

**Injectivity domain of ellipsoid of revolution. The oblate case.**

The canonical metric on an oblate ellipsoid of revolution with semi-minor axis  $\mu \in (0, 1]$  is

$$dt^2 = X d\theta^2 + (1 - X/\lambda) d\varphi^2$$

in the universal covering of the surface minus its two poles, with

$$X = \sin^2 \varphi \text{ and } \lambda = \frac{1}{1 - \mu^2} \in (1, \infty].$$

The associated Hamiltonian is

$$H = \frac{1}{2} \left( \frac{p_\theta^2}{X} + \frac{p_\varphi^2}{1 - X/\lambda} \right).$$

Extremals are parameterized by arc length on  $H = 1/2$ , and remain well defined in the singular Riemannian case  $\lambda = 1$  (flat ellipsoid— $\mu = 0$ —embedded in  $\mathbf{R}^2$  as a two-sided Poincaré disk). The Hamiltonian flow allows to define the exponential mapping on  $\mathbf{R} \times H(x_0, \cdot)^{-1}(\{1/2\})$  by

$$\exp_{x_0}(t, p_0) = x(t, x_0, p_0).$$

As a subset of the cotangent bundle, its injectivity domain is

$$I(x_0) = \{tp_0 \mid p_0 \in H(x_0, \cdot)^{-1}(\{1/2\}), t \in [0, t_{\text{cut}}(p_0)]\}.$$

(Convexity properties are identical for the domain expressed on the tangent or the cotangent bundle since the corresponding change of coordinates is linear.) The domain is convex if and only if its boundary is a convex curve, that is a curve with constant sign curvature. Because of symmetries, convexity has only to be checked on a quarter of the curve. Geodesics are parameterized by  $p_\theta$  and the cut time in the oblate case is equal to the halved period of  $\varphi$ ,

$$t_{\text{cut}}(p_0) = t_{\text{cut}}(p_\theta) = \frac{T(p_\theta)}{2}$$

with  $p_\theta \in [0, \sqrt{X_0}]$  on  $H = 1/2$  ( $X_0 = \sin^2 \varphi_0$  and  $\theta_0$  is set to 0 thanks to the symmetry of revolution). The curvature condition is expressed as a sign condition on the quantity

$$T(T + p_\theta T') + (X_0 - p_\theta^2)(2T'^2 - TT''), \quad p_\theta \in [0, \sqrt{X_0}],$$

where  $' = d/dp_\theta$  and where  $T$  implicitly also depends on the parameter  $\lambda$ . The quadrature on  $\varphi$  is parameterized by the algebraic complex curve

$$\left[ \frac{\dot{X}(\lambda - X)}{\sqrt{\lambda}} \right]^2 = 4(X - p_\theta^2)(X - 1)(X - \lambda)$$

which is of genus one excluding the equator  $p_\theta = 1$  for  $X_0 = 1$  ( $\varphi_0 = \pi/2$ ). The curve also degenerates to a rational surface for  $\lambda = \infty$  ( $\mu = 1$ , round sphere) or  $\lambda = 1$  ( $\mu = 0$ , flat ellipsoid). In the latter case, as the induced metric on the two-sided disk is flat, the injectivity domain for  $\mu$  close to 0 and  $\varphi_0$  close to  $\pi/2$

is by continuity a deformation of the union at the origin of two disjoint disks (eight-shaped domain), hence not convex. Conversely, for  $\mu$  close to 1, convexity must hold for any initial condition. For any  $\mu$  between those, convexity must also hold for an initial point close enough to the poles as the injectivity domain of poles is, up to a dilation, a circle.

Setting

$$u = X - \frac{p_\theta^2 + 1 + \lambda}{3} \text{ and } v = \frac{\dot{X}(\lambda - X)}{\sqrt{\lambda}},$$

we get the Weierstrass parameterization by  $v^2 = 4u^3 - g_2u - g_3$  with invariants rational in the parameters,

$$g_2 = \frac{4}{3}[p_\theta^4 - (\lambda + 1)p_\theta^2 + (\lambda^2 - \lambda + 1)],$$

$$g_3 = \frac{4}{27}[2p_\theta^6 - 3(\lambda + 1)p_\theta^4 - 3(\lambda^2 - 4\lambda + 1)p_\theta^2 + (2\lambda^3 - 3\lambda^2 - 3\lambda + 2)].$$

As  $X \in [p_\theta^2, 1]$ ,  $u \in [e_2, e_3]$  with

$$e_1 = \frac{2\lambda - p_\theta^2 - 1}{3}, \quad e_2 = \frac{2p_\theta^2 - 1 - \lambda}{3}, \quad e_3 = \frac{2 - p_\theta^2 - \lambda}{3},$$

and the parameterization uses the bounded component of the real cubic, that is  $z \in \omega' + \mathbf{R}$  where  $\omega\mathbf{Z} + \omega'\mathbf{Z}$  is the real rectangular lattice of periods of  $u(z) = \wp(z)$ . The time law is

$$\frac{dt}{dz} = \frac{\lambda - X}{\sqrt{\lambda}}$$

so that the period of  $\varphi$  is

$$T = \frac{4}{\sqrt{\lambda}}(e_1\omega + \eta), \quad \eta = \zeta(\omega).$$

Define  $\tau = 3T\sqrt{\lambda}/4$ , and use the derivatives of periods and quasi-periods with respect to the invariants to express the first and second order derivatives with respect to  $p_\theta$  as linear combinations in  $\omega$  and  $\eta$  with coefficients in  $\mathbf{R}(\lambda, p_\theta)$ :

$$\begin{aligned} \tau &= -(p_\theta^2 - 2\lambda + 1)\omega + 3\eta, \\ \tau' &= \frac{p_\theta}{p_\theta^2 - 1}[-(p_\theta^2 + \lambda - 2)\omega + 3\eta], \\ \tau'' &= \frac{[(2\lambda - 1)p_\theta^4 - (\lambda^2 + 1)p_\theta^2 - \lambda(\lambda - 2)]\omega + 3[(\lambda - 2)p_\theta^2 + \lambda]\eta}{(p_\theta^2 - 1)^2(p_\theta^2 - \lambda)}. \end{aligned}$$

For any  $(\lambda, X_0)$ ,

$$\kappa(p_\theta, X_0, \lambda) = \tau(\tau + p_\theta\tau') + (X_0 - p_\theta^2)(2\tau'^2 - \tau\tau'')$$

is decreasing in  $p_\theta$  on  $[0, \sqrt{X_0}]$  [to be completed]. The worst case is so obtained for  $p_\theta = \sqrt{X_0}$ , and the sign of

$$\kappa_2(X_0, \lambda) = \tau + \tau'\sqrt{X_0}$$

has to be checked for  $X_0 \in [0, 1]$ . For any  $\lambda$ , the function is decreasing [*to be completed*] and degenerates as  $X_0 \rightarrow 1$  (equator). One gets

$$\kappa_2 = \frac{\omega}{1 - X_0} [(\lambda - 1 - 3\chi) + (\lambda - 2 + 6\chi)(1 - X_0) + 2(1 - X_0)^2], \quad \chi = \frac{\eta}{\omega}.$$

Using an asymptotic up to first order of  $\chi$ ,

$$\kappa_2 = \frac{3\pi}{2\sqrt{\lambda - 1}} \left( \lambda - \frac{3}{2} \right) + o(1), \quad X_0 \rightarrow 1,$$

hence the zero for  $\lambda = 3/2$ , that is for  $\mu = 1/\sqrt{3}$ .

In conclusion, domains of injectivity of the oblate ellipsoid are convex (for any initial condition) if and only if the semi-minor axis is not less than  $1/\sqrt{3}$ . Below this limit, there are always initial conditions such that convexity is lost.