes(b*g\otimes c*g^{-1}) & \mapsto & (-1)^{\deg(c)\cdot(\deg(b)+\deg(a))}c\,{}^{g^{-1}}a\,{}^{g^{-1}}b\,. \end{array}$$

Of course, in this case, the resulting dg Λ^e-module $A*G$ is Λ. The following basic properties are used throughout the section.

Lemma 3.1.1. *Assume setting 3.1.*

(1) $\Delta\simeq (A^e)^{\mathrm{Card}(G)}$ *as dg* A^e*-modules.*
(2) $\Lambda^e\simeq \Delta^{\mathrm{Card}(G)}$ *as dg* Δ*-modules.*
(3) Let $M\in\mathrm{Mod}(\Delta)$*, then*
- *the following mapping is an isomorphism of dg* Λ^e*-modules,*

$$\begin{array}{rcl} M\otimes_\Delta \Lambda^e & \to & M*G \\ m\otimes(a*g\otimes b*h) & \mapsto & (m\cdot(a*h^{-1}\otimes b*h))*hg\,. \end{array} \tag{3.1.3}$$

- M *is a direct summand of* $M\underset{\Delta}{\otimes}\Lambda^e$ *in* $Z^0\mathrm{Mod}(\Delta)$*,*
- *if* $\mathrm{char}(\Bbbk)\nmid \mathrm{Card}(G)$ *then* M *is a direct summand of* $M\underset{A^e}{\otimes}\Delta$ *in* $Z^0\mathrm{Mod}(\Delta)$*.*

(4) The restriction-of-scalars functor $\mathrm{Mod}(\Delta)\to\mathrm{Mod}(A^e)$ *maps cofibrant objects to cofibrant objects. Moreover, for all cofibrant resolutions* $X\to A$ *in* $\mathrm{Mod}(\Delta)$*, the composite morphism* $X\otimes_\Delta\Lambda^e\to A\otimes_\Delta\Lambda^e\xrightarrow{(3.1.3)}\Lambda$ *is a cofibrant resolution in* $\mathrm{Mod}(\Lambda^e)$*.*

Proof. (1) follows from the definition of Δ and the fact that $A*g\simeq A$ as dg A-modules.

(2) follows from the direct sum decompositions

$$\Lambda^e=\bigoplus_{t\in G}(\bigoplus_{g\in G}(A*g)\otimes(A*g^{-1}t)) = \bigoplus_{t\in G}(\bigoplus_{g\in G}(A*g)\otimes(A*tg^{-1})) \tag{3.1.4}$$

where each term $\oplus_{g\in G}(A*g)\otimes(A*g^{-1}t)$ (or, $\oplus_{g\in G}(A*g)\otimes(A*tg^{-1})$) is a dg $\Delta\otimes(A^e)^{\mathrm{op}}$-submodule (or, a dg $A^e\otimes\Delta^{\mathrm{op}}$-submodule) of Λ^e freely generated by $(1*e)\otimes(1*t)$ as a dg Δ-module (or, as a dg Δ^{op}-module), and hence is isomorphic to Δ as a dg Δ-module (or, as a dg Δ^{op}-module, respectively).

(3) The mapping (3.1.3) is a well-defined morphism of dg Λ^e-modules, its inverse is

$$\begin{array}{rcl} M*G & \to & M\otimes_\Delta\Lambda^e \\ m*g & \mapsto & m\otimes((1*g)\otimes(1*e))\,. \end{array}$$

In order to prove that M is a direct summand of $M\otimes_\Delta\Lambda^e$ in $Z^0\mathrm{Mod}(\Delta)$, denote by X_t the dg $\Delta\otimes(A^e)^{\mathrm{op}}$-submodule $\oplus_{g\in G}(A*g)\otimes(A*g^{-1}t)$ of Λ^e, for all $t\in G\backslash\{e\}$. In the decomposition (3.1.4), both summands X_e and $\oplus_{t\in G\backslash\{e\}}X_t$ are dg $\Delta\otimes\Delta^{\mathrm{op}}$-submodules. Accordingly, there is a direct sum decomposition in $Z^0\mathrm{Mod}(\Delta)$

$$M=(M\otimes_\Delta X_e)\bigoplus\left(M\otimes_\Delta(\bigoplus_{t\neq e}X_t)\right)\,.$$

Since $X_e=\Delta$, the dg module M is a direct summand of $M\otimes_\Delta\Lambda^e$ in $Z^0\mathrm{Mod}(\Delta)$.

Finally, note that Lemma 2.3.1 may be applied here to the dg algebras A^e and $\Delta\simeq A^e*G$ instead of to A and $A*G$, respectively. Using part (3) of that lemma yields that M is a direct summand of $M\otimes_{A^e}\Delta$ in $Z^0\mathrm{Mod}(\Delta)$.

(4) Let $X\to A$ be a cofibrant resolution in $\mathrm{Mod}(\Delta)$.

In view of (1), the functor $\mathrm{Hom}_{A^e}(\Delta,-)\colon \mathrm{Mod}(A^e)\to\mathrm{Mod}(\Delta)$ preserves quasi-isomorphisms and component-wise surjective morphisms. Accordingly, its left adjoint, the restriction-of-scalars functor $\mathrm{Mod}(\Delta)\to\mathrm{Mod}(A^e)$, preserves cofibrant objects. Thus, $X\to A$ is a cofibrant resolution in $\mathrm{Mod}(A^e)$.

The restriction-of-scalars functor $\mathrm{Mod}(\Lambda^e)\to\mathrm{Mod}(\Delta)$ preserves quasi-isomorphisms and component-wise surjective morphisms. Accordingly, its left adjoint, the extension-of-scalars functor $-\otimes_\Delta\Lambda^e\colon\mathrm{Mod}(\Delta)\to\mathrm{Mod}(\Lambda^e)$ preserves cofibrant objects. In view of (2), it also preserves quasi-isomorphisms. Thus, $X\otimes_\Delta\Lambda^e\to A\otimes_\Delta\Lambda^e$ is a cofibrant resolution in $\mathrm{Mod}(\Lambda^e)$ and remains so after composing with the quasi-isomorphism (3.1.3). □

3.2. **Homological smoothness of A and Λ.**

Proposition 3.2.1. *Assume setting 3.1.*

(1) If Λ is homologically smooth, then so is A.
(2) The converse holds true when $\mathrm{char}(\Bbbk)$ *does not divide* $\mathrm{Card}(G)$.

Proof. (1) Assume that Λ is homologically smooth. Consider the restriction-of-scalars functors,

$$\mathcal{D}(\Lambda^e)\xrightarrow{\mathrm{res}_1}\mathcal{D}(\Delta)\xrightarrow{\mathrm{res}_2}\mathcal{D}(A^e)\,.$$

Since $\Lambda^e\simeq\Delta^{\mathrm{Card}(G)}$ (see Lemma 3.1.1, part (2)), the triangle functor res_1 maps $\mathrm{per}(\Lambda^e))$ into $\mathrm{per}(\Delta)$. Moreover, $\Lambda\in\mathrm{per}(\Lambda^e)$. Therefore,

$$\Lambda\in\mathrm{per}(\Delta)\,. \tag{3.2.1}$$

Applying part (3) of Lemma 3.1.1 to $M=A$ and using the restriction-of-scalars functor $\mathcal{D}(\Lambda^e)\to\mathcal{D}(\Delta)$ yields that $\Lambda\simeq A\otimes_\Delta\Lambda^e$ in $\mathcal{D}(\Delta)$ and that A is a direct summand of $A\otimes_\Delta\Lambda^e$ in $\mathcal{D}(\Delta)$. Therefore, it follows from (3.2.1) that

$$A\in\mathrm{per}(\Delta)\,. \tag{3.2.2}$$

Finally, $\Delta\in\mathrm{per}(A^e)$ because $\Delta\simeq(A^e)^{\mathrm{Card}(G)}$ (see Lemma 3.1.1, part (1)); accordingly, $\mathrm{res}_2(\mathrm{per}(\Delta))\subseteq\mathrm{per}(A^e)$; therefore $A\in\mathrm{per}(A^e)$ (see (3.2.2)), that is, A is homologically smooth.

(2) Assume that $\mathrm{char}(\Bbbk)$ does not divide $\mathrm{Card}(G)$ and that A is homologically smooth. Consider the extension-of-scalars functors,

$$\mathcal{D}(A^e)\xrightarrow{-\otimes^{\mathrm{L}}_{A^e}\Delta}\mathcal{D}(\Delta)\xrightarrow{-\otimes^{\mathrm{L}}_{\Delta}\Lambda^e}\mathcal{D}(\Lambda^e)\,.$$

First, $A\otimes^{\mathrm{L}}_{A^e}\Delta\in\mathrm{per}(\Delta)$ because $A\in\mathrm{per}(A^e)$. Besides, $A\otimes^{\mathrm{L}}_{A^e}\Delta\simeq A\otimes_{A^e}\Delta$ in $\mathcal{D}(\Delta)$ because $\Delta\simeq(A^e)^{\mathrm{Card}(G)}$ as dg $(A^e)^{\mathrm{op}}$-modules (see Lemma 3.1.1, part (1)). Now, A is a direct summand of $A\otimes_{A^e}\Delta$ in $\mathcal{D}(\Delta)$ (see above or part (3) of Lemma 3.1.1). Thus,

$$A\in\mathrm{per}(\Delta)\,. \tag{3.2.3}$$

Finally, $A\otimes_\Delta\Lambda^e\simeq A\otimes^{\mathrm{L}}_\Delta\Lambda^e$ because $\Lambda^e\simeq\Delta^{\mathrm{Card}(G)}$ as dg Δ^{op}-modules, and $\Lambda\simeq A\otimes_\Delta\Lambda^e$ in $\mathcal{D}(\Lambda^e)$ (see Lemma 3.1.1, parts (2) and (3)). Therefore $\Lambda\in\mathrm{per}(\Lambda^e)$ (see (3.2.3)), that is, Λ is homologically smooth. □

3.3. **Interaction between duality and extension of scalars.** This subsection establishes a relationship between $\mathrm{RHom}_{A^e}(A,A^e)$ and $\mathrm{RHom}_{\Lambda^e}(\Lambda,\Lambda^e)$. For this purpose, the following lemma makes of $\mathrm{RHom}_{A^e}(-,A^e)$ a triangle functor $\mathcal{D}(\Delta)\to\mathcal{D}(\Delta)$.

Lemma 3.3.1. *Assume setting 3.1. Let $M\in\mathrm{Mod}(\Delta)$. For all $f\in\mathrm{Hom}_{A^e}(M,A^e)$ and $g\in G$, denote by ${}^gf$ the following mapping*

$$\begin{array}{rccl} {}^gf\colon & M & \to & A^e \\ & m & \mapsto & {}^g(f(m\cdot((1*g)\otimes(1*g^{-1})))\,). \end{array} \tag{3.3.1}$$

This construction defines an action of G on $\mathrm{Hom}_{A^e}(M, A^e)$ compatible with the natural structure of dg A^e-module, and hence an additive functor

$$\mathrm{Hom}_{A^e}(-, A^e)\colon \mathrm{Mod}(\Delta) \to \mathrm{Mod}(\Delta) \tag{3.3.2}$$

such that the two following diagrams are commutative (the vertical arrows are the restriction-of-scalars functors).

$$\begin{tikzcd} \mathrm{Mod}(\Delta) \arrow[r, "\mathrm{Hom}_{A^e}(-,A^e)"] \arrow[d] & \mathrm{Mod}(\Delta) \arrow[d] \\ \mathrm{Mod}(A^e) \arrow[r, "\mathrm{Hom}_{A^e}(-,A^e)"'] & \mathrm{Mod}(A^e) \end{tikzcd} \tag{3.3.3}$$

$$\begin{tikzcd} \mathcal{D}(\Delta) \arrow[r, "\mathrm{RHom}_{A^e}(-,A^e)"] \arrow[d] & \mathcal{D}(\Delta) \arrow[d] \\ \mathcal{D}(A^e) \arrow[r, "\mathrm{RHom}_{A^e}(-,A^e)"'] & \mathcal{D}(A^e) \end{tikzcd} \tag{3.3.4}$$

Proof. The part of the statement before (3.3.3) follows from a direct computation. In particular, (3.3.3) is commutative. Following Lemma 3.1.1 (part (4)), the vertical arrows in (3.3.3) preserve cofibrant objects. The diagram (3.3.4) therefore arises from (3.3.3). □

Note that, given a quasi-isomorphism $X \to M$ in $\mathcal{D}(\Delta)$ such that X is cofibrant in $\mathrm{Mod}(A^e)$, the image of M under $\mathrm{RHom}_{A^e}(-, A^e)\colon \mathcal{D}(\Delta) \to \mathcal{D}(\Delta)$ is isomorphic to $\mathrm{Hom}_{A^e}(X, A^e)$. Indeed, there exists a cofibrant resolution $X' \to X$ in $\mathrm{Mod}(\Delta)$. Therefore, the image of M under $\mathrm{RHom}_{A^e}(-, A^e)\colon \mathcal{D}(\Delta) \to \mathcal{D}(\Delta)$ is isomorphic to $\mathrm{Hom}_{A^e}(X', A^e)$. Now, X' is cofibrant in $\mathrm{Mod}(A^e)$ because so it is in $\mathrm{Mod}(\Delta)$; actually, it follows from [17, Proposition 2.5] and from the isomorphism (3.1.1) that a dg Δ-module is homotopically projective if and only if it is so as a dg A^e-module. The quasi-isomorphism $X' \to X$ between cofibrant objects in $\mathrm{Mod}(A^e)$ therefore induces a quasi-isomorphism $\mathrm{Hom}_{A^e}(X, A^e) \to \mathrm{Hom}_{A^e}(X', A^e)$ in $\mathrm{Mod}(A^e)$, and hence also in $\mathrm{Mod}(\Delta)$. In particular, there is no risk of misunderstanding in using the same notation, $\mathrm{RHom}_{A^e}(-, A^e)$, for both functors $\mathcal{D}(A^e) \to \mathcal{D}(A^e)$ and $\mathcal{D}(\Delta) \to \mathcal{D}(\Delta)$.

It is now possible to establish the relationship between $\mathrm{RHom}_{A^e}(A, A^e)$ and $\mathrm{RHom}_{\Lambda^e}(\Lambda, \Lambda^e)$ announced at the beginning of the subsection.

Proposition 3.3.2. *Assume setting 3.1. The following diagram, where the top horizontal arrow is given by the top horizontal one of* (3.3.4)*, is commutative up to an isomorphism of functors.*

$$\begin{tikzcd} \mathcal{D}(\Delta) \arrow[r, "\mathrm{RHom}_{A^e}(-,A^e)"] \arrow[d, "-\otimes^{\mathrm{L}}_{\Delta}\Lambda^e"'] & \mathcal{D}(\Delta) \arrow[d, "-\otimes^{\mathrm{L}}_{\Delta}\Lambda^e"] \\ \mathcal{D}(\Lambda^e) \arrow[r, "\mathrm{RHom}_{\Lambda^e}(-,\Lambda^e)"'] & \mathcal{D}(\Lambda^e) \end{tikzcd} \tag{3.3.5}$$

Consequently, there are isomorphisms in $\mathcal{D}(\Lambda^e)$,

$$\mathrm{RHom}_{A^e}(A, A^e) * G \simeq \mathrm{RHom}_{A^e}(A, A^e) \otimes^{\mathrm{L}}_{\Delta} \Lambda^e \simeq \mathrm{RHom}_{\Lambda^e}(\Lambda, \Lambda^e)\,. \tag{3.3.6}$$

Proof. For the ease of reading, A and $\Bbbk G$ are identified with their respective canonical images in Λ and so are A^e and $(\Bbbk G)^e$ in Λ^e. Unless otherwise specified, "$\cdot$" denotes the product in Λ^e. For all $M \in \mathrm{Mod}(\Delta)$, there is an isomorphism of dg Λ^e-modules,

$$\Phi_M\colon \mathrm{Hom}_{A^e}(M, A^e) \otimes_{\Delta} \Lambda^e \to \mathrm{Hom}_{\Delta}(M, \Lambda^e) \tag{3.3.7}$$

such that $\Phi_M(f \otimes ((a*g)\otimes(b*h)))$ is given as follows, for all $f \in \mathrm{Hom}_{A^e}(M, A^e)$, $a, b \in A$ and $g, h \in G$,

$$\begin{array}{rcl} M & \to & \Lambda^e \\ m & \mapsto & \pm\sum_{k\in G}((b*h)\otimes(a*g))\cdot f(m\cdot(k\otimes k^{-1}))\cdot(k^{-1}\otimes k)\,, \end{array}$$

where the sign is $(-1)^{\deg(b)\cdot(\deg(a)+\deg(f))+\deg(a)\cdot\deg(f)}$. Here is a description of Φ_M^{-1}. Given $\varphi \in \mathrm{Hom}_\Delta(M, \Lambda^e)$ and $m \in M$, there is a unique decomposition $\varphi(m) = \sum_{s,t\in G}\varphi_{s,t}(m)\cdot(s\otimes t)$ in Λ^e, where $\varphi_{s,t}(m) \in A^e$; in particular, for all $s, t \in G$,

- $\varphi_{s,t}(m\cdot(a\otimes b)) = \varphi_{s,t}(m)\cdot({}^s a \otimes {}^{t^{-1}} b)$, for all $a, b \in A$, and
- $\varphi_{s,t}(m\cdot(k\otimes k^{-1})) = \varphi_{sk^{-1},kt}(m)$, for all $k \in G$;

hence, the following well-defined mapping lies in $\mathrm{Hom}_{A^e}(M, A^e)$, for all $g \in G$,

$$\varphi^{(g)}\colon \quad \begin{array}{rcl} M & \to & A^e \\ m & \mapsto & (g^{-1}\otimes 1)\cdot\varphi_{g,e}(m)\cdot(g\otimes 1)\,. \end{array}$$

With this notation, Φ_M^{-1} is given by $\varphi \mapsto \sum_{g\in G}\varphi^{(g)}\otimes(g\otimes 1)$.

This construction is natural. Since, moreover, $\mathrm{Hom}_\Delta(-, \Lambda^e) \cong \mathrm{Hom}_{\Lambda^e}(-\otimes_\Delta \Lambda^e, \Lambda^e)$, there is a functorial isomorphism of functors from $\mathrm{Mod}(A^e)$ to $\mathrm{Mod}(\Lambda^e)$,

$$\mathrm{Hom}_{A^e}(-, A^e)\otimes_\Delta \Lambda^e \xrightarrow{\cong} \mathrm{Hom}_{\Lambda^e}(-\otimes_\Delta \Lambda^e, \Lambda^e)\,. \tag{3.3.8}$$

Now, recall that $\Lambda^e \simeq \Delta^{\mathrm{Card}(G)}$ both as dg Δ-modules and as dg Δ^{op}-modules. Hence, deriving (3.3.8) yields that (3.3.5) commutes up to an isomorphism of functors, which entails (3.3.6), see subsection 3.1 for the first isomorphism of (3.3.6), and see the isomorphism (3.1.3) with $M = A$ for the second one. □

3.4. **Main theorem.** Now it is possible to prove Theorem 1.2. Here is a restatement of it.

Theorem 3.4.1. *Assume setting 3.1. Assume that* $\mathrm{char}(\Bbbk) \nmid \mathrm{Card}(G)$*, that A is homologically smooth and that* $\Sigma^d\mathrm{RHom}_{A^e}(A, A^e) \simeq A$ *in* $\mathcal{D}(\Delta)$*, then A and Λ are d-Calabi-Yau.*

Proof. By restriction of scalars, $\Sigma^d\mathrm{RHom}_{A^e}(A, A^e)$ is isomorphic to A in $\mathcal{D}(A^e)$ because this is the case in $\mathcal{D}(\Delta)$. Hence, A is d-Calabi-Yau. By Proposition 3.2.1, the dg algebra Λ is homologically smooth. Finally, since $\Sigma^d\mathrm{RHom}_{A^e}(A, A^e) \simeq A$ in $\mathcal{D}(\Delta)$, it follows from the isomorphism (3.1.3) with $M = A$ and from Proposition 3.3.2 that Λ is d-Calabi-Yau. □

Here is a necessary condition for Λ to be Calabi-Yau. No assumption on $\mathrm{char}(\Bbbk)$ is made.

Proposition 3.4.2. *Assume setting 3.1. Assume that* $\mathrm{HH}^0(A)$ *is a local* $\Bbbk$*-algebra. If Λ is d-Calabi-Yau, then A is homologically smooth and there exists $g \in G$ such that, in* $\mathcal{D}(A^e)$*,*

$$\Sigma^d\mathrm{RHom}_{A^e}(A, A^e) \simeq A_g\,,$$

where $\bullet_g$ denotes the restriction-of-scalars functor along the dg automorphism $A^e \to A^e$, $a\otimes b \mapsto {}^g a\otimes b$.

Proof. The dg algebra A is homologically smooth by Proposition 3.2.1. Since Λ is d-Calabi-Yau, there is an isomorphism $\Sigma^d\mathrm{RHom}_{\Lambda^e}(\Lambda, \Lambda^e) \xrightarrow{\sim} \Lambda$ in $\mathcal{D}(\Lambda^e)$. Using Proposition 3.3.2 and the isomorphism (3.1.3) with $M = A$ yields that $\Sigma^d\mathrm{RHom}_{A^e}(A, A^e)\otimes^{\mathrm{L}}_\Delta \Lambda^e \simeq A\otimes^{\mathrm{L}}_\Delta \Lambda^e$ in $\mathcal{D}(\Lambda^e)$. Using (3.1.3) yields that there is an isomorphism in $\mathcal{D}(A^e)$,

$$\bigoplus_{g\in G}\Sigma^d\mathrm{RHom}_{A^e}(A, A^e)_g \simeq \bigoplus_{g\in G} A_g\,. \tag{3.4.1}$$

Note that all the summands appearing in (3.4.1) lie in $\mathrm{per}(A^e)$ because they are direct summands of Λ in $\mathcal{D}(A^e)$.

Now, since $\mathrm{RHom}_{A^e}(-, A^e)$ is a duality on $\mathrm{per}(A^e)$, the endomorphism algebras of A_g and $\Sigma\mathrm{RHom}_{A^e}(A, A^e)_h$, for $g, h \in G$, are isomorphic to $\mathcal{D}(A^e)(A, A)$ ($\simeq \mathrm{HH}^0(A)$). Since $\mathrm{HH}^0(A)$ is local, it follows from (3.4.1) that $\Sigma^d\mathrm{RHom}_{A^e}(A, A^e) \simeq A_g$ in $\mathcal{D}(A^e)$ for some $g \in G$. □

3.5. **Skew group algebras of deformed Calabi-Yau completions.** Assume setting 3.1 and assume that $\mathrm{char}(\Bbbk)$ does not divide $\mathrm{Card}(G)$ and that A is homologically smooth. Hence Λ is homologically smooth. This subsection investigates when G acts on the deformed Calabi-Yau completions of A, as introduced in [24], and when the associated skew group algebras are deformed Calabi-Yau completions of Λ. For this subsection, consider a cofibrant replacement in $\mathrm{Mod}(\Delta)$

$$r_A \colon X \to A\,,$$

and consider a quasi-isomorphism in $\mathrm{Mod}(\Delta)$ which is a cofibrant replacement in $\mathrm{Mod}(A^e)$,

$$p_A \colon \Theta_A \to \mathrm{Hom}_{A^e}(X, A^e)\,.$$

Recall that r_A is a cofibrant resolution in $\mathrm{Mod}(A^e)$ (see subsection 3.1) and that dg Δ-modules are also viewed as dg A^e-modules with a compatible action of G.

The quasi-isomorphism p_A induces a quasi-isomorphism in $\mathrm{Mod}(\Bbbk)$ because A is homologically smooth,

$$\begin{array}{rcl} \Sigma^{d-2}X \otimes_{A^e} A & \to & \mathrm{Hom}_{A^e}(\Sigma^{d-1}\Theta_A, \Sigma A) \\ x \otimes a & \mapsto & (\theta \mapsto (p_A(\theta))(x) \cdot a)\,, \end{array} \tag{3.5.1}$$

where the "$\cdot$" denotes the structure action of A^e on the left dg A^e-module A. The (deformed) Calabi-Yau completions of A are defined as follows.

Definition 3.5.1 ([24])**.** Let A be a homologically smooth dg algebra. Let Θ_A be a cofibrant replacement of $\mathrm{RHom}_{A^e}(A, A^e)$ in $\mathrm{Mod}(A^e)$.

(1) The *d-Calabi-Yau completion* $\mathbf{\Pi}_d(A)$ of A is the following tensor dg algebra,

$$\mathbf{\Pi}_d(A) = T_A(\Sigma^{d-1}\Theta_A)\,. \tag{3.5.2}$$

(2) Let $D \in Z^0\mathrm{Hom}_{A^e}(\Sigma^{d-1}\Theta_A, \Sigma A)$. Denote by δ the differential of $\Sigma^{d-1}\Theta_A$. Define δ_D to be the unique square-zero skew-derivation of degree 1 on the graded algebra $T_A(\Sigma^{d-1}\Theta_A)$ whose restriction to A is equal to the differential of A and whose restriction to $\Sigma^{d-1}\Theta_A$ is given by

$$[D, \delta] \colon \Sigma^{d-1}\Theta_A \to \Sigma A \bigoplus \Sigma^d\Theta_A\,. \tag{3.5.3}$$

(3) Let $c \in \mathrm{HH}_{d-2}(A)$. Let $D \in Z^0\mathrm{Hom}_{A^e}(\Sigma^{d-1}\Theta_A, \Sigma A)$ represent c *via* (3.5.1). The *deformed d-Calabi-Yau completion* $\mathbf{\Pi}_d(A, c)$ is defined by

$$\mathbf{\Pi}_d(A, c) = (T_A(\Sigma^{d-1}\Theta_A), \delta_D)\,. \tag{3.5.4}$$

Up to quasi-isomorphisms of dg algebras, neither $\mathbf{\Pi}_d(A)$ nor $\mathbf{\Pi}_d(A, c)$ depend on the cofibrant replacement $r_A \colon X \to A$ or on the cocycle D.

Theorem 3.5.2 ([24, 37])**.** *Let A be a homologically smooth dg algebra. Let d be an integer. The Calabi-Yau completion $\mathbf{\Pi}_d(A)$ is d-Calabi-Yau (*[24, Theorem 4.8]*). For all $c \in \mathrm{HH}_{d-2}(A)$ lying in the image of the canonical morphism $\mathrm{HN}_{d-2}(A) \to \mathrm{HH}_{d-2}(A)$ the deformed Calabi-Yau completion $\mathbf{\Pi}_d(A, c)$ is d-Calabi-Yau (*[37, Theorem 3.17]*).*

In order to relate the (deformed) Calabi-Yau completions of A to those of Λ, it is necessary to endow the former ones with suitable actions of G. For this purpose, note that the mapping (3.5.1) is G-equivariant provided that $\Sigma^{d-2}X \otimes_{A^e} A$ and $\mathrm{Hom}_{A^e}(\Sigma^{d-1}\Theta_A, \Sigma A)$ are endowed with the following actions of G,

- ${}^g(x \otimes a) = {}^gx \otimes {}^ga$, for all $g \in G$, $x \in X$ and $a \in A$,

- $({}^{g}D)(\theta) = {}^{g}(D({}^{g^{-1}}\theta))$, for all $g \in G$, $D \in \mathrm{Hom}_{A^e}(\Sigma^{d-1}\Theta_A, \Sigma A)$ and $\theta \in \Sigma^{d-1}\Theta_A$.

In particular, the former action induces the natural action on the Hochschild homology $\mathrm{HH}_{d-2}(A)$ which is computed by $X \otimes_{A^e} A$.

Lemma 3.5.3. *Assume setting 3.1. Assume that* $\mathrm{char}(\Bbbk)$ *does not divide* $\mathrm{Card}(G)$ *and that A is homologically smooth. Let Θ_A be a dg Δ-module which is a cofibrant replacement of* $\mathrm{RHom}_{A^e}(A, A^e)$ *in* $\mathrm{Mod}(A^e)$.

(1) The actions of G on A and on Θ_A extend uniquely to an action by dg automorphisms on $T_A(\Sigma^{d-1}\Theta_A) = \mathbf{\Pi}_d(A)$.

(2) Let $c \in \mathrm{HH}_{d-2}(A)^G$. Then, there exists $D \in Z^0\mathrm{Hom}_{A^e}(\Sigma^{d-1}\Theta_A, \Sigma A)$ which represents c via (3.5.1) *and such that $D({}^{g}\theta) = {}^{g}(D(\theta))$ for all $g \in G$ and $\theta \in \Theta_A$. For any such D, the action of G on the graded algebra $T_A(\Sigma^{d-1}\Theta_A)$ is an action by dg automorphisms on $(T_A(\Sigma^{d-1}\Theta_A), \delta_D) = \mathbf{\Pi}_d(A, c)$ (see part (2) of Definition 3.5.1 for the definition of δ_D).*

Proof. (1) In view of (3.1.2), there is an action of G on $(\Sigma^{d-1}\Theta_A)^{\otimes_A n}$, for all $n \in \mathbb{N}\backslash\{0\}$, such that ${}^{g}(\theta_1 \otimes \cdots \otimes \theta_n) = {}^{g}\theta_1 \otimes \cdots \otimes {}^{g}\theta_n$ for all $\theta_1, \ldots, \theta_n \in \Sigma^{d-1}\Theta_A$ and $g \in G$. These actions together with the one on A do form an action of G on $T_A(\Sigma^{d-1}\Theta_A)$ by dg automorphisms. The uniqueness of this action follows from $T_A(\Sigma^{d-1}\Theta_A)$ being generated by Θ_A over A as an algebra.

(2) Note that $(H^0\mathrm{Hom}_{A^e}(\Sigma^{d-1}\Theta_A, \Sigma A))^G \cong H^0(\mathrm{Hom}_{A^e}(\Sigma^{d-1}\Theta_A, \Sigma A)^G)$ because the algebra $\Bbbk G$ is semisimple. This explains the existence of D. In particular, both the differential of A and the mapping (3.5.3) are G-equivariant. Hence, so is δ_D. Accordingly, the action of G on the graded algebra $T_A(\Sigma^{d-1}\Theta_A)$ is an action by dg automorphisms on $\mathbf{\Pi}_d(A, c)$ when the latter is taken equal to $(T_A(\Sigma^{d-1}(\Theta_A), \delta_D)$. □

Lemma 3.5.3 sets the framework in which the dg algebras $\mathbf{\Pi}_d(A) * G$ and $\mathbf{\Pi}_d(A, c) * G$ are defined. Using this, it is possible to prove Theorem 1.3. This theorem is restated below for convenience.

Theorem 3.5.4. *Assume setting 3.1. Assume that A is homologically smooth and that* $\mathrm{char}(\Bbbk) \nmid \mathrm{Card}(G)$.

(1) $A * G$ is homologically smooth and, up to appropriate choices of inverse dualising complexes of A and $A*G$, the group G acts on $\mathbf{\Pi}_d(A)$ by dg automorphisms and $\mathbf{\Pi}_d(A)*G \simeq \mathbf{\Pi}_d(A * G)$ as dg algebras. In particular, $\mathbf{\Pi}_d(A) * G$ is d-Calabi-Yau.

(2) Let $c \in \mathrm{HH}_{d-2}(A)^G$. Let $\overline{c}$ be the image of c under the mapping $\mathrm{HH}_{d-2}(A) \to \mathrm{HH}_{d-2}(A * G)$ induced by $A \to A * G$, $a \mapsto a * e$. Up to appropriate choices as in (1) and up to appropriate choices of representatives of c and $\overline{c}$, the group G acts on $\mathbf{\Pi}_d(A, c)$ by dg automorphisms and $\mathbf{\Pi}_d(A, c) * G \simeq \mathbf{\Pi}_d(A * G, \overline{c})$ as dg algebras. In particular, if c lifts to $\mathrm{HN}_{d-2}(A)$ (so that $\mathbf{\Pi}_d(A, c)$ is d-Calabi-Yau), then $\mathbf{\Pi}_d(A, c) * G$ is d-Calabi-Yau.

Proof. It follows from Proposition 3.2.1 that Λ is homologically smooth.

(1) Here, $\mathbf{\Pi}_d(A)$ is taken equal to the dg algebra $T_A(\Sigma^{d-1}\Theta_A)$ endowed with the action of G introduced in Lemma 3.5.3. Note that $\Theta_A * G$ is a cofibrant replacement of $\mathrm{RHom}_{\Lambda^e}(\Lambda, \Lambda^e)$ in $\mathrm{Mod}(\Lambda^e)$ (see subsection 3.1 and Proposition 3.3.2). Hence, $\mathbf{\Pi}_d(A * G)$ may be taken equal to the tensor dg algebra $T_\Lambda(\Sigma^{d-1}\Theta_A * G)$. To avoid confusion, the piece of notation "$*$" is kept for elementary tensors in $A * G$ and $\Theta_A * G$ but not for those in $T_A(\Sigma^{d-1}\Theta_A) * G$. The following map is therefore a morphism of dg algebras,

$$\begin{array}{rcl} \mathbf{\Pi}_d(A) * G & \longrightarrow & \mathbf{\Pi}_d(A * G) \\ a \otimes g & \longmapsto & a * g \in A * G \\ (\phi_1 \otimes \cdots \otimes \phi_n) \otimes g & \longmapsto & (\phi_1 * e) \otimes \cdots \otimes (\phi_{n-1} * e) \otimes (\phi_n * g) \quad , \end{array} \tag{3.5.5}$$

where $a \in A$, $\phi_1, \dots, \phi_n \in \Sigma^{d-1}\Theta_A$ and $g \in G$. It is an isomorphism, its inverse maps $a*g \in A*G$ onto $a \otimes g$, for all $a \in A$ and $g \in G$, and it maps $(\phi_1 * g_1) \otimes \cdots \otimes (\phi_n * g_n)$ onto

$$(\phi_1 \otimes {}^{g_1}\phi_2 \otimes \cdots \otimes {}^{g_1\cdots g_l}\phi_{l+1} \otimes \cdots \otimes {}^{g_1\cdots g_{n-2}}\phi_{n-1} \otimes {}^{g_1\cdots g_{n-1}}\phi_n) \otimes g_1 \cdots g_n$$

for all $(\phi_1 * g_1), \dots, (\phi_n * g_n) \in \Sigma^{d-1}\Theta_A * G$.

(2) In view of the functoriality of the Hochschild homology and of the negative cyclic homology, if c lifts to $\mathrm{HN}_{d-2}(A)$ then $\overline{c}$ lifts to $\mathrm{HN}_{d-2}(A * G)$. It is hence sufficient to prove the first statement of (2). Thanks to the remark following Lemma 3.3.1, it is possible to choose p_A such that it is, in addition, a cofibrant resolution in $\mathrm{Mod}(\Delta)$. Here $\mathbf{\Pi}_d(A, c)$ is taken equal to $(T_A(\Sigma^{d-1}\Theta_A), \delta_D)$ where D is as in Lemma 3.5.3. Thus, $\mathbf{\Pi}_d(A, c)$ is endowed with an action of G by dg automorphisms.

It is necessary to first describe $\mathbf{\Pi}_d(\Lambda, \overline{c})$ in the same way as $\mathbf{\Pi}_d(A, c)$. The following composite mapping is a cofibrant resolution in $\mathrm{Mod}(\Lambda^e)$,

$$r_\Lambda \colon X \otimes_\Delta \Lambda^e \xrightarrow{r_A \otimes \Lambda^e} A \otimes_\Delta \Lambda^e \xrightarrow{(3.1.3)} \Lambda\,.$$

The following composite mapping is a cofibrant resolution in $\mathrm{Mod}(\Lambda^e)$,

$$p_\Lambda \colon \Theta_A \otimes_\Delta \Lambda^e \xrightarrow{p_A \otimes \Lambda^e} \mathrm{Hom}_{A^e}(X, A^e) \otimes_\Delta \Lambda^e \xrightarrow[(3.3.7)]{\frac{1}{\mathrm{Card}(G)} \cdot \Phi_X} \mathrm{Hom}_\Delta(X, \Lambda^e) \xrightarrow{\cong} \mathrm{Hom}_{\Lambda^e}(X \otimes_\Delta \Lambda^e, \Lambda^e)\,.$$

Hence, $\mathbf{\Pi}_d(\Lambda, \overline{c})$ may be taken equal to the graded algebra $T_\Lambda(\Sigma^{d-1}\Theta_A \otimes_\Delta \Lambda^e)$ with the differential $(\delta \otimes_\Delta \Lambda^e)_{D'}$, where D' is any 0-cocycle in $\mathrm{Hom}_{\Lambda^e}(\Sigma^{d-1}\Theta_A \otimes_\Delta \Lambda^e, \Sigma\Lambda)$ which represents $\overline{c}$ *via* the following quasi-isomorphism

$$\begin{array}{rcl} \Sigma^{d-2}(X \otimes_\Delta \Lambda^e) \otimes_{\Lambda^e} \Lambda & \to & \mathrm{Hom}_{\Lambda^e}(\Sigma^{d-1}\Theta_A \otimes_\Delta \Lambda^e, \Sigma\Lambda) \\ \alpha \otimes \lambda & \mapsto & (\theta \mapsto (p_\Lambda(\theta))(\alpha) \cdot \lambda) \end{array} \tag{3.5.6}$$

where the "$\cdot$" stands for the structure action of Λ^e on Λ. Recall that δ is the differential of $\Sigma^{d-1}\Theta_A$.

In order to prove (2), it is hence sufficient to prove the following claim.

Claim 1 - D' may be chosen such that, in addition, (3.5.5) *is an isomorphism of dg algebras from* $(T_A(\Sigma^{d-1}\Theta_A), \delta_D) * G$ *to* $(T_\Lambda(\Sigma^{d-1}\Theta_A \otimes_\Delta \Lambda^e), (\delta \otimes_\Delta \Lambda^e)_{D'})$.

Note that, in view of the definition of p_λ, and up to the two canonical identifications

$$(X \otimes_\Delta \Lambda^e) \otimes_{\Lambda^e} \Lambda \cong X \otimes_\Delta \Lambda \tag{3.5.7}$$

and

$$\mathrm{Hom}_{\Lambda^e}(\Sigma^{d-1}\Theta_A \otimes_\Delta \Lambda^e, \Sigma\Lambda) \cong \mathrm{Hom}_\Delta(\Sigma^{d-1}\Theta_A, \Sigma\Lambda)\,, \tag{3.5.8}$$

the quasi-isomorphism (3.5.6) identifies with the following one

$$\begin{array}{rcl} \Sigma^{d-2}X \otimes_\Delta \Lambda & \to & \mathrm{Hom}_\Delta(\Sigma^{d-1}\Theta_A, \Sigma\Lambda) \\ x \otimes \lambda & \mapsto & (\theta \mapsto \frac{1}{\mathrm{Card}(G)} \cdot \sum_{k \in G} (p_A(\theta))({}^{k^{-1}}x) \cdot (k^{-1} \otimes k) \cdot \lambda)\,, \end{array} \tag{3.5.9}$$

where, for simplicity, A^e and $(\Bbbk G)^e$ are viewed as subspaces of Λ^e in a natural way.

Denote by $\overline{D}$ the following cocycle

$$\begin{array}{rcl} \overline{D} \colon \Sigma^{d-1}\Theta_A & \to & \Sigma\Lambda \\ \theta & \mapsto & D(\theta) * e\,. \end{array}$$

Then, the following diagram where the rightmost vertical arrow is the pre-image of $\overline{D}$ under (3.5.8) is commutative,

$$\begin{tikzcd} \Sigma^{d-1}\Theta_A * G \arrow[r, "(3.5.5)"] \arrow[d, "\overline{D}"'] & \Sigma^{d-1}\Theta_A \otimes_\Delta \Lambda^e \arrow[d] \\ \Sigma A * G \arrow[r, "(3.5.5)"'] & \Sigma\Lambda \end{tikzcd}$$

Hence, in order to prove claim 1, it is sufficient to prove the following claim.

Claim 2 - D' may be chosen such that its image under (3.5.8) *is equal to $\overline{D}$.*

In order to prove this claim, consider the following diagram

$$\begin{tikzcd} (\Sigma^{d-2}X \otimes_{A^e} A)^G \arrow[r] \arrow[d, "(3.5.1)"'] & \Sigma^{d-2}X \otimes_\Delta \Lambda \arrow[r, "(3.5.7)"] \arrow[d, "(3.5.9)"] & (\Sigma^{d-2}X \otimes_\Delta \Lambda^e) \otimes_{\Lambda^e} \Lambda \arrow[d, "p_\Lambda"] \\ \mathrm{Hom}_\Delta(\Sigma^{d-1}\Theta_A, \Sigma A) \arrow[r] & \mathrm{Hom}_\Delta(\Sigma^{d-1}\Theta_A, \Sigma\Lambda) \arrow[r, "(3.5.8)"'] & \mathrm{Hom}_{\Lambda^e}(\Sigma^{d-1}\Theta_A \otimes_\Delta \Lambda^e, \Sigma\Lambda) \end{tikzcd} \tag{3.5.10}$$

where

- the codomain of the G-invariant part of (3.5.1) is indeed equal to $\mathrm{Hom}_\Delta(\Sigma^{d-1}\Theta_A, \Sigma A)$ as a subcomplex of $\mathrm{Hom}_{A^e}(\Sigma^{d-1}\Theta_A, \Sigma A)$,
- the unlabelled top horizontal arrow is induced by the mapping $x \otimes a \mapsto x \otimes (a * e)$, and
- the unlabelled bottom horizontal arrow is given by the composition with the embedding $\Sigma A \to \Sigma\Lambda$, $a \mapsto a * e$.

Note that,

- the mapping $\mathrm{HH}_{d-2}(A)^G \to \mathrm{HH}_{d-2}(A * G)$ which assigns $\overline{c}$ to c is induced by the composition of the top horizontal arrows of (3.5.10),
- $\overline{D}$ is the image of D under the unlabelled bottom horizontal arrow of (3.5.10),
- c (or $\overline{c}$) is represented by any cocycle in $(\Sigma^{d-2}X \otimes_{A^e} A)^G$ (or, in $(\Sigma^{d-2}X \otimes_\Delta \Lambda^e) \otimes_{\Lambda^e} \Lambda$) whose image under the leftmost (or, rightmost) vertical arrow of (3.5.10) is cohomologous to D (or, to D', respectively), when viewing $\mathrm{Hom}_\Delta(\Sigma^{d-1}\Theta_A, \Sigma A)$ as a subcomplex of $\mathrm{Hom}_{A^e}(\Sigma^{d-1}\Theta_A, \Sigma A)$.

Hence, in order to prove claim 2, it is sufficient to prove the following claim.

Claim 3 - The diagram (3.5.10) *is commutative.*

Here is a proof of this claim. Given that (3.5.9) is obtained upon composing (3.5.7), (3.5.6), and (3.5.8), the rightmost square is commutative. Let $\sum_i x_i \otimes a_i \in (\Sigma^{d-2}X \otimes_{A^e} A)^G$. Hence, for all $k \in G$,

$$\sum_i x_i \otimes a_i = \sum_i {}^k x_i \otimes {}^k a_i \,. \tag{3.5.11}$$

On one hand, the image of $\sum_i x_i \otimes a_i$ under the leftmost vertical arrow of (3.5.10) is as follows

$$\begin{array}{rcl} \Sigma^{d-1}\Theta_A & \to & \Sigma A \\ \theta & \mapsto & \sum_i (p_A(\theta)(x_i)) \cdot a_i \,, \end{array}$$

which has image under the unlabelled bottom horizontal arrow as follows

$$\begin{array}{rcl} \Sigma^{d-1}\Theta_A & \to & \Sigma\Lambda \\ \theta & \mapsto & \sum_i \underbrace{((p_A(\theta)(x_i) \cdot a_i)) * e}_{=(p_A(\theta)(x_i))\cdot(a_i * e)\,.} \end{array} \tag{3.5.12}$$

On the other hand, the image of $\sum_i x_i \otimes a_i$ under the unlabelled top horizontal arrow of (3.5.10) is equal to $\sum_i x_i \otimes (a_i * e)$, which has image under the middle vertical arrow as follows

$$\begin{array}{rcl} \Sigma^{d-1}\Theta_A & \to & \Sigma\Lambda \\ \theta & \mapsto & \frac{1}{\mathrm{Card}(G)} \cdot \sum_{k\in G,\, i} (p_A(\theta)({}^{k^{-1}}x_i)) \cdot \underbrace{(k^{-1}\otimes k)\cdot(a_i * e)}_{={}^{k^{-1}}a_i * e} . \end{array} \tag{3.5.13}$$

In view of (3.5.11), the mappings (3.5.12) and (3.5.13) are equal. Thus (3.5.10) is commutative. This finishes proving claim 3, claim 2, claim 1, and, finally, (2). □

4. Application to Ginzburg dg algebras

Assume setting 1.4. It is restated below for convenience.

Setting 4.1. Let Q be a finite graded $\Bbbk$-quiver. Let G be a finite group such that $\mathrm{char}(\Bbbk) \nmid \mathrm{Card}(G)$, acting on $\Bbbk Q$ by degree preserving automorphisms in such a way that both the set of (idempotents associated with) vertices and the vector space generated by the arrows of Q are stabilised by the action. Let W be a homogeneous potential of degree $d-3$ on Q.

The purpose of this section is to prove Corollary 1.5 by applying section 3 to Ginzburg dg algebras. Subsection 4.1 gives a reminder on Ginzburg dg algebras. Subsection 4.2 proves that cyclic derivations are G-equivariant. This is used in subsection 4.3 to prove that G acts on $\mathcal{A}(Q,W)$ by dg automorphisms if W is G-invariant up to cyclic permutation. Subsection 4.4 proves Corollary 1.5.

4.1. **Reminder on Ginzburg dg algebras.** These were introduced in [15, 4.2]. The more general definition used here follows [24, 6.3].

Let Q be a finite *graded* $\Bbbk$-*quiver*, that is, Q is a quiver (or, an oriented graph) with (finite) set of vertices denoted by Q_0 and (finite) set of arrows denoted by Q_1, and the $\Bbbk Q_0$-bimodule $\Bbbk Q_1$ spanned by Q_1 has a $\mathbb{Z}$-grading. Without loss of generality, assume that every arrow of Q is homogeneous. The path algebra $\Bbbk Q$ is hence $\mathbb{Z}$-graded, idempotents e_x of vertices x have degree 0. With this grading, $\Bbbk Q$ is also viewed as a dg algebra with zero differential.

The space of *potentials* on Q is the $\mathbb{Z}$-graded vector space,

$$\frac{\Bbbk Q \otimes_{\Bbbk Q_0{}^e} \Bbbk Q_0}{\left\langle uv\otimes e_x - (-1)^{\deg(u).\deg(v)} vu \otimes e_y \mid u \in e_x \Bbbk Q e_y,\ v \in e_y \Bbbk Q e_x \text{ homogeneous},\ x,y \in Q_0 \right\rangle} . \tag{4.1.1}$$

Hence, potentials are linear combinations of oriented cycles in Q, each of which is considered up to cyclic permutation with Koszul-type signs. Note that $\Bbbk Q$ has a cofibrant resolution in $\mathrm{Mod}(\Bbbk Q^e)$ by the cone of

$$\begin{array}{rcl} \Bbbk Q \otimes_{\Bbbk Q_0} \Bbbk Q_1 \otimes_{\Bbbk Q_0} \Bbbk Q & \to & \Bbbk Q \otimes_{\Bbbk Q_0} \Bbbk Q \\ u \otimes a \otimes v & \mapsto & ua \otimes v - u \otimes av \end{array}$$

so that the Hochschild homology of $\Bbbk Q$ is computed by the cone of

$$\begin{array}{rcl} (\Bbbk Q_1 \otimes_{\Bbbk Q_0} \Bbbk Q) \otimes_{\Bbbk Q_0{}^e} \Bbbk Q_0 & \to & \Bbbk Q \otimes_{\Bbbk Q_0{}^e} \Bbbk Q_0 \\ a \otimes u \otimes 1 & \mapsto & (-1)^{\deg(a)\cdot\deg(u)} ua \otimes 1 . \end{array}$$

Hence, a homogeneous potential of degree n on Q may be viewed as an element of $\mathrm{HH}_n(\Bbbk Q)$.

Let $d \geqslant 3$ be an integer. Let W be a homogeneous potential on Q of degree $d-3$. The Ginzburg dg algebra $\mathcal{A}(Q,W)$ is defined as follows. As a graded algebra, $\mathcal{A}(Q,W)$ equals $\Bbbk\widetilde{Q}$ where $\widetilde{Q}$ is the following graded $\Bbbk$-quiver,

- $\widetilde{Q}$ and Q have the same vertices,

- as a graded $\Bbbk Q_0$-bimodule, $\Bbbk\widetilde{Q}_1$ is equal to

$$\Bbbk\widetilde{Q}_1 = \Bbbk Q_1 \bigoplus \Sigma^{d-2}\mathrm{Hom}_{\Bbbk Q_0{}^e}(\Bbbk Q_1, \Bbbk Q_0{}^e) \bigoplus \Sigma^{d-1}\Bbbk Q_0\,, \tag{4.1.2}$$

given an arrow $a\colon x \to y$ in Q denote by a^* the element of $\mathrm{Hom}_{\Bbbk Q_0{}^e}(\Bbbk Q_1, \Bbbk Q_0{}^e)$ defined by its behaviour on arrows as follows

$$a^*\colon b \in Q_1 \mapsto \begin{cases} e_x \otimes e_y & \text{if } b = a \\ 0 & \text{otherwise,} \end{cases} \tag{4.1.3}$$

and given $i \in Q_0$ denote by c_i the corresponding element of $\Sigma^{d-1}\Bbbk Q_0$; this defines a bijective $\Bbbk$-linear mapping

$$\begin{array}{rcl} \Bbbk Q_1 & \longrightarrow & \mathrm{Hom}_{\Bbbk Q_0{}^e}(\Bbbk Q_1, \Bbbk Q_0{}^e) \\ a \in Q_1 & \longmapsto & a^*\,. \end{array} \tag{4.1.4}$$

Note that the grading on $\mathcal{A}(Q,W)$ is determined by the grading on $\Bbbk Q$.

The differential on $\mathcal{A}(Q,W)$ (of degree $+1$) is uniquely determined by the following rules:

- it vanishes on $\Bbbk Q$,
- it maps $a^*\colon y \to x$ to the cyclic derivative $\partial_a W$, for all arrows $a\colon x \to y$ in Q_1,
- it maps $c_i\colon i \to i$ to $\sum\limits_{a\colon i\to\cdot} aa^* - \sum\limits_{a\colon \cdot\to i} a^*a$, where the first sum runs over all the arrows of Q starting in i and the second sum runs over all the arrows of Q arriving in i.

Recall that, when $a \in Q_1$, the cyclic derivation ∂_a is the linear mapping which takes a path p to $\partial_a p = \sum\limits_{p=p_1ap_2} (-1)^{\deg(p_1)\cdot\deg(ap_2)} p_2p_1$, where the sum runs over all decompositions $p = p_1ap_2$ with paths p_1 and p_2.

By [24, Subsections 6.1 and 6.2, and Theorem A.12], $\mathcal{A}(Q,W)$ is d-Calabi-Yau and there is a quasi-isomorphism of dg algebras

$$\mathbf{\Pi}_d(\Bbbk Q, c) \xrightarrow{\sim} \mathcal{A}(Q,W)\,, \tag{4.1.5}$$

where $c \in \mathrm{HH}_{d-2}(\Bbbk Q)$ is the image of W under Connes' boundary in Hochschild homology $\mathrm{HH}_{d-3}(\Bbbk Q) \xrightarrow{B} \mathrm{HH}_{d-2}(\Bbbk Q)$.

4.2. **Equivariant non commutative differential calculus on quiver algebras.** Assume setting 4.1. In this subsection, tensor products, quotients and Hom-spaces involving G-modules are considered as G-modules in the classical sense of group representations. Denote by $\mathrm{DR}(\Bbbk Q)$ the G-module of potentials on $\Bbbk Q$. The following lemma is essential for subsection 4.3.

Lemma 4.2.1. *Assume setting 4.1. The following $\Bbbk$-linear mapping is G-equivariant*

$$\begin{array}{ccc} \mathrm{Hom}_{\Bbbk Q_0{}^e}(\Bbbk Q_1, \Bbbk Q_0{}^e) \otimes \mathrm{DR}(\Bbbk Q) & \longrightarrow & \Bbbk Q \\ \varphi \otimes w & \longmapsto & d_{\mathcal{A}(Q,w)}(\varphi)\,, \end{array} \tag{4.2.1}$$

where $d_{\mathcal{A}(Q,w)}$ denotes the differential of the dg algebra $\mathcal{A}(Q,w)$, for all potentials w on Q.

Proof. The proof uses the interpretation ([15, Subsections 3.5 and 3.6]) of cyclic derivatives as reduced contractions ([8, Section 2]). Recall that the tensor product of two homogeneous $\Bbbk$-linear mappings f and f' is defined using the Koszul-sign rule, that is $(f \otimes f')(u \otimes v) = (-1)^{\deg(f')\cdot\deg(u)} f(u) \otimes f'(v)$ for all homogeneous u, v such that $f(u)$ and $f'(v)$ are defined. Denote by $\Delta_{\Bbbk Q}$ the natural coassociative and counital comultiplication of $\Bbbk Q$, viewed as $T_{\Bbbk Q_0}(\Bbbk Q_1)$. Denote by p the natural projection from $\Bbbk Q$ to $\Bbbk Q_1$. Denote by $-^\diamond$ the $\Bbbk$-linear mapping from $\Bbbk Q \otimes \Bbbk Q$ to $\Bbbk Q$ given by $(u \otimes v)^\diamond = (-1)^{\deg(u)\cdot\deg(v)} vu$ for all homogeneous $u, v \in \Bbbk Q$.

By considering tensors of the shape $a^* \otimes w$ where $a \in Q_1$ and $w \in \mathrm{DR}(\Bbbk Q)$, it is elementary to check that (4.2.1) is equal to the following mapping (for the ease of reading, no distinction is made between a linear combination of oriented cycles and its associated potential)

$$\begin{array}{ccc} \mathrm{Hom}_{\Bbbk {Q_0}^e}(\Bbbk Q_1, \Bbbk {Q_0}^e) \otimes \mathrm{DR}(\Bbbk Q) & \longrightarrow & \Bbbk Q \\ \varphi \otimes w & \longmapsto & \left(\left(\left(\mathrm{Id} \otimes (\varphi \circ p) \otimes \mathrm{Id}\right) \circ (\Delta_{\Bbbk Q} \otimes \mathrm{Id}) \circ \Delta_{\Bbbk Q}\right)(w)\right)^{\diamond} . \end{array}$$

This is G-equivariant, hence so is (4.2.1) □

Remark 4.2.2. The proof and the statement of Lemma 4.2.1 still hold true when G is infinite.

Remark 4.2.3. Let $g \in G$. It is elementary to check that (4.2.1) is $\langle g \rangle$-equivariant if

$$(\forall a \in Q_1) \quad {}^g a \in \{t \cdot b \mid t \in \Bbbk^\times, b \in Q_1\} . \tag{4.2.2}$$

Since G is finite, one may try to reduce the proof of Lemma 4.2.1 to that particular case using a $\Bbbk$-algebra automorphism $f \colon \Bbbk Q \to \Bbbk Q$ such that $f^{-1}({}^g f(-))$ satisfies (4.2.2) in place of g. The reduction then consists in proving that the conjugation of (4.2.1) under f^{-1} is $\langle g \rangle$-equivariant. However, this is not immediate because the conjugate of (4.2.1) under f^{-1} is not (4.2.1). Rather, its expression is governed by the cyclic chain rule ([11, Lemma 3.9]). The details of the reduction using this rule are left to the reader.

The following example illustrates the fact that conjugating the cyclic derivation by an automorphism does not yield the cyclic derivation.

Example 4.2.4. Assume that $\Bbbk = \mathbb{C}$, that $G = \{e, g\}$, that Q is the quiver

, where $\deg(a) = \deg(b) = 0$, that ${}^g a = \frac{a+b}{\sqrt{2}}$ and ${}^g b = \frac{a-b}{\sqrt{2}}$, and that W is the $\langle g \rangle$-invariant potential $a^2 + {}^g(a^2) = \frac{1}{2}(3a^2 + ab + ba + b^2)$. In Remark 4.2.3, one may assume that $f(a) = a - ib$ and $f(b) = a + ib$. Indeed, ${}^g f(a) = \frac{1+i}{2\sqrt{2}} f(b)$ and ${}^g f(b) = \frac{1-i}{2\sqrt{2}} f(a)$. Denote (4.2.1) by ι and denote its conjugate under f^{-1} by ι'. Note that f acts on $\mathrm{Hom}_{\Bbbk {Q_0}^e}(\Bbbk Q_1, \Bbbk {Q_0}^e)$ by precomposition with f^{-1}. On one hand, $\iota(a^* \otimes W) = \partial_a(W) = 3a + b$. On the other hand, $\iota'(a^* \otimes W) = f^{-1}(\iota((a^* \circ f^{-1}) \otimes f(W)))$; note that $a^* \circ f^{-1} = \frac{1}{2}(a^* - ib^*)$ and $f(W) = 3a^2 + iab + iba - b^2$; hence $\iota'(a^* \otimes W) = 4a + 2ib$. Thus ι is not invariant under conjugation by f^{-1}.

4.3. **Group actions on $\mathcal{A}(Q, W)$.** Assume setting 4.1 and assume that W is G-invariant up to cyclic permutation. Like in subsection 4.2, tensor products, quotients and Hom-spaces involving G-modules are considered as G-modules in the classical sense of group representations. In particular, $\mathrm{Hom}_{\Bbbk {Q_0}^e}(\Bbbk Q_1, \Bbbk {Q_0}^e)$ is a G-module. Denote by c the image of W under $B \colon \mathrm{HH}_{d-3}(\Bbbk Q) \to \mathrm{HH}_{d-2}(\Bbbk Q)$. Then c lies in $\mathrm{HH}_{d-2}(\Bbbk Q)^G$. The purpose of this subsection is to show that G acts on $\mathcal{A}(Q, W)$ by algebra automorphisms, that this is an action by dg automorphisms, and that, when $\mathbf{\Pi}_d(\Bbbk Q, c)$ is endowed with the action of G of Lemma 3.5.3, the following quasi-isomorphism of [24, Theorem 6.3] is G-equivariant

$$\mathbf{\Pi}_d(\Bbbk Q, c) \to \mathcal{A}(Q, W) . \tag{4.3.1}$$

Here is a reminder on this quasi-isomorphism. In $\mathrm{Mod}(\Bbbk Q^e)$, there is a cofibrant replacement of $\Bbbk Q$ equal to the cone of the following morphism

$$\begin{array}{ccc} \Bbbk Q \otimes_{\Bbbk Q_0} \Bbbk Q_1 \otimes_{\Bbbk Q_0} \Bbbk Q & \xrightarrow{\alpha} & \Bbbk Q \otimes_{\Bbbk Q_0} \Bbbk Q \\ 1 \otimes a \otimes 1 & \mapsto & a \otimes 1 - 1 \otimes a . \end{array} \tag{4.3.2}$$

Denote by $\Theta_{\Bbbk Q}$ the cylinder of the following morphism.

$$\mathrm{Hom}_{\Bbbk Q^e}(\Bbbk Q \otimes_{\Bbbk Q_0} \Bbbk Q, \Bbbk Q^e) \xrightarrow{\alpha^t} \mathrm{Hom}_{\Bbbk Q^e}(\Bbbk Q \otimes_{\Bbbk Q_0} \Bbbk Q_1 \otimes_{\Bbbk Q_0} \Bbbk Q, \Bbbk Q^e) . \tag{4.3.3}$$

Hence, $\Theta_{\Bbbk Q}$ is a cofibrant replacement of $\mathrm{RHom}_{\Bbbk Q^e}(\Bbbk Q, \Bbbk Q \otimes \Bbbk Q)$ in $\mathrm{Mod}(\Bbbk Q^e)$ and it determines the dg algebra $\mathbf{\Pi}_d(\Bbbk Q, c)$ (see part (2) of Lemma 3.5.3). Note that $\Theta_{\Bbbk Q}$ is canonically isomorphic to the cone of the following morphism

$$\Bbbk Q_0 \otimes_{\Bbbk Q_0{}^e} \Bbbk Q^e \to \mathrm{Hom}_{\Bbbk Q_0{}^e}(\Bbbk Q_1, \Bbbk Q_0{}^e) \otimes_{\Bbbk Q_0{}^e} \Bbbk Q^e \,. \tag{4.3.4}$$

Since $\mathbf{\Pi}_d(\Bbbk Q, c) = T_{\Bbbk Q}(\Sigma^{d-1}\Theta_{\Bbbk Q})$ as graded algebras, (4.3.1) is an isomorphism of graded algebras, not just a quasi-isomorphism, and it is induced by the natural inclusions

(4.3.5)

$$\Bbbk Q \hookrightarrow \mathcal{A}(Q,W),\ \Sigma^{d-2}\mathrm{Hom}_{\Bbbk Q_0{}^e}(\Bbbk Q_1, \Bbbk Q_0{}^e) \hookrightarrow \mathcal{A}(Q,W) \text{ and } \left\{ \begin{array}{ccc} \Sigma^{d-1}\Bbbk Q_0 & \hookrightarrow & \mathcal{A}(Q,W) \\ e_i & \mapsto & c_i \,. \end{array} \right.$$

Note that each one of the $\Bbbk Q^e$-modules appearing in (4.3.3) and (4.3.4) is endowed with an action of G. These actions are compatible so that the corresponding objects are dg Δ-modules. In this sense, (4.3.3) and (4.3.4) are morphisms of dg Δ-modules and their cones lie in $\mathrm{Mod}(\Delta)$.

On one hand, $\Theta_{\Bbbk Q}$ is a dg Δ-module on its own by means of Lemma 3.3.1 and this structure is used to endow $\mathbf{\Pi}_d(\Bbbk Q, c)$ with action of G (see Lemma 3.5.3). By construction, the structure of dg Δ-module of $\Sigma\Theta_{\Bbbk Q}$ coincides with the one of the cone of (4.3.3).

On the other hand, the actions of G on $\Bbbk Q$ and $\mathrm{Hom}_{\Bbbk Q_0{}^e}(\Bbbk Q_1, \Bbbk Q_0{}^e)$ endow $\Bbbk \widetilde{Q}$ with an action by automorphisms of graded algebra (recall that $\widetilde{Q}$ is the graded quiver whose path algebra equals the underlying graded algebra of $\mathcal{A}(Q,W)$, see (4.1.2)). And it follows from Lemma 4.2.1 that this is actually an action by dg automorphisms. The mappings of (4.3.5) are G-equivariant, and hence so is (4.3.1).

Lemma 4.3.1. *Assume setting 4.1. Assume that W is G-invariant up to cyclic permutation.*

- *Take $\Theta_{\Bbbk Q}$ to be the cofibrant replacement of $\mathrm{RHom}_{\Bbbk Q^e}(\Bbbk Q, \Bbbk Q^e)$ whose suspension is equal to the cone of (4.3.3) and endow $\mathbf{\Pi}_d(\Bbbk Q, c)$ with its action of G by dg automorphisms as in Lemma 3.5.3.*
- *Endow $\mathrm{Hom}_{\Bbbk Q_0^e}(\Bbbk Q_1, \Bbbk Q_0{}^e)$ with its natural action of G.*

Then,

1. *the latter action together with the one on $\Bbbk Q$ endow $\mathcal{A}(Q,W)$ with an action of G by graded algebra automorphisms. For this action, (4.3.1) is G-equivariant.*
2. *the action of G on $\mathcal{A}(Q,W)$ is an action by dg automorphisms and, taking $\mathbf{\Pi}_d(\Bbbk Q, c)$ like in part (2) of Lemma 3.5.3, the mapping (4.3.1) is a G-equivariant isomorphism of dg algebras.*

Proof. This follows from the discussion made from the beginning of the subsection. □

4.4. **Skew group algebras of Ginzburg dg algebras.** Assume setting 4.1. Assume that W is G-invariant up to cyclic permutation. Following subsection 4.3, endow $\mathcal{A}(Q,W)$ with the resulting action of G by dg automorphisms. The purpose of this subsection is to prove Corollary 1.5.

Following [34] and [9], there exists a graded $\Bbbk$-quiver Q' and a (non unital) injective homomorphism of graded algebras $\Bbbk Q' \to \Bbbk Q * G$ whose restriction-of-scalars functor is an equivalence $\mathrm{Mod}(\Bbbk Q * G) \to \mathrm{Mod}(\Bbbk Q')$. The quiver Q' is constructed in [34] when G is cyclic and in [9] in full generality. Here is a reminder of its definition.

Definition 4.4.1 ([9])**.** Let $[G\backslash Q_0]$ be a complete set of representatives of the G-orbits of vertices of Q. For each $i \in Q_0$, denote by G_i the stabiliser of i and let $[G/G_i]$ be a complete set of representatives of the cosets of G modulo G_i. Finally, for all $i \in Q_0$, let $\mathrm{irr}(G_i)$ be a complete set of representatives of the isomorphism classes of the irreducible representations of G_i; it is convenient to assume that $\rho = \Bbbk G_i e_\rho$ for some primitive idempotent e_ρ of $\Bbbk G_i$, for all $\rho \in \mathrm{irr}(G_i)$.

(1) Let ε be the following idempotent of $\Bbbk Q * G$,

$$\varepsilon = \sum_{\substack{i\in[G\backslash Q_0]\\ \rho\in\mathrm{irr}(G_i)}} e_i * e_\rho\,, \tag{4.4.1}$$

(2) Let Q' be any quiver as follows.
- Its vertices are the pairs (i,ρ) where $i\in[G\backslash Q_0]$ and $\rho\in\mathrm{irr}(G_i)$.
- For all vertices (i,ρ) and (j,τ), denote by $M(i,j;\tau)$ the following vector subspace of $\Bbbk Q * G$

$$M(i,j;\tau) = \bigoplus_{y\in[G/G_j]} (e_i \Bbbk Q_1 e_{y\cdot j}) * (y\Bbbk G_j e_\tau)\,; \tag{4.4.2}$$

by means of the multiplication, $\Bbbk Q * G$ is a representation of G and, by restriction, it is a representation of G_i; then $M(i,j;\tau)$ is a subrepresentation of this G_i-module; the vector space with basis being the family of arrows of Q' from (i,ρ) to (j,τ) is $\mathrm{Hom}_{G_i}(\rho, M(i,j;\tau))$.

There exists an isomorphism of algebras

$$\Bbbk Q' \xrightarrow{\sim} \varepsilon\cdot(\Bbbk Q * G)\cdot\varepsilon \tag{4.4.3}$$

given as follows,
- for all vertices (i,ρ) of Q', the corresponding idempotent of $\Bbbk Q'$ is mapped onto $e_i * e_\rho$,
- for all vertices (i,ρ) and (j,τ) of Q', and for all $f\in\mathrm{Hom}_{G_i}(\rho, M(i,j;\tau))$, then f is mapped onto $f(e_\rho)$. Hence, the arrow $f\colon(i,\rho)\to(j,\tau)$ of Q' may be identified with the corresponding element $f(e_\rho)$ of $(e_i * e_\rho)\cdot(\Bbbk Q * G)\cdot(e_j * e_\tau)$.

The following lemma introduces a potential W' on Q'. The purpose of this subsection is to prove that $\mathcal{A}(Q,W) * G$ and $\mathcal{A}(Q',W')$ have equivalent derived categories.

Lemma 4.4.2. *Assume setting 4.1. Assume that W is G-invariant up to cyclic permutation. There exists a homogeneous potential W' of degree $d-3$ on Q' such that the image of W under the mapping $\mathrm{HH}_{d-3}(\Bbbk Q)\to\mathrm{HH}_{d-3}(\Bbbk Q * G)$ induced by the natural embedding $\Bbbk Q\to\Bbbk Q * G$ is equal to the image of W' under the isomorphism $\mathrm{HH}_{d-3}(\Bbbk Q')\to\mathrm{HH}_{d-3}(\Bbbk Q * G)$ induced by $\Bbbk Q' \xrightarrow{(4.4.3)} \Bbbk Q * G \hookrightarrow \Bbbk Q * G$.*

Proof. This follows from standard facts on Hochschild homology, see [31, Lemma 1.1.7, p. 10]. Here are some details since they are needed for the examples. Recall that, for a graded algebra R, the piece of notation $\mathrm{DR}(R)$ stands for the quotient R by the vector subspace generated by the commutators $uv-(-1)^{\deg(u)\cdot\deg(v)}vu$, for all homogeneous $u,v\in R$.

Following [9], there exists a complete family $(\varepsilon_j)_{j\in J}$ of primitive pairwise orthogonal idempotents of $\Bbbk Q * G$ and there exists a subset $I\subseteq J$ such that,
- the image of $\Bbbk Q'\to\Bbbk Q * G$ is equal to $\varepsilon(\Bbbk Q * G)\varepsilon$, where ε denotes $\sum_{i\in I}\varepsilon_i$ and
- for all $j\in J$ there exists a unique $\alpha(j)\in I$ such that $\varepsilon_j\cdot(\Bbbk Q * G)\simeq\varepsilon_{\alpha(j)}\cdot(\Bbbk Q * G)$ as graded $\Bbbk Q * G$-modules.

For all $j\in J$, there hence exist homogeneous $a_j,b_j\in\Bbbk Q * G$ such that $\varepsilon_j = a_jb_j$ and $\varepsilon_{\alpha(j)} = b_ja_j$.

Now, since $1_{\Bbbk Q * G} = \sum_{j\in J}\varepsilon_j$, the following equalities hold true in $\mathrm{DR}(\Bbbk Q * G)$, where elements of $\Bbbk Q$ are identified with their natural images in $\Bbbk Q * G$,

$$\begin{aligned} W &= \textstyle\sum_{j\in J}\varepsilon_j W\varepsilon_j\\ &= \textstyle\sum_{j\in J} a_jb_jWa_jb_ja_jb_j\\ &= \textstyle\sum_{i\in I}\varepsilon_i\left(\sum_{j\ s.t.\ \alpha(j)=i}\pm b_jWa_j\right)\varepsilon_i\,; \end{aligned}$$

the sign is $(-1)^{\deg(b_j)\cdot(\deg(a_j)+\deg(W))}$. The last term above defines an element of $\mathrm{DR}(\varepsilon(\Bbbk Q*G)\varepsilon)$ whose image under the mapping $\mathrm{DR}(\varepsilon(\Bbbk Q*G)\varepsilon)\to\mathrm{DR}(\Bbbk Q*G)$ induced by the inclusion mapping $\varepsilon\cdot(\Bbbk Q*G)\cdot\varepsilon\to\Bbbk Q*G$ is equal to the image of W under the mapping $\mathrm{DR}(\Bbbk Q)\to\mathrm{DR}(\Bbbk Q*G)$ induced by $\Bbbk Q\to\Bbbk Q*G$. Since $\Bbbk Q'\simeq\varepsilon(\Bbbk Q*G)\varepsilon$ as graded algebras, this proves the lemma. □

The potential provided by Lemma 4.4.2 fits the requirements of Corollary 1.5 which can now be proved. This corollary is restated below for convenience.

Corollary 4.4.3. *Assume setting 4.1. Assume that W is G-invariant up to cyclic permutation.*

(1) The action of G on $\Bbbk Q$ extends to an action of G on $\mathcal{A}(Q,W)$ by dg automorphisms and $\mathcal{A}(Q,W)*G$ is d-Calabi-Yau.

(2) For all graded $\Bbbk$-quivers Q' and for all (non unital) graded algebra homomorphisms $\Bbbk Q'\to\Bbbk Q*G$ whose restriction-of-scalars functor is an equivalence from $\mathrm{Mod}(\Bbbk Q*G)$ to $\mathrm{Mod}(\Bbbk Q')$ (see [9]), there exists a homogeneous of degree $d-3$ potential W' on Q' such that $\Bbbk Q'\to\Bbbk Q*G$ extends to a (non unital) dg algebra homomorphism

$$\mathcal{A}(Q',W')\to\mathcal{A}(Q,W)*G$$

whose restriction-of-scalars functor induces an equivalence

$$\mathcal{D}(\mathcal{A}(Q,W)*G)\xrightarrow{\sim}\mathcal{D}(\mathcal{A}(Q',W'))\,.$$

Proof. Let W' be as in Lemma 4.4.2. The following diagram is commutative. In this diagram, B stands for the Connes' boundary and the horizontal arrows are induced by $\Bbbk Q\to\Bbbk Q*G$ and $\Bbbk Q'\to\Bbbk Q*G$, and the rightmost horizontal arrows are isomorphisms because $\Bbbk Q'$ and $\Bbbk Q*G$ are Morita equivalent.

$$\begin{tikzcd}
\mathrm{HH}_{d-3}(\Bbbk Q)^G \arrow[r, hook] \arrow[d] & \mathrm{HH}_{d-3}(\Bbbk Q) \arrow[r] \arrow[d, "B"'] & \mathrm{HH}_{d-3}(\Bbbk Q*G) \arrow[d, "B"'] & \mathrm{HH}_{d-3}(\Bbbk Q') \arrow[l, "\sim"'] \arrow[d, "B"] \\
\mathrm{HH}_{d-2}(\Bbbk Q)^G \arrow[r, hook] & \mathrm{HH}_{d-2}(\Bbbk Q) \arrow[r] & \mathrm{HH}_{d-2}(\Bbbk Q*G) & \mathrm{HH}_{d-2}(\Bbbk Q') \arrow[l, "\sim"]
\end{tikzcd} \tag{4.4.4}$$

This defines homology classes $c,\overline{c},c'$ as follows,

(a) $c\in\mathrm{HH}_{d-2}(\Bbbk Q)^G$ is the image of W under $B\colon\mathrm{HH}_{d-3}(\Bbbk Q)\to\mathrm{HH}_{d-2}(\Bbbk Q)$,
(b) $\overline{c}\in\mathrm{HH}_{d-2}(\Bbbk Q*G)$ is the image of c under $\mathrm{HH}_{d-2}(\Bbbk Q)\to\mathrm{HH}_{d-2}(\Bbbk Q*G)$ and
(c) $c'\in\mathrm{HH}_{d-2}(\Bbbk Q')$ is the pre-image of $\overline{c}$ under $\mathrm{HH}_{d-2}(\Bbbk Q')\xrightarrow{\sim}\mathrm{HH}_{d-2}(\Bbbk Q*G)$.

In particular,

(d) c' is the image of W' under $B\colon\mathrm{HH}_{d-3}(\Bbbk Q')\to\mathrm{HH}_{d-2}(\Bbbk Q')$.

The following realisations of the graded algebras $\mathbf{\Pi}_d(\Bbbk Q,c)$ and $\mathbf{\Pi}_d(\Bbbk Q*G,\overline{c})$ are used in this proof,

- as for $\mathbf{\Pi}_d(\Bbbk Q,c)$, take the de-suspension of the cone of (4.3.3) as a cofibrant replacement $\Theta_{\Bbbk Q}$ of $\mathrm{RHom}_{\Bbbk Q^e}(\Bbbk Q,\Bbbk Q^e)$ and
- as for $\mathbf{\Pi}_d(\Bbbk Q,\overline{c})$, take $\Theta_{\Bbbk Q}*G$ as cofibrant replacement of $\mathrm{RHom}_{(\Bbbk Q*G)^e}(\Bbbk Q*G,(\Bbbk Q*G)^e)$ (see the proof of Theorem 3.5.4).

The proof of the proposition is based on a sequence of quasi-isomorphic dg algebras. Actually, because of the choices just made, this sequence comes from a sequence of composable isomorphisms of dg algebras.

As proved in Lemma 4.3.1, there is G-equivariant isomorphism $\mathbf{\Pi}_d(\Bbbk Q,c)\xrightarrow{\sim}\mathcal{A}(Q,W)$ of dg algebras. Accordingly, there is an isomorphism of dg algebras

$$\mathbf{\Pi}_d(\Bbbk Q,c)*G\xrightarrow{\sim}\mathcal{A}(Q,W)*G\,. \tag{4.4.5}$$

Following Theorem 1.3, there is an isomorphism of dg algebras

$$\mathbf{\Pi}_d(\Bbbk Q, c) * G \xrightarrow{\sim} \mathbf{\Pi}_d(\Bbbk Q * G, \overline{c})\,. \tag{4.4.6}$$

Now, by [24, Theorem 5.8], the (non unital) algebra homomorphism $\Bbbk Q' \to \Bbbk Q * G$ extends to a (non unital) dg algebra homomorphism $\mathbf{\Pi}_d(\Bbbk Q', c') \to \mathbf{\Pi}_d(\Bbbk Q * G, \overline{c})$ whose restriction-of-scalars functor is a triangle equivalence

$$\mathcal{D}(\mathbf{\Pi}_d(\Bbbk Q', c')) \simeq \mathcal{D}(\mathbf{\Pi}_d(\Bbbk Q * G, \overline{c}))\,. \tag{4.4.7}$$

Finally, in view of point (d) above relating c' to W', it follows from [24, Theorem 6.3] that the identity mapping $\Bbbk Q' \to \Bbbk Q'$ extends to a quasi-isomorphism of dg algebras

$$\mathbf{\Pi}_d(\Bbbk Q', c') \xrightarrow{\text{qis}} \mathcal{A}(Q', W')\,. \tag{4.4.8}$$

The assertion (2) follows from (4.4.5), (4.4.6), (4.4.7), and (4.4.8). The assertion (1) follows from (2) and from the fact that $\mathcal{A}(Q', W')$ is d-Calabi-Yau. □

4.5. **Examples.** Assume setting 4.1. Assume that W is G-invariant up to cyclic permutation. This subsection illustrates the computation of a quiver with potential (Q', W') such as in Corollary 4.4.3. In the examples of this subsection, graded $\Bbbk$-quivers are concentrated in degree 0 and the resulting Ginzburg dg algebras are 3-Calabi-Yau. For convenience, for all $g \in G$ and $i \in Q_0$, the vertex of Q associated with the idempotent ${}^g e_i$ is denoted by $g \cdot i$. In some places it is also convenient to identify the elements of $\Bbbk Q$ and G with their respective images in $\Bbbk Q * G$.

The computation of a potential W' such as above is therefore made in two steps.

(1) Express W' as an element of $\varepsilon \cdot (\Bbbk Q * G) \cdot \varepsilon$, see the proof of Lemma 4.4.2.
(2) Express the result of the first step as a linear combination of paths in Q'.

Remark 4.5.1. For the second step, it is worth noting that any oriented cycle in Q with source $i_0 \in [G \backslash Q_0]$ is equal in $\Bbbk Q * G$ to a linear combination of products of the shape

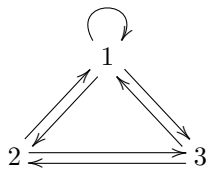
$$(\alpha_1 * \sigma_1) \cdots (\alpha_j * \sigma_j) \cdots (\alpha_t * \sigma_t) \cdot (1 * g)\,, \tag{4.5.1}$$

where there exists a sequence $i_1, \ldots, i_t \in [G \backslash Q_0]$ such that α_j is an arrow of Q from i_{j-1} to $\sigma_j \cdot i_j$ and $\sigma_j \in [G/G_j]$, for all j, and where $g \in G_{i_0}$; besides, when a given index $j \in \{1, \ldots, t-1\}$ is such that G_{i_j} is abelian, then $\sum_{\rho \in \mathrm{irr}(G_{i_j})} e_\rho$ is the unity of $\Bbbk G_i$, and hence

$$(\alpha_j * \sigma_j) \cdot (\alpha_{j+1} * \sigma_{j+1}) = \sum_{\rho \in \mathrm{irr}(G_{i_j})} (\alpha_j * \sigma_j) \cdot (e_{i_j} * e_\rho) \cdot (\alpha_{j+1} * \sigma_{j+1})\,. \tag{4.5.2}$$

Example 4.5.2. Assume that (Q, W) is the following quiver with potential

(4.5.3)



$$W = x_{12}x_{23}x_{32}x_{21} - x_{13}x_{32}x_{23}x_{31} + x_{13}x_{31}x_{11} - x_{12}x_{21}x_{11}$$

where x_{ij} denotes the unique arrow $i \to j$ of Q if any, for all vertices i, j. Recall that $\mathcal{A}(Q, W)$ is quasi-isomorphic to $H^0\mathcal{A}(Q, W)$ and that the latter is a non commutative crepant resolution of the suspended pinch-point, see [32, Section 8]. Let G be the subgroup $\langle (23) \rangle$ of order 2 of the symmetric group $\mathscr{S}_3$. Then, G acts on $\Bbbk Q$ as follows

- $\sigma \cdot i = \sigma(i)$ for all $\sigma \in G$ and $i \in \{1, 2, 3\}$,
- ${}^{(23)}x_{12} = -x_{13}$ and ${}^{(23)}x_{13} = -x_{12}$, and
- ${}^{\sigma}x_{ij} = x_{\sigma(i)\sigma(j)}$ for all $\sigma \in G$ and for all arrows x_{ij} of Q distinct from x_{12} and x_{13}.

Note that, for this action, W is invariant up to cyclic permutation. One may take the following data to define Q'.

- $[G\backslash Q_0] = \{1,2\}$, $[G/G_1] = \{\mathrm{Id}\}$ and $[G/G_2] = G$,
- $\mathrm{irr}(G_1)$ consists of the trivial representation $\rho_+ = \Bbbk G_1 \cdot (\mathrm{Id} + (23))$ of G_1 and the non trivial one $\rho_- = \Bbbk G_1 \cdot (\mathrm{Id} - (23))$,
- $\mathrm{irr}(G_2)$ consists of the trivial representation $\Bbbk$ of G_2.

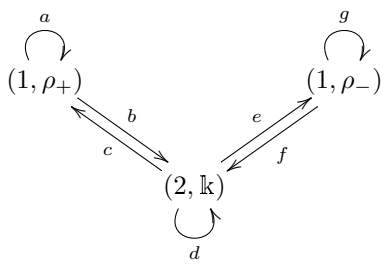
Then (see (4.4.1)), $\varepsilon = \varepsilon_+ + \varepsilon_- + \varepsilon_2$, where $\varepsilon_+ = \frac{1}{2}(e_1 * \mathrm{Id} + e_1 * (23))$, $\varepsilon_- = \frac{1}{2}(e_1 * \mathrm{Id} - e_1 * (23))$ and $\varepsilon_2 = e_2 * e$, recall that e denotes the neutral element of G. Hence (see (4.4.2)),

$$\begin{array}{rcl} M(1,1;\rho_+) & = & \mathrm{span}(x_{11}\varepsilon_+) \simeq \rho_+ \\ M(1,1;\rho_-) & = & \mathrm{span}(x_{11}\varepsilon_-) \simeq \rho_- \\ M(1,2;\Bbbk) & = & \underbrace{\mathrm{span}(\varepsilon_+ x_{12} e_2)}_{\simeq \rho_+} \bigoplus \underbrace{\mathrm{span}(\varepsilon_- x_{12} \varepsilon_2)}_{\simeq \rho_-} \end{array} \qquad \begin{array}{rcl} M(2,1;\rho_+) & = & \mathrm{span}(x_{21}\varepsilon_+) \simeq \Bbbk \\ M(2,1;\rho_-) & = & \mathrm{span}(x_{21}\varepsilon_-) \simeq \Bbbk \\ M(2,2;\Bbbk) & = & \mathrm{span}(x_{23}(23)\varepsilon_2) \simeq \Bbbk\,, \end{array}$$

note that, because of the equalities ${}^{(23)}x_{12} = -x_{13}$ and ${}^{(23)}x_{13} = -x_{12}$,

$$\begin{array}{rclrcl} \varepsilon_- x_{11} \varepsilon_+ & = & 0, & \varepsilon_+ x_{12} \varepsilon_2 & = & -\varepsilon_+ x_{13}(23)\varepsilon_2, \\ \varepsilon_+ x_{11} \varepsilon_- & = & 0, & \varepsilon_- x_{12} \varepsilon_2 & = & \varepsilon_- x_{13}(23)\varepsilon_2\,. \end{array} \tag{4.5.4}$$

Hence Q' is the following quiver



where

$$\begin{array}{rclrclrcl} a & = & \varepsilon_+ x_{11} \varepsilon_+ & b & = & \varepsilon_+ x_{12} \varepsilon_2 & c & = & \varepsilon_2 x_{21} \varepsilon_+ \\ d & = & \varepsilon_2 x_{23}(23)\varepsilon_2 & e & = & \varepsilon_2 x_{21} \varepsilon_- & f & = & \varepsilon_- x_{12} \varepsilon_2 \\ g & = & \varepsilon_- x_{11} \varepsilon_-\,. & & & & & & \end{array}$$

Now, $W = e_1 W e_1 = \varepsilon_+ W \varepsilon_+ + \varepsilon_- W \varepsilon_-$ because $e_1 = \varepsilon_+ + \varepsilon_-$. Note that $\varepsilon_- W \varepsilon_+ = \varepsilon_+ W \varepsilon_- = 0$ because W is G-invariant up to cyclic permutation. Therefore, by Lemma 4.4.2, it is possible to assume that

$$W' = \varepsilon_+ W \varepsilon_+ + \varepsilon_- W \varepsilon_-\,. \tag{4.5.5}$$

There only remains to decompose W' as a linear combination of oriented cycles in Q'. For this purpose, it is worth noting that $\sum_{\rho \in \mathrm{irr}(G_1)} e_1 * e_\rho = \varepsilon_+ + \varepsilon_-$ and $\sum_{\rho \in \mathrm{irr}(G_2)} e_2 * e_\rho = \varepsilon_2$. Using Remark 4.5.1 and (4.5.4), the computations yield,

$$\begin{array}{rcl} \varepsilon_+ x_{12} x_{23} x_{32} x_{21} \varepsilon_+ & = & (\varepsilon_+ x_{12} \varepsilon_2) \cdot (\varepsilon_2 x_{23}(23)\varepsilon_2) \cdot (\varepsilon_2 x_{23}(23)\varepsilon_2) \cdot (\varepsilon_2 x_{21} \varepsilon_+) \\ \varepsilon_- x_{12} x_{23} x_{32} x_{21} \varepsilon_- & = & (\varepsilon_- x_{12} \varepsilon_2) \cdot (\varepsilon_2 x_{23}(23)\varepsilon_2) \cdot (\varepsilon_2 x_{23}(23)\varepsilon_2) \cdot (\varepsilon_2 x_{21} \varepsilon_-) \\ \varepsilon_+ x_{13} x_{32} x_{23} x_{31} \varepsilon_+ & = & -(\varepsilon_+ x_{12} \varepsilon_2) \cdot (\varepsilon_2 x_{23}(23)\varepsilon_2) \cdot (\varepsilon_2 x_{23}(23)\varepsilon_2) \cdot (\varepsilon_2 x_{21} \varepsilon_+) \\ \varepsilon_- x_{13} x_{32} x_{23} x_{31} \varepsilon_- & = & -(\varepsilon_- x_{12} \varepsilon_2) \cdot (\varepsilon_2 x_{23}(23)\varepsilon_2) \cdot (\varepsilon_2 x_{23}(23)\varepsilon_2) \cdot (\varepsilon_2 x_{21} \varepsilon_-) \\ \varepsilon_+ x_{13} x_{31} x_{11} \varepsilon_+ & = & -(\varepsilon_+ x_{12} \varepsilon_2) \cdot (\varepsilon_2 x_{21} \varepsilon_+) \cdot (\varepsilon_+ x_{11} \varepsilon_+) \\ \varepsilon_- x_{13} x_{31} x_{11} \varepsilon_- & = & -(\varepsilon_- x_{12} \varepsilon_2) \cdot (\varepsilon_2 x_{21} \varepsilon_-) \cdot (\varepsilon_- x_{11} \varepsilon_-) \\ \varepsilon_+ x_{12} x_{21} x_{11} \varepsilon_+ & = & (\varepsilon_+ x_{12} \varepsilon_2) \cdot (\varepsilon_2 x_{21} \varepsilon_+) \cdot (\varepsilon_+ x_{11} \varepsilon_+) \\ \varepsilon_- x_{12} x_{21} x_{11} \varepsilon_- & = & (\varepsilon_- x_{12} \varepsilon_2) \cdot (\varepsilon_2 x_{21} \varepsilon_-) \cdot (\varepsilon_- x_{11} \varepsilon_-)\,. \end{array}$$

It follows from these computations and from (4.5.3) and (4.5.5) that $W' = 2(bd^2c + fd^2e - bca - feg)$.

Example 4.5.3. Assume that (Q, W) is the following quiver with potential

(4.5.6)
$$\begin{tikzcd} & 1 \arrow[dr, shift left] \arrow[dl, shift left] & \\ 2 \arrow[ur, shift left] \arrow[rr, shift left] & & 3 \arrow[ul, shift left] \arrow[ll, shift left] \end{tikzcd} \qquad W = x_{12}x_{23}x_{31} - x_{13}x_{32}x_{21}$$

where the naming of the arrows follows the same convention as in Example 4.5.2. Let G be the symmetric group $\mathscr{S}_3$ on $\{1, 2, 3\}$. It acts on $\Bbbk Q$ as follows.

- The action on the set of vertices is the natural one.
- ${}^{\sigma}x_{i,j} = (-1)^{\sigma} x_{\sigma(i)\sigma(j)}$ for all $\sigma \in G$ and $i, j \in \{1, 2, 3\}$, where $(-1)^{\sigma}$ is the sign of the permutation σ.

For this action, W is invariant up to cyclic permutation. Actually, $W = \frac{1}{3}\sum_{\sigma\in G} {}^{\sigma}(x_{12}x_{23}x_{31})$.

Take the following sets of representatives to define Q',

- $[G\backslash Q_0] = \{1\}$, $[G/G_1] = \{\mathrm{Id}, c, c^2\} = \langle c\rangle$, where $c = (123)$,
- $\mathrm{irr}(G_1)$ consists of the trivial representation $\rho_+ = \Bbbk G_1 \cdot (\mathrm{Id} + (23))$ of G_1 $(= G)$ and the non trivial one $\rho_- = \Bbbk G_1 \cdot (\mathrm{Id} - (23))$.

Then, ε is the sum of two primitive pairwise orthogonal idempotents of $\Bbbk Q * G$,

$$\varepsilon = \varepsilon_+ + \varepsilon_- ,$$

where $\varepsilon_+ = \frac{1}{2}(e_1 * \mathrm{Id} + e_1 * (23))$ and $\varepsilon_- = \frac{1}{2}(e_1 * \mathrm{Id} - e_1 * (23))$. Hence,

- $M(1, 1; \rho_+) = \mathrm{span}(x_{12}c\varepsilon_+, x_{13}c^2\varepsilon_+) = \underbrace{\mathrm{span}(\varepsilon_+x_{12}c\varepsilon_+)}_{\simeq\rho_+} \oplus \underbrace{\mathrm{span}(\varepsilon_-x_{12}c\varepsilon_+)}_{\simeq\rho_-}$,
- $M(1, 1; \rho_-) = \mathrm{span}(x_{12}c\varepsilon_-, x_{13}c^2\varepsilon_-) = \underbrace{\mathrm{span}(\varepsilon_+x_{12}c\varepsilon_-)}_{\simeq\rho_+} \oplus \underbrace{\mathrm{span}(\varepsilon_-x_{12}c\varepsilon_-)}_{\simeq\rho_-}$,

note that $(23)x_{13}c^2 = -x_{12}c(23)$ and $(23)x_{12}c = -x_{13}c^2(23)$ so that

(4.5.7)
$$\begin{array}{rclrcl} \varepsilon_+x_{12}c\varepsilon_+ & = & -\varepsilon_+x_{13}c^2\varepsilon_+ , & \varepsilon_-x_{12}c\varepsilon_+ & = & \varepsilon_-x_{13}c^2\varepsilon_+ , \\ \varepsilon_+x_{12}c\varepsilon_- & = & -\varepsilon_+x_{13}c^2\varepsilon_- , & \varepsilon_-x_{12}c\varepsilon_- & = & -\varepsilon_-x_{13}c^2\varepsilon_- . \end{array}$$

Hence Q' is the following quiver

$$\begin{tikzcd} (1, \rho_+) \arrow[loop above, "\alpha"] \arrow[r, shift left, "\beta"] & (1, \rho_-) \arrow[loop above, "\gamma"] \arrow[l, shift left, "\delta"] \end{tikzcd}$$

where

$$\alpha = \varepsilon_+x_{12}c\varepsilon_+ , \qquad \beta = \varepsilon_+x_{12}c\varepsilon_- , \qquad \gamma = \varepsilon_-x_{12}c\varepsilon_- , \qquad \delta = \varepsilon_-x_{12}c\varepsilon_+ .$$

Since $e_1 = \varepsilon_+ + \varepsilon_-$,

$$W = e_1We_1 = \varepsilon_+W\varepsilon_+ + \varepsilon_-W\varepsilon_- ,$$

here $\varepsilon_-W\varepsilon_+ = \varepsilon_+W\varepsilon_- = 0$ because W is G-invariant up to cyclic permutation. Therefore, by Lemma 4.4.2, it is possible to assume that

(4.5.8)
$$W' = \varepsilon_+W\varepsilon_+ + \varepsilon_-W\varepsilon_- .$$

There only remains to decompose W' as a linear combination of oriented cycles in Q'. For this purpose, note that $x_{12}x_{23}x_{31} = x_{12}cx_{12}cx_{12}c$, that $x_{13}x_{32}x_{21} = x_{13}c^2x_{13}c^2x_{13}c^2$ and that

$\sum_{\rho\in\mathrm{irr}(G_1)} e_1 * e_\rho = \varepsilon_+ + \varepsilon_-$. Hence, using Remark 4.5.1 and (4.5.7), the computations yield,

$$\begin{array}{rcl}
\varepsilon_+ x_{12}x_{23}x_{31}\varepsilon_+ & = & \alpha\alpha\alpha + \alpha\beta\delta + \beta\gamma\delta + \beta\delta\alpha\,, \\
\varepsilon_- x_{12}x_{23}x_{31}\varepsilon_- & = & \gamma\gamma\gamma + \gamma\delta\beta + \delta\beta\gamma + \delta\alpha\beta\,, \\
\varepsilon_+ x_{13}x_{32}x_{21}\varepsilon_+ & = & -(\alpha\alpha\alpha + \alpha\beta\delta + \beta\gamma\delta + \beta\delta\alpha)\,, \\
\varepsilon_- x_{13}x_{32}x_{21}\varepsilon_- & = & -(\gamma\gamma\gamma + \gamma\delta\beta + \delta\beta\gamma + \delta\alpha\beta)\,.
\end{array}$$

It follows from these computations and from (4.5.6) and (4.5.8) that

$$W' = 2 \sum_{w \text{ cycle of length } 3} w\,.$$

5. Application to cluster tilting objects

Assume setting 1.6. It is restated below for convenience.

Setting 5.1. Let $\Bbbk$ be a field. Let G be a finite group such that $\mathrm{char}(\Bbbk)$ does not divide $\mathrm{Card}(G)$. Let A be dg $\Bbbk$-algebra acted upon by G by dg automorphisms. Assume that A is concentrated in non positive degrees and $H^0(A)$ is finite dimensional. Assume that both A and $A * G$ are 3-Calabi-Yau.

Note that G acts on $H^0(A)$ by algebra automorphisms and that $H^0(A * G) \simeq H^0(A) * G$ as algebras. In particular, $H^0(A * G)$ is finite dimensional. These conditions ensure the existence of the associated generalised cluster categories $\mathcal{C}_A$ and $\mathcal{C}_{A*G}$. The purpose of this section is to relate these two categories by means of biadjoint functors

$$F_\lambda \colon \mathcal{C}_A \rightleftarrows \mathcal{C}_{A*G} \colon F_\cdot$$

in order to compare the cluster tilting objects of $\mathcal{C}_A$ and those of $\mathcal{C}_{A*G}$.

Such a relationship appeared first in [27]: given a Galois covering $R' \to R$ with group G, which was not assumed to be finite, where R is a finite dimensional piecewise hereditary algebra of quiver type, [27] shows that R' is piecewise hereditary, it introduces an action of G on the cluster category $C_{R'}$ of R' and a *push-down* functor from $C_{R'}$ to the cluster category C_R of R. This provides a description of the cluster tilting objects of C_R in terms of G-invariant cluster tilting objects of $C_{R'}$. In particular, it is proved in [27] that every indecomposable direct summand of any cluster tilting object of C_R is isomorphic to the image under the push-down functor of an indecomposable direct summand of a G-invariant cluster tilting object of $C_{R'}$. Recall that in that setting, R is Morita equivalent to $R' * G$ [7, Proposition 2.4].

Following [2] the generalised cluster categories of A and $A * G$ are, respectively,

$$\begin{array}{l}
\mathcal{C}_A = \mathrm{per}(A)/\mathcal{D}_{\mathrm{fd}}(A)\,, \\
\mathcal{C}_{A*G} = \mathrm{per}(A * G)/\mathcal{D}_{\mathrm{fd}}(A * G)\,.
\end{array}$$

They are Hom-finite, Krull-Schmidt and 2-Calabi-Yau, that is, the square of their suspensions are Serre functors. Denote by π_A and π_{A*G} the canonical functors

$$\begin{array}{l}
\pi_A \colon \mathrm{per}(A) \to \mathcal{C}_A\,, \\
\pi_{A*G} \colon \mathrm{per}(A * G) \to \mathcal{C}_{A*G}\,.
\end{array}$$

For simplicity, given an object T in $\mathcal{C}_A$ or $\mathcal{C}_{A*G}$, its endomorphism algebra is denoted by $\mathrm{End}(T)$. The objects $\overline{A} := \pi_A(A)$ and $\overline{A * G} := \pi_{A*G}(A*G)$ are cluster tilting in $\mathcal{C}_A$ and $\mathcal{C}_{A*G}$, respectively.

The section is organised as follows. Subsection 5.1 introduces an action of G on $\mathcal{C}_A$ and defines the adjoint functors $F_\lambda \colon \mathcal{C}_A \to \mathcal{C}_{A*G}$ and $F_\cdot \colon \mathcal{C}_{A*G} \to \mathcal{C}_A$. Subsection 5.2 proves Theorem 1.7 which establishes a first general comparison between the cluster tilting subcategories of $\mathcal{C}_A$ and those of $\mathcal{C}_{A*G}$. This comparison is strengthened in subsection 5.3 when G acts freely on the isomorphism classes of indecomposable projective $H^0(A)$-modules. Finally, these results are

applied in subsection 5.4 to prove Corollary 1.8 which relates $\mathcal{C}_A$ being acyclic to $\mathcal{C}_{A*G}$ being acyclic.

5.1. **The action of G on $\mathcal{C}_A$ and the adjunction between $\mathcal{C}_A$ and $\mathcal{C}_{A*G}$.** The action of G on $\mathcal{D}(A)$ stabilises $\mathrm{per}(A)$ and $\mathcal{D}_{\mathrm{fd}}(A)$. Accordingly, it induces a strict action of G on $\mathcal{C}_A$ by strict automorphisms of triangulated categories. For all $M \in \mathcal{C}_A$, define the *stabiliser* of M as the subgroup $\{g \in G \mid M^g \simeq M\}$ of G, it is denoted by G_M.

Consider the adjoint pair of triangle functors of subsection 2.3,

$$- \otimes^{\mathrm{L}}_A (A*G)\colon \mathcal{D}(A) \rightleftarrows \mathcal{D}(A*G)\colon \mathrm{Rres}$$

- $- \otimes^{\mathrm{L}}_A (A*G)$ maps $\mathrm{per}(A)$ into $\mathrm{per}(A*G)$, and Rres maps $\mathrm{per}(A*G)$ into $\mathrm{per}(A)$ because $A*G \simeq \oplus_{g\in G} A$ in $\mathrm{Mod}(A)$, and
- $- \otimes^{\mathrm{L}}_A (A*G)$ maps $\mathcal{D}_{\mathrm{fd}}(A)$ into $\mathcal{D}_{\mathrm{fd}}(A*G)$ because $A*G \simeq \oplus_{g\in G} A$ in $\mathrm{Mod}(A^{\mathrm{op}})$, and Rres maps $\mathcal{D}_{\mathrm{fd}}(A*G)$ into $\mathcal{D}_{\mathrm{fd}}(A)$.

Definition 5.1.1. The functors $F_\lambda\colon \mathcal{C}_A \to \mathcal{C}_{A*G}$ and $F.\colon \mathcal{C}_{A*G} \to \mathcal{C}_A$ are the triangle functors induced by $- \otimes^{\mathrm{L}}_A (A*G)\colon \mathcal{D}(A) \to \mathcal{D}(A*G)$ and $\mathrm{Rres}\colon \mathcal{D}(A*G) \to \mathcal{D}(A)$, respectively.

The functor F_λ between (generalised) cluster categories has already been considered in other situations, for instance,

- in [27, Section 4] between cluster categories $C_{R'} \to C_R$ where R is a piecewise hereditary algebra and R' is a Galois covering of R with group G,
- in [33, Section 3], where A is the Ginzburg dg algebra of any quiver with potential (Q, P) and the action of G is given by a free action on Q by quiver automorphisms,
- in [3, Corollary 3.6], where A is the Ginzburg dg algebra of any quiver with potential (Q, P), and the action of $G = \mathbb{Z}/2\mathbb{Z}$ is given by an action on Q by quiver automorphisms, the last condition is actually superfluous in that result once one knows that the skew group algebra $\mathcal{A}(Q, W) * \mathbb{Z}/2\mathbb{Z}$ is Morita equivalent to a Ginzburg dg algebra.

Proposition 5.1.2. *Assume setting 5.1.*

(1) The pairs $(F_\lambda, F.)$ and $(F., F_\lambda)$ are adjoint.
(2) The units of these adjunctions split (functorially).
(3) There exists a family $(\lambda_g\colon F_\lambda(\bullet^g) \xrightarrow{\sim} F_\lambda)_{g\in G}$ of isomorphisms of triangle functors such that the following equalities hold true, for all $M \in \mathcal{C}_A$ and $g, h \in G$,

$$\begin{cases} (\lambda_e)_M &= \mathbb{1}_M \\ (\lambda_g)_M \circ (\lambda_h)_{M^g} &= (\lambda_{gh})_M\,. \end{cases}$$

(4) For all $M \in \mathcal{C}_A$ there is a functorial isomorphism $F.F_\lambda M \simeq \oplus_{g\in G} M^g$ in $\mathcal{C}_A$.
(5) For all $M, M' \in \mathcal{C}_A$ there is an isomorphism $\oplus_{g\in G}\mathcal{C}_A(M, M'^g) \xrightarrow{\sim} \mathcal{C}_{A*G}(F_\lambda M, F_\lambda M')$ functorial in M and M'.
(6) $(F.N)^g \simeq F.N$ for all $N \in \mathcal{C}_{A*G}$.
(7) For all indecomposable $M \in \mathcal{C}_A$, if $G_M = \{e\}$, then $F_\lambda M$ is indecomposable.

Proof. (1) Consider the following subcategories of $\mathrm{per}(A)$ and $\mathrm{per}(A*G)$, respectively,

$$\begin{cases} \mathcal{F}_A = \mathcal{D}^{\leqslant 0}(A) \cap {}^{\perp}\mathcal{D}^{\leqslant -2}(A) \cap \mathrm{per}(A)\,, \\ \mathcal{F}_{A*G} = \mathcal{D}^{\leqslant 0}(A*G) \cap {}^{\perp}\mathcal{D}^{\leqslant -2}(A*G) \cap \mathrm{per}(A*G)\,, \end{cases}$$

where $\mathcal{D}^{\leqslant i}$ stands for the full subcategory of objects Z such that $H^p Z$ vanishes for $p > i$ and ${}^{\perp}\bullet$ denotes the full subcategory of objects with no non zero morphism to any object in a given

subcategory $\bullet$. Following [2, Proposition 2.9], the following restrictions of π_A and π_{A*G} are equivalences

$$\begin{cases} \mathcal{F}_A \to \mathcal{C}_A\,, \\ \mathcal{F}_{A*G} \to \mathcal{C}_{A*G}\,. \end{cases}$$

Since the pairs $(-\otimes^{\mathrm{L}}_A (A*G), \mathrm{Rres})$ and $(\mathrm{Rres}, -\otimes^{\mathrm{L}}_A (A*G))$ are adjoint, it is sufficient to prove the following assertions in order to prove that $(F_\lambda, F.)$ and $(F., F_\lambda)$ are adjoint,

(a) $X \otimes^{\mathrm{L}}_A (A*G) \in \mathcal{F}_{A*G}$ for all $X \in \mathcal{F}_A$, and
(b) $\mathrm{Rres} Y \in \mathcal{F}_A$ for all $Y \in \mathcal{F}_{A*G}$.

Here is a proof of these assertions. First, recall that $-\otimes^{\mathrm{L}}_A (A*G)$ and Rres map $\mathrm{per}(A)$ and $\mathrm{per}(A*G)$ into $\mathrm{per}(A*G)$ and $\mathrm{per}(A)$, respectively. Next, for all $i \in \mathbb{Z}$,

$$\mathrm{Rres}(\mathcal{D}^{\leqslant i}(A*G)) \subseteq \mathcal{D}^{\leqslant i}(A)\,. \tag{5.1.1}$$

Next, since A is concentrated in non positive degrees, the following holds true for all $i \in \mathbb{Z}$,

$$(\forall M \in \mathcal{D}^{\leqslant i}(A)) \;\; X \otimes^{\mathrm{L}}_A (A*G) \in \mathcal{D}^{\leqslant i}(A)\,. \tag{5.1.2}$$

Because of the adjunction $(-\otimes^{\mathrm{L}}_A (A*G), \mathrm{Rres})$, it follows from (5.1.1) that

$$(\forall X \in {}^{\perp}\mathcal{D}^{\leqslant -2}(A)) \;\; X \otimes^{\mathrm{L}}_A (A*G) \in {}^{\perp}\mathcal{D}^{\leqslant -2}(A*G)\,. \tag{5.1.3}$$

Because of the adjunction $(\mathrm{Rres}, -\otimes^{\mathrm{L}}_A (A*G))$, it follows from (5.1.2) that

$$(\forall Y \in {}^{\perp}\mathcal{D}^{\leqslant -2}(A*G)) \;\; \mathrm{Rres} Y \in {}^{\perp}\mathcal{D}^{\leqslant -2}(A)\,. \tag{5.1.4}$$

The assertions (a) and (b) above are consequences of (5.1.1), (5.1.2), (5.1.3) and (5.1.4). Thus, $(F_\lambda, F.)$ and $(F., F_\lambda)$ are adjoint.

(2), (3), (4), (5) and (6) follow from (1) and from Lemma 2.3.1.

(7) The arguments in the proof of [14, 3.5] apply literally here: for a direct sum decomposition $F_\lambda M = N \oplus N'$ such that N is non zero, there exists a non empty subset H of G such that $F.N \simeq \oplus_{g\in H} M^g$ and $F.N' \simeq \oplus_{g\in G\setminus H} M^g$ because M is indecomposable and because of (4); hence $gH \subseteq H$ for all $g \in G$ (because $(F.N)^g \simeq F.N$); accordingly, $H = G$, and hence $N' = 0$. $\square$

5.2. **Comparison of cluster tilting subcategories of $\mathcal{C}_A$ and $\mathcal{C}_{A*G}$.** This subsection compares the cluster tilting subcategories of $\mathcal{C}_A$ to the ones of $\mathcal{C}_{A*G}$. Using completely analogous, and hence omitted, arguments, the same conclusions are obtained when replacing cluster tilting subcategories by maximal rigid ones. The comparison is based on the following more general property where cluster tilting subcategories are used instead of cluster tilting objects for the ease of exposition.

Proposition 5.2.1. *Let $\mathcal{C}$ and $\mathcal{C}'$ be triangulated $\Bbbk$-linear categories which are* Hom*-finite, Krull-Schmidt and 2-Calabi-Yau. Assume that there exist triangle functors* $F\colon \mathcal{C} \rightleftarrows \mathcal{C}'\colon F'$ *such that*

- *(F, F') and (F', F) are adjoint and*
- *for all $M \in \mathcal{C}$ and $N \in \mathcal{C}'$ the unit morphisms $M \to F'FM$ and $N \to FF'N$ split in $\mathcal{C}$ and $\mathcal{C}'$, respectively.*

Then, the two assignments $\mathcal{T} \mapsto \mathrm{add}(F\mathcal{T})$ and $\mathcal{T}' \mapsto \mathrm{add}(F'\mathcal{T}')$ induce mutually inverse bijections between

- *cluster tilting subcategories $\mathcal{T}$ of $\mathcal{C}$ which are stable under $F'F$, and*
- *cluster tilting subcategories $\mathcal{T}'$ of $\mathcal{C}'$ which are stable under FF'.*

The same result holds true when replacing cluster tilting subcategories by maximal rigid ones.

Proof. By adjunction, F and F' preserve functorially finite subcategories.

Let $\mathcal{T}$ be a cluster tilting subcategory of $\mathcal{C}$ stable under $F'F$. Then, $\mathcal{C}'(FM, \Sigma FM') \simeq \mathcal{C}(M, \Sigma F'FM') = 0$ for all $M, M' \in \mathcal{T}$ (note that, by assumption, $F'FM' \in \mathcal{T}$). Hence, $\mathcal{C}'(N_1, \Sigma N_2) = 0$, for all $N_1, N_2 \in \mathrm{add}(F\mathcal{T})$. Moreover, for all $N \in \mathcal{C}'$, if $\mathcal{C}'(N', \Sigma N) = 0$ for all $N' \in \mathrm{add}(F\mathcal{T})$, then $\mathcal{C}(M, \Sigma F'N) \simeq \mathcal{C}'(FM, \Sigma N) = 0$ for all $M \in \mathcal{T}$; therefore, $F'N \in \mathcal{T}$, and hence $FF'N \in F\mathcal{T}$; since $N \in \mathrm{add}(FF'N)$, the object N lies in $\mathrm{add}(F\mathcal{T})$. Thus, $\mathrm{add}(F\mathcal{T})$ is a cluster tilting subcategory of $\mathcal{C}'$.

Mutatis mutandis, the same arguments show that, for all cluster tilting subcategories $\mathcal{T}'$ of $\mathcal{C}'$ stable under FF', the subcategory $\mathrm{add}(F'\mathcal{T}')$ of $\mathcal{C}$ is cluster tilting.

Finally, given a subcategory $\mathcal{T}$ of $\mathcal{C}$ (or $\mathcal{T}'$ of $\mathcal{C}'$) such that $\mathcal{T}$ (or $\mathcal{T}'$) is closed under direct sums, direct summands and $F'F$ (or FF'), then $\mathrm{add}(F'\mathrm{add}(F\mathcal{T})) = \mathcal{T}$ (or $\mathrm{add}(F\mathrm{add}(F'\mathcal{T}')) = \mathcal{T}'$) because, for all $M \in \mathcal{C}$ (or $N \in \mathcal{C}'$), the unit morphism $M \to F'FM$ (or $N \to FF'N$, respectively) splits. □

In view of Proposition 5.1.2, Proposition 5.2.1 applies to the current situation.

Corollary 5.2.2. *Assume setting 5.1. The assignments* $\mathcal{T} \mapsto \mathrm{add}(F_\lambda \mathcal{T})$ *and* $\mathcal{T}' \mapsto \mathrm{add}(F_\cdot \mathcal{T}')$ *induce mutually inverse bijections between*

- *cluster tilting subcategories* $\mathcal{T}$ *of* $\mathcal{C}_A$ *which are stable under the action of* G*, and*
- *cluster tilting subcategories* $\mathcal{T}'$ *of* $\mathcal{C}_{A*G}$ *which are stable under* $F_\lambda F_\cdot$.

The same result holds true if cluster tilting subcategories are replaced by maximal rigid ones.

Proof. Note that, since $F_\cdot F_\lambda M \simeq \oplus_{g\in G} M^g$, for all $M \in \mathcal{C}_A$, a full subcategory of $\mathcal{C}_A$ which is closed under direct sums and direct summands is stable under the action of G if and only if it is stable under $F_\cdot F_\lambda$. The conclusion therefore follows from Proposition 5.2.1 applied to $\mathcal{C} = \mathcal{C}_A$, $\mathcal{C}' = \mathcal{C}_{A*G}$, $F = F_\lambda$ and $F' = F_\cdot$. □

Given cluster tilting (or maximal rigid) objects T, T' of $\mathcal{C}_A$ and $\mathcal{C}_{A*G}$, respectively, such that $\mathrm{add}(T') = \mathrm{add}(F_\lambda T)$ as in Corollary 5.2.2, it is natural to ask whether there exists a relation between $\mathrm{End}(T)$ and $\mathrm{End}(T')$. It is provided by the following result.

For a better understanding, recall ([34, 3.1]) that when C is a $\Bbbk$-linear category endowed with an action of G (on the right) by automorphisms, the *skew group* category $C*G$ is the idempotent completion of the orbit category $C[G]$ (see also [10, Definition 2.3] for an approach avoiding idempotent completion). Recall that the *orbit category* $C[G]$ has the same objects as C, that, for all $M, M' \in C$, its space of morphisms $M \to M'$ is defined by $C[G](M, M') = \oplus_{g\in G} C(M, M'^g)$, and that the composition of morphisms is induced naturally by the one in C. Recall also that, for a $\Bbbk$-linear category C, the piece of notation $\mathrm{mod}(C)$ stands for the category of finitely presented contravariant functors from C to finite dimensional vector spaces.

Proposition 5.2.3. *Assume setting 5.1. Let* $\mathcal{T}$ *be a cluster tilting subcategory of* $\mathcal{C}_A$ *stable under the action of* G*. Let* $\mathcal{T}' = \mathrm{add}(F_\lambda \mathcal{T})$*. Then,*

(1) the skew group category $\mathcal{T} * G$ *is Morita equivalent to* $\mathcal{T}'$ *and*
(2) there exists a cluster tilting object $T \in \mathcal{C}_A$ *such that* $T^g = T$ *for all* $g \in G$ *and* $\mathcal{T} = \mathrm{add}(T)$*. For any such* T*, the algebras* $\mathrm{End}(T) * G$ *and* $\mathrm{End}(F_\lambda T)$ *are Morita equivalent.*

The same result holds true when cluster tilting subcategories are replaced by maximal rigid ones.

Proof. (1) It follows from Proposition 5.1.2 that the categories $\mathcal{T}[G]$ and $F_\lambda \mathcal{T}$ are equivalent. Taking idempotent completions yields that $\mathcal{T} * G$ and $\mathrm{add}(F_\lambda \mathcal{T})$ are equivalent. Hence,

$$\mathrm{mod}(\mathcal{T} * G) \simeq \mathrm{mod}(\mathcal{T}')\,. \tag{5.2.1}$$

(2) Let $\{T_1, \dots, T_n\}$ be a complete set of representatives of the G-orbits of the indecomposable objects of $\mathcal{T}$. Let T be the object $\bigoplus_{\substack{1\leqslant i\leqslant n\\ g\in G}} T_i^g$ of $\mathcal{C}_A$. Then $T^g = T$ for all $g \in G$ and $\mathcal{T} = \mathrm{add}(T)$. The assignment $(g, f) \mapsto f^{g^{-1}}$ defines an action of G on $\mathrm{End}(T)$ by algebra automorphisms, which defines $\mathrm{End}(T) * G$.

Since $\mathcal{T}$ is idempotent complete, the following holds true ([34, Proposition 3.4]),

$$\mathrm{mod}(\mathcal{T} * G) \simeq \mathrm{mod}(\mathcal{T}) * G\,. \tag{5.2.2}$$

By construction of T, the categories $\mathrm{mod}(\mathcal{T})$ and $\mathrm{mod}(\mathrm{End}(T))$ are equivalent and

$$\mathrm{mod}(\mathrm{End}(T)) * G \simeq \mathrm{mod}(\mathcal{T}) * G\,. \tag{5.2.3}$$

Note also that ([34, 3.1])

$$\mathrm{mod}(\mathrm{End}(T)) * G \simeq \mathrm{mod}(\mathrm{End}(T) * G)\,. \tag{5.2.4}$$

Finally, since $\mathcal{T}' = \mathrm{add}(F_\lambda T)$,

$$\mathrm{mod}(\mathcal{T}') \simeq \mathrm{mod}(\mathrm{End}(F_\lambda T))\,. \tag{5.2.5}$$

The conclusion therefore follows from (5.2.1), (5.2.2), (5.2.3), (5.2.4) and (5.2.5). □

5.3. **Cluster tilting objects of the first kind.** The considerations of subsection 5.2 raise the question whether a given cluster tilting or maximal rigid object of $\mathcal{C}_{A*G}$ lies in the essential image of $F_\lambda\colon \mathcal{C}_A \to \mathcal{C}_{A*G}$. For instance, $F_\lambda \overline{A} = \overline{A * G}$. An object $N \in \mathcal{C}_{A*G}$ is called here of the *first kind* with respect to F_λ if, for all indecomposable direct summands N' of N, there exists $M' \in \mathcal{C}_A$ such that $F_\lambda M' \simeq N'$. This terminology comes from the theory of Galois coverings of algebras (see [13]).

This subsection investigates cluster tilting or maximal rigid objects of $\mathcal{C}_{A*G}$ which are of the first kind with respect to F_λ. Clearly, further assumptions are needed for this investigation to be fruitful because, although $F_\lambda \overline{A} = \overline{A * G}$, the indecomposable direct summands of $\overline{A * G}$ need not lie in the essential image of F_λ. This is because F_λ need not take an indecomposable direct summand of $\overline{A}$ to an indecomposable object of $\mathcal{C}_{A*G}$. In view of Proposition 5.1.2, having some information on the stabilisers of the indecomposable direct summands of $\overline{A}$ would be useful. This is provided by the following lemma.

For all $g \in G$, the mapping $A \to A$ given by $a \mapsto {}^g a$ induces an isomorphism in $\mathcal{C}_A$ denoted by α_g,

$$\alpha_g\colon \overline{A} \to \overline{A}^g\,. \tag{5.3.1}$$

Recall [2, Proposition 2.9] that the mapping $A \to \mathrm{Hom}_A(A, A)$ given by $x \mapsto (a \mapsto xa)$ induces an isomorphism of algebras

$$H^0(A) \xrightarrow{\sim} \mathrm{End}(\overline{A})\,. \tag{5.3.2}$$

In particular, for all direct summands X of $\overline{A}$, the $\mathrm{End}(\overline{A})$-module $\mathcal{C}_A(\overline{A}, X)$ is projective and its restriction of scalars along (5.3.2) is isomorphic to $\varepsilon H^0(A)$ for some idempotent $\varepsilon \in H^0(A)$. Any idempotent of $H^0(A)$ may be obtained in this way.

Lemma 5.3.1. *Assume setting 5.1. Let $\varepsilon \in H^0(A)$ be an idempotent. Let X be a direct summand of $\overline{A}$ such that $\varepsilon H^0(A)$ is isomorphic, as an $H^0(A)$-module, to the restriction of scalars of $\mathcal{C}_A(\overline{A}, X)$ along (5.3.2). Let $g \in G$. Let Y be a direct summand of $\overline{A}$ such that $X^g \simeq Y$ (see (5.3.1)). Then, ${}^g\varepsilon H^0(A)$ is isomorphic to the restriction of scalars of $\mathcal{C}_A(\overline{A}, Y)$ along (5.3.2).*

Proof. The proof is based on the following diagram where all the arrows are isomorphisms. Note that the top horizontal arrow is given by the action of G on A and the leftmost bottom horizontal arrow is given by the action of G on $\mathcal{C}_A$.

$$\begin{tikzcd} H^0(A) \arrow[rr, "x \mapsto {}^{g^{-1}}x"] \arrow[d, "(5.3.2)"'] & & H^0(A) \arrow[d, "(5.3.2)"] \\ \mathrm{End}(\overline{A}) \arrow[r, "-^g"'] & \mathrm{End}(\overline{A}^g) \arrow[r, "\alpha_g^{-1}\circ - \circ \alpha_g"'] & \mathrm{End}(\overline{A}) \end{tikzcd} \tag{5.3.3}$$

This diagram is commutative as shown by a direct computation using that

- $\mathrm{End}(\overline{A}) \xrightarrow{-^g} \mathrm{End}(\overline{A}^g)$ is induced by the identity mapping $\mathrm{Hom}_A(A,A) \to \mathrm{Hom}_A(A,A)$,
- $\mathrm{End}(\overline{A}^g) \xrightarrow{\alpha_g^{-1}\circ - \circ \alpha_g} \mathrm{End}(\overline{A})$ is induced by the mapping $\mathrm{Hom}_A(A,A) \to \mathrm{Hom}_A(A,A)$ which assigns to any $f \in \mathrm{Hom}_A(A,A)$ the mapping $a \mapsto {}^{g^{-1}}f({}^g a)$.

On one hand,

- by assumption, the restriction of scalars of $\mathcal{C}_A(\overline{A}, X)$ along the rightmost vertical arrow of (5.3.3) is isomorphic to $\varepsilon H^0(A)$ as an $H^0(A)$-module and
- the following mapping is an isomorphism of $H^0(A)$-modules from the restriction of scalars of $\varepsilon H^0(A)$ along the top horizontal arrow of (5.3.3) to ${}^g\varepsilon H^0(A)$,

$$\begin{array}{ccc} \varepsilon H^0(A) & \to & {}^g\varepsilon H^0(A) \\ x & \mapsto & {}^g x\,. \end{array}$$

On the other hand, denoting by $\lambda\colon X \to {}^gY$ any isomorphism in $\mathcal{C}_A$,

- the following mapping is an isomorphism of $\mathrm{End}(\overline{A}^g)$-modules from the restriction of scalars of $\mathcal{C}_A(\overline{A}, X)$ along the arrow $\mathrm{End}(\overline{A}^g) \to \mathrm{End}(\overline{A})$ of (5.3.3) to $\mathcal{C}_A(\overline{A}^g, Y^g)$,

$$\begin{array}{ccc} \mathcal{C}_A(\overline{A}, X) & \to & \mathcal{C}_A(\overline{A}^g, Y^g) \\ f & \mapsto & \lambda \circ f \circ \alpha_g^{-1}\,, \end{array}$$

- and the following mapping is an isomorphism of $\mathrm{End}(\overline{A})$-modules from the restriction of scalars of $\mathcal{C}_A(\overline{A}^g, Y^g)$ along the arrow $\mathrm{End}(\overline{A}) \to \mathrm{End}(\overline{A}^g)$ of (5.3.3) to $\mathcal{C}_A(\overline{A}, Y)$,

$$\begin{array}{ccc} \mathcal{C}_A(\overline{A}^g, Y^g) & \to & \mathcal{C}_A(\overline{A}, Y) \\ f & \mapsto & f^{g^{-1}}\,. \end{array}$$

Since the diagram (5.3.3) commutes, it follows that the restriction of scalars of $\mathcal{C}_A(\overline{A}, Y)$ along (5.3.2) is isomorphic to ${}^g\varepsilon H^0(A)$. □

The main results of this subsection prove that being of the first kind with respect to F_λ is an invariant property within a mutation class of cluster tilting or of maximal rigid objects. This fact is based on the following lemma. It is already proved in other contexts for Galois coverings instead of skew group algebras, namely, in [28, Lemma 3.2] in the context of module categories, in [30, Lemma 4.2] in the context of derived categories of finite dimensional algebras and in [27, Lemma 6.2] in the context of cluster categories of piecewise hereditary algebras.

Lemma 5.3.2. *Assume setting 5.1. Let $\Delta\colon X \to M \to Y \to \Sigma X$ be a triangle in $\mathcal{C}_{A*G}$, such that*

- *there exists $\widetilde{X} \in \mathcal{C}_A$ such that $F_\lambda \widetilde{X} \simeq X$,*
- *there exists a direct sum decomposition into indecomposable objects $M = M_1 \oplus \cdots \oplus M_t$ in $\mathcal{C}_{A*G}$, and there exist $\widetilde{M_1}, \ldots, \widetilde{M_t} \in \mathcal{C}_A$ such that $F_\lambda \widetilde{M_i} \simeq M_i$ for all i,*
- *$\mathcal{C}_{A*G}(Y, \Sigma M) = 0$.*

Then, there exist $\widetilde{Y} \in \mathcal{C}_A$ and $g_1, \dots, g_t \in G$, and there exists a triangle $\widetilde{X} \to \bigoplus_{i=1}^t \widetilde{M}_i^{g_i} \to \widetilde{Y} \to \Sigma\widetilde{X}$ in $\mathcal{C}_A$ whose image under F_λ is isormorphic to Δ. In particular, $F_\lambda\widetilde{Y} \simeq Y$.

Proof. The proof of [30, Lemma 4.2] applies here provided that the occurrences of $\mathcal{D}^b(\operatorname{mod}\mathcal{C})$ and $\mathcal{D}^b(\operatorname{mod} A)$ there are replaced by $\mathcal{C}_A$ and $\mathcal{C}_{A*G}$ here. Note that, by Proposition 5.1.2, the functor $F_\lambda\colon \mathcal{C}_A \to \mathcal{C}_{A*G}$ has all the needed properties of the functor that is also denoted by F_λ in the proof of [30, Lemma 4.2]. □

As explained at the beginning of subsection 5.3, it is preferable to have some control on the stabilisers of the indecomposable direct summands of the cluster tilting or maximal rigid objects of $\mathcal{C}_A$. For this purpose, the following lemma is useful. Like Lemma 5.3.2, the following result is already known in the framework of Galois coverings in [30, Proposition 4.6] for tilting complexes over piecewise hereditary algebras and in [27, Proposition 6.7] for cluster tilting objects of cluster categories of piecewise hereditary algebras.

Lemma 5.3.3. *Assume the setting of Lemma 5.3.2. Assume, moreover, that*

- *X is indecomposable and $X \not\simeq M_i$ for all $i \in \{1, \dots, t\}$,*
- *$\mathcal{C}_{A*G}(M, \Sigma X) = 0$,*
- *the stabiliser of $\widetilde{X}$ is trivial.*

Then, for all triangles $\widetilde{X} \to \oplus_{i=1}^t \widetilde{M}_i^{g_i} \to \widetilde{Y} \to \Sigma\widetilde{Y}$ of $\mathcal{C}_A$ whose images under F_λ are isomorphic to Δ, the stabiliser of $\widetilde{Y}$ is trivial.

Proof. The proof of [27, Proposition 6.7] may be adapted to the current situation. Here are the details. Let $\widetilde{\Delta}\colon \widetilde{X} \to \oplus_{i=1}^t \widetilde{M}_i^{g_i} \to \widetilde{Y} \to \Sigma\widetilde{X}$ be a triangle of $\mathcal{C}_A$ whose image under F_λ is isomorphic to Δ. Let $w\colon \widetilde{Y}^g \to \widetilde{Y}$ be an isomorphism in $\mathcal{C}_{A*G}$, where $g \in G$. For all $h \in G$, the space $\mathcal{C}_A(\widetilde{M}_i^h, \Sigma\widetilde{X})$ is isomorphic to a subspace of $\mathcal{C}_{A*G}(M, \Sigma X)$ (see Proposition 5.1.2), since the latter is zero, there exist morphisms u, v, u', v' fitting into commutative diagrams as follows

$$\begin{tikzcd}
\widetilde{X}^g \arrow[r] \arrow[d, "u"'] & \bigoplus_{i=1}^t \widetilde{M}_i^{g_i g} \arrow[r] \arrow[d, "v"'] & \widetilde{Y}^g \arrow[r] \arrow[d, "w"'] & \Sigma\widetilde{X}^g \arrow[d, "\Sigma u"] \\
\widetilde{X} \arrow[r] & \bigoplus_{i=1}^t \widetilde{M}_i^{g_i} \arrow[r] & \widetilde{Y} \arrow[r] & \Sigma\widetilde{X}
\end{tikzcd}
\qquad
\begin{tikzcd}
\widetilde{X} \arrow[r] \arrow[d, "u'"'] & \bigoplus_{i=1}^t \widetilde{M}_i^{g_i} \arrow[r] \arrow[d, "v'"'] & \widetilde{Y} \arrow[r] \arrow[d, "w^{-1}"'] & \Sigma\widetilde{X} \arrow[d, "\Sigma u'"] \\
\widetilde{X}^g \arrow[r] & \bigoplus_{i=1}^t \widetilde{M}_i^{g_i g} \arrow[r] & \widetilde{Y}^g \arrow[r] & \Sigma\widetilde{X}^g
\end{tikzcd}$$

Therefore, the following diagram commutes

$$\begin{tikzcd}
\widetilde{X} \arrow[r] \arrow[d, "u\circ u' - \mathbb{1}"'] & \bigoplus_{i=1}^t \widetilde{M}_i^{g_i} \arrow[r] \arrow[d, "v\circ v' - \mathbb{1}"'] & \widetilde{Y} \arrow[r] \arrow[d, "0"'] & \Sigma\widetilde{X} \arrow[d, "\Sigma(u\circ u' - \mathbb{1})"] \\
\widetilde{X} \arrow[r] & \bigoplus_{i=1}^t \widetilde{M}_i^{g_i} \arrow[r] & \widetilde{Y} \arrow[r] & \Sigma\widetilde{X}
\end{tikzcd}$$

and hence $\Sigma(u \circ u' - \mathbb{1})$ factors through $\Sigma\widetilde{X} \to \oplus_{i=1}^t \Sigma\widetilde{M}_i^{g_i}$. Note that $\widetilde{X} \not\simeq \widetilde{M}_i^{g_i}$ because $X \not\simeq M_i$, $X \simeq F_\lambda\widetilde{X}$ and $F_\lambda\widetilde{M}_i \simeq M_i$, for all $i \in \{1, \dots, t\}$. Since $\widetilde{X}$ is indecomposable, it follows that $\Sigma(u \circ u' - \mathbb{1})$ lies in the radical of the algebra $\operatorname{End}(\Sigma\widetilde{X})$, and hence $u \circ u'$ is an automorphism of $\widetilde{X}$. Thus, $u\colon \widetilde{X}^g \to \widetilde{X}$ is an isomorphism. Given that $\widetilde{X}$ has trivial stabiliser, it follows that $g = e$. This proves that $\widetilde{Y}$ has trivial stabiliser. □

Combining Lemmas 5.3.2 and 5.3.3 yields the following result. Recall that, given cluster tilting objects $T, T' \in \mathcal{C}_{A*G}$, the object T' is called a *mutation* of T if there exist indecomposable direct summands X and Y of T and T', respectively, such that the set of isomorphism classes of indecomposable direct summands of T not isomorphic to X is equal to the set of isomorphism

classes of indecomposable direct summands of T' not isomorphic to Y. In such a case, there exists a triangle $X \to M \to Y \to \Sigma X$ in $\mathcal{C}_{A*G}$, where $M \in \mathrm{add}(T) \cap \mathrm{add}(T')$. The *mutation class* of T is the transitive closure of $\{T\}$ under mutation. The same concepts are defined for maximal rigid objects. See [22] for more details.

Lemma 5.3.4. *Assume setting 5.1. Let $T, T' \in \mathcal{C}_{A*G}$ be cluster tilting objects in the same mutation class. The following assertions are equivalent.*

(i) All indecomposable direct summands of T lie in the essential image of F_λ.
(ii) All indecomposable direct summands of T' lie in the essential image of F_λ.

When these assertions are true, then the following ones are equivalent.

(iii) $G_X = \{e\}$ for all indecomposable objects $X \in \mathcal{C}_A$ such that $F_\lambda X$ is isomorphic to a direct summand of T.
(iv) $G_X = \{e\}$ for all indecomposable objects $X \in \mathcal{C}_A$ such that $F_\lambda X$ is isomorphic to a direct summand of T'.

The same result holds true when cluster tilting objects are replaced by maximal rigid ones.

Proof. This follows from Lemmas 5.3.2 and 5.3.3. □

Using Lemma 5.3.4, it is possible to prove a first result answering the question stated at the beginning of subsection 5.3 and regarding cluster tilting objects of $\mathcal{C}_{A*G}$ which are of the first kind with respect to F_λ. Note that the action of G on $H^0(A)$ defines an action on the finite dimensional $H^0(A)$-modules like the action of G on A defines an action on dg A-modules.

Proposition 5.3.5. *Assume setting 5.1. Assume that G acts freely on the isomorphism classes of indecomposable projective $H^0(A)$-modules. Then, for all basic cluster tilting objects $T = T_1 \oplus \cdots \oplus T_n$ of $\mathcal{C}_{A*G}$ in the mutation class of $\overline{A*G}$ and where $T_1, \ldots, T_n$ are indecomposable,*

(1) there exist $\widetilde{T}_1, \ldots, \widetilde{T}_n \in \mathcal{C}_A$ such that $F_\lambda \widetilde{T}_i \simeq T_i$ and $G_{\widetilde{T}_i} = \{e\}$, for all i.

Moreover, for any such $\widetilde{T}_1, \ldots \widetilde{T}_n$, denoting $\bigoplus_{1 \leqslant i \leqslant n, g \in G} \widetilde{T}_i^g$ by $\widetilde{T}$,

(2) $\widetilde{T}$ is a cluster tilting object in $\mathcal{C}_{A*G}$ such that $\mathrm{add}(T) = F_\lambda \mathrm{add}(\widetilde{T})$ and $\widetilde{T}^g = \widetilde{T}$ for all $g \in G$, and
(3) F_λ induces a Galois covering with group G from $\mathrm{End}(\widetilde{T})$ to $\mathrm{End}(T)$.

The same result holds true if cluster tilting objects are replaced by maximal rigid ones.

Proof. (1) First, the stated property holds true when $\mathrm{add}(T) = \mathrm{add}(\overline{A*G})$. Indeed,

- $F_\lambda \overline{A} = \overline{A*G}$,
- by assumption, every indecomposable direct summand of $\overline{A}$ has a trivial stabiliser (see Lemma 5.3.1) and consequently,
- $F_\lambda M$ is isomorphic to an indecomposable direct summand of $\overline{A*G}$ for all indecomposable direct summands M of $\overline{A}$ (see Proposition 5.1.2, part (7)).

Next, it follows from Lemma 5.3.4 that property (1) is preserved within a mutation class.

(2) By construction,

- $\mathrm{add}(T) = F_\lambda \mathrm{add}(\widetilde{T})$ and $\widetilde{T}^g = \widetilde{T}$ for all $g \in G$. Moreover,
- $F_\lambda F_\cdot(F_\lambda \widetilde{T}) \simeq F_\lambda(\widetilde{T}^{\mathrm{Card}(G)})$ (see Proposition 5.1.2, part (4)), and hence $\mathrm{add}(F_\lambda \widetilde{T})$ is stable under $F_\lambda F_\cdot$.

Therefore, $\mathrm{add}(\widetilde{T})$ is a cluster tilting subcategory (see Corollary 5.2.2), and hence $\widetilde{T}$ is a cluster tilting object of $\mathcal{C}_A$.

(3) This was proved already in other contexts ([28, Lemma 2.2] and [30, Lemma 4.8]) and may be proved similarly here. Here are some explanations for the ease of reading.

Here, $\mathrm{End}(\widetilde{T})$ and $\mathrm{End}(T)$ are identified with the categories whose objects sets are equal to the sets $\{\widetilde{T}_i^g \mid 1 \leqslant i \leqslant n\,, g \in G\}$ and $\{T_1, \ldots, T_n\}$, respectively. For all $i \in \{1, \ldots, n\}$, let $\mu_i \colon F_\lambda \widetilde{T}_i \to T_i$ be an isomorphism in $\mathcal{C}_{A*G}$. Then, define a functor

$$\mathrm{End}(\widetilde{T}) \quad \to \quad \mathrm{End}(T) \tag{5.3.4}$$

as follows, where $g, h \in G$, $i, j \in \{1, \ldots, n\}$,

- the object $\widetilde{T}_i^g$ is mapped onto T_i,
- any morphism $u \colon \widetilde{T}_i^g \to \widetilde{T}_j^h$ is mapped onto the following composite morphism

$$T_i \xrightarrow{\mu_i^{-1}} F_\lambda \widetilde{T}_i \xrightarrow{(\lambda_g)^{-1}_{\widetilde{T}_i}} F_\lambda \widetilde{T}_i^g \xrightarrow{F_\lambda u} F_\lambda \widetilde{T}_j^h \xrightarrow{(\lambda_h)_{\widetilde{T}_j}} F_\lambda \widetilde{T}_j \xrightarrow{\mu_j} T_j\,,$$

  recall that λ_g, λ_h are introduced in Proposition 5.1.2 (part (3)).

In view of Proposition 5.1.2 (parts (3) and (5)) and since $\widetilde{T}_i^g$ has a trivial stabiliser, for all $g \in G$ and $i \in \{1, \ldots, n\}$, this construction does yield a Galois covering with group G. □

When the cluster tilting objects of $\mathcal{C}_{A*G}$ form a mutation class, Proposition 5.3.5 provides the following refinement of Corollary 5.2.2. Note that this hypothesis is known to hold true when, for instance, $\mathcal{C}_{A*G}$ is acyclic ([6, Proposition 3.5]).

Corollary 5.3.6. *Assume setting 5.1. Assume that G acts freely on the isomorphism classes of indecomposable projective $H^0(A)$-modules and that the cluster tilting objects of $\mathcal{C}_{A*G}$ form a mutation class. Then,*

(1) statements (1), (2) and (3) of Proposition 5.3.5 hold true for all basic cluster tilting objects of $\mathcal{C}_{A*G}$,
(2) the mapping $\mathcal{T} \mapsto F_\lambda \mathcal{T}$ induces a bijection from the set of cluster tilting subcategories of $\mathcal{C}_A$ stable under the action of G to the set of cluster tilting subcategories of $\mathcal{C}_{A*G}$.

When cluster tilting objects are replaced by maximal rigid ones both in the hypotheses and the conclusion, then (1) and (2) still hold true and, in addition,

(3) given $N \in \mathcal{C}_{A*G}$ indecomposable, the following assertions are equivalent.
  (i) N is rigid.
  (ii) There exists a maximal rigid object $T \in \mathcal{C}_A$ such that $T^g \simeq T$ for all $g \in G$ and there exists an indecomposable direct summand M of T such that $N \simeq F_\lambda M$.

Proof. (1) follows from Proposition 5.3.5. (2) follows from (1) and Corollary 5.2.2. (3) follows from (2). □

Parts of Corollary 5.3.6 are proved in the following more particular cases,

- in [27, Section 6], when replacing $\mathcal{C}_{A*G}$ by the cluster category of a piecewise hereditary algebra R of quiver type and $\mathcal{C}_A$ by the cluster category of R', and assuming that R' is a Galois covering with group G of R,
- in [33, Subsection 7.4], where A is given by a quiver with potential (Q, W), where the action of G is given by a free action on Q by quiver automorphisms such that W is G-invariant up to cyclic permutation, and assuming that $\mathcal{C}_{A*G}$ is acyclic and with finitely many isomorphism classes of indecomposable objects.

5.4. **The acyclic case.** This subsection proves Corollary 1.8 relating $\mathcal{C}_A$ being acyclic to $\mathcal{C}_{A*G}$ being acyclic. Cluster categories of Dynkin quivers are particular because these are exactly the Hom-finite, Krull-Schmidt and 2-Calabi-Yau categories with only finitely many isomorphism classes of indecomposable objects ([1, Corollary 6.6]). For these, there is a general phenomenon as follows.

Proposition 5.4.1. *Assume the setting of Proposition 5.2.1. The following assertions are equivalent.*

(i) $\mathcal{C}$ has finitely many isomorphism classes of indecomposable objects.
(ii) $\mathcal{C}'$ has finitely many isomorphism classes of indecomposable objects.

Proof. It suffices to prove "$(i) \Longrightarrow (ii)$". Assume (i). Let $M \in \mathcal{C}$ such that $\mathcal{C} = \mathrm{add}(M)$. Then $\mathcal{C}' = \mathrm{add}(FM)$ by assumption on the unit of the adjoint pair (F, F'). Thus, (ii) holds true. □

Using subsections 5.2 and 5.3 together with Proposition 5.4.1, it is possible to prove Corollary 1.8. This corollary is restated below for convenience.

Corollary 5.4.2. *Assume setting 5.1.*

(1) The following assertions are equivalent.
  (i) $\mathcal{C}_A$ is equivalent to the cluster category of a Dynkin quiver.
  (ii) $\mathcal{C}_{A*G}$ is equivalent to the cluster category of a Dynkin quiver.
(2) If $\mathcal{C}_A$ is acyclic and has infinitely many isomorphism classes of indecomposable objects, then
  (a) there exists a cluster tilting object $T \in \mathcal{C}_A$ such that $\mathrm{End}(T)$ is hereditary and $T^g = T$ for all $g \in G$ and
  (b) for any such T, there exists a cluster tilting object $T' \in \mathcal{C}_{A*G}$ such that the algebras $\mathrm{End}(T) * G$ and $\mathrm{End}(T')$ are Morita equivalent.
  Consequently, $\mathcal{C}_{A*G}$ is acyclic.
(3) Assume that G acts freely on the set of isomorphism classes of indecomposable projective $H^0(A)$-modules. If $\mathcal{C}_{A*G}$ is acyclic, then so is $\mathcal{C}_A$.

Proof. (1) follows from Propositions 5.1.2 and 5.4.1 and from [1, Corollary 6.6].

(2) The following arguments are adapted from [12] where it is proved that the skew group algebras of piecewise hereditary algebras are piecewise hereditary.

Since $\mathcal{C}_A$ is acyclic, there exists a unique connected component Γ of the Auslander-Reiten quiver of $\mathcal{C}_A$ having only finitely many τ-orbits, where τ $(= \Sigma)$ is the Auslander-Reiten translation. Note that Γ is the repetitive quiver of a finite acyclic quiver because $\mathcal{C}_A$ has infinitely many isomorphism classes of indecomposable objects.

The action of G preserves the Auslander-Reiten structure of $\mathcal{C}_A$. In particular, it stabilises Γ. Following [12, Definition 4.1.1], let $M_0 \in \Gamma$ and denote by $\mathcal{X}$ the full subquiver of Γ consisting of the vertices $M \in \Gamma$ such that

- there exists $g \in G$ and there exists a path from M_0^g to M in Γ and
- any such path is sectional.

Note that $\mathcal{X}$ is stable under the action of G up to isomorphisms. Moreover ([12, Proposition 4.1.4]) $\mathcal{X}$ is a section of Γ, that is, $\mathcal{X}$ is a connected quiver, it is convex in Γ and it intersects each τ-orbit of Γ exactly once. Let $\{T_i\}_{i\in\{1,\dots,n\}}$ be a complete family of representatives of G-orbits of indecomposable objects of $\mathrm{add}(\mathcal{X})$ and denote by T the object $\bigoplus_{\substack{1\leqslant i\leqslant n\\ g\in G}} T_i^g$ of $\mathcal{C}_A$. Then, $\mathrm{add}(\mathcal{X}) = \mathrm{add}(T)$ and $T^g = T$ for all $g \in G$. Since $\mathcal{X}$ is a section of Γ, the object T is a cluster tilting object of $\mathcal{C}_A$ and $\mathrm{End}(T)$ is a hereditary algebra, isomorphic to $\Bbbk\mathcal{X}$. This proves (a). Note that (b) follows from (a) and from Proposition 5.2.3 taking $T' = F_\lambda T$.

Finally, it follows from (b) that $\mathrm{End}(T)$ and $\mathrm{End}(F_\lambda T)$ have the same global dimension ([34, Theorem 1.3]). Hence, $\mathrm{End}(F_\lambda T)$ is hereditary. Thus, $\mathcal{C}_{A*G}$ is acyclic (see [25]).

(3) There exists a cluster tilting object $T' \in \mathcal{C}_{A*G}$ such that $\mathrm{End}(T')$ is a hereditary algebra. In particular, the cluster tilting objects of $\mathcal{C}_{A*G}$ form a mutation class (see [6, Proposition 3.5]). By Corollary 5.3.6, there exists a cluster tilting object $T \in \mathcal{C}_A$ such that $T^g \simeq T$ for all $g \in G$

and such that $T' \simeq F_\lambda T$. By Proposition 5.2.3, the algebras $\mathrm{End}(T) * G$ and $\mathrm{End}(T')$ are Morita equivalent. Since $\mathrm{End}(T')$ is hereditary, so is $\mathrm{End}(T)$. Thus, $\mathcal{C}_A$ is acyclic (see [25]). □

6. Application to higher Auslander-Reiten theory

This section applies the study of subsection 3.5 on Calabi-Yau completions to the interplay between skew group algebras and d-representation infinite algebras in the sense of [19]. For the sake of comprehensiveness, the corresponding interplay for d-representation finite algebras is presented in subsection 6.1. The case of d-representation infinite algebras is presented in subsection 6.2. The setting is the following.

Setting 6.1. Let A be a finite dimensional algebra. Let d be a non negative integer. Let G be a finite group acting on A by algebra automorphisms. Assume that $\mathrm{char}(\Bbbk) \nmid \mathrm{Card}(G)$.

Until the end of the text, the category of finitely generated modules over a finite dimensional algebra R is denoted by $\mathrm{mod}(R)$.

6.1. ***d*-representation finite algebras.** Given a finite dimensional algebra R, recall ([20]) that a *d-cluster tilting subcategory* of $\mathrm{mod}(R)$ is a functorially finite subcategory $\mathcal{C}$ such that

$$\left\{ \begin{array}{rcl} \mathcal{C} & = & \{M \in \mathrm{mod}(R) \mid (\forall X \in \mathcal{C})\ (\forall i = 1, \dots, d-1)\ \mathrm{Ext}^i_R(X, M) = 0\} \\ & = & \{M \in \mathrm{mod}(R) \mid (\forall X \in \mathcal{C})\ (\forall i = 1, \dots, d-1)\ \mathrm{Ext}^i_R(M, X) = 0\}\,, \end{array} \right.$$

and a *d-cluster tilting module* is a module $M \in \mathrm{mod}(R)$ such that $\mathrm{add}(M)$ is a d-cluster tilting subcategory. The algebra R is called *d-representation finite* when such an M exists and when R has global dimension at most d. When such an M exists and when d is equal to the global dimension of R, then $\mathrm{add}(M)$ is necessarily equal to the following subcategory denoted by $\mathcal{I}_R$ (see [18, Proposition 0.2]),

$$\mathcal{I}_R = \mathrm{add}(\{\tau_d^k I \mid k \in \mathbb{N},\ I \in \mathrm{mod}(A) \text{ injective}\})\,, \tag{6.1.1}$$

where $\tau_d := D\mathrm{Ext}^d_R(-, R) \colon \mathrm{mod}(R) \to \mathrm{mod}(R)$ is the higher Auslander-Reiten translation ($D := \mathrm{Hom}_{\Bbbk}(-, \Bbbk)$).

The aim of this subsection is to prove that, when A has global dimension d, then A is d-representation finite if and only if so is $A * G$. This is based on the following lemma.

Lemma 6.1.1. *Assume setting 6.1. Denote by F_λ and by F_* the extension-of-scalars functor $\mathrm{mod}(A) \to \mathrm{mod}(A*G)$ and the restriction-of-scalars functor $\mathrm{mod}(A*G) \to \mathrm{mod}(A)$, respectively.*

(1) $\tau_d F_\lambda M \simeq F_\lambda \tau_d M$ for all finite dimensional A-modules M.
(2) $\tau_d F_* N \simeq F_* \tau_d N$ for all finite dimensional A-modules N.

Proof. (1) Let $M \in \mathrm{mod}(A)$. By adjunction, there is an isomorphism $D(\mathrm{Hom}_{A*G}(F_\lambda M, A*G)) \simeq D(\mathrm{Hom}_A(M, A*G))$ in $\mathrm{mod}(A*G)$. The following composite mapping is bijective,

$$D(\mathrm{Hom}_A(M, A*G)) \to \bigoplus_{g \in G} D(\mathrm{Hom}_A(M, A^g)) \to \bigoplus_{g \in G} D(\mathrm{Hom}_A(M, A))\,, \tag{6.1.2}$$

where the leftmost arrow is induced by the mapping $\oplus_{g \in G} A^g \to A * G$ given by $(a_g)_{g \in G} \mapsto \sum_{g \in G} a_g * g$ and the rightmost arrow is induced by the mappings $A \to A^g$ given by $a \mapsto {}^g a$, for all $g \in G$. Let $\varphi \in D(\mathrm{Hom}_A(M, A*G))$. If its image under (6.1.2) is denoted by $(\lambda_g)_{g \in G}$, then

$$(\forall g \in G)\ (\forall f \in \mathrm{Hom}_A(M, A))\ \ \lambda_g(f) = \varphi \left(\begin{array}{ccc} M & \to & A * G \\ m & \mapsto & {}^g f(m) * g \end{array} \right).$$

Given $a \in A$, if $(\mu_g)_{g \in G}$ denotes the image of $\varphi \cdot (a * e)$ under (6.1.2), then $\mu_g(f) = \lambda_g({}^{g^{-1}} a \cdot f)$ for all $g \in G$ and $f \in \mathrm{Hom}_A(M, A)$; in other words, $\mu_g = \lambda_g \cdot ({}^{g^{-1}} a)$ for all $g \in G$. Besides,

given $h \in H$, if $(\nu_g)_{g\in G}$ denotes the image of $\varphi \cdot (1 * h)$ under (6.1.2), then $\nu_g(f) = \lambda_{hg}(f)$ for all $f \in \mathrm{Hom}_A(M, A)$; in other words, $\nu_g = \lambda_{hg}$ for all $g \in G$.

It follows from these considerations that the composition of the following mapping with (6.1.2) is an isomorphism in $\mathrm{mod}(A * G)$ from $D(\mathrm{Hom}_A(M, A * G))$ to $D(\mathrm{Hom}_A(M, A)) \otimes_A A * G$.

$$\begin{array}{ccc} \bigoplus_{g\in G} D(\mathrm{Hom}_A(M, A)) & \to & D(\mathrm{Hom}_A(M, A)) \otimes_A A * G \\ (\lambda_g)_{g\in G} & \mapsto & \sum_{g\in G} \lambda_g \otimes (1 * g^{-1}) \,. \end{array} \tag{6.1.3}$$

Therefore, there is a functorial isomorphism in $\mathrm{mod}(A * G)$,

$$D(\mathrm{Hom}_{A*G}(F_\lambda M, A * G)) \simeq F_\lambda D(\mathrm{Hom}_A(M, A)) \,. \tag{6.1.4}$$

Now, the isomorphism $\tau_d F_\lambda M \simeq F_\lambda \tau_d M$ follows from (6.1.4) after deriving because F_λ is exact and preserves projective modules.

(2) Let $N \in \mathrm{mod}(A * G)$. By adjunction, there is an isomorphism in $\mathrm{mod}(A)$,

$$D(\mathrm{Hom}_A(F_\cdot N, A)) \simeq D(\mathrm{Hom}_{A*G}(N, A * G)) \,. \tag{6.1.5}$$

And the isomorphism $\tau_d F_\cdot N \simeq F_\cdot \tau_d N$ follows from (6.1.5) after deriving also because $F_\cdot$ is exact and preserves projective modules. □

Here is the first main result of this section.

Proposition 6.1.2. *Assume setting 6.1. The following assertions are equivalent.*

(i) A is d-representation finite.
(ii) $A * G$ is d-representation finite.

Proof. Denote by F_λ and by $F_\cdot$ the extension-of-scalars functor $\mathrm{mod}(A) \to \mathrm{mod}(A * G)$ and the restriction-of-scalars functor $\mathrm{mod}(A * G) \to \mathrm{mod}(A)$, respectively. Note that $A * G$ has global dimension d. Since $F_\lambda D(A) \simeq D(A * G)$ as $A * G$-modules and $F_\cdot D(A * G) \simeq D(A)^{\mathrm{Card}(G)}$ as A-modules, it follows from Lemma 6.1.1 that

$$\left\{ \begin{array}{rcl} \mathrm{add}(F_\lambda \mathcal{I}_A) & = & \mathcal{I}_{A*G} \\ \mathrm{add}(F_\cdot \mathcal{I}_{A*G}) & = & \mathcal{I}_A \,. \end{array} \right. \tag{6.1.6}$$

Assume (i) and let $M \in \mathrm{mod}(A)$ be a d-cluster tilting module such that $\mathrm{add}(M) = \mathcal{I}_A$. By construction, $\mathcal{I}_A$ is stable under the action of G. Hence, so is $\mathrm{add}(M)$. Therefore, $\mathrm{add}(M)$ is stable under $\mathrm{add}(F_\cdot F_\lambda)$ (see Lemma 2.3.1, part (4)). Hence, the proof of Proposition 5.2.1 can be adapted here to prove that $\mathrm{add}(F_\lambda M)$ is a d-cluster tilting subcategory of $\mathrm{mod}(A * G)$ provided that Lemma 2.3.1 is used instead of Proposition 5.1.2. Therefore, $(i) \Longrightarrow (ii)$.

Conversely, assume (ii) and let $N \in \mathrm{mod}(A * G)$ be a d-cluster tilting module such that $\mathrm{add}(N) = \mathcal{I}_{A*G}$. Since $\mathcal{I}_{A*G}$ is stable under $\mathrm{add}(F_\lambda F_\cdot)$ ((6.1.6)), the proof of Proposition 5.2.1 can also be adapted here to prove that $\mathrm{add}(F_\cdot N)$ is a d-cluster tilting subcategory of $\mathrm{mod}(A)$. Therefore $(ii) \Longrightarrow (i)$. □

6.2. ***d*-representation infinite algebras.** Let R be a finite dimensional $\Bbbk$-algebra. Following [19], it is called *d-representation infinite* when it has global dimension at most d and $\mathbb{S}_d^{-i} R$ is an R-module for all $i \in \mathbb{N}$, where $\mathbb{S}_d$ denotes $(- \otimes_R^{\mathrm{L}} DR) \circ \Sigma^{-d}$. This entails that R has global dimension equal to d. When R has global dimension at most d, it is d-representation infinite if and only if the cohomology of $\mathbf{\Pi}_{d+1}(R)$ is concentrated in degree 0. Recall that the Calabi-Yau completion $\mathbf{\Pi}_{d+1}(R)$ is also called the *derived* $(d + 1)$*-preprojective* algebra and its 0-th cohomology is isomorphic to the $(d + 1)$*-preprojective algebra* $\Pi_{d+1}(R)$ of R which is defined by $\Pi_{d+1}(R) = T_R(\mathrm{Ext}_R^d(DR, R))$ ([21]).

Proposition 6.2.1. *Assume setting 6.1.*

(1) G acts on the derived $d+1$-preprojective algebra $\mathbf{\Pi}_{d+1}(A)$ by dg automorphisms and $\mathbf{\Pi}_{d+1}(A)*G$ is quasi-isomorphic to the derived $d+1$-preprojective algebra $\mathbf{\Pi}_{d+1}(A*G)$.
(2) G acts on the $d+1$-preprojective algebra $\Pi_{d+1}(A)$ by algebra automorphisms and the skew group algebra $\Pi_{d+1}(A)*G$ is isomorphic to the $d+1$-preprojective algebra $\Pi_{d+1}(A*G)$.
(3) The following assertions are equivalent.
(i) A is d-representation infinite.
(ii) $A*G$ is d-representation infinite.

Proof. (1) is a rephrasing of Theorem 1.3. (2) and (3) follow from (1). □

Note that subsection 3.5 applies here. Since A has global dimension to d, there exists a bounded complex P of finitely generated projective A-bimodules endowed with a compatible action of G and with G-equivariant differential, and there exists a G-equivariant quasi-isomorphism $P \to A$ in $\mathrm{Mod}(A^e)$. Then, there is a compatible action of G on $\mathrm{Hom}_{A^e}(P, A^e)$ such that $({}^g f)(p) = {}^g(f({}^{g^{-1}}p))$ for all $p \in P$, $g \in G$ and $f \in \mathrm{Hom}_{A^e}(P, A^e)$. Hence, there is an isomorphism $\mathrm{Hom}_{A^e}(P, A^e) \simeq \mathrm{RHom}_{A^e}(A, A^e)$ in $\mathcal{D}(\Delta)$ and, since P is bounded, $\mathrm{Hom}_{A^e}(P, A^e)$ is cofibrant in $\mathrm{Mod}(A^e)$. Therefore, one may assume that

- $\mathbf{\Pi}_{d+1}(A) = T_A(\Sigma^d \mathrm{Hom}_{A^e}(P, A^e))$ as dg algebras respectively endowed with an action of G by dg automorphisms, and $\mathbf{\Pi}_{d+1}(A*G) \underset{\text{qis}}{\simeq} T_A(\Sigma^d \mathrm{Hom}_{A^e}(P, A^e)) * G$, and, applying $H^0(-)$, that
- $\Pi_{d+1}(A) \simeq T_A(H^d \mathrm{Hom}_{A^e}(P, A^e))$ as algebras respectively endowed with an action of G by algebra automorphisms, and $\Pi_{d+1}(A*G) \simeq T_A(H^d \mathrm{Hom}_{A^e}(P, A^e)) * G$.

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