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Introduction

An important task in dynamical systems, and the whole point of ergodic theory, is to study the measures invariant under some transformation. We will focus here on physical measures, which are empirical measures for a set of initial conditions of non-zero Lebesgue measure (see Subsection 1.2 for a definition of empirical mesures). To obtain non-trivial results, some assumptions must be made on those transformations, for instance on their smoothness. We shall study here maps which are piecewise differentiable, with Hölder derivative, and uniformly expanding (that is, with positive and bounded away from zero Lyapunov exponents) on some compact space.

In one dimension, a most interesting result was found in 1973 by A. Lasota and J.A. Yorke [LY73]: a piecewise twice differentiable uniformly expanding map on an interval admits a finite number of physical invariant measures, whose densities have bounded variation, and (up to taking an iterate of the transformation) they are mixing. This is obtained via the study of the Perron-Frobenius operator on the space of functions with bounded variation: the fixed points of this operator are densities of invariant measures, and the existence of a spectral gap provides the property of mixing at an exponential rate. To sum it up, the quasi-compactness of the Perron-Frobenius operator, along with some classical arguments, is enough to deduce many valuable ergodic properties of the system.

Problems arise when we weaken those assumptions. A possible generalization is to obtain results in higher dimensions, where general answers were obtained only recently, despite some early results on particular systems [Kel79]. Actually, a naive generalization of the theorem proved by A. Lasota and J.A. Yorke does not hold in dimension 2 or higher, as is shown by the counter-examples of M. Tsujii [Tsu00] and J. Buzzi [Buz01]: some additional assumptions must be made. A first strategy is to require a lower bound on the angles made by the discontinuities; P. Góra and A. Boyarski [GB89] made a successful attempt in 1989, although the sufficient condition they found on the expanding rate is far from optimum. Another strategy is to study the combinatorial complexity, i.e. the way the space is cut by the discontinuities and the number of times they overlap; it was used for instance in 2000 by J.W. Cowieson [Cow00] and B. Saussol [Sau00], and we will follow this approach.

In all those studies, a major problem is to find a suitable space of functions on which the Perron-Frobenius operator may act. For piecewise smooth maps, the function spaces we use should show some regularity (otherwise, no spectral gap will be found), but not too much, since discontinuities must be allowed. For piecewise twice differentiable maps, the space of functions with bounded variation is suitable, as was shown by J.W. Cowieson [Cow00] (although he proved only the existence of invariant measures, and not stronger properties). For less smooth maps, B. Saussol worked with spaces of functions with bounded oscillation [Sau00]. Following the method used in [BG09], we will work here mainly with Sobolev spaces.

The approach followed in this article has many advantages. We work with usual and well-known spaces, instead of ad hoc function spaces. Moreover, the dynamic does not need to be studied in detail: this gives a very robust method, which can be more or less easily adapted to prove similar results for other function spaces or dynamical systems, since it relies only on the core properties of the function spaces.

This paper is the outcome of a master's thesis, supervised by V. Baladi. The main goal is to simplify the proofs of the previous paper from V. Baladi and S. Gouëzel [BG09], restricting ourselves to the study of expanding maps. Most of the setting and proofs in sections 1 to 6 are directly adapted from that work (a notable exception being Lemma 4.1, corresponding to Lemma 31 in [BG09] but whose proof is adapted from another article [BG10]). There are multiple gains: the setting is simpler, the function spaces used more standard (classical Sobolev spaces) and most of the proofs much shorter (some of which are even found in the literature). We will also deal with a limit case, when the transformation is piecewise differentiable with a Lipschitz derivative, and prove a new result involving functions with bounded variation - basically a stronger version of J.W. Cowieson's theorem [Cow00].

The definitions and setting are explained in Section 1, as well as the main results: a bound on the essential spectral radius of transfer operators, Theorem 1.3, and its consequence on the existence of finitely many mixing physical measures with densities in appropriate Sobolev spaces, and on the rate of mixing for Hölder test functions (Corollary 1.5 and Theorem 1.6). We shall also present an application of those result on a class of piecewise affine maps. Section 2 presents extensively the Sobolev spaces and their basic properties, as well as the space of functions with bounded variation. Section 3 contains the main lemmas, which exploit the properties of the Sobolev spaces; they are used in Section 4 to prove Theorem 1.3. Section 5 contains the proof of Theorem 1.6. Finally, in Section 6 we will deal with a limit case, where the transformation is piecewise differentiable with Lipschitz derivative and one works on functions with bounded variation, and a discussion on the previous similar results by J.W. Cowieson [Cow00] and B. Saussol [Sau00]. This last section also highlights the fact that our main theorems can be quite easily adapted to non-Sobolev function spaces.

1 Setting and results

If B is a Banach space, we denote the norm of an element f of B by $\|f\|_B$. In this paper, a map defined on a closed subset of a manifold is said to be $\mathcal{C}^k$ or $\mathcal{C}^\infty$ if it admits an extension to a neighborhood of this closed subset, which is $\mathcal{C}^k$ or $\mathcal{C}^\infty$ in the usual sense. For $\alpha \in (0,1)$, a map is said to be $\mathcal{C}^\alpha$ if it is α-Hölder, $\mathcal{C}^{1+\alpha}$ if it is $\mathcal{C}^1$ with α-Hölder derivative, and $\mathcal{C}^{1+\mathrm{Lip}}$ if it is $\mathcal{C}^1$ with Lipschitz derivative.

Let X be a Riemannian manifold of dimension d, and let X_0 be a compact subset of X. We call $\mathcal{C}^1$ hypersurface with boundary a codimension-one $\mathcal{C}^1$ submanifold of X with boundary (i.e., every point of this set has a neighborhood diffeomorphic either to $\mathbb{R}^{d-1}$ or $\mathbb{R}^{d-2} \times [0, +\infty)$).

Definition 1.1 (Piecewise $\mathcal{C}^{1+\alpha}$ and $\mathcal{C}^{1+\mathrm{Lip}}$ expanding maps)**.**
For $\alpha > 0$, we say that a map $T : X_0 \to X_0$ is a piecewise $\mathcal{C}^{1+\alpha}$ expanding map (respectively piecewise $\mathcal{C}^{1+\mathrm{Lip}}$ expanding map) if:

- *There exists a finite number of disjoint open subsets $O_1, \dots, O_I$ of X_0, covering Lebesgue-almost all X_0, whose boundaries are unions of finitely many compact $\mathcal{C}^1$ hypersurfaces with boundary.*
- *For $1 \leq i \leq I$, there exists a $\mathcal{C}^{1+\alpha}$ (respectively $\mathcal{C}^{1+\mathrm{Lip}}$) map T_i defined on a neighborhood of $\overline{O_i}$, which is a diffeomorphism onto its image, such that T*

coincides with T_i on O_i.

- *For any $x \in X_0$ and whenever $T_i(x)$ is defined, $\lambda(x) := \inf_{v \in \mathcal{T}_x} \frac{|DT_i(x)v|}{|v|} > 1$.*

Since we choose to study the asymptotical combinatorial complexity, we have to quantify it. It will be done the following way.

Let choose a non-zero integer n.

Let $\mathbf{i} = (i_0, \dots, i_{n-1}) \in \{1, \dots, I\}^n$. We define inductively sets $O_\mathbf{i}$ by $O_{(i_0)} = O_i$, and

$$O_{(i_0,\dots,i_{n-1})} = \{x \in O_{i_0} \mid T_{i_0} x \in O_{(i_1,\dots,i_{n-1})}\}. \tag{1.1}$$

Let also $T^n_\mathbf{i} = T_{i_{n-1}} \circ \cdots \circ T_{i_0}$; it is defined on a neighborhood of $O_\mathbf{i}$.

For any $x \in X_0$ and whenever $T^n_\mathbf{i}(x)$ is defined, we denote :

$$\lambda_n(x) := \inf_{v \in \mathcal{T}_x} \frac{|DT^n_\mathbf{i}(x)v|}{|v|} > 1.$$

We define the complexity at the beginning

$$D^b_n = \max_{x \in X_0} \mathrm{Card}\{\mathbf{i} = (i_0, \dots, i_{n-1}) \mid x \in \overline{O_\mathbf{i}}\}, \tag{1.2}$$

and the complexity at the end

$$D^e_n = \max_{x \in X_0} \mathrm{Card}\{\mathbf{i} = (i_0, \dots, i_{n-1}) \mid x \in \overline{T^n(O_\mathbf{i})}\}. \tag{1.3}$$

For $T(x) = 2x \bmod 1$ on $[0, 1]$ we have $D^e_n \geq 2^n$, but fortunately this quantity plays no role when T is piecewise $\mathcal{C}^{1+\mathrm{Lip}}$, where we can take p as close to 1 as needed in Corollary 1.5 (cf. Corollary 6.1), or work in the space of functions with bounded variation - see Theorem 1.4 or Cowieson's Theorem (referenced here as Theorem 6.2).

For generic piecewise expanding maps, the complexity D^b_n increases subexponentially, and therefore does not play an important role in the spectral formula (1.5) below (see [Cow02]). However, some pathologic cases may arise when the iterates of the map T cut some open set Ω too often, and then map it onto itself. In spite of the expansion, these numerous "cut-and-fold" may make the images of Ω arbitrarily small, so that the physical measure starting from any point of Ω can be, for instance, a Dirac measure. This phenomenon is central in the examples given by M. Tsuji [Tsu00] and J. Buzzi [Buz01]. A control on the combinatorial complexities ensures that the expansion do beat the cuts.

1.1 Spectral results

The results about the physical measures will be obtained through the study of transfer operators (or Perron-Frobenius operators); we now define them.

Definition 1.2 (Transfer operator)**.**

Let $\alpha > 0$. For all $g : X_0 \to \mathbb{C}$ such that the restriction of g to any O_i is $\mathcal{C}^\alpha$ (in the sense that it admits a $\mathcal{C}^\alpha$ extension to some neighborhood of $\overline{O_i}$), we define the transfer operator $\mathcal{L}_g$ on $\mathbb{L}^\infty(X_0)$ by:

$$\mathcal{L}_g u(x) = \sum_{y \in T^{-1}(\{x\})} g(y)u(y). \tag{1.4}$$

These operators will act on Sobolev spaces $\mathcal{H}_p^t$, for $1 < p < \infty$ and $0 < t < \min(1/p, \alpha)$, or on functions with bounded variation. As for now, we will just recall that the Sobolev spaces H_p^t on $\mathbb{R}^d$ are defined as $\{u \in \mathbb{L}^p : \mathcal{F}^{-1}((1+|\xi|^2)^{t/2}(\mathcal{F}u)(\xi)) \in \mathbb{L}^p\}$, with $\|u\|_{H_p^t} = \left\|\mathcal{F}^{-1}((1+|\xi|^2)^{t/2}(\mathcal{F}u)(\xi))\right\|_{\mathbb{L}^p}$, where $\mathcal{F}$ denotes the Fourier transform. Using charts, one can define a Sobolev space $\mathcal{H}_p^t$ on X. Here, t is an index for regularity; as t increases, $\mathcal{H}_p^t$ contains more regular functions. The Sobolev and "bounded variation" spaces will be defined precisely in Section 2.

The first result is about the essential spectral radius of $\mathcal{L}_g$ when acting on $\mathcal{H}_p^t$; then, an application of this result to $\mathcal{L}_{1/|\det DT|}$, the Perron-Frobenius operator, will lead to the proof (under some conditions) of the existence of finitely many mixing physical measures.

Theorem 1.3 (Spectral theorem for piecewise $\mathcal{C}^{1+\alpha}$ expanding maps)**.**

Let $\alpha \in (0, 1]$, and let T be a piecewise $\mathcal{C}^{1+\alpha}$ uniformly expanding map. Choose $p \in (1, +\infty)$, and $0 < t < \min(1/p, \alpha)$. Let $g : X_0 \mapsto \mathbb{C}$ be a function such that the restriction of g to any O_i admits a $\mathcal{C}^\alpha$ extension to $\overline{O_i}$. Then $\mathcal{L}_g$ acts continuously on $\mathcal{H}_p^t$, where its essential spectral radius is at most

$$\lim_{n\to+\infty} (D_n^b)^{\frac{1}{p}\frac{1}{n}} (D_n^e)^{(1-\frac{1}{p})\frac{1}{n}} \|g^{(n)}| \det DT^n|^{\frac{1}{p}} \lambda_n^{-t}\|_{\mathbb{L}^\infty}^{\frac{1}{n}}, \tag{1.5}$$

where $g^{(n)} = \prod_{i=0}^{n-1} g \circ T^i$, and the limit exists by submultiplicativity.

When we say that $\mathcal{L}_g$ acts continuously on $\mathcal{H}_p^t$, we mean that, for any $u \in \mathcal{H}_p^t \cap \mathbb{L}^\infty(\text{Leb})$, the function $\mathcal{L}_g u$, which is essentially bounded, still belongs to $\mathcal{H}_p^t$ and satisfies $\|\mathcal{L}_g u\|_{\mathcal{H}_p^t} \leq C \|u\|_{\mathcal{H}_p^t}$. Since $\mathbb{L}^\infty$ is dense in $\mathcal{H}_p^t$ (by Theorem 3.2/2 in [Tri77]), the operator $\mathcal{L}_g$ can be extended to a continuous operator on $\mathcal{H}_p^t$. The restriction $0 < t < 1/p$ is exactly designed so that the space H_p^t is stable under multiplication by characteristic functions of nice sets (see Lemma 3.2). We also provide a version of this theorem for piecewise $\mathcal{C}^{1+\text{Lip}}$ maps, including piecewise $\mathcal{C}^2$ maps. We cannot use the function space H_1^1 (which for instance contains only continuous functions in dimension 1); we will instead use the space of functions with bounded variation $\mathbb{BV}$, which will be defined rigorously in Section 2:

Theorem 1.4 (Spectral theorem for piecewise $\mathcal{C}^{1+\text{Lip}}$ expanding maps)**.**

Let be T a piecewise $\mathcal{C}^{1+\text{Lip}}$ expanding map. Let $g : X_0 \mapsto \mathbb{C}$ be a function such that the restriction of g to any O_i admits a Lipschitz extension to $\overline{O_i}$. Then $\mathcal{L}_g$ acts continuously on $\mathbb{BV}(X_0)$, where its essential spectral radius is at most

$$\lim_{n\to+\infty} (D_n^b)^{\frac{1}{n}} \|g^{(n)}| \det DT^n| \lambda_n^{-1}\|_{\mathbb{L}^\infty}^{\frac{1}{n}}. \tag{1.6}$$

1.2 Physical measures

The empirical measure of T with initial condition $x \in X_0$ is the weak limit, if it exists, of $1/n \sum_{k=0}^{n-1} \delta_{T^k x}$; such a measure is always T-invariant. A physical measure of T is an invariant probability measures μ which is an empirical measure for a set of initial conditions of non-zero Lebesgue measure; the basin of a physical measure μ is the set of all initial conditions whose corresponding empirical measure is μ.

In the following and for any finite measure μ and integrable function f, the expression $\langle \mu, f \rangle$ will be used for the integral of f against μ. We may see any integrable function u as the density (with respect to Lebesgue measure Leb) of some finite measure, and use $\langle u, f \rangle$ instead of $\langle u \, \mathrm{dLeb}, f \rangle$. For all $f \in \mathbb{L}^\infty$ and $u \in \mathcal{H}_p^t$, we have $\langle u, f \circ T \rangle = \langle \mathcal{L}_{1/|\det DT|} u, f \rangle$; a nonnegative and nontrivial fixed point u of $\mathcal{L}_{1/|\det DT|}$ in $\mathcal{H}_p^t$ then corresponds (up to normalization) to an invariant probability measure $\mu_u = u \, \mathrm{dLeb}$ whose density with respect to the Lebesgue measure is in $\mathcal{H}_p^t$. By Birkhoff's theorem, when such a measure μ_u is ergodic, it is physical, since $1/n \sum_{k=0}^{n-1} \delta_{T^k x}$ converges weakly to μ_u for Lebesgue-almost every point in the support of u.

Theorem 1.3 implies in this setting:

Corollary 1.5.

Under the hypotheses of Theorem 1.3, assume that

$$\lim_{n \to +\infty} (D_n^b)^{\frac{1}{p}\frac{1}{n}} \cdot (D_n^e)^{(1-\frac{1}{p})\frac{1}{n}} \cdot \left\| \lambda_n^{-t} |\det DT^n|^{\frac{1}{p}-1} \right\|_{\mathbb{L}^\infty}^{\frac{1}{n}} < 1. \tag{1.7}$$

Then the essential spectral radius of $\mathcal{L}_{1/|\det DT|}$ acting on $\mathcal{H}_p^t$ is strictly smaller than 1.

If (1.7) holds, we will be able to prove the existence of a spectral gap, i.e. of some $\tau < 1$ such that $\{z \in Spec(\mathcal{L}_{1/|\det DT|}) : \tau < |z| < 1\} = \emptyset$, together with a nice repartition of the eigenvalues on the unit cicle. The number τ can be larger the essential spectral radius, as is shown for instance by G. Keller and H.H. Rugh in [KR04] in a 1-dimensional setting. This will imply the following:

Theorem 1.6.

Under the assumptions of Theorem 1.3, if (1.7) *holds, then T has a finite number of physical measures whose densities are in $\mathcal{H}_p^t$, which are ergodic, and whose basins cover Lebesgue almost all X_0. Moreover, if μ is one of these measures, there exist an integer k and a decomposition $\mu = \mu_1 + \cdots + \mu_k$ such that T sends μ_j to μ_{j+1} for $j \in \mathbb{Z}/k\mathbb{Z}$, and the probability measures $k\mu_j$ are mixing at an exponential rate for T^k and α-Hölder test functions.*

The above will be proved in Section 5, although the arguments are classical. We shall also obtain its bounded variation counterpart:

Theorem 1.7.

Under the assumptions of Theorem 1.4, assume that:

$$\lim_{n \to +\infty} (D_n^b)^{\frac{1}{n}} \cdot \left\| \lambda_n^{-1} \right\|_{\mathbb{L}^\infty}^{\frac{1}{n}} < 1. \tag{1.8}$$

Then T has a finite number of physical measures whose densities are in $\mathbb{BV}(X_0)$, which are ergodic, and whose basins cover Lebesgue almost all X_0. Moreover, if μ is one of these measures, there exist an integer k and a decomposition $\mu = \mu_1 + \cdots + \mu_k$ such that T sends μ_j to μ_{j+1} for $j \in \mathbb{Z}/k\mathbb{Z}$, and the probability measures $k\mu_j$ are mixing at an exponential rate for T^k and α-Hölder test functions.

1.3 Piecewise affine maps

We now describe an explicit class of maps for which the assumptions of the previous theorems are satisfied. Let $A_1, ..., A_N$ be $d \times d$ matrices with no eigenvalue of modulus smaller or equal to 1, such that any two of these matrices commute. Let X_0 be a (non necessarily connected) polyhedral region of $\mathbb{R}^d$, and define a map T on X_0 by cutting it into finitely many polyhedral subregions $O_1, \dots, O_N$, applying A_i to each open set O_i, and then mapping $A_1O_1, \dots, A_NO_N$ back into X_0 by translations. It is obviously piecewise $\mathcal{C}^\infty$ and uniformly expanding. Let λ be the lowest modulus of all the eigenvalues of all the matrices A_i; we get easily $\lambda_n \geq \lambda^n$.

Proposition 1.8.

Under those assumptions, the essential spectral radius of $\mathcal{L}_{1/|\det DT|}$ acting on $\mathbb{BV}$ is at most λ^{-1}, thus strictly smaller than 1. Therefore, T satisfies the conclusions of Theorem 1.7.

Proof.

Let K be the total number of the sides of the polyhedra O_i. Around any point x, the boundaries of the sets $O_{(i_0,\dots,i_{n-1})}$ are preimages of theses sides by one of the maps $A_{i_0}, \dots, \prod_{l=0}^{n-1} A_{i_l}$. Hence, there are at most $J(n) = K \sum_{k=1}^{n} \mathrm{Card}\{\prod_{l=0}^{k-1} A_{i_l} : \{i_0, ..., i_{k-1}\} \in \{1, .., N\}^k\}$ such preimages. Since all the A_i commute, there are at most k^N different maps which can be written as $\prod_{l=0}^{k-1} A_{i_l}$, and $J(n)$ grows polynomially. Following the claim p.105 in [Buz97], $D_n^b \leq 2J(n)^d$. This quantity grows subexponentially.

By Theorem 1.4, the essential spectral radius of $\mathcal{L}_{1/|\det DT|}$ acting on $\mathbb{BV}$ is bounded by λ^{-1}. □

This proposition can also be deduced from Cowieson's theorem [Cow00] whenever the polyhedral domain is connected.

2 Sobolev and bounded variation spaces

It is now time to define precisely the function spaces we work with. First, we will present the Sobolev spaces $\mathcal{H}_p^t$ ($p \in (0, 1)$ and $0 \leq t$) and some of their properties. Since they have been thoroughly studied in the 1960s-1970s, we may exploit many ready-to-use results. They are spaces of $\mathbb{L}^p$ functions which satisfy some regularity condition; since the transfer operator of expanding dynamics tends to improve the regularity, this may imply the existence of a spectral gap for these operators. We show two equivalent ways to define these spaces; the first is via the Fourier transform, the second via interpolation theory. Then, we will present the space of functions with bounded variation, which is nicer than the Sobolev space $\mathcal{H}_1^1$ in our setting.

2.1 Definition via Fourier transform

Definition 2.1 (Local spaces H_p^t)**.**

For $1 < p < \infty$, $t \geq 0$, $\xi \in \mathbb{R}^d$, put $a_t(\xi) = (1 + |\xi|^2)^{t/2}$. We define the space H_p^t of functions in $\mathbb{R}^d$ as the subspace of functions $u \in \mathbb{L}^p$ such that $\mathcal{F}^{-1}(a_t \mathcal{F} u) \in \mathbb{L}^p$ with its canonical norm, i.e., the $\mathbb{L}^p$ norm of the expression above.

Since a_t increases at infinity, the condition $\mathcal{F}^{-1}(a_t \mathcal{F}u) \in \mathbb{L}^p$ can be understood as a condition on the decay at infinity of $\mathcal{F}u$, hence on the regularity of u.

If $0 \leq t < \alpha$, we shall see that H_p^t is invariant under composition by $\mathcal{C}^{1+\alpha}$ diffeomorphisms: this is Lemma 3.3. Hence, we can glue such spaces locally together in appropriate coordinate patches, to define a space $\mathcal{H}_p^t$ of functions on the manifold:

Definition 2.2 (Sobolev spaces on X_0)**.**

Let $0 \leq t < \alpha$. Fix a finite number of $C^{1+\alpha}$ charts $\kappa_1, \ldots, \kappa_J$ whose domains of definition cover a compact neighborhood of X_0, and a partition of unity $\rho_1, \ldots, \rho_J$, such that the support of ρ_j is compactly contained in the domain of definition of κ_j, and $\sum \rho_j = 1$ on X. The space $\mathcal{H}_p^t$ is then the space of functions u supported on X_0 such that $(\rho_j u) \circ \kappa_j^{-1}$ belongs to H_p^t for all j, endowed with the norm

$$\|u\|_{\mathcal{H}_p^t} = \sum_{j=1}^{J} \left\| (\rho_j u) \circ \kappa_j^{-1} \right\|_{H_p^t} . \tag{2.1}$$

Changing the charts and the partition of unity gives an equivalent norm on the same space of functions by Lemma 3.1. To fix ideas, we shall view the charts and partition of unity as fixed.

The Fourier transform approach also provides some effective theorems, for instance Fourier multipliers theorems (the following is e.g. Theorem 2.4/2 in [Tri77]):

Theorem 2.3 (Marcinkiewicz multiplier theorem)**.**

Let $b \in \mathcal{C}^d(\mathbb{R}^d)$ satisfy $|\zeta^\gamma D^\gamma b(\zeta)| \leq K$ for all multi-indices $\gamma \in \{0,1\}^d$, and all $\zeta \in \mathbb{R}^d$. For all $p \in (1, \infty)$, there exists a constant $C_{p,d}$ which depends only on p and d such that, for any $u \in \mathbb{L}^p$,

$$\left\| \mathcal{F}^{-1}(b \mathcal{F}u) \right\|_{\mathbb{L}^p} \leq C_{p,d} K \|u\|_{\mathbb{L}^p} . \tag{2.2}$$

2.2 Definition via interpolation theory

Now, let us present the complex interpolation theory, developped by J.L. Lions, A.P. Calderón and S.G. Krejn (see e.g. §1.9 in [Tri78]).

A pair $(\mathcal{B}_0, \mathcal{B}_1)$ of Banach spaces is called an interpolation couple if they are both continuously embedded in a linear Hausdorff space $\mathcal{B}$. For any interpolation couple $(\mathcal{B}_0, \mathcal{B}_1)$, we let $L(\mathcal{B}_0, \mathcal{B}_1)$ be the space of all linear operators $\mathcal{L}$ mapping $\mathcal{B}_0 + \mathcal{B}_1$ to itself so that $\mathcal{L}_{|\mathcal{B}_j}$ is continuous from $\mathcal{B}_j$ to itself for $j = 0, 1$. For an interpolation couple $(\mathcal{B}_0, \mathcal{B}_1)$ and $0 < \theta < 1$, we next define $[\mathcal{B}_0, \mathcal{B}_1]_\theta$ the complex interpolation space of parameter θ.

Set $S = \{z \in \mathbb{C} \mid 0 < \Re z < 1\}$, and introduce the vector space

$$\begin{aligned} F(\mathcal{B}_0, \mathcal{B}_1) = \{ & f : S \to \mathcal{B}_0 + \mathcal{B}_1, \text{ analytic, extending continuously to } \overline{S}, \\ & \text{with } \sup_{z \in \overline{S}} \|f(z)\|_{\mathcal{B}_0 + \mathcal{B}_1} < \infty, t \mapsto f(it) \in \mathcal{C}_b(\mathbb{R}, \mathcal{B}_0), \\ & \text{and } t \mapsto f(1+it) \in \mathcal{C}_b(\mathbb{R}, \mathcal{B}_1) \}. \end{aligned}$$

Then, we define the following norm on this space:

$$\|f\|_{F(\mathcal{B}_0, \mathcal{B}_1)} := \max(\left\| \|f(it)\|_{\mathcal{B}_0} \right\|_\infty , \left\| \|f(1+it)\|_{\mathcal{B}_1} \right\|_\infty)$$

The complex interpolation space is defined for $\theta \in (0,1)$ by

$$[\mathcal{B}_0, \mathcal{B}_1]_\theta = \{u \in \mathcal{B}_0 + \mathcal{B}_1 \mid \exists f \in F(\mathcal{B}_0, \mathcal{B}_1) \text{ with } f(\theta) = u\},$$

normed by

$$\|u\|_{[\mathcal{B}_0,\mathcal{B}_1]_\theta} = \inf_{f(\theta)=u} \|f\|_{F(\mathcal{B}_0,\mathcal{B}_1)}.$$

The main idea is that the $[\mathcal{B}_0, \mathcal{B}_1]_\theta$ spaces are intermediates between $\mathcal{B}_0$ and $\mathcal{B}_1$, "close to $\mathcal{B}_0$" for low parameters θ and "close to $\mathcal{B}_1$" for high parameters. Usually, the embedding can be done for example in $\mathcal{S}'$, the dual space of the space of $\mathcal{C}^\infty$ rapidly decaying functions. We mention here the following key result (Theorem 1.(c) of §2.4.2 in [Tri78]):

Proposition 2.4 (Interpolation of Sobolev spaces)**.**
For any t_0, $t_1 \in \mathbb{R}_+$, p_0, $p_1 \in (1,\infty)$ and $\theta \in (0,1)$, the interpolation space $[H^{t_0}_{p_0}, H^{t_1}_{p_1}]_\theta$ is equal to H^t_p for $t = t_0(1-\theta) + t_1\theta$ and $1/p = (1-\theta)/p_0 + \theta/p_1$.
In particular, for all $t \in (0,1)$ and $p \in (1,\infty)$, $H^t_p = [\mathbb{L}^p, H^1_p]_t$.

Remark 2.5 (Alternative definition of Sobolev spaces)**.**
By Proposition 2.4, we have another way to define the $\mathcal{H}^t_p$ spaces on a (compact) Riemannian manifold X, this time intrinsically. Since there exists a Lebesgue measure Leb *on X, the spaces $\mathbb{L}^p$ are well defined. Moreover, the space $\mathcal{H}^1_p$ is none other than $\{u \in \mathbb{L}^p : \nabla u \in \mathbb{L}^p\}$ with the norm $\|u\|_{\mathcal{H}^1_p} = \|u\|_{\mathbb{L}^p} + \|\nabla u\|_{\mathbb{L}^p}$, the derivative being taken in a weak sense, and defined via the Levi-Civita connection (see Theorem 1.(b) of §2.3.3 in [Tri78]). It is not difficult to see that $\mathcal{H}^t_p = [\mathbb{L}^p(X_0), \mathcal{H}^1_p(X_0)]_t$, with a norm equivalent to the one previously defined (Remark 2 p.321 in [Tri92]).*

Here, the main interest of interpolation theory will be to derive inequalities for operators in the H^t_p spaces from inequalities in the $\mathbb{L}^p$ and H^1_p spaces, where the manipulations are much easier. This will be made possible thanks to the following property (see e.g. §1.9 in [Tri78]): for any interpolation couple $(\mathcal{B}_0, \mathcal{B}_1)$, $\theta \in (0,1)$ and $\mathcal{L} \in L(\mathcal{B}_0, \mathcal{B}_1)$, we have

$$\|\mathcal{L}\|_{[\mathcal{B}_0,\mathcal{B}_1]_\theta \to [\mathcal{B}_0,\mathcal{B}_1]_\theta} \leq \|\mathcal{L}\|^{1-\theta}_{\mathcal{B}_0\to\mathcal{B}_0} \|\mathcal{L}\|^{\theta}_{\mathcal{B}_1\to\mathcal{B}_1} \tag{2.3}$$

Remark 2.6 (Sobolev spaces with negative parameter)**.**
The dual spaces of the Sobolev spaces are well known (see e.g. §4.8.1 in [Tri78]): for any $p \in (1,\infty)$ and $t \in \mathbb{R}_+$, $\mathcal{H}^t_p$ is reflexive and $(\mathcal{H}^t_p)^* = \mathcal{H}^{-t}_{p^*}$, where $1/p + 1/p^* = 1$ and the $\mathcal{H}^{-t}_{p^*}$ is defined via the Fourier transform the same way as the $\mathcal{H}^t_p$ spaces. Such Sobolev spaces with negative parameter will appear (although briefly), for instance in the proof of Lemma 3.5.

2.3 Functions with bounded variation

In order to define the space of functions with bounded variation, we proceed in the same way as in Subsection 2.1: we first define this space on open sets of $\mathbb{R}^d$, and then use the charts to define such a space on a Riemannian manifold.

Definition 2.7 (Functions with bounded variation)**.**

Let us denote the Lebesgue measure on $\mathbb{R}^d$ by Leb. *Let Ω be an open set of $\mathbb{R}^d$. For $u \in \mathbb{L}^1(\Omega)$, we define the variation of u, $V(u)$, by*

$$V(u) = d^{-1} \sup_{\substack{\varphi \in \mathcal{C}_c^\infty(\Omega, \mathbb{R}^d) \\ \|\varphi\|_\infty \leq 1}} \int_\Omega u.div(\varphi) \, \mathrm{dLeb}\,.$$

The space of functions with bounded variation on Ω, denoted $\mathbb{BV}(\Omega)$, is the subspace of functions u of $\mathbb{L}^1(\Omega)$ such that $V(u) < +\infty$, endowed with the norm $\|u\|_{\mathbb{BV}(\Omega)} = \|u\|_{\mathbb{L}^1} + V(u)$.

Remark 2.8.

Formally, we have (see e.g. Remark 1.8 in [Giu84]):

$$V(u) = d^{-1} \int_\Omega |\nabla(u)|.$$

The derivative being taken in a weak sense, $|\nabla(u)|$ may be a measure with no density with respect to the Lebesgue measure. For $\mathcal{C}^1$ functions, it can be taken literally, with

$$V(u) = d^{-1} \int_\Omega \sum_{i=1}^d \left| \frac{\partial u}{\partial x_i} \right| \mathrm{dLeb}\,.$$

As for the Sobolev spaces, we may define the space of functions with bounded variation on a compact set X_0 of a Riemannian manifold X starting from its definition on $\mathbb{R}^d$ (this space also behaves well under composition by $\mathcal{C}^{1+\mathrm{Lip}}$ diffeomorphisms, as is shown by Lemma 6.6) and transporting by the charts.

Definition 2.9 (Functions with bounded variation on X_0)**.**

Fix a finite number of $C^{1+\mathrm{Lip}}$ charts $\kappa_1, \ldots, \kappa_J$ whose domains of definition cover a compact neighborhood of X_0, and a partition of unity $\rho_1, \ldots, \rho_J$, such that the support of ρ_j is compactly contained in the domain of definition of κ_j, and $\sum \rho_j = 1$ on X. The space $\mathbb{BV}$ of functions with bounded variation on X_0 (sometimes also denoted $\mathbb{BV}(X_0)$) is the space of functions u supported on X_0 such that $(\rho_j u) \circ \kappa_j^{-1}$ belongs to $\mathbb{BV}(\mathbb{R}^d)$ for all j, endowed with the norm

$$\|u\|_{\mathbb{BV}} = \sum_{j=1}^J \left\| (\rho_j u) \circ \kappa_j^{-1} \right\|_{\mathbb{BV}(\mathbb{R}^d)}. \tag{2.4}$$

Since we can neither use the Fourier transform nor the interpolation theory here, we will need different tools for the part of our proof relying on functions with bounded variation. We choose to use two theorems dealing with the approximation of functions with bounded variation by $\mathcal{C}^\infty$ functions, and then to work with smooth functions (for which the computations can be done explicitly). They are respectively a variation on Theorem 1.17 and Theorem 1.9 in [Giu84]. The first is obtained via convolution with smooth kernels (say, gaussian or $\mathcal{C}^\infty$ with compact support), the second follows directly from the definition of the variation.

Theorem 2.10 (Approximation of functions with bounded variation)**.**

For any function u in $\mathbb{BV}(\mathbb{R}^d)$, there exists a sequence of $\mathcal{C}^\infty$ functions $(u_n)_{n\in\mathbb{N}}$ which converges in $\mathbb{L}^1$ to u, and such that $(\|u_n\|_{\mathbb{BV}(\mathbb{R}^d)})_{n\in\mathbb{N}}$ converges to $\|u\|_{\mathbb{BV}(\mathbb{R}^d)}$.

Theorem 2.11 (Control of the approximation)**.**

Let $u \in \mathbb{BV}(\mathbb{R}^d)$, and $(u_n)_{n\in\mathbb{N}}$ a sequence of functions with bounded variation which converges to u in $\mathbb{L}^1$. Then $\|u\|_{\mathbb{BV}(\mathbb{R}^d)} \leq \liminf_{n\to+\infty} \|u_n\|_{\mathbb{BV}(\mathbb{R}^d)}$.

3 Elementary inequalities

Our goal in this section is to prove several continuity results for operations in the H^t_p spaces, for instance multiplication by $\mathcal{C}^\alpha$ functions or by characteristic functions of intervals. They show the convenient properties of the spaces we are working with, provided we take good values of t and p, and are virtually sufficient to get the main theorems.

Hence, obtaining versions of the main theorem for other parameters or function spaces can be reduced to the obtention of those results of continuity; see for instance Section 6 for the corresponding results in the $\mathbb{BV}$ space.

3.1 First inequalities

The first inequalities are continuity results for some linear operations; they were proven in the 60s and 70s.

We first deal with the multiplication by smooth enough functions, which will be necessary primarily when we multiply u and g, and also when it comes to study what happens at a small scale (we will multiply the functions u with smooth functions with small supports). A proof of the following lemma can be found in §4.2.2 of [Tri92].

Lemma 3.1 (Multiplication by $\mathcal{C}^\alpha$ functions)**.**

Let $0 < t < \alpha$ be real numbers. There exists $C_{t,\alpha}$ such that, for all $p \in (1, +\infty)$, for all $u \in H^t_p(\mathbb{R}^d)$ and $g \in \mathcal{C}^\alpha(\mathbb{R}^d)$,

$$\|gu\|_{H^t_p} \leq C_{t,\alpha} \|g\|_{\mathcal{C}^\alpha} \|u\|_{H^t_p}.$$

The next inequality (Corollary I.4.2 from [Str67]) is central for our study, since it tells us that $H^t_p(\mathbb{R}^d)$ behaves well in a "piecewise setting" up to constraints on the parameters t and p. We note that one has to combine Corollary I.4.2 from [Str67] with its Corollary I.3.7 to show that the constant $C_{t,p}$ in the following lemma does not depend on Ω otherwise than via L.

Lemma 3.2 (Multiplication by characteristic functions of nice sets)**.**

Let $0 < t < 1/p < 1$ be real numbers; let $L \geq 1$. There exists $C_{t,p}$ such that for each measurable subset Ω of $\mathbb{R}^d$ whose intersection with almost every line parallel to some coordinate axis has at most L connected components, for all $u \in H^t_p(\mathbb{R}^d)$,

$$\|1_\Omega u\|_{H^t_p} \leq C_{t,p} L \|u\|_{H^t_p}.$$

3.2 Composition with $\mathcal{C}^{1+\alpha}$ diffeomorphisms

The injection from $\mathcal{H}^t_p$ into $\mathbb{L}^p$ is compact, t being positive and X_0 compact (in the case of compact subset of some $\mathbb{R}^d$, this is for instance Lemma 2.2 in [Bal]; the corresponding result for compact subset of Riemannian manifolds follows immediatly). Thus, Lemma 3.3 (a version of Lemma 24 in [BG09]) is essential in the way it gives the decomposition of $\|\mathcal{L}_g u\|_{\mathcal{H}^t_p}$ into a part bounded by $\|u\|_{\mathcal{H}^t_p}$, and a part bounded by $\|u\|_{\mathbb{L}^p}$. By Hennion's theorem [Hen93], the part bounded by $\|u\|_{\mathbb{L}^p}$ will not affect our estimation of the essential spectral radius of $\mathcal{L}_g$, so that only the part bounded by $\|u\|_{\mathcal{H}^t_p}$ matters.

When we apply Lemma 3.3, F will be the local inverse of an iterate of T, i.e. some $T_{\mathbf{i}}^{-n}$ (or rather the maps on some open set of $\mathbb{R}^d$ obtained by transporting $T_{\mathbf{i}}^{-n}$ via the charts), and A will be a local approximation of $DT_{\mathbf{i}}^{-n}$. If A is such an approximation in a neighborhood of some x, then, T being expanding, we can write $\|A\| \leq \tilde{\lambda_n}^{-1}$ where $\tilde{\lambda_n}$ is the essential infimum of λ_n on this neighborhood of x.

Lemma 3.3 (Composition with smooth diffeomorphisms)**.**

Let $F \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^d)$ be a diffeomorphism and $A \in GL_d(\mathbb{R})$ such that, for all $z \in \mathbb{R}^d$, $\|A^{-1} \circ DF(z)\| \leq 2$ and $\|DF(z)^{-1} \circ A\| \leq 2$.

Then, for all $t \in [0,1]$, for all $p \in (1, +\infty)$, there exist constants $C_{t,p}$ and $C'_{t,p}$, which do not depend on F nor on A, such that, for all $u \in H_p^t$,

$$\|u \circ F\|_{H_p^t} \leq C_{t,p} |\det A|^{-\frac{1}{p}} \|A\|^t \|u\|_{H_p^t} + C'_{t,p} |\det A|^{-\frac{1}{p}} \|u\|_{\mathbb{L}^p}$$

Proof.

We will write $u \circ F = u \circ A \circ A^{-1} \circ F$, and put $\tilde{F} = A^{-1} \circ F$. First we deal with $u \circ A$, and then with $u \circ \tilde{F}$.

First step: $u \circ A$

We want to estimate $\|u \circ A\|_{H_p^t} = \|\mathcal{F}^{-1}(a_t \mathcal{F}(u \circ A))\|_{\mathbb{L}^p}$, where $a_t(\xi) = (1 + |\xi|^2)^{t/2}$. A change of variables gives $\mathcal{F}^{-1}(a_t \mathcal{F}(u \circ A)) = \mathcal{F}^{-1}(a_t \circ {}^tA \mathcal{F}(u)) \circ A$.

If $|\xi| > 1/\|A\|$, we have

$$(a_t \circ {}^tA)(\xi) = (1 + |{}^tA\xi|^2)^{t/2} \leq (1 + \|A\|^2|\xi|^2)^{t/2} \leq 2^{t/2}\|A\|^t a_t(\xi).$$

On the other hand, if $|\xi| \leq 1/\|A\|$, we have

$$(a_t \circ {}^tA)(\xi) \leq (1 + \|A\|^2|\xi|^2)^{t/2} \leq 2^{t/2}.$$

Finally, we get $a_t \circ {}^tA \leq 2^{t/2}\|A\|^t a_t + 2^{t/2}$, and by the same means the same kind of upper bound for the $\xi^\gamma D^\gamma \left(\dfrac{a_t \circ {}^tA}{\|A\|^t a_t + 1} \right)$ for all $\gamma \in \{0,1\}^d$. By Theorem 2.3, there exist $C_{t,p}$ and $C'_{t,p}$ such that

$$\begin{aligned} \left\|\mathcal{F}^{-1}(a_t \circ {}^tA\mathcal{F}(u))\right\|_{\mathbb{L}^p} &\leq C_{t,p} |\det A|^{-1/p} \|A\|^t \left\|\mathcal{F}^{-1}(a_t \mathcal{F}(u))\right\|_{\mathbb{L}^p} \\ &\quad + C'_{t,p} |\det A|^{-1/p} \left\|\mathcal{F}^{-1}(\mathcal{F}(u))\right\|_{\mathbb{L}^p} \\ &\leq C_{t,p} |\det A|^{-1/p} \|A\|^t \|u\|_{H_p^t} + C'_{t,p} |\det A|^{-1/p} \|u\|_{\mathbb{L}^p}. \end{aligned}$$

The following estimate ensues:

$$\|u \circ A\|_{H_p^t} \leq C_{t,p} |\det A|^{-1/p} \|A\|^t \|u\|_{H_p^t} + C'_{t,p} |\det A|^{-1/p} \|u\|_{\mathbb{L}^p}. \tag{3.1}$$

This inequality ends the first part of the proof.

Second step: $u \circ \tilde{F}$

We will use interpolation. Notice that, since $\|D\tilde{F}\| \geq 1/2$ everywhere, $|\det \tilde{F}^{-1}| \geq 2^d$ and $\left\|u \circ \tilde{F}\right\|_{\mathbb{L}^p} \leq 2^{d/p} \|u\|_{\mathbb{L}^p}$ for all $u \in \mathbb{L}^p$. We also get the same way $\left\|u \circ \tilde{F}\right\|_{H_p^1} \leq C_p 2^{d/p+1} \|u\|_{H_p^1}$ for some C_p, since the H_p^1 norm is equivalent to $\|u\|_{\mathbb{L}^p} + \|Du\|_{\mathbb{L}^p}$ (see for instance §2.3.3 in [Tri78]).

Hence, by Proposition 2.4, there exists $C_{t,p}$ such that, for all $t \in [0,1]$, for all $u \in H_p^t$,

$$\left\| u \circ \tilde{F} \right\|_{H_p^t} \leq C_{t,p} 2^{d/p} \|u\|_{H_p^t} . \tag{3.2}$$

The lemma follows immediately from (3.1) and (3.2). □

Since the injection $H_p^t \to \mathbb{L}^p$ is continuous (t being non-negative), a consequence of (3.1) is that, for all $p \in (1, +\infty)$ and $t \in [0,1]$, there exists a constant $C_{t,p}$ such that, for all $u \in H_p^t(\mathbb{R}^d)$, for all $A \in GL_d(\mathbb{R})$,

$$\|u \circ A\|_{H_p^t} \leq C_{t,p} \sup\{\|A\|, 1\} |\det A|^{-1/p} \|u\|_{H_p^t}. \tag{3.3}$$

3.3 Localization

For the proof of Theorem 1.3, we work locally, on small open sets, and the compactness of X_0 allows us to work globally with a finite number of such sets. However, we need to control the intersection multiplicity of those sets. This control will be obtained with a partition of the unity and a "zoom" to get some regularity when working at small scales.

Lemma 3.4 (Localization principle)**.**
Let $\eta \in \mathcal{C}_c^\infty(\mathbb{R}^d, \mathbb{R})$, and write, for all $x \in \mathbb{R}^d$ and $m \in \mathbb{Z}^d$, $\eta_m(x) = \eta(x+m)$. Then, there exists a constant $C_{t,p,\eta}$ such that, for all $u \in H_p^t$,

$$\left(\sum_{m \in \mathbb{Z}^d} \|\eta_m u\|_{H_p^t}^p \right)^{\frac{1}{p}} \leq C_{t,p,\eta} \|u\|_{H_p^t} . \tag{3.4}$$

This lemma is Theorem 2.4.7 from [Tri92].

The constant $C_{t,p,\eta}$ depends on t, p, on the support of η and on its $\mathcal{C}^k$ norm for some large enough k. The proof we give of a localization principle in Section 6 will make explicit a kind of dependence of $C_{\mathbb{BV},\eta}$ (the corresponding bound for the space of functions with bounded variation) in η.

Lemma 3.5.
Let $1 < p < +\infty$ and $t \in \mathbb{R}_+$. There exists a constant $C_{t,p}$ such that, for any functions $v_1, \ldots, v_l$ with compact support in $\mathbb{R}^d$, belonging to H_p^t, there exists a constant $C_{t,p,\{v\}}$, depending only on the supports of the functions v_i, with

$$\left\| \sum_{i=1}^l v_i \right\|_{H_p^t}^p \leq C_{t,p} m^{p-1} \sum_{i=1}^l \|v_i\|_{H_p^t}^p + C_{t,p,\{v\}} \sum_{i=1}^l \|v_i\|_{\mathbb{L}^p}^p ,$$

where m is the intersection multiplicity of the supports of the v_i's, i.e. $m = \sup_{x \in \mathbb{R}^d} \mathrm{Card}\{i \mid x \in \mathrm{Supp}(v_i)\}$.

Proof.
Let B be the operator acting on functions by $Bv = \mathcal{F}^{-1}((1+|\xi|^2)^{t/2} \mathcal{F} v)$, so that $\|v\|_{H_p^t} = \|Bv\|_{\mathbb{L}^p}$.

By Lemma 2.7 in [Bal], for any compact K and any neighborhood K' of K, there exist $C_{K,K',t,p} > 0$ and a function $\Psi : \mathbb{R}^d \to [0,1]$ equal to 1 on K and vanishing on the complement of K', such that for any $v \in H^t_p$ with support in K,

$$\|\Psi Bv - Bv\|_{\mathbb{L}_p} \leq C_{K,K',t,p} \|v\|_{H^{t-1}_p} . \tag{3.5}$$

Let $v_1, \ldots, v_l$ be functions with compact supports whose intersection multiplicity is m. Choose neighborhoods $K'_1, \ldots, K'_l$ of the supports of the v_is whose intersection multiplicity is also m, and functions $\Psi_1, \ldots, \Psi_l$ as above. Then

$$\left\| \sum_i v_i \right\|^p_{H^t_p} = \left\| \sum_i Bv_i \right\|^p_{\mathbb{L}^p} \leq \left\| \sum_i \Psi_i Bv_i \right\|^p_{\mathbb{L}^p} + C_{K,K',t,p} \sum_i \|v_i\|^p_{H^{t-1}_p} . \tag{3.6}$$

Since $t \leq 1$, the inclusion $\mathbb{L}^p \to H^{t-1}_p$ is continuous, and $\|v_i\|^p_{H^{t-1}_p} \leq C_p \|v_i\|^p_{\mathbb{L}^p}$.

By convexity, the inequality $(x_1 + \cdots + x_m)^p \leq m^{p-1} \sum x_i^p$ holds for any non-negative numbers $x_1, \ldots, x_m$. Since the multiplicity of the K'_i is at most m, this yields

$$\left| \sum_i \Psi_i Bv_i \right|^p \leq m^{p-1} \sum_i |Bv_i|^p.$$

Integrating this inequality and using (3.6), we get the lemma. □

4 Proof of the main theorem

Before proving Theorem 1.3, we need to show that T does not "hack" too much the functions, or in other words that its discontinuities are not so bad that functions in $\mathcal{H}^t_p$ do not stay in this space when composed by T. The following lemma will allow us to apply Lemma 3.2. For once, its proof is adapted from [BG10] instead of [BG09].

Lemma 4.1.

For $1 \leq i \leq I$, let L_i be the number of smooth boundary components of O_i, and $L = \max_i L_i$.

For any $n \geq 1$, for any $\mathbf{i} = (i_0, \ldots, i_{n-1})$, for any $x \in \overline{O_{\mathbf{i}}}$, for any j such that $x \in \operatorname{Supp} \rho_j$, there exists a neighborhood O' of x and an orthogonal matrix M such that the intersection of $M\kappa_j(O' \cap O_{\mathbf{i}})$ with almost any line parallel to a coordinate axis has at most Ln components.

Proof.

If x is in the interior of some $O_{\mathbf{i}}$, we may just take for O' a ball small enough so that $O' \subset O_{\mathbf{i}}$, and $M = Id$. Let assume that x belongs to a backwards image of a $\mathcal{C}^1$ hypersurface of some ∂O_{i_k} by $T^k_{\mathbf{i}}$, with $k \leq n$.

On a small neighbourhood of x, the smooth boundary components of $O_{\mathbf{i}}$ are close to their tangent hyperplanes in x. We can choose d orthogonal vectors of norm 1 all transverse to all those hyperplanes. The smooth boundary components are $\mathcal{C}^1$, so that the vectors we chose are transverse to the boundary components (an not only to the hyperplanes) on a small enough ball around x, that we shall denote O'.

We have $O_{\mathbf{i}} = \bigcap_{k=0}^{n-1} T^{-k} O_{i_k}$, each of the $T^{-k}O_{i_k}$ being an open set bounded by at most L hypersurfaces; hence, $O_{\mathbf{i}}$ is bounded by at most Ln preimages of those hypersurfaces under some T^{-k}. By construction, each line parallel to any coordinate axis in the new basis intersects any of these backward images of hypersurfaces in at most one point, so that their intersection with $O_{\mathbf{i}} \cap O'$ has at most Ln connected components. All we need is to take for M the change-of-basis natrix from the new orthogonal basis to the canonical one. □

We now have all the tools we need to prove the main theorem.

Proof of Theorem 1.3.

Let p and t be as in the assumptions of the theorem. Let $n > 0$, and $r_n > 1$ (the precise value of r_n will be chosen later). We define a dilation R_n on $\mathbb{R}^d$ by $R_n(z) = r_n z$. Let $\|u\|_n$ be another norm on $\mathcal{H}_p^t$, given by

$$\|u\|_n = \sum_{j=1}^{J} \left\| (\rho_j u) \circ \kappa_j^{-1} \circ R_n^{-1} \right\|_{H_p^t}. \tag{4.1}$$

The norm $\|u\|_n$ is of course equivalent to the usual norm on $\mathcal{H}_p^t$, but we look at the space X_0 at a smaller scale. Functions are much flatter at this new scale, so that we have no problem using Lemma 3.1, which involves their $\mathcal{C}^\alpha$ norm: we will replace it by their $\mathbb{L}^\infty$ norm. This will also enable us to use partitions of unity (η_m) with very small supports without spoiling the estimates. The use of this "zooming" norm is similar to the good choice of ϵ_0 in [Sau00].

We will prove that there exists $C_{t,p,\eta,\alpha}$ such that, if n is fixed and r_n is large enough, then there exists $C_{t,p,n}$ such that:

$$\left\| \mathcal{L}_g^n u \right\|_n^p \leq C_{t,p,n} \|u\|_{\mathbb{L}^p}^p + C_{t,p,\eta,\alpha} D_n^b (D_n^e)^{p-1} (Ln)^p \left\| |\det DT^n| \lambda_n^{-tp} |g^{(n)}|^p \right\|_{\mathbb{L}^\infty} \|u\|_n^p. \tag{4.2}$$

We have already observed that, since X_0 is compact and $t > 0$, the injection of $\mathcal{H}_p^t$ into $\mathbb{L}^p$ is compact (see e.g. Lemma 2.2 in [Bal]). Hence, by Hennion's theorem (actually Corollary 1 in [Hen93]), the essential spectral radius of $\mathcal{L}_g^n$ acting on $\mathcal{H}_p^t$ (for either $\|u\|_{\mathcal{H}_p^t}$ or $\|u\|_n$, since these norms are equivalent) is at most

$$\left[C_{t,p,\eta,\alpha} (Ln)^p D_n^b (D_n^e)^{p-1} \left\| |\det DT^n| \lambda_n^{-tp} |g^{(n)}|^p \right\|_{\mathbb{L}^\infty} \right]^{\frac{1}{p}}. \tag{4.3}$$

Taking the power $1/n$ and letting n tend to $+\infty$, we obtain Theorem 1.3 since the quantity $(C_{t,p,\eta,\alpha}^{1/p}(Ln)^p)^{1/n}$ converges to 1.

It remains to prove (4.2), for large enough r_n. The estimate will be subdivided into three steps:

1. Decomposing u into a sum of functions $v_{j,m}$ with small supports and well controlled $\|\cdot\|_n$ norms.
2. Estimating each term $(1_{O_{\mathbf{i}}} g^{(n)} v_{j,m}) \circ T_{\mathbf{i}}^{-n}$, for i of length n.
3. Adding all terms to obtain $\mathcal{L}_g^n u$.

First step:

For $1 \le j \le J$ and $m \in \mathbb{Z}^d$, let $\tilde{v}_{j,m} = \eta_m \cdot (\rho_j u) \circ \kappa_j^{-1} \circ R_n^{-1}$, where $\eta_m(x) = \eta(x+m)$, with $\eta : \mathbb{R}^d \to [0,1]$ a compactly supported $\mathcal{C}^\infty$ function so that $\sum_{m\in\mathbb{Z}^d} \eta_m = 1$. Since the intersection multiplicity of the supports of the functions η_m is bounded, this is also the case for the $\tilde{v}_{j,m}$. Moreover, if j is fixed, we get, using Lemma 3.4,

$$\begin{aligned}\sum_{m\in\mathbb{Z}^d} \|\tilde{v}_{j,m}\|_{H_p^t}^p &= \sum_{m\in\mathbb{Z}^d} \left\|\eta_m \cdot (\rho_j u) \circ \kappa_j^{-1} \circ R_n^{-1}\right\|_{H_p^t}^p \\ &\le C_{t,p,\eta} \left\|(\rho_j u) \circ \kappa_j^{-1} \circ R_n^{-1}\right\|_{H_p^t}^p \le C_{t,p,\eta} \|u\|_n^p .\end{aligned} \tag{4.4}$$

Since R_n expands the distances by a factor r_n while the size of the supports of the functions η_m is uniformly bounded, the supports of the functions

$$v_{j,m} = \tilde{v}_{j,m} \circ R_n \circ \kappa_j = \eta_m \circ R_n \circ \kappa_j \cdot (\rho_j u)$$

are arbitrarily small if r_n is large enough. Finally

$$u = \sum_j \rho_j u = \sum_{j,m} v_{j,m}. \tag{4.5}$$

Second step:

Fix $j, k \in \{1, \dots, J\}$, $m \in \mathbb{Z}^d$ and $\mathbf{i} = (i_0, \dots, i_{n-1})$. We will prove that

$$\begin{aligned}&\left\|(\rho_k (g^{(n)} 1_{O_\mathbf{i}} v_{j,m}) \circ T_\mathbf{i}^{-n}) \circ \kappa_k^{-1} \circ R_n^{-1}\right\|_{H_p^t} \\ &\le C_{t,p,n} \|u\|_{\mathbb{L}^p} + C_{t,p}(Ln) \left\| |\det DT^n|^{\frac{1}{p}} g^{(n)} \lambda_n^{-t} \right\|_{\mathbb{L}^\infty} \|\tilde{v}_{j,m}\|_{H_p^t} .\end{aligned} \tag{4.6}$$

First, if the support of $v_{j,m}$ is small enough (which can be ensured by taking r_n large enough), there exists a neighborhood O of this support and a matrix M satisfying the conclusion of Lemma 4.1 for all x in $O_\mathbf{i}$. Therefore, the intersection of $R_n(M(\kappa_j(O \cap O_\mathbf{i})))$ with almost any line parallel to a coordinate axis has at most Ln connected components. Hence, Lemma 3.2 implies that the multiplication by $1_{O\cap O_\mathbf{i}} \circ \kappa_j^{-1} \circ M^{-1} \circ R_n^{-1}$ sends H_p^t into itself, with a norm bounded by $C_{t,p}Ln$. Using the fact that M and R_n commute, the properties of M, and (3.3), we get

$$\left\|1_{O_\mathbf{i}} \circ \kappa_j^{-1} \circ R_n^{-1} \cdot \tilde{v}_{j,m}\right\|_{H_p^t} \le C_{t,p} Ln \|\tilde{v}_{j,m}\|_{H_p^t} . \tag{4.7}$$

Next, let

$$\tilde{v}_{j,k,m} = ((\rho_k \circ T_\mathbf{i}^n) 1_{O_\mathbf{i}}) \circ \kappa_j^{-1} \circ R_n^{-1} \cdot \tilde{v}_{j,m}$$

(we suppress i from the notation for simplicity). Let also χ be a $\mathcal{C}^\infty$ function supported in the neighborhood O of the support of $v_{j,m}$ with $\chi \equiv 1$ on this support. Up to taking larger r_n we may ensure that $\left\|(\chi(\rho_k \circ T_\mathbf{i}^n)) \circ \kappa_j^{-1} \circ R_n^{-1}\right\|_{C^\alpha} \le 2$. Then Lemma 3.1 and (4.7) imply

$$\|\tilde{v}_{j,k,m}\|_{H_p^t} \le C_{t,p,\alpha} Ln \|\tilde{v}_{j,m}\|_{H_p^t} . \tag{4.8}$$

In addition, putting $G = \kappa_j \circ T_\mathbf{i}^{-n} \circ \kappa_k^{-1}$, we have

$$((\rho_k \circ T_{\mathbf{i}}^n) 1_{O_{\mathbf{i}}} v_{j,m}) \circ T_{\mathbf{i}}^{-n} \circ \kappa_k^{-1} \circ R_n^{-1} = \tilde{v}_{j,k,m} \circ R_n \circ \kappa_j \circ T_{\mathbf{i}}^{-n} \circ \kappa_k^{-1} \circ R_n^{-1} \qquad (4.9)$$
$$= \tilde{v}_{j,k,m} \circ R_n \circ G \circ R_n^{-1}.$$

Applying Lemma 3.3 to $F = R_n \circ G \circ R_n^{-1}$, we get (for some point x in the support of $v_{j,m}$, and some matrix A of the form $DG(R_n^{-1}(x))$ for some x, so that $\|A\| \leq \lambda_n^{-1}(x)$)

$$\left\|\tilde{v}_{j,k,m} \circ R_n \circ G \circ R_n^{-1}\right\|_{H_p^t} \leq C_{t,p,n} \|u\|_{\mathbb{L}^p} \qquad (4.10)$$
$$+ C_{t,p} |\det A|^{-\frac{1}{p}} \lambda_n(x)^{-t} \|\tilde{v}_{j,k,m}\|_{\mathcal{H}_p^t}.$$

The constant $C_{t,p,n}$ also depends on the choice of A. Let χ be a $\mathcal{C}^\infty$ function supported in O' with $\chi \equiv 1$ on the support of $v_{j,m} \circ T_{\mathbf{i}}^{-n}$. For $\delta > 0$, we can ensure by increasing r_n that the $\mathcal{C}^\alpha$ norm of $(\chi g^{(n)}) \circ T_{\mathbf{i}}^{-n} \circ \kappa_k^{-1} \circ R_n^{-1}$ is bounded by $|g^{(n)}(x)| + \delta$ for some x in the support of $v_{j,m}$. Choosing $\delta > 0$ small enough, we deduce from (4.10), Lemma 3.1 and (4.8)

$$\left\|(\rho_k (g^{(n)} 1_{O_{\mathbf{i}}} v_{j,m}) \circ T_{\mathbf{i}}^{-n}) \circ \kappa_k^{-1} \circ R_n^{-1}\right\|_{H_p^t}$$
$$\leq C_{t,p,n} \|u\|_{\mathbb{L}^p} + C_{t,p,\alpha} Ln \left\| |\det DT^n|^{\frac{1}{p}} g^{(n)} \lambda_n^{-t} \right\|_{\mathbb{L}_\infty} \|\tilde{v}_{j,m}\|_{H_p^t}.$$

This proves (4.6).

Third step:

We have $\mathcal{L}_g^n u = \sum_{j,m} \sum_{\mathbf{i}} (1_{O_{\mathbf{i}}} g^{(n)} v_{j,m}) \circ T_{\mathbf{i}}^{-n}$. (Note that only finitely many terms in this sum are nonzero by compactness of the support of each ρ_j.) We claim that the intersection multiplicity of the supports of the functions $(1_{O_{\mathbf{i}}} g^{(n)} v_{j,m}) \circ T_{\mathbf{i}}^{-n}$ is bounded by $C_\eta D_n^e$. Indeed, this follows from the fact that any point $x \in X_0$ belongs to at most D_n^e sets $\overline{T_{\mathbf{i}}^n(O_{\mathbf{i}})}$, and that the intersection multiplicity of the supports of the functions $v_{j,m}$ is bounded.

To estimate $\left\|\mathcal{L}_g^n u\right\|_n$, we have to bound each term $\left\|(\rho_k \mathcal{L}_g^n u) \circ \kappa_k^{-1} \circ R_n^{-1}\right\|_{H_p^t}$, for $1 \leq k \leq J$. Let us fix such a k. By Lemma 3.5, we have

$$\left\|(\rho_k \mathcal{L}_g^n u) \circ \kappa_k^{-1} \circ R_n^{-1}\right\|_{H_p^t}^p$$
$$\leq C_{t,p,n} \|u\|_{\mathbb{L}^p}^p + C_{t,p,\alpha} (C_{t,p,\eta} D_n^e)^{p-1} \sum_{j,m,\mathbf{i}} \left\|(\rho_k (1_{O_{\mathbf{i}}} g^{(n)} v_{j,m}) \circ T_{\mathbf{i}}^{-n}) \circ \kappa_k^{-1} \circ R_n^{-1}\right\|_{H_p^t}^p.$$

We can bound each term in the sum using (4.6) and the convexity inequality $(a+b)^p \leq 2^{p-1}(a^p + b^p)$. Moreover, for any (j,m), the number of parameters i for which the corresponding term is nonzero is bounded by the number of sets $\overline{O_{\mathbf{i}}}$ intersecting the support of $v_{j,m}$. Choosing r_n large enough, we can ensure that the supports of the $v_{j,m}$ are small enough so that this number is bounded by D_n^b. Together with (4.4), this concludes the proof of (4.2), and of Theorem 1.3. □

5 Existence of finitely many physical measures

In this section, we prove Theorem 1.6. In view of Theorem 1.4, and with a few minor adaptations, the same proof leads to Theorem 1.7. This proof is a simplified version of Appendix B in [BG09], which is itself adapted from [BKL02].

Proof.

First step: the spectrum

First, note that the Lebesgue measure Leb on the manifold X_0 is in $(\mathcal{H}_p^t)^*$ and is a fixed point for $\mathcal{L}^*$, so that 1 is an eigenvalue of $\mathcal{L}^*$ and thus is in the spectrum of $\mathcal{L}$. By Corollary 1.5 the essential spectral radius ρ_{ess} of $\mathcal{L}$ is strictly smaller than 1, and 1 is an eigenvalue of $\mathcal{L}$.

We also have the inclusions $\mathcal{H}_p^t \subset \mathbb{L}^p \subset \mathbb{L}^1$, so that $\mathbb{L}^\infty \subset \mathcal{H}_{p^*}^{-t}$ and any essentially bounded function on X_0 can be seen as a linear functional on $\mathcal{H}_p^t$.

In the following, when γ denotes an eigenvalue of $\mathcal{L}$ of modulus strictly bigger than the essential spectral radius, E_γ will denote the corresponding finite-dimensional eigenspace and $\pi_\gamma : \mathcal{H}_p^t \to E_\gamma$ the corresponding canonical projection. Consider such an eigenvalue γ; since $\mathbb{L}^\infty \cap \mathcal{H}_p^t$ is dense is $\mathcal{H}_p^t$, its image by π_γ in E_γ is a dense subspace of E_γ, and therefore is E_γ itself.

Let us first prove that $\mathcal{L}$ has no eigenvalue of modulus strictly bigger than 1, nor a nontrivial Jordan block for an eigenvalue of modulus 1. Otherwise, let γ be an eigenvalue of $\mathcal{L}$ of maximal modulus, with a Jordan block of maximal size d. Therefore, there exists a bounded function f such that $n^{-d}\sum_{i=0}^{n-1}\gamma^{-i}\mathcal{L}^i f$ converges to a nonzero limit u. For any $g \in \mathbb{L}^\infty$,

$$\langle u, g\rangle := \int_{X_0} ug \,\mathrm{dLeb} = \lim_{n\to+\infty} \frac{1}{n^d}\sum_{i=0}^{n-1}\gamma^{-i}\langle \mathcal{L}^i f, g\rangle = \lim_{n\to+\infty} \frac{1}{n^d}\sum_{i=0}^{n-1}\gamma^{-i}\int f\cdot g\circ T^i \,\mathrm{dLeb}\,.$$

If $|\gamma| > 1$ or $d \geq 2$, this quantity converges to 0 when $n \to +\infty$ since $\int f\cdot g\circ T^i \,\mathrm{dLeb}$ is uniformly bounded. This contradicts the fact that u is nonzero.

Let us take any eigenvalue γ of modulus 1. Since γ is of maximal modulus and $\rho_{ess} < 1$, the eigenprojection is given by

$$\pi_\gamma f = \lim_{n\to+\infty} \frac{1}{n}\sum_{i=0}^{n-1}\gamma^{-i}\mathcal{L}^i f, \tag{5.1}$$

where the convergence holds in $\mathcal{H}_p^t$.

Let $u \in E_\gamma$. Obviously, $E_\gamma \subset \mathcal{H}_p^t \subset \mathbb{L}^p \subset \mathbb{L}^1$ and u can be seen as the density of a finite complex measure with respect to the Lebesgue measure.

For any $f \in \mathbb{L}^\infty \cap \mathcal{H}_p^t$ and any non-negative measurable bounded function g,

$$\begin{aligned}
|\langle \pi_\gamma f, g\rangle| &\leq \|f\|_{\mathbb{L}^\infty} \lim_{n\to+\infty} \frac{1}{n}\sum_{i=0}^{n-1}\int \left|g\circ T^i\right| \mathrm{dLeb} \\
&\leq \|f\|_{\mathbb{L}^\infty} \lim_{n\to+\infty} \left|\frac{1}{n}\sum_{i=0}^{n-1}\int g\mathcal{L}^i 1 \,\mathrm{dLeb}\right| \\
&\leq \|f\|_{\mathbb{L}^\infty} \left|\langle \pi_1 1, g\rangle\right|.
\end{aligned}$$

This means that the measures $u\,\mathrm{dLeb}$, where u belongs to some E_γ with $|\gamma| = 1$, are all absolutely continuous with respect to the reference measure $\mu = \pi_1 1\,\mathrm{dLeb}$, with bounded density.

For any $u \in E_\gamma$, we write $u\,\mathrm{dLeb} = \varphi_u \mu$, where $\varphi_u \in \mathbb{L}^\infty(\mu)$ is defined μ-almost everywhere. The equation $\mathcal{L}u = \gamma u$ translates into $T^*(\varphi_u\mu) = \gamma\varphi_u\mu$. Hence, since μ is invariant,

$$\begin{aligned}\int |\varphi_u \circ T - \gamma^{-1}\varphi_u|^2 \,\mathrm{d}\mu &= \int |\varphi_u|^2 \circ T \,\mathrm{d}\mu + \int |\gamma^{-1}\varphi_u|^2 \,\mathrm{d}\mu - 2\Re \int \overline{\varphi_u} \circ T\gamma^{-1}\varphi_u \,\mathrm{d}\mu \\ &= 2\int |\varphi_u|^2 \,\mathrm{d}\mu - 2\Re \int \gamma^{-1}\overline{\varphi_u}\,\mathrm{d}T^*(\varphi_u\mu) = 0.\end{aligned}$$

Let $F_\gamma = \{\varphi \in \mathbb{L}_\infty(\mu) \mid \varphi \circ T = \gamma^{-1}\varphi\}$. The map $\Phi_\gamma : u \to \varphi_u$ is an injective morphism from E_γ to F_γ. We now show that Φ_γ is also surjective, or in other words that the functions $\varphi_u\pi_1 1$ always belong to $\mathcal{H}^t_p$.

Let be $\varphi \in F_\gamma$. Since the continuous functions are dense in $\mathbb{L}^1(\mu)$, and any continuous function can be uniformly approximated (thus approximated in $\mathbb{L}^1(\mu)$, all our measures being finite) by a sequence of $\mathcal{C}^1$ functions, $\mathcal{C}^\alpha \subset \mathcal{H}^t_p$ is dense in $\mathbb{L}^1(\mu)$. We choose an approximation (φ_m) of φ by $\mathcal{C}^\alpha$ functions in $\mathbb{L}^1(\mu)$, and put $u_m = \pi_\gamma(\varphi_m\pi_1 1)$.

The functions (u_m) belong to $\mathcal{H}^p_t(\mathrm{Leb})$ by Lemma 3.1. Moreover, $|\mathcal{L}u| \le \mathcal{L}|u|$ for any $u \in \mathcal{H}^p_t$, so that $\|\mathcal{L}u\|_{\mathbb{L}^1(\mathrm{Leb})} \le \|u\|_{\mathbb{L}^1(\mathrm{Leb})}$.

For any $f \in \mathcal{C}^\alpha$, we have:

$$\|\pi_\gamma(f\pi_1 1)\|_{\mathbb{L}^1(\mathrm{Leb})} \le \liminf_{n\to+\infty} \frac{1}{n}\sum_{i=0}^{n-1} \left\|\mathcal{L}^i(f\pi_1 1)\right\|_{\mathbb{L}^1(\mathrm{Leb})} \le \|f\|_{\mathbb{L}^1(\mu)}$$

Hence, $f \to \pi_\gamma(f\pi_1 1)$ is continuous from $\mathcal{C}^\alpha$ (endowed with the semi-norm $\|\cdot\|_{\mathbb{L}^1(\mu)}$) to E_γ (endowed with the norm $\|\cdot\|_{\mathbb{L}^1(\mathrm{Leb})}$). As a consequence, (u_m) is a Cauchy sequence in E_γ, which is of finite dimension and thus complete. The sequence (u_m) converges to some $u \in E_\gamma$. Then, we have $\Phi_\gamma(u) = \varphi$, and Φ_γ is an isomorphism.

The eigenvalues of $\mathcal{L}$ of modulus 1 are exactly those γ such that F_γ is not reduced to 0. We have $\varphi^n_\gamma \in F_{\gamma^n}$ for all n whenever $\varphi_\gamma \in F_\gamma$, so that this set is an union of groups. Since $\mathcal{L}$ only has a finite number of eigenvalues of modulus 1, this implies that these eigenvalues are roots of unity. In particular, there exists $N > 0$ such that $\gamma^N = 1$ for any eigenvalue γ.

We now have a good description of the spectrum of $\mathcal{L}$: its radius is 1, its essential radius strictly smaller than 1, and the eigenvalues of modulus 1 form a finite group, none of them having a nontrivial Jordan block. The measures corresponding to those eigenvalues are all absolutely continuous with bounded density with respect to a reference invariant measure, $\pi_1 1\,\mathrm{dLeb}$, and are all absolutely continuous with respect to the Lebesgue measure.

Second step: mixing physical measures

Let us now assume that 1 is the only eigenvalue of $\mathcal{L}$ of modulus 1 (in the general case, this will be true for $\mathcal{L}^N$, so we will be able to deduce the general case from this particular case). Under this assumption, $\mathcal{L}^n u$ converges to $\pi_1 u$ for any $u \in \mathcal{H}^t_p$.

Consider the subset of F_1 (the T-invariant functions of $\mathbb{L}(\mu)^\infty$) given by the nonnegative functions whose integral with respect to the measure μ is 1. It is

nonempty, since it contains the function $\mu(X_0)^{-1}$. It is a bounded convex subset of F_1, whose extremal points are of the form 1_B for some minimal invariant set B defined μ-almost everywhere. Such extremal points are automatically linearly independent; we can also assume that they are subsets of $\mathrm{Supp}(\mu)$. Since F_1 is finite-dimensional, there is only a finite number of them, say $1_{B_1}, \dots, 1_{B_l}$, and a function belongs to F_1 if and only if it can be written as $\varphi = \sum \alpha_i 1_{B_i}$ for some scalars $\alpha_1, \dots, \alpha_l$. The decomposition of the function $1 \in F_1$ is given by $1 = \sum 1_{B_i}$, hence the sets B_i cover the whole space up to a set of zero measure for μ. Moreover, since B_i is minimal, the measure $\mu_i := \dfrac{1_{B_i}\mu}{\mu(B_i)}$ is an invariant ergodic probability measure.

Let $u_i = \Phi_1^{-1}(1_{B_i}) = 1_{B_i}\pi_1 1 \in \mathcal{H}_p^t$, then any element of E_1 is a linear combination of the u_i. In particular, this applies to $\pi_1(fu_i)$ for any $f \in \mathcal{C}^\alpha$ (we recall that $fu_i \in \mathcal{H}_p^t$ by Lemma 3.1, and that 1_{B_i} is T-invariant in $\mathbb{L}^\infty(\mu)$); we now compute $\pi_1(fu_i)$.

$$\begin{aligned}\langle \pi_1(fu_i), 1_{B_j}\rangle &= \lim_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} \int \mathcal{L}^k(fu_i)\cdot 1_{B_j}\,\mathrm{dLeb} = \lim_{n\to+\infty}\frac{1}{n}\sum_{k=0}^{n-1}\int f\cdot 1_{B_i}\cdot 1_{B_j}\circ T^k\,\mathrm{d}\mu \\ &= \int_{B_i\cap B_j} f\,\mathrm{d}\mu = \mu(B_i)\delta_{i,j}\int f\,\mathrm{d}\mu_i\end{aligned}$$

Hence, we get:

$$\pi_1(fu_i) = \left(\int f\,\mathrm{d}\mu_i\right) u_i \tag{5.2}$$

This enables us to deduce that each measure μ_i is exponentially mixing, as follows. Let $\delta < 1$ be the spectral radius of $\mathcal{L} - \pi_1$; we have $\|\mathcal{L}^n - \pi_1\| = O(\delta^n)$. Then, if f, g are $\mathcal{C}^\alpha$ functions, and following (5.2),

$$\begin{aligned}\int f\cdot g\circ T^n\,\mathrm{d}\mu_i &= \frac{1}{\mu(B_i)}\langle \mathcal{L}^n(fu_i), g\rangle = \frac{1}{\mu(B_i)}\langle \pi_1(fu_i), g\rangle + O(\delta^n) \\ &= \left(\int f\,\mathrm{d}\mu_i\right)\frac{1}{\mu(B_i)}\langle u_i, g\rangle + O(\delta^n) \\ &= \left(\int f\,\mathrm{d}\mu_i\right)\left(\int g\,\mathrm{d}\mu_i\right) + O(\delta^n).\end{aligned}$$

Since $\mathcal{C}^\alpha$ is dense $\mathbb{L}^1$, every μ_i is mixing and thus ergodic. Those μ_i are ergodic measures, and absolutely continuous with respect to the Lebesgue measure, so that they are also physical measures.

Third step: basins

We now turn to the relationship between Lebesgue measure and the measures μ_i. For any function $f \in \mathbb{L}^\infty(\mathrm{Leb}) \cap \mathcal{H}_p^t$, let us write

$$\pi_1(f) = \sum_{i=1}^{l} a_i(f) u_i. \tag{5.3}$$

We shall need to describe the coefficients $a_i(f)$. Let be $1 \le i \le l$, and $f \in \mathbb{L}^\infty(\mathrm{Leb}) \cap \mathcal{H}_p^t$. We recall that the sets B_i are T-invariant and disjoint for the Lebesgue measure

(except perhaps for a set a zero Lebesgue measure).

$$\lim_{n\to+\infty}\int_{X_0} f\cdot\frac{1}{n}\sum_{k=0}^{n-1}1_{B_i}\circ T^k\,\mathrm{dLeb}=\lim_{n\to+\infty}\frac{1}{n}\sum_{k=0}^{n-1}\langle 1_{B_i},\mathcal{L}^k f\rangle=\langle 1_{B_i},\pi_1 f\rangle=a_i(f)\mu(B_i)$$

Moreover, all the $1/n\sum_{k=0}^{n-1}1_{B_i}\circ T^k$ are bounded by 1 ; let h_i be one of their weak $\mathbb{L}^2(\mathrm{Leb})$ limits. The function h_i is T-invariant Leb-almost everywhere, and satisfies for all $f\in\mathbb{L}^\infty(\mathrm{Leb})\cap\mathcal{H}_p^t$:

$$a_i(f)=\frac{1}{\mu(B_i)}\int f h_i\,\mathrm{dLeb}$$

Since $a_i(1)=1$, we have $\int h_i\,\mathrm{dLeb}=\mu(B_i)$.

Let us now compute $\int h_i h_j\,\mathrm{dLeb}$. To begin with, h_j being invariant,

$$a_j(1_{B_i}\circ T^n)=\int 1_{B_i}\circ T^n\cdot h_j\,\mathrm{dLeb}=\int 1_{B_i}\circ T^n\cdot h_j\circ T^n\,\mathrm{dLeb}=a_j(1_{B_i}\mathcal{L}^n 1).$$

The computation now gives :

$$\begin{aligned}\int h_i h_j\,\mathrm{dLeb}=\mu(B_j)a_j(h_i)&=\mu(B_j)\lim_{n\to+\infty}\frac{1}{n}\sum_{k=0}^{n-1}a_j(1_{B_i}\circ T^k) \qquad (5.4)\\ &=\mu(B_j)\lim_{n\to+\infty}a_j\left(1_{B_i}\frac{1}{n}\sum_{k=0}^{n-1}\mathcal{L}^k 1\right)=\mu(B_j)a_j(u_i)=\mu(B_i)\delta_{i,j}\end{aligned}$$

Taking $i=j$, we get $\int h_i^2\,\mathrm{dLeb}=\mu(B_i)=\int h_i\,\mathrm{dLeb}$. Since h_i takes its values in $[0,1]$, this shows that there exists a subset C_i of X_0 such that $h_i=1_{C_i}$, with $\int 1_{C_i}\,\mathrm{dLeb}=\mu(B_i)$. Moreover, (5.4) shows that $\mathrm{Leb}(C_i\cap C_j)=0$ if $i\neq j$. For any function $f\in\mathbb{L}^\infty(\mathrm{Leb})\cap\mathcal{H}_p^t$,

$$a_i(f)=\frac{1}{\mu(B_i)}\int_{C_i} f\,\mathrm{dLeb}$$

Moreover,

$$\int_{X_0}\sum_i 1_{C_i}\,\mathrm{dLeb}=\sum_i\mu(B_i)=\mu(X_0)=\mathrm{Leb}(X_0)$$

This shows that the sets C_i form a partition of the space modulo a set of zero Lebesgue measure. We have proved that

$$\pi_1(f)=\sum_{i=1}^{l}\frac{1}{\mu(B_i)}\left(\int_{C_i} f\,\mathrm{dLeb}\right)u_i, \qquad (5.5)$$

which means that the sets C_i form a partition of almost all X_0 into ergodic basins for the physical measures. □

6 Piecewise $\mathcal{C}^{1+Lip}$ maps and functions with bounded variation

For now on, we will assume that the transformation T is a piecewise $\mathcal{C}^{1+Lip}$ uniformly expanding map, and work with the space of functions with bounded variation. We prove the existence of a spectral gap for the operator $\mathcal{L}_{1/|DetDT|}$ in this space under simple assumptions, and compare our results with previous works by J.W. Cowieson [Cow00] and B. Saussol [Sau00].

In order to simplify the notations, in this section λ_n denotes the essential infimum of the $\lambda_n(x)$.

As an introduction and a last tribute to our work with Sobolev spaces, we first prove a corollary to Corollary 1.5.

Corollary 6.1.
If T is a piecewise $\mathcal{C}^2$ expanding map with piecewise $\mathcal{C}^1$ boundaries and if:

$$\lim_{n\to+\infty}(D_n^b)^{\frac{1}{n}}\lambda_n^{-\frac{1}{n}}<1,$$

then there exist $0<t_0<1/p_0<1$ such that, for all $0<t<1/p<1$ such that $t_0\leq t$ and $1/p_0\leq 1/p$, the essential spectral radius of $\mathcal{L}_{1/|DetDT|}$ acting on $\mathcal{H}_p^t$ is strictly smaller than 1.

Proof.
Let us put $\epsilon = 1-\lim_{n\to+\infty}(D_n^b)^{\frac{1}{n}}\lim_{n\to+\infty}\lambda_n^{-\frac{1}{n}}$.

We first choose $0<t_0<1$ such that $\lim_{n\to+\infty}(D_n^b)^{\frac{1}{n}}\lim_{n\to+\infty}\lambda_n^{-\frac{t_0}{n}}<1-\epsilon/2$.

Then, we choose $1<p_0<1/t_0$ such that $\lim_{n\to+\infty}(D_n^e)^{(1-\frac{1}{p_0})\frac{1}{n}}<1+\epsilon/2$.

Let $1>t\geq t_0$ and $1<p\leq p_0$ such that $t<1/p<1$. The essential spectral radius of $\mathcal{L}_{1/|DetDT|}$ acting on $\mathcal{H}_p^t$ is at most

$$\lim_{n\to+\infty}(D_n^b)^{\frac{1}{p}\frac{1}{n}}\cdot(D_n^e)^{\frac{1}{n}(1-\frac{1}{p})}\cdot\left\|\lambda_n^{-t}|\det DT^n|^{\frac{1}{p}-1}\right\|_{\mathbb{L}^\infty}^{\frac{1}{n}}$$

$$\leq\lim_{n\to+\infty}(D_n^b)^{\frac{1}{n}}\cdot(D_n^e)^{(1-\frac{1}{p_0})\frac{1}{n}}\cdot\lambda_n^{-\frac{t_0}{n}}\leq(1-\epsilon/2)(1+\epsilon/2)<1$$

□

Corollary 6.1 can be seen as an asymptotic result; even if we work with parameters t and p close to 1, we do not work on $\mathcal{H}_1^1$. Indeed, this result does not hold when replacing $\mathcal{H}_p^t$ by $\mathcal{H}_1^1$. Since T is piecewise continuous, we must allow discontinuities in the function spaces we let $\mathcal{L}_{1/|DetDT|}$ act on, and for instance in dimension one every function of $\mathcal{H}_1^1$ has a continuous version (because its derivative is a function in $\mathbb{L}^1$). Here, the problem comes from the fact that the characteristic function of an interval is in general not in $H_1^1(\mathbb{R})$, since its derivative - in general a sum of atomic measures - is not in $\mathbb{L}^1$. Therefore, we have to work in a broader space, for instance the space of functions with bounded variation; this will lead us to an alternative proof of a theorem shown by J.W. Cowieson in 2000 [Cow00]. We will also have a closer look at the results proved by B. Saussol in 2000 [Sau00], which may be linked to this work in a similar fashion.

6.1 Cowieson's theorem

We recall here J.W. Cowieson's results [Cow00], and compare them to what we prove later.

Theorem 6.2 (Cowieson, 2000)**.**
Let X be a bounded and connected open set of $\mathbb{R}^d$ with piecewise $\mathcal{C}^2$ boundaries, and $T : X \mapsto X$ be a piecewise $\mathcal{C}^2$ and uniformly expanding map with piecewise $\mathcal{C}^2$ boundaries.

If there exists $n \in \mathbb{N}$ such that $\lambda_n > D_n^b$, then T admits an absolutely continuous invariant probability measure whose density is in $\mathbb{BV}(\overline{X})$.

This theorem is indeed a consequence of Theorem 1.7. First, notice that $(\lambda_n)_{n\in\mathbb{N}}$ and $(D_n^b)_{n\in\mathbb{N}}$ are respectively supermultiplicative and submultiplicative. Thus, the limits $\lim_{n\to+\infty} \lambda_n^{\frac{1}{n}}$ and $\lim_{n\to+\infty} (D_n^b)^{\frac{1}{n}}$ exist, and are respectively equal to $\sup_{n\in\mathbb{N}} \lambda_n^{\frac{1}{n}}$ and $\inf_{n\in\mathbb{N}} (D_n^b)^{\frac{1}{n}}$. Therefore, $\lim_{n\to+\infty} \lambda_n^{-\frac{1}{n}} (D_n^b)^{\frac{1}{n}} < 1$ if and only if there exists some $n \in \mathbb{N}$ such that $\lambda_n > D_n^b$, and the condition stated in Cowieson's theorem and condition (1.8) are equivalent.

The differences between Cowieson's theorem and our results are the following. To begin with, our setting is less restrictive: the domain does not need to be the closure of a bounded, connected open set of some $\mathbb{R}^d$, but a compact subset of a Riemannian manifold, and the boundaries do not need to be piecewise $\mathcal{C}^2$ but piecewise $\mathcal{C}^1$. The transformation T also only needs to be piecewise $\mathcal{C}^{1+\mathrm{Lip}}$, not piecewise $\mathcal{C}^2$. Then, the condition on the expansion rate and the combinatorial complexity is the same. At last, we shall prove that the essential spectral radius of $\mathcal{L}_{1/|\det DT|}$ acting on $\mathbb{BV}$ is strictly smaller than 1. Moreover, once the spectral gap is proven, we will get not only the existence of an absolutely continuous invariant measure with bounded variation, but far stronger results, as in Theorem 1.7.

We recall once more that, for generic piecewise expanding maps, the complexity D_n^b increases subexponentially (Theorem 1.1 in [Cow02]), so that the consequences of Theorem 1.7 hold generically.

6.2 Basic lemmas

We need to adapt the lemmas from Sections 3 and 4 to this new setting in order to get Theorem 1.4 and subsequently Theorem 1.7. They are:

- A Fubini-type property, which allows us to work in a 1-dimensional setting: Lemma 6.3.
- The continuity of the multiplication by Lipschitz functions (equivalent of Lemma 3.1): Lemma 6.4.
- The continuity of the multiplication by characteristic functions of nice sets (equivalent of Lemma 3.2): Lemma 6.5.
- The effect of composition with $\mathcal{C}^{1+\mathrm{Lip}}$ diffeomorphisms (equivalent of Lemma 3.3): Lemma 6.6.
- A localization lemma (equivalent of Lemma 3.4): Lemma 6.7.

- A summation lemma (equivalent to Lemma 3.5).
- The compactness of the injection from $\mathbb{BV}(X_0)$ to $\mathbb{L}^1(X_0)$.

In may arguments below, we will work in fact with $\mathcal{C}^1$ functions, and then proceed by approximations to get the results for functions with bounded variation, the approximation being done with Theorems 2.10 and 2.10. We hope that the quite explicit computations involved will help to explain the properties described by the many lemmas in Section 3 and Section 4.

We start with the Fubini-like theorem, which allows us to reduce the problem to dimension 1 in the proofs of some following lemmas. Let $u \in \mathbb{BV}(\mathbb{R}^d)$, $i \in 1..d$, $x \in \mathbb{R}^{d-1}$ and $t \in \mathbb{R}$.

We put $x = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_d)$. We will denote by $u_{x,i}$ the map $t \mapsto u(x_1, ..., x_{i-1}, t, x_{i+1}, ..., x_d)$, and by u_i the map $x \mapsto \|u_{x,i}\|_{\mathbb{BV}(\mathbb{R})}$.

Lemma 6.3 (Fubini-type property)**.**
For all $u \in \mathcal{C}^1 \cap \mathbb{BV}(\mathbb{R}^d)$ and $i \in 1..d$, then $u_i \in \mathbb{L}^1$ and

$$\|u\|_{\mathbb{BV}(\mathbb{R}^d)} = d^{-1} \sum_{i=1}^{d} \|u_i\|_{\mathbb{L}^1}$$

Proof.

Let $u \in \mathcal{C}^1$. Fubini-Tonelli theorem gives, for all $i \in 1..d$, $\|u\|_{\mathbb{L}^1} = \left\| \|u_{.,i}\|_{\mathbb{L}^1} \right\|_{\mathbb{L}^1}$, hence

$$\|u\|_{\mathbb{L}^1} = d^{-1} \sum_{i=1}^{d} \left\| \|u_{.,i}\|_{\mathbb{L}^1} \right\|_{\mathbb{L}^1}$$

On the other hand,

$$V(u) = d^{-1} \int_{\mathbb{R}^d} \sum_{i=1}^{d} \left| \frac{\partial u}{\partial x_i} \right| \mathrm{dLeb} = d^{-1} \sum_{i=1}^{d} \int_{\mathbb{R}^{d-1}} V(u_{.,i}) \, \mathrm{dLeb} = d^{-1} \sum_{i=1}^{d} \|V(u_{.,i})\|_{\mathbb{L}^1} .$$

Then, all is left is to sum the equalities. □

For any Lipschitz function $u \in \mathrm{Lip}(\mathbb{R}^d, \mathbb{C})$, we put $\|u\|_{\mathrm{Lip}} = \|u\|_{\mathbb{L}^\infty} + L(u)$, where $L(u)$ is the best Lipschitz constant for u. With this lemma, we can easily deduce the following:

Lemma 6.4 (Multiplication by Lipschitz functions)**.**
For all $u \in \mathbb{BV}(\mathbb{R}^d)$, for all $g \in \mathrm{Lip}(\mathbb{R}^d, \mathbb{C})$,

$$\|gu\|_{\mathbb{BV}(\mathbb{R}^d)} \leq \|g\|_{\mathrm{Lip}} \|u\|_{\mathbb{BV}(\mathbb{R}^d)}$$

Proof.

Assume first that $u \in \mathcal{C}^1$. Thanks to the Fubini-type property of Lemma 6.3, we just have to show that $\|gu\|_{\mathbb{BV}(\mathbb{R})} \leq \|g\|_{\mathrm{Lip}} \|u\|_{\mathbb{BV}(\mathbb{R})}$. Since g is continuous, this will be an application of the Leibniz formula (see e.g. Proposition 3.2 in [Bal00]; the derivative of g has to be taken in a weak sense):

$$\begin{aligned}\|gu\|_{\mathbb{BV}(\mathbb{R})} &= \int_{\mathbb{R}} |gu| \,\mathrm{dLeb} + \int_{\mathbb{R}} |D(gu)| \,\mathrm{dLeb} \\ &\leq \int_{\mathbb{R}} |gu| \,\mathrm{dLeb} + \int_{\mathbb{R}} |gDu| \,\mathrm{dLeb} + \int_{\mathbb{R}} |uDg| \\ &\leq \|g\|_{\mathbb{L}^\infty} \int_{\mathbb{R}} |u| \,\mathrm{dLeb} + \|g\|_{\mathbb{L}^\infty} \int_{\mathbb{R}} |Du| \,\mathrm{dLeb} + \|Dg\|_{\mathbb{L}^\infty} \int_{\mathbb{R}} |u| \,\mathrm{dLeb} \\ &\leq \|g\|_{\mathrm{Lip}} \|u\|_{\mathbb{BV}(\mathbb{R})}\end{aligned}$$

For any $u \in \mathbb{BV}(\mathbb{R})$, let be a sequence of $\mathcal{C}^1$ functions $(u_n)_{n\in\mathbb{N}}$ approaching u in the sense of Theorem 2.10.

Then, since

$$\|gu_n - gu\|_{\mathbb{L}^1} \leq \|g\|_{\mathbb{L}^\infty} \|u_n - u\|_{\mathbb{L}^1},$$

we see that (gu_n) is a sequence of $\mathbb{BV}(\mathbb{R})$ functions which converges to gu in $\mathbb{L}^1$. Theorem 2.11 tells us that:

$$\|gu\|_{\mathbb{BV}(\mathbb{R})} \leq \liminf_{n\to+\infty} \|gu_n\|_{\mathbb{BV}(\mathbb{R})} \leq \|g\|_{\mathrm{Lip}} \liminf_{n\to+\infty} \|u_n\|_{\mathbb{BV}(\mathbb{R})} \leq \|g\|_{\mathrm{Lip}} \|u\|_{\mathbb{BV}(\mathbb{R})}$$

□

Lemma 6.5 (Multiplication by characteristic functions of nice sets)**.**

Let O be a measurable subset of $\mathbb{R}^d$ whose intersection with almost every line parallel to a coordinate axis has at most L connected components. Then, for all $u \in \mathbb{BV}$,

$$\|1_O u\|_{\mathbb{BV}} \leq L \|u\|_{\mathbb{BV}}.$$

Proof.

Assume that $u \in \mathcal{C}^1$. Using again the Fubini-type property of Lemma 6.3, we just have to show that $\|1_O u\|_{\mathbb{BV}(\mathbb{R})} \leq L \|u\|_{\mathbb{BV}(\mathbb{R})}$ if O is an union of intervals, or even that $\|1_O u\|_{\mathbb{BV}(\mathbb{R})} \leq \|u\|_{\mathbb{BV}(\mathbb{R})}$ if O is an interval, and then sum the contributions of the different intervals. Let us put $O = [a, b]$ (the case $O = \mathbb{R}_+$ is even easier). Then:

$$\|1_O u\|_{\mathbb{L}^1} \leq \|u\|_{\mathbb{L}^1},$$

$$\begin{aligned}V(1_O u) &= |u(a)| + \int_{[a,b]} |u'(x)| \,\mathrm{d}x + |u(b)| \\ &= \left|\int_{(-\infty,a]} u'(x) \,\mathrm{d}x\right| + \int_{[a,b]} |u'(x)| \,\mathrm{d}x + \left|\int_{[b,+\infty)} u'(x) \,\mathrm{d}x\right| \leq V(u).\end{aligned}$$

Summing those inequalities ends this proof for $\mathcal{C}^1$ functions. For a general function in $\mathbb{BV}(\mathbb{R})$, and using the same trick as in the end of the proof of Lemma 6.4, we just have to show that, if (u_n) converges to u in $\mathbb{L}^1$, then $(1_O u_n)$ converges to $1_O u$ in $\mathbb{L}^1$, which is trivial. □

The next lemma deals with the composition with $\mathcal{C}^{1+\mathrm{Lip}}$ diffeomorphisms.

Lemma 6.6 (Composition with $\mathcal{C}^{1+\mathrm{Lip}}$ diffeomorphisms)**.**

Let $F \in \mathcal{C}^{1+\mathrm{Lip}}(\mathbb{R}^d, \mathbb{R}^d)$ be a diffeomorphism and let $A \in GL_d(\mathbb{R})$ such that, for all $z \in \mathbb{R}^d$, $\|A^{-1} \circ DF(z)\| \leq 2$ and $\|DF(z)^{-1} \circ A\| \leq 2$. Then, for all $u \in \mathcal{C}^1(\mathbb{R}^d)$,

$$\|u \circ F\|_{\mathbb{BV}(\mathbb{R}^d)} \leq 2^{d+1} |\det A|^{-1} \|A\| \, \|u\|_{\mathbb{BV}(\mathbb{R}^d)} + 2^{d+1} |\det A|^{-1} \, \|u\|_{\mathbb{L}^1} .$$

Proof.

Assume first that u is $\mathcal{C}^1$. We put, as in the proof of Lemma 3.3, $u \circ F = u \circ A \circ A^{-1} \circ F$. Let us deal with $u \circ A$ first.

We have $\|u \circ A\|_{\mathbb{L}^1} \leq |\det A|^{-1} \, \|u\|_{\mathbb{L}^1}$ and $V(u \circ A) \leq |\det A|^{-1} \|A\| V(u)$, so that

$$\|u \circ A\|_{\mathbb{BV}(\mathbb{R}^d)} \leq |\det A|^{-1} \|A\| \, \|u\|_{\mathbb{BV}(\mathbb{R}^d)} + |\det A|^{-1} \, \|u\|_{\mathbb{L}^1} . \tag{6.1}$$

We now deal with $u \circ A^{-1} \circ F$; we write $\tilde{F} = A^{-1} \circ F$. Since the derivative of $\tilde{F}$ is everywhere bounded between $1/2$ and 2, $|\det D(\tilde{F}^{-1})| \leq 2^d$ and a change of variables gives $\left\| u \circ \tilde{F} \right\|_{\mathbb{L}^1} \leq 2^d \, \|u\|_{\mathbb{L}^1}$ and $V(u \circ \tilde{F}) \leq 2^{d+1} V(u)$, so that

$$\left\| u \circ \tilde{F} \right\|_{\mathbb{BV}(\mathbb{R}^d)} \leq 2^{d+1} \, \|u\|_{\mathbb{BV}(\mathbb{R}^d)} \tag{6.2}$$

Together, (6.1) and (6.2) give

$$\|u \circ F\|_{\mathbb{BV}(\mathbb{R}^d)} \leq 2^{d+1} |\det A|^{-1} \|A\| \, \|u\|_{\mathbb{BV}(\mathbb{R}^d)} + 2^{d+1} |\det A|^{-1} \, \|u\|_{\mathbb{L}^1} .$$

Now, let us take any function u in $\mathbb{BV}(\mathbb{R}^d)$, and use the same trick as in the end of the proof of Lemma 6.4. Since $\left\| u \circ \tilde{F} \right\|_{\mathbb{L}^1} \leq 2^d \, \|u\|_{\mathbb{L}^1}$ holds for any $\mathbb{L}^1$ function, if (u_n) converges to u in $\mathbb{L}^1$, then $(u_n \circ \tilde{F})$ converges to $u \circ \tilde{F}$ in $\mathbb{L}^1$. Hence, inequality (6.2) holds for any function in $\mathbb{BV}(\mathbb{R}^d)$.

Inequality (6.1) can be obtained with a slighly different method: $\|u \circ A\|_{\mathbb{L}^1} \leq |\det A|^{-1} \, \|u\|_{\mathbb{L}^1}$ holds obviously for any u in $\mathbb{L}^1$. We just have to prove that $V(u \circ A) \leq |\det A|^{-1} \|A\| V(u)$ for any u in $\mathbb{BV}(\mathbb{R}^d)$; the approximation by $\mathcal{C}^1$ function still works (the reader can easily check that Theorem 2.11 is still true if one writes $V(\cdot)$ instead of $\|\cdot\|_{\mathbb{BV}}$).

Therefore, Lemma 6.6 holds for any function in $\mathbb{BV}(\mathbb{R}^d)$. □

A consequence of (6.1) is that, for all $u \in \mathbb{BV}(\mathbb{R}^d)$ and $A \in GL_d(\mathbb{R})$, $\|u \circ A\|_{\mathbb{BV}} \leq 2 \max(\|A\|, 1) |\det A|^{-1} \|u\|_{\mathbb{BV}}$.

The next result to prove is the equivalent of the first localization principle, Lemma 3.4.

Lemma 6.7 (Localization principle)**.**

Let be $\eta \in \mathcal{C}^1_c(\mathbb{R}^d, \mathbb{R})$, and write, for all $x \in \mathbb{R}^d$ and $m \in \mathbb{Z}^d$, $\eta_m(x) = \eta(x + m)$. Then, there exists a constant C_η such that, for all $u \in \mathbb{BV}(\mathbb{R}^d)$,

$$\sum_{m \in \mathbb{Z}^d} \|\eta_m u\|_{\mathbb{BV}} \leq C_\eta \, \|u\|_{\mathbb{BV}} .$$

Proof.

We define on $\mathbb{R}^d$ a relation by $x < y$ if and only if $x_i < y_i$ for all $i \in \{1, ..., d\}$, and another relation $\leq$ the same way. For $k \in \mathbb{Z}^d$, $0 < k$, we define $O_k = \{x \in \mathbb{R}^d : 0 < x < k\}$. Up to a translation, we may assume that there exists some

$k = (k_1, ..., k_d) \in \mathbb{Z}^d$ such that $\mathrm{Supp}(\eta) \subset O_k$. We also note, for $\lambda \in \mathbb{Z}^d$, by λk the vector $(\lambda_1 k_1, .., \lambda_d k_d)$.

We point out that, for any u in $\mathbb{BV}(\mathbb{R}^d)$ and λ in $\mathbb{R}^d$, the sum $\sum_{\lambda \in \mathbb{Z}^d} \|u\|_{\mathbb{BV}(\lambda k + l + O_k)}$ is equal to $\|u\|_{\mathbb{BV}(\bigcup_{\lambda \in \mathbb{Z}^d} \lambda k + l + O_k)}$, which is at most $\|u\|_{\mathbb{BV}(\mathbb{R}^d)}$ (the $\mathbb{L}^1$ norms are the same, and the inequality of the variations comes directly from Definition 2.7)

We now split the sum, and then apply Lemma 6.4:

$$\begin{aligned}
\sum_{m \in \mathbb{Z}^d} \|\eta_m u\|_{\mathbb{BV}(\mathbb{R}^d)} &= \sum_{\lambda \in \mathbb{Z}^d} \sum_{\substack{l \in \mathbb{Z}^d \\ 0 \leq l < k}} \|\eta_{\lambda k + l} u\|_{\mathbb{BV}(\lambda k + l + O_k)} \\
&\leq \|\eta\|_{\mathcal{C}^1} \sum_{\substack{l \in \mathbb{Z}^d \\ 0 \leq l < k}} \sum_{\lambda \in \mathbb{Z}^d} \|u\|_{\mathbb{BV}(\lambda k + l + O_k)} \\
&\leq \|\eta\|_{\mathcal{C}^1} \sum_{\substack{l \in \mathbb{Z}^d \\ 0 \leq l < k}} \|u\|_{\mathbb{BV}(\mathbb{R}^d)} \\
&= \|\eta\|_{\mathcal{C}^1} \left(\prod_{i=1}^{d} k_i \right) \|u\|_{\mathbb{BV}(\mathbb{R}^d)} .
\end{aligned}$$

This is the lemma, with $C_\eta = \|\eta\|_{\mathcal{C}^1} \prod_{i=1}^{d} k_i$. □

Since the space $\mathbb{BV}$ make us work morally with $p = 1$ and $t = 1$, an equivalent of Lemma 3.5 for functions with bounded variation is that, for any functions $v_1, \ldots, v_l$ with compact support in $\mathbb{R}^d$ and belonging to $\mathbb{BV}(\mathbb{R}^d)$,

$$\left\| \sum_{i=1}^{l} v_i \right\|_{\mathbb{BV}(\mathbb{R}^d)} \leq \sum_{i=1}^{l} \|v_i\|_{\mathbb{BV}(\mathbb{R}^d)} .$$

This is just the triangular inequality for the $\mathbb{BV}(\mathbb{R}^d)$ norm, so that there is nothing to prove here.

Finally, it is a well-known fact that, for any open and bounded set with nice boundaries (for instance piecewise $\mathcal{C}^1$ boundaries) Ω of some $\mathbb{R}^d$, the canonical immersion of $\mathbb{BV}(\Omega)$ into $\mathbb{L}^1(\Omega)$ is compact; see for instance Theorem 1.19 in [Giu84]. The compactness of the canonical immersion of $\mathbb{BV}(X_0)$ into $\mathbb{L}^1(X_0)$ ensues.

6.3 Saussol's theorem

In sections 6.1 and 6.2, we have seen how to obtain Theorem 1.7, that is, a version of Theorem 1.6 when T is a piecewise $\mathcal{C}^{1+\mathrm{Lip}}$ uniformly expanding map, in the limit $p = 1$ and $t = 1$. This involves the space of functions with bounded variation, since the Sobolev space $\mathcal{H}^1_1$ is not suitable for this task (for instance, it is not large enough in dimension 1). One might want to know suitable spaces when one works instead with a piecewise $\mathcal{C}^{1+\alpha}$ uniformly expanding map, and wants to get to the limit $t = \alpha$ (in this setting, Theorem 1.3 tells nothing about the essential spectral radius of the Perron-Frobenius operator on the $\mathcal{H}^\alpha_p$ spaces).

The last part of this article is a study of a previous result by B. Saussol [Sau00], which is very close to our own results and gives us another example of function spaces to which our method could be applied, with the parameters $t = \alpha$ and $p = 1$. B. Saussol proved the existence of a spectral gap of $\mathcal{L}_{1/|\det DT|}$ when acting on spaces of functions with bounded oscillation $\mathbb{V}_\alpha$, when T is a piecewise $\mathcal{C}^{1+\alpha}$ uniformly expanding map.

First, let us present the functions with bounded oscillation. Let X_0 be a compact set of some $\mathbb{R}^d$. For any Borel set $S \in \mathcal{B}(\mathbb{R}^d)$ such that $\mathrm{Leb}(S) > 0$, and any $f \in \mathbb{L}^1(\mathbb{R}^d)$, we denote the essential infimum of f on S by $\mathrm{Einf}_S f$, and its essential supremum by $\mathrm{Esup}_S f$. Next, we choose some $\epsilon_0 > 0$ and $\alpha \in (0, 1]$, and define for all $f \in \mathbb{L}^1(\mathbb{R}^d)$:

$$\mathrm{osc}(f, S) = \mathrm{Esup}_S f - \mathrm{Einf}_S f \tag{6.3}$$

$$|f|_\alpha = \sup_{0<\epsilon\leq\epsilon_0} \epsilon^{-\alpha} \int_{\mathbb{R}^d} \mathrm{osc}(f, B(x, \epsilon))\, \mathrm{d}x \tag{6.4}$$

Then, we define $\mathbb{V}_\alpha(\mathbb{R}^d) = \{f \in \mathbb{L}^1(\mathbb{R}^d) : |f|_\alpha < +\infty\}$ and $\mathbb{V}_\alpha = \mathbb{V}_\alpha(X_0) = \{f \in \mathbb{V}_\alpha(\mathbb{R}^d)) : \mathrm{Supp}\, f \subset X_0\}$, both endowed with the norm $\|f\|_{\mathbb{V}_\alpha} = \|f\|_{\mathbb{L}^1} + |f|_\alpha$.

Remark 6.8.

Different choices of ϵ_0 lead to different norms, although they are all equivalent: the space $\mathbb{V}_\alpha$ does not depend on ϵ_0.

We have already encountered $\mathbb{V}_1(\mathbb{R})$: it is the same space as $\mathbb{BV}(\mathbb{R})$ [Kel85]. Hence, the results of Saussol are a generalization of the previous theorem by A. Lasota and J.A. Yorke [LY73].

Remark 6.9.

We endow $\mathcal{C}_\alpha$, the set of α-Hölder functions on X_0, with the norm $\|f\|_\alpha = \|f\|_{\mathbb{L}^1} + |f|'_\alpha$, where:

$$|f|'_\alpha = \sup_{x,y\in X_0} \frac{|f(x) - f(y)|}{|x - y|^\alpha}$$

For $f \in \mathcal{C}_\alpha$, $x \in X_0$ and $\epsilon > 0$, we get easily the following inequalities:

$$\begin{aligned} |f|_\alpha &= \sup_{0<\epsilon\leq\epsilon_0} \epsilon^{-\alpha} \int_{\mathbb{R}^d} \mathrm{osc}(f, B(x, \epsilon))\, \mathrm{d}x \\ &\leq \sup_{0<\epsilon\leq\epsilon_0} \epsilon^{-\alpha} (2\epsilon)^\alpha \,\mathrm{Leb}(X_0) |f|'_\alpha \\ &\leq 2^\alpha \,\mathrm{Leb}(X_0) |f|'_\alpha \end{aligned}$$

Hence, for all $\alpha \in (0, 1]$, we have a continuous inclusion from $\mathcal{C}_\alpha$ into $\mathbb{V}_\alpha$. However, $\mathbb{V}_\alpha$ is much larger, so that functions belonging to this space may present discontinuities.

The setting studied by B. Saussol is more general than the one we presented in Section 1: for instance, it allows under some conditions boundaries whose Hausdorff dimension is strictly larger than $d-1$, or maps whose sets of continuity are countably many. However, one of these conditions is very abstract, and the use of more flexible conditions leads naturally to our setting.

In the following, we consider that $\alpha \in (0,1]$ is fixed, and we put $\gamma_d = \mathrm{Leb}(B(0,1))$. Here is the main theorem, an adaptation of Theorem 5.1 and Lemma 2.1 in [Sau00] with the additional use of Hennion's theorem [Hen93]):

Theorem 6.10 (Saussol's theorem)**.**

Let T be a piecewise $\mathcal{C}^{1+\alpha}$ uniformly expanding map. $\mathcal{L}_{1/|\det DT|}$ acts continuously on $\mathbb{V}_\alpha$, and its essential spectral radius is at most:

$$\lambda^{-\alpha} + \frac{4\gamma_d D_1^b}{(\lambda-1)\gamma_{d-1}}. \tag{6.5}$$

This theorem naturally leads to a result of existence of physical measures (stated in a different way in [Sau00]) similar to Theorem 1.6:

Theorem 6.11.

Let t be a piecewise $\mathcal{C}^{1+\alpha}$ uniformly expanding map such that the bound 6.5 for the essential spectral radius is smaller than 1. Then T has a finite number of physical measures whose densities are in $\mathbb{V}_\alpha$, which are ergodic, and whose basins cover Lebesgue almost all X_0. Moreover, if μ is one of these measures, there exist an integer k and a decomposition $\mu = \mu_1 + \cdots + \mu_k$ such that T sends μ_j to μ_{j+1} for $j \in \mathbb{Z}/k\mathbb{Z}$, and the probability measures $k\mu_j$ are mixing at an exponential rate for T^k and α-Hölder test functions.

Since the setting and conclusion are already familiar to the reader, we will look more closely at the upper bound of the essential spectral radius. The estimate 6.5 is rather rough, and for $\alpha = 1$ and small d is worse than the one given in Theorem 1.7. However, just like in the proof of Theorem 1.3, one may iterate the transformation to get better estimates of the essential spectral radius ρ_{ess} of the Perron-Frobenius operator, and then take the limit as the number of iterations grows to infinity:

$$\rho_{ess} \leq \sup\left\{ \lim_{n\to+\infty} \lambda_n^{-\frac{1}{n}} (D_n^b)^{\frac{1}{n}}; \lim_{n\to+\infty} \lambda_n^{-\frac{\alpha}{n}} \right\} \tag{6.6}$$

Proof.

The expression 6.5 appears in a Lasota-Yorke type inequality in the proof of Saussol's theorem. When using Hennions's theorem [Hen93], one can get a better bound on the essential spectral radius of an operator by iterating this operator. In other words, we have for all positive integer n:

$$\rho_{ess} \leq \left(\lambda_n^{-\alpha} + \frac{4\gamma_d D_n^b}{(\lambda_n - 1)\gamma_{d-1}} \right)^{\frac{1}{n}}.$$

Let us put $C = \sup\left\{ \lim_{n\to+\infty} \lambda_n^{-\frac{1}{n}} (D_n^b)^{\frac{1}{n}}; \lim_{n\to+\infty} \lambda_n^{-\frac{\alpha}{n}} \right\}$. Let $\epsilon > 0$. Obviously,

$$\lim_{n\to+\infty} \left(\lambda_n^{-\alpha} + \frac{4\gamma_d D_n^b}{(\lambda_n - 1)\gamma_{d-1}} \right) (C+\epsilon)^{-n} = 0.$$

Hence,

$$\limsup_{n\to+\infty} \left(\lambda_n^{-\alpha} + \frac{4\gamma_d D_n^b}{(\lambda_n - 1)\gamma_{d-1}} \right)^{\frac{1}{n}} (C+\epsilon)^{-1} \leq 1.$$

Since this is true for all $\epsilon > 0$, inequality 6.6 ensues. □

Clearly, a sufficient condition for the conclusions of Theorem 6.11 to hold is $\lim_{n\to+\infty} \lambda_n^{-\frac{1}{n}} (D_n^b)^{\frac{1}{n}} < 1$.

The $\mathbb{V}_\alpha$ spaces were constructed to satisfy the same properties that we needed (they include the characteristic functions of nice enough sets, the multiplication by α-Hölder functions behaves nicely, and the injection into $\mathbb{L}^1$ is compact), and one should be able to adapt the different lemmas as we did for the space of functions with bounded variation. Finally, our method would probably give a different (and worse) estimate of the essential spectral radius of the Perron-Frobenius operator, such as (we take $t = \alpha$ and $p = 1$ in Theorem 1.3):

$$\rho_{ess} \leq \lim_{n\to+\infty} (D_n^b)^{\frac{1}{n}} \cdot \lambda_n^{-\frac{\alpha}{n}}$$

B. Saussol also gives a lower bound on the spectral gap via the study of the cones in $\mathbb{V}_\alpha$ (for another example of cone contraction method, see e.g. [Bal00]), and an upper bound on the number of ergodic physical measures. Such features could perhaps be adapted to our current setting, with Sobolev spaces or the space of functions with bounded variation.

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