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STABLE GROUND STATES FOR THE GRAVITATIONAL VLASOV-MANEV SYSTEM

MOHAMMED LEMOU, FLORIAN MÉHATS, AND CYRIL RIGAUT

ABSTRACT. In this work, we prove the orbital stability of ground state stationary solutions to the so-called Vlasov-Manev (VM) system. This system is a kinetic model which has a similar Vlasov structure as the classical Vlasov-Poisson system, but is coupled to a potential in $-1/r - 1/r^2$ (Manev potential) instead of the usual gravitational potential in $-1/r$, and in particular the potential field does not satisfy a Poisson equation but a fractional-Laplacian equation. The ground states are constructed as minimizers of the Hamiltonian, and the orbital stability is deduced both from the compactness of minimizing sequences and the rigidity of the flow. In driving this analysis, there are two mathematical obstacles: the first one is related to the possible blow-up of solutions to the VM system, which we overcome by imposing a sub-critical condition on the constraints of the variational problem. The second difficulty (and the most important) is related to the nature of the Euler-Lagrange equations (fractional-Laplacian equations) to which classical results for the Poisson equation do not extend. In this paper we prove the uniqueness of the minimizer under equimeasurability constraints, without using Euler-Lagrange equations.

1. Introduction and main results

In this paper, we study the stability of steady states to the Vlasov-Manev (VM) model for gravitational systems. In this mean field kinetic model, the usual Newtonian interaction potential is replaced by the so-called Manev potential. This potential, which was studied in particular by Manev in the 1920' in order to explain some observed phenomena in planetary systems [16, 17, 18, 19], corrects the Newtonian gravitational potential as follows:

$$U(x) = -\frac{1}{4\pi|x|} - \frac{\kappa}{2\pi^2|x|^2},$$

where κ is a positive constant. Further physical studies of this potential can be found in [7].

Taking into account this correction, the standard Vlasov-Poisson system is replaced by the following Vlasov-Manev system:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x \phi_f \cdot \nabla_v f = 0, & (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3, \\ f(t = 0, x, v) = f_0(x, v) \geq 0, \end{cases} \quad (1.1)$$

in which $f = f(t, x, v) \geq 0$ is a distribution function and ϕ_f the associated potential defined as follows. We have

$$\phi_f(t, x) = \phi_f^P + \kappa \phi_f^M, \quad (1.2)$$

where ϕ_f^P and ϕ_f^M are respectively the Poisson potential and the Manev potential of f given by:

$$\phi_f^P(t, x) = - \int_{\mathbb{R}^3} \frac{\rho_f(t, y)}{4\pi|x-y|} dy, \quad \phi_f^M(t, x) = - \int_{\mathbb{R}^3} \frac{\rho_f(t, y)}{2\pi^2|x-y|^2} dy, \quad (1.3)$$

and ρ_f is the density associated to the distribution function f and given by

$$\rho_f(t, x) = \int_{\mathbb{R}^3} f(t, x, v) dv.$$

Note that the two potentials satisfy:

$$\Delta \phi_f^P = \rho_f \quad \text{and} \quad \sqrt{-\Delta} \phi_f^M = -\rho_f,$$

and in particular system (1.1) reduces to the well-known gravitational Vlasov-Poisson system in the case $\kappa = 0$.

To our knowledge, the only existing mathematical analysis of the Vlasov-Manev model is due to Bobylev, Dukes, Illner and Victory [5, 6]. In these works, the local existence of regular solutions is proved and questions of global existence and finite-time blow-up are discussed.

We now give some basic properties of the Vlasov-Manev system (1.1). Sufficiently regular solutions to (1.1) on a time interval $[0, T]$ satisfy the conservation of the so-called Casimir functionals:

$$\forall t \in [0, T], \quad \|j(f(t))\|_{L^1} = \|j(f_0)\|_{L^1} \quad (1.4)$$

and the conservation of the Hamiltonian

$$\forall t \in [0, T], \quad \mathcal{H}(f(t)) = \mathcal{H}(f_0),$$

where j is a smooth real-valued function with $j(0) = 0$, and

$$\mathcal{H}(f(t)) = \| |v|^2 f(t) \|_{L^1} - E_{pot}(f(t)). \quad (1.5)$$

The potential energy E_{pot} is defined by

$$\begin{aligned} E_{pot}(f(t)) &= - \int_{\mathbb{R}^3} \phi_f(t, x) \rho_f(t, x) dx \\ &= - \int_{\mathbb{R}^3} \phi_f^P(t, x) \rho_f(t, x) dx - \kappa \int_{\mathbb{R}^3} \phi_f^M(t, x) \rho_f(t, x) dx \\ &= E_{pot}^P(f(t)) + \kappa E_{pot}^M(f(t)) \end{aligned}$$

and is controled thanks to standard interpolation inequalities:

$$0 \leq E_{pot}^P(f) \leq C_1 \| |v|^2 f \|_{L^1}^{\frac{1}{2}} \| f \|_{L^1}^{\frac{7p-9}{6(p-1)}} \| f \|_{L^p}^{\frac{p}{3(p-1)}}, \quad (1.6)$$

$$0 \leq E_{pot}^M(f) \leq C_2 \| |v|^2 f \|_{L^1} \| f \|_{L^1}^{\frac{p-3}{3(p-1)}} \| f \|_{L^p}^{\frac{2p}{3(p-1)}}, \quad (1.7)$$

for all $p \geq 3$.

Our aim in this paper is to prove the orbital stability of ground states for the Vlasov-Manev problem using variational techniques. While this question has not been studied in the past for the the VM system, it has attracted considerable attention in the case of the Vlasov-Poisson system ($\kappa = 0$), both in physics (see [1, 2], [3] and the references therein) and mathematics community [22, 8, 9, 10, 11, 20, 13]. The extension of the standard variational approach to the case of VM system faces two mathematical difficulties. The first one is that the Hamiltonian may be unbounded from below, and this prevents from obtaining ground states as minimizers

of this Hamiltonian. This phenomenon is also related to the possible finite-time blow-up of solutions to the VM system [5], and a similar situation has been encountered in the stability analysis of the relativistic Vlasov-Poisson system [12]. As in [12], this difficulty will be overcome thanks to a homogeneity-breaking property which allows to impose a sub-critical condition on the constraints of the variational problem. The second difficulty (probably the most important) is related to the nature of the Euler-Lagrange equations to which classical results for the Poisson equation do not extend. In the classical VP case, a complete stability result is generally obtained by using both the Euler-Lagrange equation (which is equivalent to a non linear Poisson equation) and the rigidity of the flow. In the present case, the Euler-Lagrange equation is a fractional-Laplacian equation, and this prevents from using ODE techniques. Nevertheless, we prove the uniqueness of the minimizer under equimeasurable constraints by a new argument which completely avoids ODE techniques.

In order to state our main results, let us make precise our assumptions. Consider a function $j : \mathbb{R} \rightarrow \mathbb{R}^+$ satisfying the following hypotheses.

(H1) j is a \mathcal{C}^1 function, strictly convex and even, with $j(0) = j'(0) = 0$.

(H2) There exists $p > 3$ such that

$$j(t) \geq Ct^p \quad \forall t \geq 0. \quad (1.8)$$

(H3) There exist $p_1, p_2 > 3$ such that

$$p_1 \leq \frac{tj'(t)}{j(t)} \leq p_2 \quad \forall t > 0. \quad (1.9)$$

We note that (H3) is equivalent to the nondichotomy condition:

$$b^{p_1}j(t) \leq j(bt) \leq b^{p_2}j(t), \quad \forall b \geq 1, t \geq 0. \quad (1.10)$$

For a function j satisfying (H1), (H2) and (H3), we define the corresponding energy space

$$\mathcal{E}_j = \{f \geq 0 \text{ such that } \|f\|_{\mathcal{E}_j} := \|f\|_{L^1} + \|j(f)\|_{L^1} + \||v|^2 f\|_{L^1} < +\infty\}. \quad (1.11)$$

From the interpolation inequality (1.7), the following constant is strictly positive:

$$K_j^M = \inf_{f \in \mathcal{E}_j \setminus \{0\}} \frac{\||v|^2 f\|_{L^1} \|f\|_{L^1}^{\frac{p-3}{3(p-1)}} \|j(f)\|_{L^1}^{\frac{2p}{3(p-1)}}}{E_{pot}^M(f)}. \quad (1.12)$$

In our first result, we establish the existence of ground states for the Vlasov-Manev problem under a subcritical condition, and then apply concentration-compactness techniques to prove the compactness of all minimizing sequences.

Theorem 1.1 (Existence of ground states). *Let j be a function satisfying (H1), (H2), (H3). Let $M_1 > 0$, $M_j > 0$ such that*

$$\kappa M_1^{\frac{p-3}{3(p-1)}} M_j^{\frac{2p}{3(p-1)}} < K_j^M, \quad (1.13)$$

where K_j^M is defined by (1.12), and let

$$\mathcal{F}(M_1, M_j) = \{f \in \mathcal{E}_j, \|f\|_{L^1} = M_1, \|j(f)\|_{L^1} = M_j\}.$$

Then any minimizing sequence of the problem

$$I(M_1, M_j) = \inf_{f \in \mathcal{F}(M_1, M_j)} \mathcal{H}(f), \quad (1.14)$$

where \mathcal{H} is the Hamiltonian defined by (1.5), is relatively strongly compact in the energy space \mathcal{E}_j up to a space translation shift in x . Let Q_j be a minimizer of (1.14). Then Q_j is a compactly supported stationary solution to (1.1) and takes the form

$$Q_j(x, v) = (j')^{-1} \left(\frac{\frac{|v|^2}{2} + \phi_{Q_j}(x) - \lambda}{\mu} \right)_+ \quad (1.15)$$

where λ and μ are negative constants. Moreover, $\phi_{Q_j}(x)$ is spherically symmetric (up to a translation shift), nondecreasing and belongs to $\mathcal{C}^{0,\alpha}$, for all $\alpha \in (0, 1)$. In (1.15), we used the notation $a_+ = \max(a, 0)$.

Notice that in the case $\kappa = 0$, condition (1.13) is always satisfied. In this case the Vlasov-Manev system (1.1) is nothing but the classical Vlasov-Poisson system, for which it is already known that minimizers of the two constraints problem (1.14) always exist and the minimizing sequences are compact, see [11]. In [12], the orbital stability in the case of the VP system has been proved thanks to a uniqueness result of these minimizers which was based on a combination of the Poisson equation and the rigidity of the flow.

Now we state our second main result, the uniqueness of the minimizer under an equimeasurability condition. The following uniqueness result is more general than the one obtained in [12] in the sense that it completely avoids the use of the Euler-Lagrange equation. The only properties of the minimizer that we use are: their equimeasurability and the fact they are all expressed as a given function of their microscopic energy. In particular, our proof avoids the usual ODE techniques, which in fact, are useless here since the Euler-Lagrange equation is a fractional-Laplacian equation.

Theorem 1.2 (Uniqueness of the minimizer under equimeasurability condition). *Let $F \in \mathcal{C}^0(\mathbb{R}, \mathbb{R}_+)$, strictly decreasing on \mathbb{R}_- , such that $F(\mathbb{R}_-) = \mathbb{R}_+$ and $F(\mathbb{R}_+) = \{0\}$. We define $Q_0(x, v) = F(\frac{|v|^2}{2} + \psi_0(x))$ and $Q_1(x, v) = F(\frac{|v|^2}{2} + \psi_1(x))$ on $\mathbb{R}^3 \times \mathbb{R}^3$, where ψ_0 and ψ_1 are two nondecreasing continuous radially symmetric potentials such that the sets $\{x \in \mathbb{R}^3, \psi_0(x) < 0\}$ and $\{x \in \mathbb{R}^3, \psi_1(x) < 0\}$ are bounded. Then the equimeasurability of Q_0 and Q_1 for the Lebesgue measure in \mathbb{R}^6*

$$\forall t > 0, \text{meas}\{(x, v) \in \mathbb{R}^6, Q_0(x, v) > t\} = \text{meas}\{(x, v) \in \mathbb{R}^6, Q_1(x, v) > t\} \quad (1.16)$$

implies that $Q_0 = Q_1$. In particular, two equimeasurable minimizers of (1.14) under the subcritical condition (1.13) are equal.

Our last result is the stability of the ground states through the Vlasov-Manev flow.

Theorem 1.3 (Orbital stability of ground states). *Any minimizer Q_j of (1.14) under the condition (1.13) is orbitally stable under the flow (1.1). More precisely, given $\varepsilon > 0$, there exists $\delta(\varepsilon)$ such that the following holds true.*

Consider f_0 a smooth function with $\|f_0 - Q_j\|_{\mathcal{E}_j} \leq \delta(\varepsilon)$, and let $f(t)$ be a local classical solution to (1.1) on a time interval $[0, T]$, with initial data f_0 . Then there

exists a translation shift $x(t) \in \mathbb{R}^3$ such that, for all $t \in [0, T]$,

$$\|f(t, x + x(t), v) - Q_j\|_{\mathcal{E}_j} < \varepsilon.$$

Remark. The goal of this article is to prove the stability of classical solutions to the Vlasov-Manev model, and not to solve the Cauchy problem. For smooth initial data decaying fast enough at the infinity, the local existence and the uniqueness of regular solutions to (1.1) has been proved in [6]. The global existence of classical solutions is an open problem. Our result shows that the solutions remain in the vicinity of the ground state Q_j (up to a translation shift), but does not a priori exclude a possible blow-up of some derivative of f .

The outline of the paper is as follows. Section 2 deals with the proof of Theorem 1.1. After preliminary technical results concerning some properties of the infimum $I(M_1, M_j)$ (Subsection 2.1), we apply in Subsection 2.2 the P.-L. Lions concentration-compactness principle in order to prove the convergence of minimizing sequences. Then we characterize the ground states: Euler-Lagrange equation, regularity and spherical symmetry. Section 3 is devoted to the proof of stability of the ground state through the Vlasov-Manev flow, in two steps. In Subsection 3.1, we prove the uniqueness of the ground state in the class of equimeasurable functions, Theorem 1.2. This uniqueness result is crucial to obtain the orbital stability, Theorem 1.3, which we prove in Subsection 3.2.

2. Existence of ground states

This section is devoted to the proof of Theorem 1.1.

2.1. Properties of the infimum. We start with a lemma concerning some monotonicity properties of the infimum defined by (1.14).

Lemma 2.1 (Monotonicity properties of the infimum $I(M_1, M_j)$). *Let j be a real-valued function satisfying Assumptions (H1)–(H3), let $M_1 > 0$ and $M_j > 0$ such that (1.13) holds, and let $I(M_1, M_j)$ be defined by (1.14). Then we have*

$$-\infty < I(M_1, M_j) < 0. \quad (2.1)$$

Moreover the following nondichotomy condition holds true: for all $0 < \alpha < 1$ and $0 \leq \beta \leq 1$,

$$I(\alpha M_1, \beta M_j) + I((1 - \alpha)M_1, (1 - \beta)M_j) > I(M_1, M_j). \quad (2.2)$$

Proof. Step 1. The infimum is finite and negative.

We first prove (2.1). From (1.6), we define K_j^P by

$$K_j^P = \inf_{f \in \mathcal{E}_j \setminus \{0\}} \frac{\| |v|^2 f \|_{L^1}^{\frac{1}{2}} \| f \|_{L^1}^{\frac{7p-9}{6(p-1)}} \| j(f) \|_{L^1}^{\frac{p}{3(p-1)}}}{E_{pot}^P(f)} > 0. \quad (2.3)$$

Let $f \in \mathcal{F}(M_1, M_j)$. Then from (1.12) and (2.3), we have

$$\begin{aligned} \mathcal{H}(f) &\geq \| |v|^2 f \|_{L^1} - \frac{1}{K_j^P} \| |v|^2 f \|_{L^1}^{\frac{1}{2}} M_1^{\frac{7p-9}{6(p-1)}} M_j^{\frac{1}{3(p-1)}} - \frac{\kappa}{K_j^M} \| |v|^2 f \|_{L^1} M_1^{\frac{p-3}{3(p-1)}} M_j^{\frac{2}{3(p-1)}} \\ &\geq \| |v|^2 f \|_{L^1} \left(1 - \frac{\kappa}{K_j^M} M_1^{\frac{p-3}{3(p-1)}} M_j^{\frac{2}{3(p-1)}} \right) - \frac{1}{K_j^P} \| |v|^2 f \|_{L^1}^{\frac{1}{2}} M_1^{\frac{7p-9}{6(p-1)}} M_j^{\frac{1}{3(p-1)}}. \end{aligned}$$

Now the subcritical condition (1.13) implies that

$$1 - \frac{\kappa}{K_j^M} M_1^{\frac{p-3}{3(p-1)}} M_j^{\frac{2}{3(p-1)}} > 0.$$

Thus $\mathcal{H}(f)$ is bounded from below, which proves that $I(M_1, M_j)$ is finite. To prove that $I(M_1, M_j)$ is negative, we use a rescaling argument. For $\lambda > 0$ and $f \in \mathcal{F}(M_1, M_j)$, we consider the rescaled function $\tilde{f}(x, v) = f(\frac{x}{\lambda}, \lambda v)$. Then \tilde{f} belongs to $\mathcal{F}(M_1, M_j)$ and we have

$$\begin{aligned} \mathcal{H}(\tilde{f}) &= \frac{1}{\lambda^2} \| |v|^2 f \|_{L^1} - \frac{1}{\lambda} E_{pot}^P(f) - \frac{\kappa}{\lambda^2} E_{pot}^M(f) \\ &\sim -\frac{1}{\lambda} E_{pot}^P(f) \text{ as } \lambda \rightarrow +\infty, \end{aligned}$$

where $E_{pot}^P(f)$ is positive. The property (2.1) follows.

Step 2. The nondichotomy condition.

We now claim the following monotonicity properties: for all $0 < k \leq 1$,

$$I(M_1, kM_j) \geq k^{\frac{1}{3(p_2-1)}} I(M_1, M_j) \quad (2.4)$$

and

$$I(kM_1, M_j) \geq k^{\frac{4p_1-6}{3(p_1-1)}} I(M_1, M_j). \quad (2.5)$$

Proof of (2.4). Let $k \in (0, 1]$ and $f \in \mathcal{F}(M_1, kM_j)$. From Appendix A, consider the unique rescaled function $\tilde{f}(x, v) = \alpha f(\alpha^{1/3}x, v)$ in $\mathcal{F}(M_1, M_j)$. From (A.2), we deduce in particular that $\alpha \geq 1$ and

$$\alpha^{p_1-1} \leq \frac{1}{k} \leq \alpha^{p_2-1}.$$

Then, we get

$$\mathcal{H}(\tilde{f}) = \| |v|^2 f \|_{L^1} - \alpha^{\frac{1}{3}} E_{pot}^P(f) - \alpha^{\frac{2}{3}} \kappa E_{pot}^M(f) \leq \| |v|^2 f \|_{L^1} - \alpha^{\frac{1}{3}} (E_{pot}^P(f) + \kappa E_{pot}^M(f))$$

and

$$I(M_1, M_j) \leq \mathcal{H}(\tilde{f}) \leq \| |v|^2 f \|_{L^1} - \left(\frac{1}{k}\right)^{\frac{1}{3(p_2-1)}} (E_{pot}^P(f) + \kappa E_{pot}^M(f)) \leq \left(\frac{1}{k}\right)^{\frac{1}{3(p_2-1)}} \mathcal{H}(f).$$

This result holds for all $f \in \mathcal{F}(M_1, kM_j)$ and $k \in (0, 1]$, which proves (2.4).

Proof of (2.5). Similarly, we take $f \in \mathcal{F}(kM_1, M_j)$ and set

$$\tilde{f}(x, v) = \alpha f(\alpha^{1/3}k^{1/3}x, v)$$

the unique rescaled function in $\mathcal{F}(M_1, M_j)$, which implies that $\alpha \leq 1$ with

$$\alpha^{p_2-1} \leq k \leq \alpha^{p_1-1}.$$

Thus,

$$\begin{aligned} \mathcal{H}(\tilde{f}) &= \frac{1}{k} \| |v|^2 f \|_{L^1} - \alpha^{\frac{1}{3}} k^{-5/3} E_{pot}^P(f) - \alpha^{\frac{2}{3}} k^{-4/3} \kappa E_{pot}^M(f) \\ &\leq \frac{1}{k} \| |v|^2 f \|_{L^1} - \left(\frac{1}{k}\right)^{\frac{5p_1-6}{3(p_1-1)}} E_{pot}^P(f) - \left(\frac{1}{k}\right)^{\frac{4p_1-6}{3(p_1-1)}} \kappa E_{pot}^M(f). \end{aligned}$$

From $k \leq 1$ and $p > 3$, we conclude that

$$I(M_1, M_j) \leq \mathcal{H}(\tilde{f}) \leq \left(\frac{1}{k}\right)^{\frac{4p_1-6}{3(p_1-1)}} \mathcal{H}(f),$$

and (2.5) follows.

We now prove (2.2). Let $0 < \alpha < 1$ and $0 \leq \beta \leq 1$. Then (2.4) and (2.5) imply

$$I(\alpha M_1, \beta M_j) \geq \alpha^{\frac{4p_1-6}{3(p_1-1)}} \beta^{\frac{1}{3(p_2-1)}} I(M_1, M_j),$$

and a similar inequality with $(1 - \alpha)$ and $(1 - \beta)$. As we have $I(M_1, M_j) < 0$, we only have to show that

$$\alpha^{\frac{4p_1-6}{3(p_1-1)}} \beta^{\frac{1}{3(p_2-1)}} + (1 - \alpha)^{\frac{4p_1-6}{3(p_1-1)}} (1 - \beta)^{\frac{1}{3(p_2-1)}} < 1,$$

which holds true since $p_2 > 1$ and $\frac{4p_1-6}{3(p_1-1)} > 1$. The proof of Lemma 2.1 is complete. \square

2.2. Proof of Theorem 1.1. This subsection is devoted to the proof of Theorem 1.1, adapting arguments from [11] and [12].

Step 1. Compactness of minimizing sequences.

Let $M_1, M_j > 0$. From Lemma 2.1, we know that $I(M_1, M_j)$ is finite and negative. Consider a minimizing sequence f_n of (1.14):

$$\|f_n\|_{L^1} = M_1, \quad \|j(f_n)\|_{L^1} = M_j \quad \text{and} \quad \lim_{n \rightarrow +\infty} \mathcal{H}(f_n) = I(M_1, M_j). \quad (2.6)$$

Let

$$\rho_n(x) = \int_{\mathbb{R}^3} f_n(x, v) dv.$$

From the concentration compactness lemma [14, 15], up to a subsequence, one of the three following possibilities occurs (B_R being the ball of radius R centered at the origin in \mathbb{R}^3):

(i) *Compactness:* there exists $y_n \in \mathbb{R}^3$ such that

$$\forall \varepsilon > 0, \exists R < +\infty \text{ such that } \forall n \geq 1, \int_{y_n + B_R} \rho_n(x) dx \geq M_1 - \varepsilon. \quad (2.7)$$

(ii) *Vanishing:*

$$\forall R < +\infty, \lim_{n \rightarrow +\infty} \sup_{y \in \mathbb{R}^3} \int_{y + B_R} \rho_n(x) dx = 0. \quad (2.8)$$

(iii) *Dichotomy:* there exists $m_1 \in (0, M_1)$ such that for all $\varepsilon > 0$, there exist $n_0 \geq 1$ and sequences $(\rho_n^1)_{n \geq 1}, (\rho_n^2)_{n \geq 1}, (w_n)_{n \geq 1} \in L^1_+(\mathbb{R}^3)$ satisfying, for all $n \geq n_0$,

$$\left\{ \begin{array}{l} \rho_n^1 \rho_n^2 = \rho_n^1 w_n = \rho_n^2 w_n = 0 \text{ almost everywhere,} \\ \rho_n = \rho_n^1 + \rho_n^2 + w_n \text{ with } 0 \leq \rho_n^1, \rho_n^2, w_n \leq \rho_n \\ \text{dist}(\text{Supp}(\rho_n^1), \text{Supp}(\rho_n^2)) \rightarrow +\infty \text{ as } n \rightarrow +\infty, \\ \|\rho_n - \rho_n^1 - \rho_n^2\|_{L^1} \leq \varepsilon, \quad \left| \int_{\mathbb{R}^3} \rho_n^1(x) dx - m_1 \right| + \left| \int_{\mathbb{R}^3} \rho_n^2(x) dx - (M_1 - m_1) \right| < \varepsilon. \end{array} \right. \quad (2.9)$$

We claim that only compactness can occur. To show this claim, we need some preliminary results. First, we know that $\mathcal{H}(f_n)$ is bounded. Consequently, the inequality

$$\mathcal{H}(f_n) \geq \| |v|^2 f_n \|_{L^1} \left(1 - \frac{\kappa}{K_j^M} M_1^{\frac{p-3}{3(p-1)}} M_j^{\frac{2}{3(p-1)}} \right) - \frac{1}{K_j^P} \| |v|^2 f_n \|_{L^1}^{\frac{1}{2}} M_1^{\frac{7p-9}{6(p-1)}} M_j^{\frac{1}{3(p-1)}},$$

and the subcritical condition (1.13) imply that the kinetic energy of f_n is also bounded. Thus f_n is bounded in the energy space \mathcal{E}_j and a standard interpolation inequality shows that (ρ_n) is a bounded sequence in $L^r(\mathbb{R}^3)$ for all $r \in [1, q]$ with $q = \frac{5p-3}{3p-1} > \frac{3}{2}$ (recall that the real number p has been defined in Assumption (H2)).

Vanishing cannot occur. Assume that vanishing occurs and pick $\varepsilon > 0$. We denote $k_P = 1$, $k_M = 2$ and introduce the index $i \in \{P, M\}$. For $0 < \eta < R$, we decompose the Poisson and Manev potential energies into a sum of three terms, as

$$\begin{aligned} E_{pot}^i(f_n) &= \int_{|x-y|>R} \frac{\rho_n(x)\rho_n(y)}{|x-y|^{k_i}} dx dy + \int_{|x-y|<\eta} \frac{\rho_n(x)\rho_n(y)}{|x-y|^{k_i}} dx dy \\ &\quad + \int_{\eta < |x-y| < R} \frac{\rho_n(x)\rho_n(y)}{|x-y|^{k_i}} dx dy. \end{aligned} \quad (2.10)$$

The first term in (2.10) can be estimated as follows. For R large enough, we have

$$\int_{|x-y|>R} \frac{\rho_n(x)\rho_n(y)}{|x-y|^{k_i}} dx dy \leq \frac{M_1^2}{R^{k_i}} \leq \varepsilon.$$

Let us now estimate the second term in (2.10). Let $K_\eta(x) = \frac{\mathbf{1}_{|x|<\eta}(x)}{|x|^{k_i}}$ and $r = \frac{q}{2(q-1)} < \frac{3}{2}$. Then, K_η belongs to $L^r(\mathbb{R}^3)$ with

$$\|K_\eta\|_r \leq C\eta^{\beta_i}, \quad \beta_i = \frac{3 - k_i r}{r} > 0.$$

Furthermore, since r satisfies $\frac{1}{r} + \frac{1}{q} = 1 + \frac{1}{q'}$, we deduce from the Young inequality that

$$\|K_\eta * \rho_n\|_{q'} \leq C \|K_\eta\|_r \|\rho_n\|_q \leq C\eta^{\beta_i} \|\rho_n\|_q.$$

Hence, from the Hölder inequality, for η small enough we have

$$\int_{|x-y|<\eta} \frac{\rho_n(x)\rho_n(y)}{|x-y|^{k_i}} dx dy = \|\rho_n(K_\eta * \rho_n)\|_1 \leq C\eta^{\beta_i} \|\rho_n\|_q^2 < \varepsilon,$$

since $\|\rho_n\|_q$ is bounded.

In order to estimate the third term in (2.10), we write

$$\begin{aligned} \int_{\eta < |x-y| < R} \frac{\rho_n(x)\rho_n(y)}{|x-y|^{k_i}} dx dy &\leq \frac{1}{\eta^{k_i}} \int_{\mathbb{R}^3} \rho_n(y) \left(\int_{|x-y|<R} \rho_n(x) dx \right) dy \\ &\leq \frac{M_1}{\eta^{k_i}} \sup_{y \in \mathbb{R}^3} \int_{y+B_R} \rho_n(x) dx. \end{aligned}$$

Since we assume that vanishing occurs, we have

$$\sup_{y \in \mathbb{R}^3} \int_{y+B_R} \rho_n(x) dx \rightarrow 0$$

and the third term converges to zero as $n \rightarrow +\infty$, for each fixed R and η . We finally obtain $E_{pot}^i(f_n) \rightarrow 0$ as $n \rightarrow +\infty$, for $i \in \{P, M\}$. Thus

$$E_{pot}(f_n) \rightarrow 0.$$

Now, passing to the limit in

$$\mathcal{H}(f_n) = \||v|^2 f_n\|_{L^1} - E_{pot}(f_n) \geq -E_{pot}(f_n),$$

and, using (2.6), we get $I(M_1, M_j) \geq 0$ which contradicts (2.1). We have proved that vanishing cannot occur.

Dichotomy cannot occur. Suppose that dichotomy occurs and pick $\varepsilon > 0$. We have a decomposition $\rho_n = \rho_n^1 + \rho_n^2 + w_n$, where ρ_n^1 , ρ_n^2 and w_n are disjointly supported. We define then $f_n^1 = f_n \mathbf{1}_{\{x \in \text{Supp}(\rho_n^1)\}}$ and similarly f_n^2 and f_n^w . Now we write, for $i \in \{P, M\}$,

$$\begin{aligned} E_{pot}^i(f_n) - E_{pot}^i(f_n^1) - E_{pot}^i(f_n^2) &= 2 \int_{\mathbb{R}^6} \frac{\rho_n^1(x) \rho_n^2(y)}{|x-y|^{k_i}} dx dy + 2 \int_{\mathbb{R}^6} \frac{\rho_n(x) w_n(y)}{|x-y|^{k_i}} dx dy \\ &\quad - \int_{\mathbb{R}^6} \frac{w_n(x) w_n(y)}{|x-y|^{k_i}} dx dy. \end{aligned} \quad (2.11)$$

Let us treat the first term in (2.11). From

$$d_n = \text{dist}(\text{Supp}(\rho_n^1), \text{Supp}(\rho_n^2)) \rightarrow +\infty,$$

we deduce that

$$\int_{\mathbb{R}^6} \frac{\rho_n^1(x) \rho_n^2(y)}{|x-y|^{k_i}} dx dy \leq \frac{M_1^2}{(d_n)^{k_i}} \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

We now treat the second and third terms in (2.11). We define $q_{crit,P} = \frac{6}{5}$ and $q_{crit,M} = \frac{3}{2}$ and we recall that, by standard interpolation inequalities, $\|w_n\|_{q_{crit,i}} \leq \|\rho_n\|_{q_{crit,i}}$ is bounded. Then, from the Hardy-Littlewood-Sobolev inequality, we have

$$\int_{\mathbb{R}^6} \frac{\rho_n(x) w_n(y)}{|x-y|^{k_i}} dx dy \leq C \|\rho_n\|_{q_{crit,i}} \|w_n\|_{q_{crit,i}} \leq C \|w_n\|_{q_{crit,i}}$$

and

$$\int_{\mathbb{R}^6} \frac{w_n(x) w_n(y)}{|x-y|^{k_i}} dx dy \leq C \|w_n\|_{q_{crit,i}}^2 \leq C \|w_n\|_{q_{crit,i}}.$$

From the Hölder inequality, we get

$$\|w_n\|_{q_{crit,i}} \leq \|w_n\|_1^{\theta_i} \|w_n\|_q^{1-\theta_i} \leq C \|w_n\|_1^{\theta_i} \leq C \varepsilon^{\theta_i}$$

for $\theta_i = \frac{q - q_{crit,i}}{q_{crit,i}(q-1)} > 0$ and for n large enough. We finally get, for $i \in \{P, M\}$,

$$\limsup_{n \rightarrow +\infty} |E_{pot}^i(f_n) - E_{pot}^i(f_n^1) - E_{pot}^i(f_n^2)| \leq C \varepsilon^{\theta_i},$$

and thus,

$$\limsup_{n \rightarrow +\infty} |E_{pot}(f_n) - E_{pot}(f_n^1) - E_{pot}(f_n^2)| \leq C(\varepsilon^{\theta_1} + \kappa \varepsilon^{\theta_2}) \leq C \varepsilon^{\theta_2}.$$

Hence,

$$\liminf_{n \rightarrow +\infty} (\mathcal{H}(f_n) - \mathcal{H}(f_n^1) - \mathcal{H}(f_n^2)) \geq \||v|^2 f_n^w\|_{L^1} - C \varepsilon^{\theta_2}.$$

As $\lim_{n \rightarrow +\infty} \mathcal{H}(f_n) = I(M_1, M_j)$, we get

$$I(M_1, M_j) \geq \limsup_{n \rightarrow +\infty} (\mathcal{H}(f_n^1) + \mathcal{H}(f_n^2)) - C \varepsilon^{\theta_2}. \quad (2.12)$$

Now, from the dichotomy assumption, the boundness of f_n in \mathcal{E}_j , and from the disjoint support property, we have

$$M_j^1 + M_j^2 \leq M_j, \quad (2.13)$$

with

$$M_j^1 = \limsup_{n \rightarrow +\infty} \|j(f_n^1)\|_{L^1} \quad \text{and} \quad M_j^2 = \limsup_{n \rightarrow +\infty} \|j(f_n^2)\|_{L^1}. \quad (2.14)$$

Let \tilde{f}_n^1 be the following rescaled function in $\mathcal{F}(m_1, M_j^1)$ defined thanks to Appendix A, Lemma A.1:

$$\tilde{f}_n^1(x, v) = \gamma_n f_n^1 \left(\frac{\gamma_n^{1/3}}{\lambda_n^{1/3}} x, v \right).$$

Then

$$\mathcal{H}(f_n^1) = \mathcal{H}(\tilde{f}_n^1) + (1 - \lambda_n) \| |v|^2 f \|_1 - (1 - \lambda_n^{5/3} \gamma_n^{1/3}) E_{pot}^P(f_n^1) - (1 - \lambda_n^{4/3} \gamma_n^{2/3}) \kappa E_{pot}^M(f_n^1),$$

where, by interpolation inequalities, the potential and kinetic energies are uniformly bounded. From the expression (A.2) of λ_n and γ_n , from the dichotomy condition (2.9) and from (2.14), we conclude that there exists $C_\varepsilon > 0$ such that, up to a subsequence extraction and for n large enough, we have

$$\mathcal{H}(f_n^1) \geq \mathcal{H}(\tilde{f}_n^1) - C_\varepsilon \geq I(m_1, M_j^1) - C_\varepsilon \quad \text{with} \quad \lim_{\varepsilon \rightarrow 0} C_\varepsilon = 0.$$

A similar result holds for f_n^2 which gives, from (2.12),

$$\begin{aligned} I(M_1, M_j) &\geq \limsup_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow +\infty} (\mathcal{H}(f_n^1) + \mathcal{H}(f_n^2)) \\ &\geq I(m_1, M_j^1) + I((M_1 - m_1), M_j^2). \end{aligned} \quad (2.15)$$

Now, from the nondichotomy condition (2.2), we deduce

$$I(m_1, M_j^1) + I((M_1 - m_1), M_j^2) > I(M_1, M_j^1 + M_j^2) \geq I(M_1, M_j), \quad (2.16)$$

where we used (2.4) coupled to (2.13). Since (2.16) contradicts (2.15), we have proved that dichotomy cannot occur.

Finally, we conclude from the concentration-compactness principle that compactness occurs. Therefore, from the boundedness of f_n in \mathcal{E}_j and the Dunford-Pettis theorem, we deduce that the sequence

$$\bar{f}_n(x, v) = f_n(x + y_n, v),$$

is weakly relatively compact in L^1 . Up to a subsequence, we have

$$\bar{f}_n \rightharpoonup f \text{ in } L^1 \text{ and } L^p \text{ with } f \in \mathcal{E}_j \quad (2.17)$$

and in particular

$$\|f\|_{L^1} = M_1.$$

Now we claim that

$$E_{pot}^P(\bar{f}_n) \rightarrow E_{pot}^P(f) \quad \text{and} \quad E_{pot}^M(\bar{f}_n) \rightarrow E_{pot}^M(f). \quad (2.18)$$

Indeed, for $i \in \{P, M\}$, we have

$$E_{pot}^i(\bar{f}_n - f) = \int_{\mathbb{R}^6} \frac{(\bar{\rho}_n(x) - \rho(x))(\bar{\rho}_n(y) - \rho(y))}{|x - y|^{k_i}} dx dy.$$

We decompose this integral similarly as in (2.10) for the proof of nonvanishing. There exist $\beta_i > 0$ and two positive constants C_1, C_2 such that, for $0 < \eta < R$,

$$E_{pot}^i(\bar{f}_n - f) \leq C_1 \eta^{\beta_i} + \frac{C_2}{R^{k_i}} + \int_{\mathbb{R}^3} (\bar{\rho}_n(x) - \rho(x)) v_n(x) dx,$$

where

$$v_n = (\bar{\rho}_n - \rho) * h_R^\eta \quad \text{with } h_R^\eta(x) = \frac{\mathbf{1}_{\{\eta < |x| < R\}}(x)}{|x|^{k_i}}.$$

The sequence v_n is bounded in $L^1 \cap L^\infty$. From $h_R^\eta \in L^\infty$ and from the convergence $\bar{\rho}_n \rightarrow \rho$ in L^1 , we get

$$\forall x \in \mathbb{R}^3, \quad v_n(x) \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

Since, in addition, there is no mass loss:

$$\begin{aligned} \|\bar{\rho}_n * h_R^\eta\|_1 &= \left(\int_{\mathbb{R}^3} h_R^\eta(x) dx \right) \left(\int_{\mathbb{R}^3} \bar{\rho}_n(y) dy \right) \\ &= \left(\int_{\mathbb{R}^3} h_R^\eta(x) dx \right) \left(\int_{\mathbb{R}^3} \rho(y) dy \right) = \|\rho * h_R^\eta\|_1, \end{aligned}$$

we have in fact $v_n \rightarrow 0$ in L^1 , and also in L^q . From the Hölder inequality, for all η and R we have

$$\int_{\mathbb{R}^3} (\bar{\rho}_n(x) - \rho(x)) v_n(x) dx \rightarrow 0,$$

which implies $E_{pot}^i(\bar{f}_n - f) \rightarrow 0$ and also (2.18). Thus, by Fatou's lemma,

$$\mathcal{H}(f) \leq I(M_1, M_j) \quad \text{and} \quad \|j(f)\|_{L^1} \leq M_j. \quad (2.19)$$

Moreover, we have from (2.4),

$$0 > I(M_1, M_j) \geq \mathcal{H}(f) \geq I(M_1, \|j(f)\|_{L^1}) \geq \left(\frac{\|j(f)\|_{L^1}}{M_j} \right)^{\frac{1}{3(p_2-1)}} I(M_1, M_j).$$

Together with (2.19), this implies that $\|j(f)\|_{L^1} = M_j$. Hence $f \in \mathcal{F}(M_1, M_j)$ and $\mathcal{H}(f) = I(M_1, M_j)$ and we finally get

$$\|\bar{f}_n\|_{L^1} \rightarrow \|f\|_{L^1}, \quad \||v|^2 \bar{f}_n\|_{L^1} \rightarrow \||v|^2 f\|_{L^1}, \quad \|j(\bar{f}_n)\|_{L^1} \rightarrow \|j(f)\|_{L^1}.$$

From standard convexity argument, see [4], we conclude that $\bar{f}_n \rightarrow f$ in L^1 , $|v|^2 \bar{f}_n \rightarrow |v|^2 f$ in L^1 and $j(\bar{f}_n - f) \rightarrow 0$ in L^1 . We have proved the strong convergence, in the \mathcal{E}_j sense, of the subsequence \bar{f}_n to the minimizer f .

Step 2. Euler-Lagrange equation for the minimizer.

Let Q be a minimizer of (1.14). Our goal in this step is to derive the Euler-Lagrange equation satisfied by Q . Let $\varepsilon > 0$. We introduce the set

$$S_\varepsilon = \{(x, v) \in \mathbb{R}^6, Q(x, v) \geq \varepsilon\},$$

and pick a compactly supported function $g \in L^\infty(\mathbb{R}^6)$ such that $g \geq 0$ almost everywhere in $\mathbb{R}^6 \setminus S_\varepsilon$. Then,

$$\text{for all } t \in [0, \frac{\varepsilon}{\|g\|_\infty}], \quad f_t = Q + tg \in \mathcal{E}_j \setminus \{0\}.$$

Similarly as in Appendix A, there exists a unique pair (γ_t, η_t) positive such that the function \tilde{f}_t defined by

$$\tilde{f}_t(x, v) = \gamma_t f_t \left(x, \left(\frac{\gamma_t}{\eta_t} \right)^{\frac{1}{3}} v \right)$$

belongs to $\mathcal{F}(M_1, M_j)$, which is equivalent to

$$\eta_t = \frac{M_1}{\|f_t\|_1} \quad \text{and} \quad \frac{1}{\gamma_t} \|j(\gamma_t f_t)\|_1 = \frac{M_j}{M_1} \|f_t\|_1. \quad (2.20)$$

By differentiating the first equality, we obtain for $t \rightarrow 0$

$$\eta_t = 1 - \frac{\int_{\mathbb{R}^6} g}{M_1} t + o(t),$$

Let, for all $t \in [0, \frac{\varepsilon}{\|g\|_\infty}]$ and for all $\gamma \in \mathbb{R}_+^*$,

$$G(\gamma, t) = \frac{1}{\gamma} \int_{\mathbb{R}^6} j(\gamma f_t) - \frac{M_j}{M_1} \int_{\mathbb{R}^6} f_t.$$

Then G is clearly a \mathcal{C}^1 function of $t \in [0, \frac{\varepsilon}{\|g\|_\infty}]$ and $\gamma \in \mathbb{R}_+^*$. Moreover, from Appendix A, we get

$$\frac{\partial G}{\partial \gamma}(\gamma, t) > 0.$$

This implies that $t \mapsto \gamma_t$ is a \mathcal{C}^1 function and, by differentiating (2.20), we obtain as $t \rightarrow 0$

$$\gamma_t = 1 + \left(\frac{M_j}{M_1 C_Q} \int_{\mathbb{R}^6} g - \frac{1}{C_Q} \int_{\mathbb{R}^6} j'(Q)g \right) t + o(t), \quad \text{with } C_Q = \|j'(Q)Q\|_1 - M_j > 0.$$

Since Q is a minimizer of (1.14) and since \tilde{f}_t belongs to $\mathcal{F}(M_1, M_j)$, we have

$$\lim_{t \rightarrow 0} \frac{\mathcal{H}(\tilde{f}_t) - \mathcal{H}(Q)}{t} \geq 0,$$

where, by Appendix A,

$$\begin{cases} \left\| |v|^2 \tilde{f}_t \right\|_1 - \left\| |v|^2 Q \right\|_1 = \frac{\eta_t^{5/3}}{\gamma_t^{2/3}} \left(\left\| |v|^2 Q \right\|_1 + t \int_{\mathbb{R}^6} |v|^2 g \right) - \left\| |v|^2 Q \right\|_1, \\ E_{pot}(\tilde{f}_t) - E_{pot}(Q) = \eta_t^2 \left(E_{pot}(Q) - 2t \int_{\mathbb{R}^6} \phi_Q g + t^2 E_{pot}(g) \right) - E_{pot}(Q), \end{cases}$$

which finally implies, after straightforward calculations using the above expansions of η_t and γ_t near 0, that

$$\int_{\mathbb{R}^6} \left(\frac{|v|^2}{2} + \phi_Q - \lambda - \mu j'(Q) \right) g \geq 0, \quad (2.21)$$

with

$$\mu = -\frac{\left\| |v|^2 Q \right\|_1}{3C_Q} \quad \text{and} \quad \lambda = -\frac{1}{M_1} \left(E_{pot}(Q) - \frac{\left\| |v|^2 Q \right\|_1}{6} \left(5 - \frac{2M_j}{C_Q} \right) \right).$$

One can remark that μ and λ are negative. Indeed, the equality $I(M_1, M_j) = \||v|^2 Q\|_1 - E_{pot}(Q)$ gives

$$\lambda = \frac{1}{M_1} \left(I(M_1, M_j) - \frac{\||v|^2 Q\|_1}{6C_Q} \int_{\mathbb{R}^6} (j'(Q)Q - 3j(Q)) \right),$$

where $I(M_1, M_j) < 0$ and, from (H3), $j'(Q)Q - 3j(Q) \geq 0$. The equality (2.21) holds for all $g \in L^\infty(\mathbb{R}^6)$ compactly supported on S_ε , which implies that

$$\frac{|v|^2}{2} + \phi_Q - \lambda - \mu j'(Q) = 0 \quad \text{on } S_\varepsilon$$

and thus on the support of Q . Similarly, out of the support of Q , as $g \geq 0$, we have

$$\frac{|v|^2}{2} + \phi_Q - \lambda \geq 0.$$

We finally get, for all $(x, v) \in (\mathbb{R}^3)^2$,

$$Q(x, v) = (j')^{-1} \left(\frac{\frac{|v|^2}{2} + \phi_{Q_j}(x) - \lambda}{\mu} \right)_+.$$

Step 3. Regularity of the potential ϕ_Q and compact support of Q

Let us prove that ϕ_Q belongs to $\mathcal{C}^{0,\alpha}$, for all $\alpha \in (0, 1)$. Using the expression of Q , we get

$$\rho_Q(x) = \int_{\mathbb{R}^3} (j')^{-1} \left(\frac{\frac{|v|^2}{2} + \phi_{Q_j}(x) - \lambda}{\mu} \right)_+ dv. \quad (2.22)$$

Passing to the spherical velocity coordinate $u = |v|$ and performing the change of variable $q = \frac{u^2}{2|\mu|}$, we get

$$\rho_Q(x) = 4\pi\sqrt{2}|\mu|^{\frac{3}{2}} \int_0^{+\infty} (j')^{-1} (k(x) - q)_+ \sqrt{q} dq, \quad (2.23)$$

where $k(x) = \frac{\phi_Q(x) - \lambda}{\mu}$. We remark that the support of ρ_Q is contained in $\{x \in \mathbb{R}^3, k(x) \geq 0\}$ and that $k(x)_+ \leq \phi_Q(x)/\mu$. Moreover, from (H2) and (H3), for all $s \geq 0$ we have

$$(j')^{-1}(s) \leq C s^{\frac{1}{p-1}}.$$

Therefore

$$\rho_Q(x) \leq C \int_0^{k(x)_+} (k(x) - q)^{\frac{1}{p-1}} \sqrt{q} dq \leq C (k(x)_+)^{\frac{3}{2} + \frac{1}{p-1}} \leq C |\phi_Q(x)|^{\frac{3}{2} + \frac{1}{p-1}}. \quad (2.24)$$

As $Q \in L^1 \cap L^p$ with $p > 3$ and $|v|^2 Q \in L^1$, interpolation inequalities give $\rho_Q \in L^1 \cap L^{q_0}$ with $q_0 > \frac{3}{2}$. Let us prove that $\rho_Q \in L^r$ for some $r > 3$. We assume that $q_0 < 3$, otherwise the claim is already proved.

Let us start a bootstrap argument. If $\rho_Q \in L^1 \cap L^{q_k}$ with $\frac{3}{2} < q_k < 3$, then from the Hardy-Littlewood-Sobolev inequality, we deduce that ϕ_Q^P belongs to all L^s with

$3 < s \leq \infty$ and that ϕ_Q^M belongs to all L^q with $\frac{3}{2} < q \leq \frac{3q_k}{3-q_k}$. Hence, from (2.24), ρ_Q belongs to $L^{q_{k+1}}$ with

$$q_{k+1} \left(\frac{3}{2} + \frac{1}{p-1} \right) = \frac{3q_k}{3-q_k}. \quad (2.25)$$

Recall that, by Assumption (H2), we have $p > 3$. Hence, a simple analysis of the sequence (q_k) defined by (2.25) with an initial term $q_0 \in (\frac{3}{2}, 3)$ shows that there exists k_0 such that $q_{k_0} > 3$. This proves the claim: there exists $r > 3$ such that $\rho_Q \in L^r$.

Consequently, from Lemma C.1 in Appendix C, we deduce that the Manev potential ϕ_Q^M belongs to $\mathcal{C}^{0,\alpha}$ for all $\alpha \in (0, 1 - \frac{3}{r})$. Since this function converges to 0 at the infinity, we have $\phi_Q^M \in L^\infty$, and then $\phi_Q \in L^\infty$. Thus (2.24) gives $\rho_Q \in L^\infty$. Finally, using again Lemma C.1, we get $\phi_Q^M \in \mathcal{C}^{0,\alpha}$ for all $\alpha \in (0, 1)$. From Sobolev embeddings, the Poisson potential ϕ_Q^P also belongs to all $\mathcal{C}^{0,\alpha}$. We have proved the regularity of the potential stated in Theorem 1.1.

Next, from the regularity of ϕ_Q and the fact that this function converges to 0 as $|x| \rightarrow +\infty$, one deduces that

$$\text{Supp}(Q) = \left\{ (x, v) \in \mathbb{R}^6, \quad \frac{|v|^2}{2} + \phi_Q(x) - \lambda \leq 0 \right\}$$

is compact.

Step 4. The functions ρ_Q and ϕ_Q are spherically symmetric and monotone.

Consider a minimizer Q of (1.14), continuous and compactly supported thanks to the previous step, and denote by Q^{*x} its symmetric rearrangement with respect to the x variable only. We have clearly $Q^{*x} \in \mathcal{F}(M_1, M_j)$ and $\int |v|^2 Q dx dv = \int |v|^2 Q^{*x} dx dv$. Moreover, by the Riesz theorem, we have

$$\int_{\mathbb{R}^6} Q(x, v) Q(y, w) g(|x-y|) dx dy \leq \int_{\mathbb{R}^6} Q^{*x}(x, v) Q^{*x}(y, w) g(|x-y|) dx dy \quad (2.26)$$

for all $(v, w) \in \mathbb{R}^3 \times \mathbb{R}^3$, where $g(r) = \frac{1}{r} + \frac{\kappa}{r^2}$. Therefore, by integrating this inequality with respect to v and w , one gets

$$E_{pot}(Q) \leq E_{pot}(Q^{*x}),$$

which means that $\mathcal{H}(Q^{*x}) \leq \mathcal{H}(Q)$: Q^{*x} is also a minimizer of (1.14). Hence, we must have the equality in the above inequalities: $E_{pot}(Q) = E_{pot}(Q^{*x})$ and, even more, we have an equality in (2.26) for all v, w . We are then in a situation of equality in the Riesz inequality: since the function g is strictly decreasing, we deduce that, for all v, w , there exists a translation shift $x_0(v, w)$ such that

$$Q(x, v) = Q^{*x}(x + x_0(v, w), v) \quad \text{and} \quad Q(x, w) = Q^{*x}(x + x_0(v, w), w). \quad (2.27)$$

Let v be such that $Q(\cdot, v) \not\equiv 0$. Q being compactly supported, we integrate the first inequality in (2.27) against x and obtain

$$\begin{aligned} \int_{\mathbb{R}^3} x Q(x, v) dx &= \int_{\mathbb{R}^3} x Q^{*x}(x + x_0(v, w), v) dx \\ &= \int_{\mathbb{R}^3} x Q^{*x}(x, v) dx - x_0(v, w) \int_{\mathbb{R}^3} Q^{*x}(x, v) dx. \end{aligned}$$

Hence, we have the expression

$$x_0(v, w) = \frac{\int_{\mathbb{R}^3} x(Q(x, v) - Q^{*x}(x, v)) dx}{\int_{\mathbb{R}^3} Q^{*x}(x, v) dx}$$

and $x_0(v, w)$ is independent of w . Similarly, using the second equality in (2.27), one can prove that x_0 is independent of v . We have proved finally that there exists $x_0 \in \mathbb{R}^3$ such that

$$Q(x, v) = Q^{*x}(x + x_0, v) = Q^{*x}(|x + x_0|, v), \quad \forall v \in \mathbb{R}^3.$$

Consequently, up to a translation shift, ρ_Q is a nonincreasing function of $|x|$.

Let us now prove that ϕ_Q is a nondecreasing function of $r = |x|$. Since the function j is strictly increasing and $\mu < 0$, the expression (2.22) shows clearly that $\phi_Q(r)$ is nondecreasing on the compact support of the nonincreasing function $\rho_Q(r)$. Let $[0, R_Q]$ be this compact support.

For $|x| = r > R_Q$, we have

$$\phi_Q^P(x) = - \int_{\mathbb{R}^3} \frac{\rho_Q(y)}{4\pi|x-y|} dy \quad \text{and} \quad \phi_Q^M(x) = - \int_{\mathbb{R}^3} \frac{\rho_Q(y)}{2\pi^2|x-y|^2} dy.$$

Similarly as in Appendix B, we find

$$\phi_Q^P(x) = -\frac{M_1}{4\pi r} \quad \text{and} \quad \phi_Q^M(x) = -\frac{1}{\pi} \int_0^{R_Q} \frac{s\rho_Q(s)}{r} \ln\left(\frac{r+s}{r-s}\right) ds.$$

Since the function $r \mapsto \frac{1}{r} \ln\left(1 + \frac{2s}{r-s}\right)$ is positive and decreasing, ϕ_Q is nondecreasing on $[R_Q, +\infty)$. The proof of Theorem 1.1 is complete. \square

3. Orbital stability of the ground states

To prove the orbital stability result stated in Theorem 1.3, we first need to prove the uniqueness of the minimizer under equimeasurability constraints which are inherited from the invariance properties of the Vlasov-Manev flow. This uniqueness result is at the heart of our stability analysis and is quite robust in the sense that its proof does not use the nature of the Euler-Lagrange equation. Technically, the uniqueness proof only uses the fact that a minimizer is some given function of the microscopic energy, and does not use the equation satisfied by the potential itself (a non linear fractional-Laplacian equation in the present case).

3.1. Uniqueness of the minimizer under equimeasurability condition. This subsection is devoted to the proof of Theorem 1.2.

Let

$$Q_0(x, v) = F\left(\frac{|v|^2}{2} + \psi_0(x)\right), \quad Q_1(x, v) = F\left(\frac{|v|^2}{2} + \psi_1(x)\right)$$

be the functions defined in Theorem 1.2. For $i \in \{0, 1\}$ and for all $\tau < 0$, we define

$$a_{\psi_i}(\tau) = \text{meas} \left\{ \frac{|v|^2}{2} + \psi_i(x) < \tau \right\}.$$

From the equimeasurability of Q_0 and Q_1 and the properties of the function F , we have

$$\forall \tau < 0, \quad a_{\psi_0}(\tau) = a_{\psi_1}(\tau). \quad (3.1)$$

For $i \in \{0, 1\}$, we define

$$\mu_{\psi_i}(\lambda) = \text{meas}_{\mathbb{R}^3} \{x \in \mathbb{R}^3, \psi_i(x) < \lambda\}$$

for all $\lambda < 0$ and we have then for all $\tau < 0$,

$$a_{\psi_i}(\tau) = \int_{\mathbb{R}^3} \mu_{\psi_i} \left(\tau - \frac{|v|^2}{2} \right) dv.$$

Passing to the spherical velocity coordinate $u = |v|$ and performing the change of variable $w = \tau - u^2/2$, we obtain

$$a_{\psi_i}(\tau) = 4\pi\sqrt{2} \int_{-\infty}^{\tau} \mu_{\psi_i}(w) \sqrt{\tau - w} dw. \quad (3.2)$$

We claim that the expression (3.2) and the equality (3.1) imply that,

$$\text{for almost all } \lambda < 0, \quad \mu_{\psi_0}(\lambda) = \mu_{\psi_1}(\lambda). \quad (3.3)$$

Hence, as ψ_0 and ψ_1 are continuous and nondecreasing, we have $\psi_0 = \psi_1$ on the set

$$\{x \in \mathbb{R}^3, \psi_0(x) < 0\} = \{x \in \mathbb{R}^3, \psi_1(x) < 0\},$$

which immediatly gives $Q_0 = Q_1$.

Proof of (3.3) from (3.1) and (3.2). By differentiating with respect to τ the function a_{ψ_i} defined by (3.2), one gets

$$\forall \tau < 0, \quad a'_{\psi_i}(\tau) = 2\pi\sqrt{2} \int_{-\infty}^{\tau} \frac{\mu_{\psi_i}(w)}{\sqrt{\tau - w}} dw. \quad (3.4)$$

Now, remarking that, for $w < \lambda$, the following integral is constant:

$$I(\lambda, w) = \int_w^{\lambda} \frac{d\tau}{\sqrt{(\lambda - \tau)(\tau - w)}} = \pi,$$

one deduces from the Fubini theorem that

$$\int_{-\infty}^{\lambda} \frac{a'_{\psi_i}(\tau)}{\sqrt{\lambda - \tau}} d\tau = 2\pi\sqrt{2} \int_{-\infty}^{\lambda} \mu_{\psi_i}(w) I(\lambda, w) dw = 2\pi^2\sqrt{2} \int_{-\infty}^{\lambda} \mu_{\psi_i}(w) dw.$$

Thus, from $a_{\psi_0} = a_{\psi_1}$, we deduce that $\mu_{\psi_0}(\lambda) = \mu_{\psi_1}(\lambda)$ for almost all $\lambda < 0$, and the proof of (3.3) is complete.

End of the proof of Theorem 1.2. Let Q_0, Q_1 be two equimeasurable radially symmetric minimizers of (1.14). From Theorem 1.1, there exist $\lambda_0, \mu_0, \lambda_1, \mu_1 < 0$ such that, for $i \in \{0, 1\}$,

$$Q_i(x, v) = (j')^{-1} \left(\frac{\frac{|v|^2}{2} + \phi_{Q_i}(x) - \lambda_i}{\mu_i} \right)_+. \quad (3.5)$$

We define then, for $i \in \{0, 1\}$,

$$\tilde{Q}_i(x, v) = Q_i \left(\frac{x}{|\mu_i|^{1/2}}, |\mu_i|^{1/2} v \right).$$

The function \tilde{Q}_0 and \tilde{Q}_1 are still equimeasurable and satisfy

$$\tilde{Q}_i(x, v) = (j')^{-1} \left(-\frac{|v|^2}{2} - \psi_i(x) \right)_+ \quad \text{with} \quad \psi_i(x) = \frac{\phi_{Q_i} \left(\frac{x}{|\mu_i|^{1/2}} \right) - \lambda_i}{|\mu_i|}.$$

Since ϕ_{Q_i} is continuous nondecreasing and converges to 0 as $r \rightarrow +\infty$, the function ψ_i is continuous, nondecreasing and the set $\{x \in \mathbb{R}^3, \psi_0(x) < 0\}$ is bounded. From the previous step, we can thus conclude that

$$\tilde{Q}_0 = \tilde{Q}_1.$$

Let $\alpha = \sqrt{\frac{\mu_1}{\mu_0}}$. Then

$$Q_1(x, v) = Q_0\left(\frac{x}{\alpha}, \alpha v\right). \quad (3.6)$$

It remains to prove that $\alpha = 1$. To this aim, let us derive a virial identity satisfied by the minimizers Q of (1.14), using a rescaling argument. For $\lambda > 0$, we set $f_\lambda(x, v) = Q(\lambda x, \frac{v}{\lambda})$, which implies $f_\lambda \in \mathcal{F}(M_1, M_j)$ and

$$\mathcal{H}(f_\lambda) = \lambda^2 \| |v|^2 Q \|_{L^1} - \lambda E_{pot}^P(Q) - \lambda^2 \kappa E_{pot}^M(Q).$$

This function of λ has a strict global minimizer in $\lambda = 1$, which yields the following Manev virial identity:

$$\| |v|^2 Q \|_{L^1} - \frac{1}{2} E_{pot}^P(Q) - \kappa E_{pot}^M(Q) = 0. \quad (3.7)$$

Moreover we recall that Q satisfies

$$\| |v|^2 Q \|_{L^1} - E_{pot}^P(Q) - \kappa E_{pot}^M(Q) = I(M_1, M_j).$$

Combining this two equalities, we get

$$\frac{1}{2} E_{pot}^P(Q) = -I(M_1, M_j). \quad (3.8)$$

Let us now use this identity for the two minimizers Q_0 and Q_1 . From (3.8) and (2.1), one deduces that

$$E_{pot}^P(Q_0) = E_{pot}^P(Q_1) > 0.$$

Moreover, from (3.6) and Appendix A, one gets

$$E_{pot}^P(Q_1) = \alpha E_{pot}^P(Q_0).$$

This yields $\alpha = 1$, which ends the proof of Theorem 1.2. \square

3.2. Orbital stability of the minimizers, proof of Theorem 1.3. In this subsection, we prove Theorem 1.3.

Let Q be a minimizer of (1.14) and assume that Theorem 1.3 is false. Then there exist $\varepsilon > 0$ and sequences $f_0^n \in \mathcal{E}_j$, $t_n > 0$, such that

$$\lim_{n \rightarrow +\infty} \|f_0^n - Q\|_{\mathcal{E}_j} = 0, \quad (3.9)$$

and

$$\forall n \geq 0, \forall x_0 \in \mathbb{R}^3, \|f^n(t_n, x, v) - Q(x + x_0, v)\|_{\mathcal{E}_j} \geq \varepsilon, \quad (3.10)$$

where $f^n(t, x, v)$ is a solution to (1.1) with initial data f_0^n .

Let $g_n(x, v) = f_n(t_n, x, v)$. From (3.9), one deduces that

$$\lim_{n \rightarrow +\infty} \mathcal{H}(f_0^n) = I(M_1, M_j), \quad \lim_{n \rightarrow +\infty} \|f_0^n\|_{L^1} = M_1, \quad \lim_{n \rightarrow +\infty} \|j(f_0^n - Q_0)\|_{L^1} = 0. \quad (3.11)$$

In particular, f_0^n converges to Q in the strong L^p topology and hence almost everywhere, up to a subsequence.

Now, by the conservation properties of the Vlasov-Manev system (1.1), we have

$$\lim_{n \rightarrow +\infty} \mathcal{H}(g_n) = I(M_1, M_j), \quad \lim_{n \rightarrow +\infty} \|g_n\|_{L^1} = M_1, \quad \lim_{n \rightarrow +\infty} \|j(g_n)\|_{L^1} = M_j, \quad (3.12)$$

and, for all $t \geq 0$,

$$\text{meas}\{(x, v) \in \mathbb{R}^3, g_n(x, v) > t\} = \text{meas}\{(x, v) \in \mathbb{R}^3, f_0^n(x, v) > t\}. \quad (3.13)$$

From Appendix A, let us define

$$\bar{g}_n(x, v) = \gamma_n g_n \left(\frac{\gamma_n^{1/3}}{\lambda_n^{1/3}} x, v \right)$$

such that $\|\bar{g}_n\|_{L^1} = M_1$ and $\|j(\bar{g}_n)\|_{L^1} = M_j$. Then, from (3.12),

$$\gamma_n \rightarrow 1, \quad \mu_n \rightarrow 1, \quad (3.14)$$

and

$$\lim_{n \rightarrow +\infty} \mathcal{H}(\bar{g}_n) = \lim_{n \rightarrow +\infty} \mathcal{H}(g_n) = I(M_1, M_j).$$

Hence \bar{g}_n is a minimizing sequence of (1.14). From Theorem 1.1, \bar{g}_n is relatively strongly compact in \mathcal{E}_j and converges to a ground state Q_1 , up to a subsequence and up to a translation shift. Hence, by (3.14), we have

$$g_n \rightarrow Q_1 \in \mathcal{E}_j \quad (3.15)$$

up to a subsequence and up to a translation shift.

Let us now prove that the equimeasurability (3.13) and the L^1 convergences of g_n and f_0^n imply the equimeasurability of Q and Q_1 . Indeed, we remark that, for $t > 0$ and $0 < \varepsilon < t$,

$$\begin{cases} \{g_n > t\} \subset (\{|g_n - Q_1| < \varepsilon\} \cap \{Q_1 > t - \varepsilon\}) \cup \{|g_n - Q_1| \geq \varepsilon\}, \\ \{g_n > t\} \supset \{|g_n - Q_1| < \varepsilon\} \cap \{Q_1 > t + \varepsilon\}. \end{cases}$$

By passing to the limit as $n \rightarrow +\infty$, one gets

$$\begin{cases} \limsup_{n \rightarrow +\infty} \text{meas}\{g_n > t\} \leq \text{meas}\{Q_1 > t - \varepsilon\}, \\ \liminf_{n \rightarrow +\infty} \text{meas}\{g_n > t\} \geq \text{meas}\{Q_1 > t + \varepsilon\}. \end{cases}$$

Finally, passing to the limit as $\varepsilon \rightarrow 0$, we have $\text{meas}\{g_n > t\} \rightarrow \text{meas}\{Q_1 > t\}$ for almost all $t > 0$ and similarly $\text{meas}\{f_0^n > t\} \rightarrow \text{meas}\{Q > t\}$ for almost all $t > 0$. Remarking now that, the functions $t \mapsto \text{meas}\{Q > t\}$ and $t \mapsto \text{meas}\{Q_1 > t\}$ are right-continuous, we obtain the equimeasurability of Q and Q_1 .

Therefore, one can use the characterization of ground states in Theorem 1.1 and the property of uniqueness of minimizers given by Theorem 1.2, to conclude that, $Q = Q_1$, up to a space translation shift. Finally, (3.15) contradicts (3.10) and the proof of Theorem 1.3 is complete. \square

Appendix A. Rescalings

Let $f \in \mathcal{E}_j$ and let $\gamma > 0$, $\lambda > 0$ and $\mu > 0$. Then the rescaled function \tilde{f} defined by $\tilde{f}(x, v) = \gamma f(\frac{x}{\lambda}, \mu v)$ satisfies the following identities.

Norms

$$\|\tilde{f}\|_1 = \gamma \frac{\lambda^3}{\mu^3} \|f\|_1 \quad ; \quad \|j(\tilde{f})\|_1 = \frac{\lambda^3}{\mu^3} \|j(\gamma f)\|_1 \quad ; \quad \left\| |v|^2 \tilde{f} \right\|_1 = \gamma \frac{\lambda^3}{\mu^5} \left\| |v|^2 f \right\|_1.$$

Functions

$$\rho_{\tilde{f}}(x) = \frac{\gamma}{\mu^3} \rho_f \left(\frac{x}{\lambda} \right) \quad ; \quad \phi_{\tilde{f}}^P(x) = \gamma \frac{\lambda^2}{\mu^3} \phi_f^P \left(\frac{x}{\lambda} \right) \quad ; \quad \phi_{\tilde{f}}^M(x) = \gamma \frac{\lambda}{\mu^3} \phi_f^M \left(\frac{x}{\lambda} \right).$$

Potential energy

$$E_{pot}^P(\tilde{f}) = \gamma^2 \frac{\lambda^5}{\mu^6} E_{pot}^P(f) \quad ; \quad E_{pot}^M(\tilde{f}) = \gamma^2 \frac{\lambda^4}{\mu^6} E_{pot}^M(f).$$

Lemma A.1. *Let $f \in \mathcal{E}_j \setminus \{0\}$ and $M_1, M_j > 0$. Then there exists an unique pair of positive constants (γ, λ) such that the rescaled function \tilde{f} defined by*

$$\tilde{f}(x, v) = \gamma f \left(\frac{\gamma^{1/3}}{\lambda^{1/3}} x, v \right) \quad (\text{A.1})$$

belongs to $\mathcal{F}(M_1, M_j)$. Moreover, γ and λ satisfy

$$\lambda = \frac{M_1}{\|f\|_1} \quad \text{and} \quad \min(\gamma^{p_1-1}, \gamma^{p_2-1}) \leq \frac{M_j \|f\|_1}{M_1 \|j(f)\|_1} \leq \max(\gamma^{p_1-1}, \gamma^{p_2-1}). \quad (\text{A.2})$$

Proof. The rescaling (A.1) gives immediately

$$\|\tilde{f}\|_1 = \lambda \|f\|_1 \quad \text{and} \quad \|j(\tilde{f})\|_1 = \frac{\lambda}{\gamma} \|j(\gamma f)\|_1.$$

Hence, we have $\tilde{f} \in \mathcal{F}(M_1, M_j)$ as soon as

$$\lambda = \frac{M_1}{\|f\|_1} \quad \text{and} \quad \frac{\|j(\gamma f)\|_1}{\gamma \|j(f)\|_1} = \frac{M_j \|f\|_1}{M_1 \|j(f)\|_1}.$$

The first parameter λ is then uniquely determined. Notice also that (A.2) is a direct consequence of the nondichotomy condition (1.10). It remains to prove the existence of a unique suitable γ .

Consider now the function of $\gamma \in \mathbb{R}_+^*$ defined by

$$h(\gamma) = \frac{\|j(\gamma f)\|_1}{\gamma \|j(f)\|_1}.$$

From the nondichotomy condition (1.10), we have

$$\lim_{\gamma \rightarrow 0} h(\gamma) = 0, \quad \lim_{\gamma \rightarrow +\infty} h(\gamma) = +\infty.$$

Moreover, from a direct calculation, one gets

$$h'(\gamma) = \frac{\|j'(\gamma f)f\|_1}{\gamma \|j(f)\|_1} - \frac{\|j(\gamma f)\|_1}{\gamma^2 \|j(f)\|_1} \geq (p_1 - 1) \frac{\|j(\gamma f)\|_1}{\gamma^2 \|j(f)\|_1} > 0,$$

where we used Assumption (H3) on the function j . Hence, there exists a unique $\gamma \in \mathbb{R}_+^*$ such that

$$h(\gamma) = \frac{M_j \|f\|_1}{M_1 \|j(f)\|_1}$$

and the Lemma is proved. \square

Appendix B. Decay at the infinity of radially symmetric potentials

Proposition B.1. *For all $k > 3$, there exist a positive constant C such that for all $f \in \mathcal{E}_j$ radially symmetric and for all $|x| > 1$*

$$|\phi_f(x)| \leq \frac{C}{|x|^{2-\frac{3}{k}}} \|\rho_f\|_{k'}.$$

Proof. Recall that

$$\phi_f^P(x) = - \int_{\mathbb{R}^3} \frac{\rho_f(y)}{4\pi|x-y|} dy \quad \text{and} \quad \phi_f^M(x) = - \int_{\mathbb{R}^3} \frac{\rho_f(y)}{2\pi^2|x-y|^2} dy.$$

Passing to the spherical coordinate $s = |y|$ and $x \cdot y = 2rs \cos \theta$, one gets

$$\begin{cases} \phi_f^P(x) = - \int_0^{+\infty} \int_0^\pi \frac{\rho_f(s) \sin \theta}{2(s^2 + r^2 - 2rs \cos \theta)^{1/2}} s^2 ds d\theta = - \int_0^{+\infty} \rho_f(s) g_r^P(s) s^2 ds, \\ \phi_f^M(x) = - \frac{1}{\pi} \int_0^{+\infty} \int_0^\pi \frac{\rho_f(s) \sin \theta}{s^2 + r^2 - 2rs \cos \theta} s^2 ds d\theta = - \frac{1}{\pi} \int_0^{+\infty} \rho_f(s) g_r^M(s) s^2 ds, \end{cases}$$

where

$$g_r^P(s) = \frac{\mathbf{1}_{\{s < r\}}(s)}{r} + \frac{\mathbf{1}_{\{s > r\}}(s)}{s} \quad \text{and} \quad g_r^M(s) = \frac{1}{sr} \ln \left| \frac{r+s}{r-s} \right|.$$

We have immediately $g_r^P \in L^k((0, +\infty), s^2 ds)$ for all $k > 3$ and a simple analysis gives that $g_r^M \in L^k((0, +\infty), s^2 ds)$ for all $k > 3/2$. Moreover, we remark that

$$g_r^P(s) = \frac{1}{r} g_1^P\left(\frac{s}{r}\right) \quad \text{and} \quad g_r^M(s) = \frac{1}{r^2} g_1^M\left(\frac{s}{r}\right),$$

which implies

$$\|g_r^P\|_{L^k((0, +\infty), s^2 ds)} \leq \frac{C_P}{r^{1-\frac{3}{k}}} \quad \text{and} \quad \|g_r^M\|_{L^k((0, +\infty), s^2 ds)} \leq \frac{C_M}{r^{2-\frac{3}{k}}}.$$

We finally conclude by using the Hölder inequality. Remark that, thanks to interpolation inequalities, under Assumption (H2), $f \in \mathcal{E}_j$ implies that $\rho_f \in L^1 \cap L^{3/2}((0, +\infty), s^2 ds)$. \square

Appendix C. Hölder continuity of the Manev potential

Lemma C.1. *Let $\rho \in L^1(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$ with $q > 3$. Then the function ϕ defined by*

$$\phi(x) = \int_{\mathbb{R}^3} \frac{\rho(y)}{|x-y|^2} dy$$

belongs to $\mathcal{C}^{0,\gamma}$ for all $\gamma \in (0, 1 - \frac{3}{q})$.

Proof. Let $x_1, x_2 \in \mathbb{R}^3$, $x_1 \neq x_2$ and let $\alpha = 1 - \frac{3}{q} \in (0, 1)$. Let $\varepsilon \in (0, \alpha)$ and $R > 0$. The function

$$v_R(x) : x \mapsto \frac{\mathbf{1}_{B_R}(x)}{|x|^{2+\varepsilon}}$$

satisfies

$$\|v_R(x)\|_{\frac{3}{2+\alpha}} = CR^{\alpha-\varepsilon}.$$

Hence, by the Hölder inequality, $v_R * \rho$ belongs to L^∞ . Let us decompose

$$\phi(x_1) - \phi(x_2) = I_1 + I_2$$

with

$$I_1 = \int_{\mathbb{R}^3} \rho(y) \left(\frac{\mathbf{1}_{B_R}(x_1 - y)}{|x_1 - y|^2} - \frac{\mathbf{1}_{B_R}(x_2 - y)}{|x_2 - y|^2} \right) dy$$

and

$$I_2 = \int_{\mathbb{R}^3} \rho(y) \left(\frac{\mathbf{1}_{B_R^c}(x_1 - y)}{|x_1 - y|^2} - \frac{\mathbf{1}_{B_R^c}(x_2 - y)}{|x_2 - y|^2} \right) dy.$$

Setting $\delta = \frac{|x_1 - x_2|}{2}$ and $X = \frac{x_2 - x_1}{|x_2 - x_1|}$. Performing the change of variable $u = \frac{1}{\delta}(y - \frac{x_1 + x_2}{2})$, one gets

$$\begin{aligned} I_1 &= \delta \int_{\mathbb{R}^3} \bar{\rho}(u) \left(\frac{\mathbf{1}_{B_{R/\delta}}(-X - u)}{|-X - u|^2} - \frac{\mathbf{1}_{B_{R/\delta}}(X - u)}{|X - u|^2} \right) dy \\ &= \delta (v_{R/\delta} * \bar{\rho})(-X) - \delta (v_{R/\delta} * \bar{\rho})(X), \end{aligned}$$

where $\bar{\rho}(u) = \rho(\delta u + \frac{x_1 + x_2}{2})$, which implies $\|\bar{\rho}\|_r = \delta^{-3/r} \|\rho\|_r$. Notice that

$$\|v_{R/\delta} * \bar{\rho}\|_\infty \leq C \|\rho\|_r \left(\frac{R}{\delta} \right)^{\alpha - \varepsilon} \delta^{-3/r} = C \|\rho\|_r R^{\alpha - \varepsilon} \delta^{\varepsilon - 1}.$$

We have then $|I_1| \leq C_1 \delta^\varepsilon R^{\alpha - \varepsilon}$. Let us now estimate I_2 . We have

$$|I_2| \leq \frac{C}{R^2} \|\rho\|_1.$$

Finally, we have proved that

$$|\phi(x_1) - \phi(x_2)| \leq C \left(|x_2 - x_1|^\varepsilon R^{\alpha - \varepsilon} + \frac{1}{R^2} \right).$$

After an optimization, for all $\varepsilon \in (0, \alpha)$, one finds

$$|\phi(x_1) - \phi(x_2)| \leq C |x_2 - x_1|^\gamma$$

with $\gamma = \frac{2\varepsilon}{2 + \alpha - \varepsilon}$. We conclude by remarking that γ converge to α as $\varepsilon \rightarrow \alpha$. \square

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