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Total-Coloring of Plane Graphs with Maximum Degree Nine

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Abstract

The central problem of the total-colorings is the total-coloring conjecture, which asserts that every graph of maximum degree $\Delta$ admits a $(\Delta + 2)$-total-coloring. Similar to edge-colorings—with Vizing’s edge-coloring conjecture—this bound can be decreased by 1 for plane graphs of higher maximum degree. More precisely, it is known that if $\Delta \geq 10$, then every plane graph of maximum degree $\Delta$ is $(\Delta + 1)$-totally-colorable. On the other hand, such a statement does not hold if $\Delta \leq 3$. We prove that every plane graph of maximum degree 9 can be 10-totally-colored.

1 Introduction

Given a graph $G = (V, E)$ and a positive integer $k$, a $k$-total-coloring of $G$ is a mapping $\lambda : V \cup E \to \{1, 2, \ldots, k\}$ such that

(i) $\lambda(u) \neq \lambda(v)$ for every pair $u, v$ of adjacent vertices,

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(ii) $\lambda(v) \neq \lambda(e)$ for every vertex $v$ and every edge $e$ incident to $v$,

(iii) $\lambda(e) \neq \lambda(e')$ for every pair $e, e'$ of incident edges.

This notion was independently introduced by Behzad [3] in his doctoral thesis and Vizing [15]. It is now a prominent notion in graph coloring, to which a whole book is devoted [17]. Both Behzad and Vizing made the celebrated total-coloring conjecture, stating that every graph of maximum degree $\Delta$ admits a $(\Delta + 2)$-total-coloring. Notice that every such graph cannot be totally-colored with less than $\Delta + 1$ colors, and that a cycle of length 5 cannot be 3-totally-colored. The best general bound so far has been obtained by Molloy and Reed [10], who established that every graph of maximum degree $\Delta$ can be $(\Delta + 10^{26})$-totally-colored. Moreover, the conjecture has been shown to be true for several special cases, namely for $\Delta = 3$ by Rosenfeld [11] and Vijayaditya [14], and then for $\Delta \in \{4, 5\}$ by Kostochka [9].

Another natural subclass to consider is the one of planar graphs, which has attracted a considerable amount of attention and for which several results were obtained. First, Borodin [5] proved that if $\Delta \geq 9$, then every plane graph of maximum degree $\Delta$ fulfills the conjecture. This result can be extended to the case where $\Delta = 8$ by the use of the four color theorem [1, 2], combined with Vizing’s theorem about edge coloring—the reader can consult the book by Jensen and Toft [8] for more details. Elsewhere, Sanders and Zhao [12] solved the case $\Delta = 7$ of the total-coloring conjecture for plane graphs. So the only open case regarding plane graphs is $\Delta = 6$. Interestingly, $\Delta = 6$ is also the only remaining open case for Vizing’s edge-coloring conjecture, after Sanders and Zhao [13] resolved the case $\Delta = 7$.

However, plane graphs with high maximum degree allow a stronger assertion. More precisely, Borodin [5] showed that if $\Delta \geq 14$, then every plane graph with maximum degree $\Delta$ is $(\Delta + 1)$-totally-colorable, and asked whether 14 could be decreased. Borodin, Kostochka, and Woodall extended this result to the case where $\Delta \geq 12$ [6], and later to $\Delta = 11$ [7]. Recently, Wang [16] established the result for $\Delta = 10$. On the other hand, this bound is not true if $\Delta \leq 3$. The complete graphs $K_2, K_4$ and the cycles of length $3k + 2$ with $k \geq 1$ are examples of plane graphs that cannot be $(\Delta + 1)$-totally-colored. We continue along those lines and establish the following theorem.

**Theorem 1.** Every plane graph of maximum degree 9 is 10-totally-colorable.

So, the values of $\Delta$, for which it is not known whether all plane graphs of maximum degree $\Delta$ are $(\Delta + 1)$-totally-colorable, are now 4, 5, 6, 7, and 8. Recall that the case where $\Delta = 6$ is even open for the total-coloring
conjecture. We also note that if $\Delta \geq 3$, then every outerplane graph with maximum degree $\Delta$ can be $(\Delta + 1)$-totally-colored [19]. Another result of the same type is that every Halin graph of maximum degree 4 admits a 5-total-coloring [18]. Note also that the complete $r$-partite balanced graph $K_{r\times n}$, whose maximum degree $\Delta$ is $n(r - 1)$, admits a $(\Delta + 2)$-total-coloring, and the cases where this bound can be decreased by 1 have been characterized [4].

We prove Theorem 1 by contradiction. From now on, we let $G = (V, E)$ be a minimum counterexample to the statement of Theorem 1, in the sense that the quantity $|V| + |E|$ is minimum. In particular, every proper subgraph of $G$ is 10-totally-colorable. First, we establish various structural properties of $G$ in section 2. Then, relying on these properties, we use the discharging method in section 3 to obtain a contradiction.

In what follows, a vertex of degree $d$ is called a $d$-vertex. A vertex is a $(\leq d)$-vertex if its degree is at most $d$; it is a $(\geq d)$-vertex if its degree is at least $d$. If $f$ is a face of $G$, then the degree of $f$ is its length, i.e., the number of its incident vertices. The notions of $d$-face, $(\leq d)$-face, and $(\geq d)$-face are defined analogously as for the vertices. Moreover, if a vertex $v$ is adjacent to a $d$-vertex $u$, then we say that $u$ is a $d$-neighbor of $v$. A cycle of length 3 is called a triangle. For integers $a, b, c$, an $(\leq a, \leq b, \leq c)$-triangle is a triangle $xyz$ of $G$ with $\deg(x) \leq a, \deg(y) \leq b$, and $\deg(z) \leq c$. The notions of $(a, \leq b, \leq c)$-triangles, $(a, b, \geq c)$-triangles, and so on are defined analogously.

2 Reducible configurations

In this section, we establish some structural properties of the graph $G$. We prove that some planar graphs are reducible configurations; i.e., they cannot be subgraphs of $G$.

For convenience, we sometimes define configurations by depicting them in figures. In all of the figures of this paper, 2-vertices are represented by small black bullets, 3-vertices by black triangles, 4-vertices by black squares, and white bullets represent vertices whose degree is at least the one shown on the figure.

Let $\lambda$ be a (partial) 10-total-coloring of $G$. For each element $x \in V \cup E$, we define $C(x)$ to be the set of colors (with respect to $\lambda$) of vertices and edges incident or adjacent to $x$. Also, we set $F(x) := \{1, 2, \ldots, 10\} \setminus C(x)$.

If $x \in V$, then we define $E(x)$ to be the set of colors of the edges incident to $x$. Moreover, $\lambda$ is nice if only some $(\leq 4)$-vertices are not colored. Observe that every nice coloring can be greedily extended to a 10-total-coloring of $G$, since $|C(v)| \leq 8$ for each $(\leq 4)$-vertex $v$; i.e., $v$ has at most 8 forbidden colors. Therefore, in the rest of this paper, we shall always suppose that such vertices
are colored at the very end. More precisely, every time we consider a partial coloring of $G$, we uncolor all $(\leq 4)$-vertices, and implicitly color them at the very end of the coloring procedure of $G$. We make the following observation about nice colorings and use it implicitly throughout this paper.

**Observation.** Let $uv$ be an edge with $\deg(v) \leq 4$. There exists a nice coloring $\lambda$ of $G - e$, in which $u$ is colored and $v$ is uncolored. Moreover, it then suffices to properly color the edge $e$ with a color from $\{1, 2, \ldots, 10\}$ to extend $\lambda$ to a nice coloring of $G$.

We now study the structural properties of $G$ in a series of lemmas.

**Lemma 2.** The graph $G$ has the following properties:

(i) the minimum degree is at least 2;

(ii) if $vu$ is an edge with $\deg(v) \leq 4$, then $\deg(u) \geq 11 - \deg(v)$;

(iii) a 9-vertex is adjacent to at most one 2-vertex;

(iv) a triangle incident to a 3-vertex must also contain a 9-vertex;

(v) there is no $(4, \leq 7, \leq 8)$-triangle;

(vi) a triangle contains at most one $(\leq 5)$-vertex.

**Proof.** (i) Suppose that $v$ is a 1-vertex, and let $u$ be its neighbor. By the minimality of $G$, the graph $G - v$ admits a nice coloring in which $u$ is colored. Since the degree of $u$ in $G - v$ is at most 8, we obtain $|C(vu)|\leq 9$. Thus, the edge $vu$ can be properly colored, which yields a nice coloring of $G$.

(ii) Suppose that $vu \in E$ with $\deg(v) \leq 4$ and $\deg(u) \leq 10 - \deg(v)$. There exists a nice coloring of $G' := G - vu$, in which $u$ is colored and $v$ is uncolored. Therefore, $|C(vu)| \leq \deg(v) - 1 + \deg(u) - 1 + 1 \leq 9$. Hence we can color properly the edge $vu$, thereby obtaining a nice coloring of $G$.

(iii) Suppose that $v$ is a 9-vertex adjacent to two 2-vertices $x$ and $y$. Let $x'$ be the neighbor of $x$ different from $v$, and let $y'$ be the neighbor of $y$ different from $v$. Notice that we may have $x' = y'$. By the previous assertion, $x'$ and $y'$ are 9-vertices. It is enough to consider the following two possibilities.

$v$ is adjacent to neither $x'$ nor $y'$. Then, we construct the graph $G'$ by first removing $x$ and $y$, and then adding the edge $vx'$. If $y' \neq x'$, then we additionally add the edge $vy'$. Note that $G'$ is a simple plane graph of maximum degree 9 with fewer vertices and edges than $G$. Therefore, it admits a nice coloring $\lambda$ by the minimality of $G$. We easily modify $\lambda$ to obtain a nice coloring of $G$. First, put $\lambda(xx') := \lambda(vy) := \lambda'(vx')$. 

Now, if \( x' \neq y' \), then we put \( \lambda(vx) := \lambda(y'v) := \lambda(v'y) \). See Figure 1(a) for an illustration. And, if \( x' = y' \), then we note that each of the edges \( yy' \) and \( vx \) has at most 9 forbidden colors. Thus, both of them can be colored and the obtained 10-total-coloring of \( G \) is nice.

\( v \) is adjacent to \( x' \). Thus \( vxx' \) is a triangle. Consider a nice coloring of \( G - vy \). To extend it to \( G \), it suffices to properly color the edge \( vy \). If this cannot be done greedily, then \( |C(vy)| = 10 \), and up to a permutation of the colors, we can assume that the coloring is the one shown in Figure 1(b). If \( a \neq 10 \), then recolor \( vx \) with 10 and color \( vy \) with 5 to obtain a nice coloring of \( G \). And if \( a = 10 \), then we interchange the colors of \( vx' \) and \( xx' \), and afterwards color \( vy \) with 4.

(iv) By (ii), a 3-vertex has only \((\geq 8)\)-neighbors. Thus we may suppose that \( vuw \) is a \((3,8,8)\)-triangle, with \( u \) being the 3-vertex. Consider a nice coloring of \( G - vu \). To extend it to \( G \), again it suffices to properly color the
edge $vu$. If we cannot do this greedily, then it means that $|C(vu)| = 10$. Thus, up to a permutation of the colors, the coloring is the one shown in Figure 1(c). If the edge $wu$ can be properly recolored, then we do so, and afterwards color the edge $vu$ with 10, which gives a nice coloring of $G$. So we deduce that $|C(wu)| = 9$. Consequently, $\{a, b, c, d, e, f, g\} = \{1, 2, 3, 4, 5, 6, 8\}$. Thus we obtain $9 \notin C(vw)$. So, we can recolor $vw$ with 9 and color $vu$ with 7 to conclude the proof.

(v) By (ii), it is enough to prove that there is no $(4, 7, \delta)$-triangle in $G$ for $\delta \in \{7, 8\}$. Suppose that $vwu$ is such a triangle with $w$ having degree $\delta$ and $u$ degree 4. Consider a nice coloring of $G - vu$. It is sufficient to properly color the edge $vu$ to obtain a nice coloring of $G$. Again, $|C(vu)| = 10$, so up to a permutation of the colors, we assume that the coloring is the one of Figure 1(d). If the edge $wu$ can be properly recolored, then do so, and color $vu$ with 8 to obtain a nice coloring of $G$. Thus, we deduce that $|C(wu)| = 9$. Therefore, $\{1, 2, 3, 4, 5, 7\} \subset \{a, b, c, d, e, f, g\}$. From this we infer that $|C(vw)| \leq 6 + \delta - 6 = \delta \leq 8$. Thus, the edge $vw$ can be properly recolored, and so the edge $vu$ can be colored with 6, yielding a nice coloring of $G$.

(vi) Let $vwu$ be a triangle with $\deg(u) = \deg(w) = 5$. Consider a total-coloring of $G - u$, and uncolor the vertex $w$. Observe that $|F(wv)| \geq 1$ and $|F(w)| \geq 1$. Furthermore, these two sets must actually be equal and of size 1; otherwise we can extend the coloring to $G$. Up to a permutation of the colors, the coloring is the one shown in Figure 1(e), with $\{A, B, C, D\} = \{1, 2, 3, 4\}$. Notice that the colors of the edges $vu$ and $vw$ can be safely interchanged. Now, the vertex $w$ can be properly colored with 6, and the edge $uw$ with 10.

Lemma 3. For the graph $G$, the following assertions hold.

(i) There is no $(5, 6, 6)$-triangle.

(ii) A 6-vertex has at most two 5-neighbors.

(iii) Suppose that $v$ is a 7-vertex, and let $x_1$ be one of its neighbors. If $v$ and $x_1$ have at least two common neighbors, then at most one of them has degree 4.

(iv) Suppose that $vwu$ and $vwu'$ are two triangles with $\deg(u) = 2$. Then, $\deg(u') \geq 4$.

(v) Suppose that $v$ is a 9-vertex incident to a $(2, 9, 9)$-triangle. Then it is not incident to a $(\leq 3, \geq 8, 9)$-triangle.
(vi) The configuration of Figure 2(a) is reducible.

(vii) The configuration of Figure 2(b) is reducible.

Proof. (i) Suppose on the contrary that \(G\) contains a \((5, 6, 6)\)-triangle \(uvw\) with \(u\) being of degree 5. The proof is in two steps. In the first step, we prove the existence of a 10-total-coloring of \(G\) in which only \(u\) is uncolored. And in the second step, we establish that such a coloring can be extended to \(G\). Consider a nice coloring of \(G - vu\), and uncolor the vertex \(u\). Our only goal in the first step is to properly color the edge \(vu\). If we cannot do this greedily, then \(|C(vu)| = 10\), and thus we can assume that the coloring is the one of Figure 3(a). We infer that \(\{6, 7, 8, 9, 10\} = \{a, b, c, d, e\}\); otherwise we can choose a color \(\alpha \in \{6, 7, 8, 9, 10\} \setminus \{a, b, c, d, e\}\), recolor \(uw\) with \(\alpha\), and color \(vu\) with 4. Consequently, we have \(C(vw) = \{4, 6, 7, 8, 9, 10\}\). Thus, we can recolor \(vw\) with 1, and color \(vu\) with 5.

For the second step, consider a partial 10-total-coloring of \(G\) such that only \(u\) is not colored. If we cannot greedily extend it to \(G\), then without loss of generality the coloring is the one of Figure 3(b). Note that if \(|C(vu)| \leq 8\), then we can recolor \(vu\), and color \(u\) with 5. Thus, we infer that \(\{a, b, c, d, e\} \supset \{7, 8, 9, 10\}\). Similarly, \(\{e, f, g, h, i\} \supset \{6, 8, 9, 10\}\). Observe that \(|C(v)| = 9\); otherwise we just properly recolor \(v\), and color \(u\) with 6.

We assert that we can assume that \(e \in \{1, 2, 3\}\). If it is not the case, then \(e \in \{8, 9, 10\}\), say \(e = 10\). By what precedes, \(|C(vw)| \leq 12 - 4 = 8\) and \(\{4, 5, 6, 7, 8, 9\} \subset C(vw)\). Thus at least one color among 1, 2, 3 can be used to recolor \(vw\), which proves the assertion. Therefore, \(\{a, b, c, d\} = \{7, 8, 9, 10\}\) and \(\{f, g, h, i\} = \{6, 8, 9, 10\}\). Thus \(vw\) can be recolored by every color of \(\{1, 2, 3\}\). So, if there exists a color \(\alpha \in \{1, 2, 3\} \setminus \{A, B, C, D\}\), we can recolor \(vw\) with a color of \(\{1, 2, 3\}\) different from \(\alpha\), recolor \(v\) with \(\alpha\), and color \(u\)
Figure 3: Configurations for the proofs of Lemmas 3 and 4.
with 6. Hence $\{1, 2, 3\} \subseteq \{A, B, C, D\}$. Now, recall that $|C(v)| = 9$, thus $4 \in \{A, B, C, D\}$. Consequently, we can interchange safely the colors of $vu$ and $wu$, recolor $v$ with 5, and finally color $u$ with 6.

(ii) Suppose that $v$ is a 6-vertex with three 5-neighbors $x_1, x_2, x_3$. By Lemma 2(vi), these three vertices are pairwise nonadjacent. Let $\lambda$ be a nice coloring of $G - vx_1$, and uncolor the edges $vx_2$ and $vx_3$ as well as the vertices $v, x_1, x_2, x_3$. Notice that $|C(x_i)| \leq 8$ and $|C(vx_i)| \leq 7$ for each $i \in \{1, 2, 3\}$. Moreover, $|C(v)| \leq 6$. Recall that $F := \{1, 2, \ldots, 10\} \setminus C(x)$ for every $x \in V \cup E$. Observe that for each $i \in \{1, 2, 3\}$, we have $F(v) \cap F(x_i) \subseteq F(vx_i)$. Hence, we infer that $F(v) \cap (F(vx_i) \cup F(x_i)) = F(v) \cap F(vx_i)$. Consequently, there exists a color $\alpha \in F(v)$ such that, after setting $\lambda(v) := \alpha$, it holds that $|F(x_3)| \geq 2$ and $|F(vx_3)| \geq 3$. If we color properly $x_1, vx_1, x_2$, and $vx_2$, then we will be able to color greedily $x_3$ and $vx_3$, and hence the proof would be complete. Observe that if $\alpha$ does not belong to $F(x_1)$ or to $F(vx_2)$, then the coloring can be extended greedily to $x_1, x_2, vx_1, vx_2$—just color $x_1$ or $vx_2$ last, respectively. Therefore we assume that $\alpha$ belongs to these two lists. Uncolor $v$ and color $x_1$ and $vx_2$ with $\alpha$. With respect to this coloring, note that $|F(vx_1)| \geq 2$, $|F(v)| \geq 3$, $|F(x_2)| \geq 1$, $|F(vx_3)| \geq 3$, and $|F(x_3)| \geq 2$. Hence, we can color $x_2$. Now, if there exists $\beta \in F(vx_1) \cap F(x_3)$, then we let $\lambda(vx_1) := \lambda(x_3) := \beta$, and afterwards greedily color $v$ and $vx_3$. So, $F(vx_1) \cap F(x_3) = \emptyset$. If there exists $\kappa \in F(v) \cap F(x_3)$, then we set $\lambda(v) := \kappa$, and afterwards we greedily color $x_3, vx_3$, and $vx_1$ in this order. This is possible since $\kappa \not\in F(vx_1)$. Otherwise, greedily coloring $vx_1, v, vx_3$, and $x_3$ in this order yields a nice coloring of $G$.

(iii) Suppose that the statement is false, so the graph $G$ contains the configuration of Figure 3(c). Consider a nice coloring $\lambda$ of $G - vx_1$. If it cannot be extended to $G$, then $|C(vx_7)| = 10$. Furthermore, $|C(vx_2)| = 9$: otherwise we can color the edge $vx_7$ with $\lambda(vx_2)$ and greedily recolor the edge $vx_2$, thereby obtaining a nice coloring of $G$. Therefore, we can assume that the coloring is the one shown in Figure 3(c). Then a nice coloring of $G$ is obtained by interchanging the colors of the edges $vx_7x_1$ and $vx_1$, recoloring $vx_2$ with 1, and coloring $vx_7$ with 2, as shown in Figure 3(d).

(iv) Suppose on the contrary that $G$ contains the configuration of Figure 3(e). Consider a nice coloring of $G - vx_9$. If the edge $vx_9$ cannot be greedily colored, then $|C(vx_9)| = 10$. Thus we may assume that the coloring is the one shown in Figure 3(e). Notice that $a = 10$; otherwise we recolor $vx_2$ with 10 and color $vx_9$ with 2. So, the recoloring in Figure 3(f) is nice.

(v) Suppose that $G$ contains the configuration of Figure 3(g), and consider a nice coloring $\lambda$ of $G - vx_9$. Without loss of generality, we may assume that it is the one of Figure 3(g). Observe that $10 \in \{a, b\}$; otherwise we obtain a nice coloring of $G$ by setting $\lambda(vx_9) := 10$ and $\lambda(vx_9) := 6$. Now, we consider
two cases regarding $b$.

$b = 10$. If $a \neq 7$, then we can interchange the colors of the edges $x_6x_7$ and $vx_7$, and color $vx_9$ with 7 to obtain a nice coloring of $G$. And if $a = 7$, then we interchange the colors of the edges $x_9x_8$ and $vx_8$, and then we let $\lambda(vx_6) := 8$ and $\lambda(vx_9) := 6$.

$b \neq 10$. In this case, $a = 10$. We interchange the colors of $x_9x_8$ and $vx_8$. Similar to before, we deduce that $b = 8$. Now, the previous case applies with 8 playing the role of color 10.

(vi) Suppose on the contrary that $G$ contains the configuration of Figure 2(a). Up to a permutation of the colors, every nice coloring of $G -vx_9$ is as the one of Figure 2(a). Note that $d = 10$; otherwise recolor $vx_8$ with 10 and color $vx_9$ with 8. Similarly, $a = 10$. Now, interchange the colors of the edges $x_1x_2$ and $vx_2$. If $b \neq 2$, then the obtained coloring extends to $G$ by coloring $vx_9$ with 2. If $b = 2$, then interchange the colors of the edges $x_9w$ and $x_1w$, thereby obtaining a nice coloring of $G - vx_9$. Since $d = 10 \neq 2$, observe that we can extend it to $G$ as before; i.e., we recolor $vx_8$ with 2 and color $vx_9$ with 8.

(vii) Suppose that $G$ contains the configuration of Figure 2(b). Consider a nice coloring of $G - vx_9$. Without loss of generality, we may assume that it is the one of Figure 2(b). Note that $10 \in \{a, b\}$; otherwise recolor $vx_5$ with 10 and color $vx_9$ with 5. By symmetry, we can assume that $a = 10$. Interchange the colors of the edges $x_5x_4$ and $vx_4$. If $b \neq 4$, then we have a nice coloring of $G - vx_9$, and we extend it to $G$ by coloring $vx_9$ with 4. Otherwise, $b = 4$, we interchange the colors of the edges $x_5x_6$ and $vx_6$, and color $vx_9$ with 6, which yields a nice coloring of $G$.

Lemma 4. The configuration of Figure 3(h) is reducible.

Proof. Consider a nice coloring of $G - vx_9$. If it cannot be greedily extended to $G$, then $|C(vx_9)| = 10$, and so we can assume that the coloring is the one of Figure 3(h). First, we note that if $a \neq 7$, then $10 \in \{b, c\}$; otherwise, we recolor $vx_7$ by 10, and color $vx_9$ with 7. Similarly, if $a \neq 2$, then $10 \in \{d, e\}$. We now split the proof into three cases.

$a \notin \{6, 8\}$. Since $a$ is different from either 2 or 7, we may assume that $a \neq 7$. As mentioned above, we must have $10 \in \{b, c\}$. Moreover, if we interchange the colors of the edges $x_9x_8$ and $vx_8$, then we deduce as before that $8 \in \{b, c\}$, the color 8 playing the role of color 10. Hence $\{b, c\} = \{8, 10\}$. Now, interchange the colors of the edges $x_7x_6$ and $vx_6$, and
Figure 4: Coloring and recoloring for the proof of Lemma 5.

and color \( vx_9 \) with 6. If \( b = 10 \), then the obtained coloring is proper, and if \( b = 8 \), then we additionally interchange the colors of the edges \( x_9x_8 \) and \( vx_8 \) to obtain the desired coloring.

\( a = 8 \). In this case \( 10 \in \{b, c\} \). By interchanging the colors of the edges \( x_9x_8 \) and \( vx_8 \), and also of \( x_9x_1 \) and \( vx_1 \), we infer that \( 1 \in \{b, c\} \). Hence \( \{b, c\} = \{1, 10\} \). Similar to the previous case, interchange the colors of \( x_7x_6 \) and \( vx_6 \), and afterwards color \( vx_9 \) with 6. If \( b = 10 \), then the obtained coloring of \( G \) is proper, and if \( b = 1 \), then it suffices to additionally interchange the colors of the edges \( x_9x_1 \) and \( vx_1 \), and also of \( x_9x_8 \) and \( vx_8 \), to obtain a nice coloring of \( G \).

\( a = 6 \). Then, \( 10 \in \{d, e\} \). Note that the colors of the edges \( x_9x_8 \) and \( vx_8 \) can be interchanged safely, because \( a \neq 8 \). Therefore, as \( a \neq 2 \), we infer that \( 8 \in \{d, e\} \), and hence \( \{d, e\} = \{8, 10\} \). We interchange now the colors of the edges \( x_2x_3 \) and \( vx_3 \), and color \( vx_9 \) with 3. If \( e = 10 \), then the obtained coloring of \( G \) if proper. And, if \( e = 8 \), then it suffices to interchange the colors of the edges \( x_9x_8 \) and \( vx_8 \) to obtain the desired coloring.

**Lemma 5.** If \( uvz \) is a triangle with an 8-vertex \( v \) and a 3-vertex \( u \), then \( v \) has no 3-neighbor distinct from \( u \).

**Proof.** Suppose that \( v \) is an 8-vertex that contradicts the lemma. Let \( u \) and \( w \) be two 3-neighbors of \( v \), and assume that \( vuz \) is a triangle. We consider a nice coloring of \( G - vu \). If we cannot extend it to \( G \), then, without loss of generality, we may assume that the coloring is the one shown in Figure 4(a). Observe that \( \{a, b\} = \{9, 10\} \); otherwise we obtain the desired coloring by
Lemma 6. The configuration of Figure 5(a) is reducible.

Proof. Consider a nice coloring of $G - vx_2$. Up to a permutation of the colors, it is the one of Figure 5(a). Note that 10 $\in \{a, b\}$; otherwise we obtain a nice coloring of $G$ by coloring $vx_2$ with 10. We split the proof into two cases, regarding the value of $b$.

Case 1. $b = 10$. If $a = 4$, then apply the recolorings of Figures 5(b) and (c), regarding whether $d$ is 3.

Suppose now that $a \neq 4$. In this case, we deduce that $d = 10$; otherwise, we can recolor $vx_4$ with 10, and color $vx_2$ with 4. If $c \neq 5$, then the desired coloring can be obtained as follows. If $a \neq 5$, then interchange the colors of the edges $x_4x_5$ and $vx_5$, and color $vx_2$ with 5, and if $a = 5$, then the recoloring of Figure 5(d) is nice.

We may assume now that $c = 5$. Interchange the colors of the edges $x_4x_5$ and $vx_5$, and also of the edges $x_4x_3$ and $vx_3$. If $a \neq 3$, then it suffices to
color $vx_2$ with 3. And, if $a = 3$, then additionally interchange the colors of the edges $x_2x_1$ and $vx_1$, recolor $vx_4$ with 1, and color $vx_2$ with 4 to obtain the sought coloring.

Case 2. $b \neq 10$. Therefore, $a = 10$. First, note that 10 $\in \{c, d\}$; otherwise, we recolor $vx_4$ with 10, and color $vx_2$ with 4. Either the obtained coloring of $G$ is nice, or $b = 4$. In the latter case, we additionally interchange the colors of $x_2x_3$ and $x_4x_3$ to obtain the desired coloring.

Suppose now that $c = 10$. Then, $b = 4$; otherwise, we uncolor $vx_4$, color $vx_2$ with 4, and apply Case 1 to the obtained coloring with the role of the vertex $x_2$. Now, interchange the colors of $x_4x_3$ and $vx_3$. The obtained coloring is nice if $d \neq 3$, and we extend it to $G$ by coloring $vx_2$ with 3. And, if $d = 3$, then we additionally interchange the colors of $x_4x_5$ and $vx_5$, and color $vx_2$ with 5.

Finally, assume that $c \neq 10$, and hence $d = 10$. Up to interchanging the colors of $x_2x_3$ and $x_4x_3$, we may assume that $b \neq 5$. Interchange the colors of $x_4x_5$ and $vx_5$. If $c \neq 5$, the obtained coloring is nice, and we extend it to $G$ by coloring $vx_2$ with 5. And, if $c = 5$, then we additionally interchanhe the colors of $x_4x_3$ and $vx_3$, and color $vx_2$ with 3. \hfill \qed

**Lemma 7.** The configuration of Figure 6(a) is reducible.

Our proof of Lemma 7 uses the following result. Given a coloring $\lambda$ and a vertex $v$, recall that $E(v)$ is the set of colors assigned to the edges incident to $v$. Let $E'(v) := \{1, 2, \ldots, 10\} \setminus (E(v) \cup \{\lambda(v)\})$.

**Lemma 8.** Suppose that $G$ contains the configuration of Figure 6(b). Then, for every nice coloring $\lambda$ of $G - vx_2$, it holds that $E'(v) \cup \{\lambda(vx_6)\} \subseteq E(x_2)$. 

Figure 6: Configurations for Lemmas 7 and 8.
Proof. Up to a permutation of the colors, the coloring \( \lambda \) is the one of Figure 6(b). Notice that \( \mathcal{E}'(v) = \{10\} \), \( \lambda(vx_6) = 6 \), and \( \mathcal{E}(x_2) = \{a, b\} \). First, \( 10 \in \{a, b\} \); otherwise, we just color \( vx_2 \) with 10. By symmetry, we may assume that \( a = 10 \). Thus, to finish the proof, it remains only to prove that \( b = 6 \). Suppose on the contrary that \( b \neq 6 \). Note that \( 10 \in \{c, d\} \); otherwise, we recolor \( vx_6 \) with 10 and color \( vx_2 \) with 6. By symmetry, we may assume that \( d = 10 \). We consider two possibilities regarding the value of \( b \).

\( b = 1 \). Interchange the colors of the edges \( x_6x_7 \) and \( vx_7 \). The obtained coloring of \( G \) is nice if \( c \neq 7 \), and if \( c = 7 \) we additionally interchange the colors of \( x_6x_5 \) and \( vx_5 \). Now, coloring \( vx_2 \) with 7 or 5 yields a nice coloring of \( G \), a contradiction.

\( b \neq 1 \). In this case, \( c = 1 \). Indeed, if \( c \neq 1 \), then we recolor \( vx_6 \) with 1, interchange the colors of \( x_2x_1 \) and \( vx_1 \), and color \( vx_2 \) with 6 to obtain a nice coloring of \( G \). Now, if \( b \neq 7 \), then interchange the colors of \( x_6x_7 \) and \( vx_7 \) and color \( vx_2 \) with 7. And, if \( b = 7 \), then interchange the colors of \( x_6x_5 \) and \( vx_5 \), and also of \( x_2x_1 \) and \( vx_1 \), and color \( vx_2 \) with 5.

Proof of Lemma 7. Consider a nice coloring \( \lambda \) of \( G - vx_2 \). Up to a permutation of the colors, we assume that the coloring is the one of Figure 6(a). By Lemma 8, we have \( \{a, b\} = \{6, 10\} \). We consider two cases.

\( a = 10 \) and \( b = 6 \). If there exists a color \( \alpha \in \{1, 10\} \setminus \{e, f, g\} \), then recolor \( vx_4 \) with \( \alpha \), and color \( vx_2 \) with 4. The obtained coloring is nice if \( \alpha = 10 \). If \( \alpha = 1 \), then it suffices to additionally interchange the colors of \( x_2x_1 \) and \( vx_1 \). Thus, \( \{1, 10\} \subset \{e, f, g\} \).

Suppose that \( 6 \notin \{e, f, g\} \). We start by interchanging the colors of the edges \( x_2x_3 \) and \( x_4x_3 \). If \( e = 10 \), then we additionally interchange the colors of \( x_2x_1 \) and \( vx_1 \). Observe that the obtained coloring does not fulfill the conclusion of Lemma 8, a contradiction. Hence, \( \{e, f, g\} = \{1, 6, 10\} \) and so \( e \in \{1, 10\} \). We interchange the colors of \( x_4x_3 \) and \( vx_3 \), and color \( vx_2 \) with 3. Either this coloring of \( G \) is nice, or \( e = 1 \) and hence additionally interchanging the colors of \( x_2x_1 \) and \( vx_1 \) yields a nice coloring of \( G \).

\( a = 6 \) and \( b = 10 \). If there exists \( \alpha \in \{3, 10\} \setminus \{f, g\} \), then recolor \( vx_4 \) with \( \alpha \), and color \( vx_2 \) with 4. If the obtained coloring is not nice, then \( \alpha = 3 \) and hence interchanging the colors of \( x_2x_3 \) and \( vx_3 \) yields a nice coloring of \( G \), a contradiction. Observe that we may assume that \( f = 3 \).
Figure 7: Reducible configurations of Lemma 9. We assume that the degree of $v$ in $G$ is 8.

and $g = 10$. Indeed, if it is not the case, then we interchange the colors of $x_2x_3$ and $vx_3$ and obtain the desired condition, with 3 playing the role of color 10.

Furthermore, $e = 5$; otherwise, we interchange the colors of $x_4x_5$ and $vx_5$, and color $vx_2$ with 5. Now, observe that $d = 10$; otherwise, we recolor $vx_6$ with 10, $vx_4$ with 6, and color $vx_2$ with 4 to obtain a nice coloring of $G$. Finally, we interchange the colors of $x_6x_7$ and $vx_7$. If $c = 7$, then we additionally interchange the colors of $x_6x_5$ and $vx_5$. Now, coloring $vx_2$ with 7 or 5 yields a nice coloring of $G$, a contradiction.

Lemma 9. The configurations of Figure 7 are reducible.

Proof. Consider a nice coloring of $G - vu$. We may assume that the coloring is the one of Figure 7. Let $\alpha \in \{1, 7, 9, 10\} \setminus \{a,b,c\}$. We recolor $vx_3$ with $\alpha$ and color $vu$ with 3. The obtained coloring of $G$ is nice unless $\alpha \in \{1, 7\}$. If $\alpha = 1$, then we additionally interchange the colors of $uw$ and $vw$. And if $\alpha = 7$, then we interchange the colors of $ut$ and $vt$. \hfill \Box

Lemma 10. A 6-vertex incident to 6 triangles is not adjacent to two 5-vertices.

Proof. Suppose that $v$ is a 6-vertex. We let $x_1, x_2, \ldots, x_6$ be its neighbors, such that $x_i$ is adjacent to $x_{i+1}$ if $i \in \{1,2,\ldots,5\}$ and $x_6$ is adjacent to $x_1$. We also assume that $x_6$ is a 5-vertex, and we let $w$ be the other 5-vertex. By symmetry and Lemma 2(vi), we may assume that $w \in \{x_2, x_3\}$. The
proof is in two steps. In the first step, we show that there exists a partial 10-total-coloring of \( G \) in which only \( x_6 \) is uncolored. In the second step, we show how to extend it to a 10-total-coloring of \( G \).

Given a total-coloring and an element \( x \in V \cup E \), recall that \( C(x) \) is the set of colors of all the elements of \( V \cup E \) incident or adjacent to \( x \). Recall also that if \( x \in V \), then \( E(x) \) is the set of colors of all the edges incident to \( x \).

Let \( \lambda \) be a total-coloring of \( G - vx_6 \), in which, furthermore, we uncolor the vertex \( x_6 \). Our goal is to properly color the edge \( vx_6 \). Note that \( |C(vx_6)| = 10 \); otherwise, the edge \( vx_6 \) can be greedily colored. Without loss of generality, we may assume that the coloring is the one shown in Figure 8(a).

We want to color \( vx_6 \) with \( \lambda(vw) \). Recall that \( w \) is either \( x_2 \) or \( x_3 \). We set \( \mathcal{E} := \mathcal{E}(w) \cup \{\lambda(w)\} \). If there exists a color \( \alpha \in \{7, 8, 9, 10\} \setminus \mathcal{E} \), then we set \( \lambda(vx_6) := \lambda(vw) \) and \( \lambda(vw) := \alpha \). Furthermore, if \( 1 \notin \mathcal{E} \), then we interchange the colors of \( x_6x_1 \) and \( vx_1 \), color \( vx_6 \) with \( \lambda(vw) \), and recolor \( vw \) with 1. Thus, \( 1 \in \mathcal{E} \). Similarly, we deduce that \( 5 \in \mathcal{E} \). Finally, note that either 2 or 3 belongs to \( \mathcal{E} \), according to whether \( w \) is \( x_2 \) or \( x_3 \). Consequently, this shows that \( |\mathcal{E}| \geq 7 \). But \( w \) has degree 5, thus \( |\mathcal{E}| = 6 \), a contradiction. This concludes the first step.

Suppose now that we are given a partial 10-total-coloring of \( G \) in which only \( x_6 \) is not colored. If we cannot extend it to \( G \), then, without loss of generality, we may assume that the coloring is the one shown in Figure 8(b). If there exists a color \( \alpha \in \{2, 4, 6, 10\} \setminus \{a, b, c, d, e\} \), then re-
color $vx_6$ with $\alpha$ and color $x_6$ with 7 to obtain a 10-total-coloring of $G$. Hence, $\{2, 4, 6, 10\} \subset \{a, b, c, d, e\}$. Suppose that $a \notin \{2, 4, 6\}$. In this case, $\{b, c, d, e\} = \{2, 4, 6, 10\}$, and thus $e \in \{2, 4, 10\}$. Interchange the colors of the edges $x_6x_5$ and $vx_5$. Now, if $a \neq 5$, then the obtained coloring is proper, and we extend it to $G$ by coloring $x_6$ with 5. And, if $a = 5$, then we additionally interchange the colors of $x_6x_1$ and $vx_1$, and color $v$ with 9. Consequently, we obtain $a \in \{2, 4, 6\}$.

If $9 \notin \{b, c, d, e\}$, then we can apply a similar recoloring. More precisely, we can interchange the colors of the edges $x_6x_1$ and $vx_1$. The obtained coloring is proper and can be extended to $G$ by coloring $x_6$ with 9. So $9 \in \{b, c, d, e\}$, and hence $5 \notin \mathcal{E}(v)$. We interchange the colors of $x_6x_5$ and $vx_5$, and color $x_6$ with 5. Either the obtained 10-total-coloring of $G$ is proper, or $e = 9$. In the latter case, we additionally interchange the colors of $x_6x_1$ and $vx_1$ to obtain the sought contradiction.

**Lemma 11.** The configuration of Figure 9(a) is reducible.

**Proof.** Consider a nice coloring of $G - vx_9$. Without loss of generality, it is the one of Figure 9(a). First, note that $a = 10$; otherwise, we can recolor the edge $vx_8$ with 10, and color $vx_9$ with 8. Next, we infer that $b = 7$; otherwise, we can interchange the colors of $x_8x_7$ and $vx_7$, and color $vx_9$ with 7. Now, observe that $10 \in \{c, d\}$; otherwise, we recolor $vx_2$ with 10, and color $vx_9$ with 2. Furthermore, $7 \in \{c, d\}$; otherwise, we interchange the colors of $x_8w$ and $x_9w$, and also of $x_8x_7$ and $vx_7$, recolor $vx_2$ with 7, and color $vx_9$ with 10. Thus, $\{c, d\} = \{7, 10\}$. If $d = 7$ and $c = 10$, then we just interchange the colors of the edges $x_2x_1$ and $vx_1$, and color $vx_9$ with 1. And, if $d = 10$ and $c = 7$, then the recoloring shown in Figure 9(b) is a nice coloring of $G$. 

![Figure 9](image-url)
3 Discharging part

Recall that $G = (V, E)$ is a minimum counterexample to the statement of Theorem 1, in the sense that $|V| + |E|$ is minimum. We obtain a contradiction by using the discharging method. Here is an overview of the proof. We fix a planar embedding of $G$. Each vertex and face of $G$ is assigned an initial charge. The total sum of the charges is negative by Euler’s formula. Then, some redistribution rules are applied, and vertices and faces send or receive some charge according to these rules. The total sum of the charges is not changed during this step, but at the end we infer that the charge of each vertex and face is nonnegative, a contradiction.

Initial charge We assign a charge to each vertex and face. For every $x \in V \cup F$, we define the initial charge $ch(x)$ to be $\deg(x) - 4$, where $\deg(x)$ is the degree of $x$ in $G$. By Euler’s formula the total sum is

$$\sum_{v \in V} ch(v) + \sum_{f \in F} ch(f) = -8.$$ 

Rules We need the following definitions to state the discharging rules. A 2-vertex is bad if it is not incident to a $(\geq 5)$-face. A triangle is bad if it contains a vertex of degree at most 4. Recall that a triangle with vertices $x, y,$ and $z$ is a $(\deg(x), \deg(y), \deg(z))$-triangle.

- Rule R0. A $(\geq 5)$-face sends 1 to each incident 2-vertex.
- Rule R1. A 5-vertex $v$ sends $1/5$ to each incident triangle.
- Rule R2. A 6-vertex sends $13/35$ to each incident $(5, 6, \geq 7)$-triangle, $1/3$ to each incident $(6, 6, 6)$-triangle, and $2/7$ to each incident $(6, \geq 6, \geq 7)$-triangle.
- Rule R3. A 7-vertex sends $1/2$ to each incident bad triangle, $3/7$ to each incident nonbad $(\leq 7, \leq 7, 7)$-triangle, and $1/3$ to each incident nonbad triangle containing a $(\geq 8)$-vertex.
- Rule R4. A 8-vertex sends

(i) $1/3$ to each adjacent 3-vertex,

(ii) $1/2$ to each incident bad triangle,

(iii) $7/15$ to each incident $(5, \leq 7, 8)$-triangle and each incident $(6, 6, 8)$-triangle,

(iv) $2/5$ to each incident $(5, \geq 8, 8)$-triangle, each incident $(6, 7, 8)$-triangle, and each incident $(6, 8, 8)$-triangle,
1. 1/3 to each incident (6, 8, 9)-triangle and each incident (≥7, ≥7, 8)-triangle.

Rule R5. A 9-vertex sends

(i) 1 to each adjacent bad 2-vertex and 1/2 to each adjacent nonbad 2-vertex,
(ii) 1/3 to each adjacent 3-vertex,
(iii) 1/2 to each incident bad triangle and each incident (5, ≤7, 9)-triangle,
(iv) 3/7 to each incident (6, 6, 9)-triangle,
(v) 2/5 to each incident (5, ≥8, 9)-triangle and each incident (6, ≥7, 9)-triangle,
(vi) 1/3 to each incident (≥7, ≥7, 9)-triangle.

In what follows, we prove that the final charge ch∗(x) of every x ∈ V ∪ F is nonnegative. Hence, we obtain

\[-8 = \sum_{v \in V} ch(v) + \sum_{f \in F} ch(f) = \sum_{v \in V} ch^*(v) + \sum_{f \in F} ch^*(f) ≥ 0,\]

a contradiction. This contradiction establishes the theorem.

Final charge of faces Let f be a d-face. Our goal is to show that ch∗(f) ≥ 0. By Lemma 2(ii) and (iii), f is incident to at most \(\left\lfloor \frac{d}{3} \right\rfloor\) vertices of degree 2. Therefore, if d ≥ 5, then by Rule R0 we obtain ch∗(f) ≥ d − 4 − \(\left\lfloor \frac{d}{3} \right\rfloor\) = \(\left\lceil \frac{2d}{3} \right\rceil\) − 4 ≥ 0. A 4-face neither sends nor receives any charge, so its charge stays 0.

Finally, let f = xyz be a triangle with deg(x) ≤ deg(y) ≤ deg(z). The initial charge of f is −1, and we assert that its final charge ch∗(f) is at least 0. We consider several cases and subcases according to the degrees of x, y, and z.

deg(x) = 2. Then both y and z have degree 9 by Lemma 2(ii), and hence f receives 1/2 from each of y and z by Rule R5(iii).

deg(x) = 3. In this case, by Lemma 2(ii) and (iv), we infer that deg(y) ≥ 8 and deg(z) = 9. Thus, f receives \(\frac{1}{2} + \frac{1}{2} = 1\) by Rules R4(ii) and R5(iii).
deg(x) = 4. Then, by Lemma 2(ii) and (v), deg(y) ≥ 7 and deg(z) ≥ 8. Hence, by Rules R3, R4(ii), and R5(iii), f receives \( \frac{1}{2} + \frac{1}{2} = 1 \) from y and z.

deg(x) = 5. According to Lemma 2(vi), deg(y) ≥ 6, and by Lemma 3(i), deg(z) ≥ 7. By Rule R1, f receives 1/5 from x, so we need only to show that it receives at least 4/5 from y and z together. Consider the following subcases.

deg(z) = 7. By Rule R3, z sends 3/7 to f, and by Rules R2 and R3, y sends at least 13/35. Thus, f receives at least \( \frac{13}{35} + \frac{3}{7} = \frac{4}{5} \) from y and z, as needed.

deg(z) = 8. If deg(y) ≤ 7, then z sends 7/15 to f by Rule R4(iii), and y sends at least 1/3 by Rules R2 and R3. And, if deg(y) = 8, then both y and z send 2/5 to f by Rule R4(iv). So, in both cases f receives 4/5 from y and z together.

deg(z) = 9. Suppose first that deg(y) ≤ 7. Then, by Rule R5(iii), z sends 1/2 to f. Moreover, by Rules R2 and R3, y sends at least 1/3 to f, which proves the assertion. Now, if deg(y) ≥ 8, then according to Rules R4(iv) and R5(v) f receives 2/5 from each of y and z, as needed.

deg(x) = 6. First, if deg(z) = 6, then f receives 1/3 from each of its vertices by Rule R2. So we assume that deg(z) ≥ 7. In this case, f receives 2/7 from x by Rule R2. Hence, we need only to show that y and z send at least 5/7 to f in total. We consider several cases, regarding the degree of z.

deg(z) = 7. Then f receives 3/7 from z by Rule R3, and at least 2/7 from y by Rules R2 and R3, as desired.

deg(z) = 8. If deg(y) = 6, then z sends 7/15 by Rule R4(iii) and y sends 2/7 by Rule R2. And, if deg(y) ≥ 7, then y sends at least 1/3 by Rules R3 and R4(iv), and z sends at least 2/5 by Rule R4(iv).

deg(z) = 9. If deg(y) = 6, then f receives 2/7 from y by Rule R2 and 3/7 from z by Rule R5(iv). And, if deg(y) ≥ 7, then f receives at least 1/3 from y by Rules R3, R4(v), and R5(v), and at least 2/5 from z by Rule R5(v), which yields the result.

deg(x) ≥ 7. The assertion follows from Rules R3, R4(v), and R5(vi).
Final charge of vertices  Let $v$ be an arbitrary vertex of $G$. We have $\deg(v) \geq 2$ by Lemma 2(i). For every positive integer $d$, we define $v_d$ to be the number of $d$-neighbors of $v$, and $f_d$ to be the number of $d$-faces incident to $v$. Let $x_1, x_2, \ldots, x_{\deg(v)}$ be the neighbors of $v$ in clockwise order. We prove that the final charge of $v$ is nonnegative. To do so, we consider several cases, regarding the degree of $v$.

If $\deg(v) = 2$, then its two neighbors are 9-vertices by Lemma 2(ii). If $v$ is bad, then it receives 1 from each of its two 9-neighbors by Rule R5(i), while otherwise it receives at least 1 from its incident faces by Rule R0, and 1/2 from each of its two 9-neighbors by Rule R5(i). Thus, in both cases its final charge is at least 0.

If $\deg(v) = 3$, then all of its neighbors have degree at least 8, so by Rules R4(i) and R5(ii) it receives 1/3 from each of its neighbors, setting its final charge to 0. If $\deg(v) = 4$, then it neither sends nor receives anything, so its charge stays 0. If $v$ is a 5-vertex, then by Rule R1 it sends $1/5$ to each of its at most five incident triangles; therefore, its final charge is nonnegative.

Suppose now that $v$ is a 6-vertex. All of its neighbors have degree at least 5 by Lemma 2(ii). Note that if $f_3 \leq 5$, then, according to Rule R2, \( \chi^*(v) \geq 2 - 5 \cdot \frac{13}{35} > 0 \). So, we assume now that $f_3 = 6$; i.e., $v$ is incident to 6 triangles. Thus, we infer from Lemma 10 that $v_5 \leq 1$. If $v_5 = 0$, then following Rule R2, the vertex $v$ sends at most $6 \cdot \frac{1}{3} = 2$; therefore, its final charge is at least 0. And, if $v_5 = 1$, then let $x_1$ be the unique 5-neighbor of $v$. By Lemma 3(i), it holds that $\deg(x_2) \geq 7$ and $\deg(x_5) \geq 7$. Consequently, $vx_3x_2$ and $vx_5x_6$ are two $(6, \geq 6, \geq 7)$-triangles. Thus, \( \chi^*(v) \geq 2 - 2 \cdot \frac{13}{35} - 2 \cdot \frac{1}{3} - 2 \cdot \frac{2}{7} = \frac{2}{105} > 0 \) by Rule R2.

Suppose that $v$ is a 7-vertex. If $f_3 \leq 6$, then $\chi^*(v) \geq 3 - 6 \cdot \frac{1}{2} = 0$ by Rule R3. So, we assume now that $f_3 = 7$. We consider several cases, according to the number of 4-neighbors of $v$. Note that, by Lemmas 2(ii) and 3(iii), the vertex $v$ has at most two such neighbors; i.e., $v_4 \leq 2$.

$v_4 = 0$. According to Rule R3, we have $\chi^*(v) \geq 3 - 7 \cdot \frac{3}{7} = 0$.

$v_4 = 1$. Let $x_1$ be this 4-neighbor. So, $x_2$ and $x_7$ both are 9-vertices by Lemma 2(v). According to Rule R3, the vertex $v$ sends 1/3 to each of $vx_2x_3$ and $vx_5x_6$. Furthermore, $v$ is incident to exactly two bad triangles and sends at most 3/7 to each nonbad triangle. Therefore, we obtain $\chi^*(v) \geq 3 - 2 \cdot \frac{1}{7} - 3 \cdot \frac{3}{7} - 2 \cdot \frac{1}{3} = \frac{1}{21} > 0$.

$v_4 = 2$. Without loss of generality, we assume that $x_1$ has degree 4. According to Lemmas 2(ii) and 3(iii), the other 4-neighbor of $v$ must be $x_4$ or $x_5$, say $x_4$ by symmetry. By Lemma 2(v), the vertices $x_2, x_7, x_3,$ and
all have degree 9. Note that \( x_6 \) has degree at least 5. Consequently, 
\[
\text{ch}^*(v) = 3 - 4 \cdot \frac{1}{2} - 3 \cdot \frac{1}{3} = 0.
\]

Suppose now that \( v \) is an 8-vertex. If \( v_3 = 0 \), then \( \text{ch}^*(v) \geq 4 - 8 \cdot \frac{1}{2} = 0 \) by Rule R4. Thus, we assume now that \( x_1 \) is a 3-vertex. Notice that, by Lemma 5, if a 3-neighbor of \( v \) is on a triangle, then \( v_3 = 1 \). Therefore, \( v_3 + f_3 \leq 9 \). If \( f_3 \leq 6 \), then we obtain \( \text{ch}^*(v) \geq 4 - 6 \cdot \frac{1}{2} - 3 \cdot \frac{1}{3} = 0 \). If \( f_3 = 7 \), then we infer from Lemma 5 that \( v_3 \leq 1 \), and so \( \text{ch}^*(v) \geq 4 - 7 \cdot \frac{1}{2} - \frac{1}{3} = \frac{1}{6} > 0 \).

Now we suppose that \( f_3 = 8 \), and thus \( v_3 = 1 \). According to Lemma 2(iv), it holds that \( \text{deg}(x_2) = 9 \) and \( \text{deg}(x_9) = 9 \). Moreover, by Lemma 9, all the vertices but \( x_1 \) have degree at least 5. Thus, by Rule R4, we infer that \( \text{ch}^*(v) \geq 4 - \frac{1}{3} - 2 \cdot \frac{1}{2} - 4 \cdot \frac{7}{15} - 2 \cdot \frac{2}{5} = 0 \).

Finally, suppose that \( v \) is a 9-vertex. By Lemma 2(iii), \( v \) is adjacent to at most one 2-vertex. We consider two cases.

**Case 1.** \( v_2 = 0 \). Suppose first that \( v \) is incident to a \((\geq 4)\)-face; i.e., \( f_3 \leq 8 \).

If \( f_3 = 8 \), then \( v_3 \leq 3 \) by Lemmas 4 and 6, and hence \( \text{ch}^*(v) \geq 5 - 8 \cdot \frac{1}{2} - 3 \cdot \frac{1}{3} = 0 \). If \( f_3 \leq 7 \), then we assert that \( f_3 + v_3 \leq 12 \). Indeed, if \( v_3 \geq 6 \), then, as two 3-vertices are not adjacent, we infer that \( f_3 \leq 2(9 - v_3) \), which yields the assertion. So if \( f_3 \leq 6 \), then we obtain \( \text{ch}^*(v) \geq 5 - 6 \cdot \frac{1}{2} - 6 \cdot \frac{1}{3} = 0 \). If \( f_3 = 7 \), then we can see that \( v_3 \leq 4 \) by Lemmas 4 and 6. Consequently, \( \text{ch}^*(v) \geq 5 - 7 \cdot \frac{1}{2} - 4 \cdot \frac{1}{3} = \frac{1}{6} > 0 \). Now assume that \( f_3 = 9 \). Then, \( v_3 \leq 2 \) according to Lemma 4. If \( v_3 \leq 1 \), then \( \text{ch}^*(v) \geq 5 - 9 \cdot \frac{1}{2} - \frac{1}{3} = \frac{1}{6} > 0 \). Assume now that \( v_3 = 2 \). Without loss of generality, say that \( x_1 \) is a 3-vertex; thus both \( x_2 \) and \( x_9 \) are \((\geq 8)\)-vertices. Note that by Lemma 6, both \( x_3 \) and \( x_8 \) are \((\geq 4)\)-vertices. Therefore, up to symmetry, it suffices to consider the following two cases.

**Case 2.** \( v_2 = 1 \). Let \( x_1 \) be the 2-neighbor. Observe that by Lemma 3(vii), \( v \) cannot have a 3-neighbor on two triangles. Moreover, \( x_1 \) cannot be incident to two triangles, so \( f_3 \leq 8 \). We consider the following possibilities.

**x₄ is the second 3-neighbor.** Then \( \text{deg}(x_3) \geq 8 \), so by Rule R5(vi) the vertex \( v \) sends \( \frac{1}{3} \) to \( v x_2 x_3 \). Hence, we infer that \( \text{ch}^*(v) \geq 5 - 2 \cdot \frac{1}{3} - 8 \cdot \frac{1}{2} - \frac{1}{3} = 0 \).

**x₅ is the second 3-neighbor.** In this case, \( \text{deg}(x_4) \geq 8 \) and \( \text{deg}(x_9) \geq 8 \). Furthermore, \( \text{deg}(x_3) \geq 5 \) by Lemma 7. Consequently, \( x_3 x_2 v \) and \( x_3 x_4 v \) are both \((\geq 5, \geq 8, 9)\)-triangles. Hence, \( v \) sends at most \( \frac{2}{5} \) to each of them by Rule R5(vi). So, \( \text{ch}^*(v) \geq 5 - 2 \cdot \frac{1}{3} - 7 \cdot \frac{1}{2} - 2 \cdot \frac{2}{5} = \frac{1}{5} > 0 \).

**Case 2.** \( v_2 = 1 \). Let \( x_1 \) be the 2-neighbor. Observe that by Lemma 3(vii), \( v \) cannot have a 3-neighbor on two triangles. Moreover, \( x_1 \) cannot be incident to two triangles, so \( f_3 \leq 8 \). We consider the following possibilities.

**x₁ is on a triangle.** Let this triangle be \( v x_1 x_2 \). From Lemma 3(iv) and (v), we infer that \( f_3 + v_3 \leq 8 \). So, \( \text{ch}^*(v) \geq 5 - 1 - 8 \cdot \frac{1}{2} = 0 \).
$x_1$ is bad but not on a triangle. In this case, $x_1$ is on two 4-faces; therefore, in particular, $f_3 \leq 7$. Note that by Lemma 3(vi), either one vertex among $x_2, x_9$ has degree at least 4, or $f_3 \leq 5$. Besides, according to Lemma 3(vii) there is no 3-neighbor on two triangles. Observe also that if both $vx_2x_3$ and $vx_8x_9$ are triangles, then Lemma 11 implies that $v_3 \leq 6$. Let us consider several cases regarding the value of $f_3$.

$f_3 \leq 4$. Then $f_3 + v_3 \leq 10$; otherwise, we obtain a contradiction by Lemma 3(vi) and (vii). Thus, $\chi^*(v) \geq 5 - 1 - 4 \cdot \frac{1}{2} - 6 \cdot \frac{1}{3} = 0$.

$f_3 = 5$. Using Lemma 3(vi) and (vii), a small case-analysis shows that $v_3 \leq 5$. Moreover, if $v_3 = 5$, then the obtained configuration is the one of Lemma 11, which is reducible. And, if $v_3 \leq 4$, then we obtain $\chi^*(v) \geq 5 - 1 - 5 \cdot \frac{1}{2} - 4 \cdot \frac{1}{3} = \frac{1}{6} > 0$.

$f_3 = 6$. In this case, $v_3 \leq 3$ by Lemma 3(vi) and (vii). Thus, $\chi^*(v) \geq 5 - 1 - 6 \cdot \frac{1}{2} - 3 \cdot \frac{1}{3} = 0$.

$f_3 = 7$. By Lemma 3(vi) and (vii), $v$ has at most one 3-neighbor, namely $x_2$ or $x_9$. Thus, $\chi^*(v) \geq 5 - 1 - 7 \cdot \frac{1}{2} - \frac{1}{3} = \frac{1}{6} > 0$.

$x_1$ is neither bad nor on a triangle. Notice that $f_3 \leq 7$. Let us again consider several cases regarding the value of $f_3$.

$f_3 \leq 5$. In this case, $f_3 + v_3 \leq 11$ by Lemma 3(vii). So, $\chi^*(v) \geq 5 - \frac{1}{2} - 5 \cdot \frac{1}{2} - 6 \cdot \frac{1}{3} = 0$.

$f_3 = 6$. Similarly as before, we infer that $v_3 \leq 4$, and hence $\chi^*(v) \geq 5 - \frac{1}{2} - 6 \cdot \frac{1}{2} - 4 \cdot \frac{1}{3} = \frac{1}{6} > 0$.

$f_3 = 7$. By Lemma 3(vii) the vertex $v$ has at most two 3-neighbors, namely $x_2$ and $x_9$. Thus, $\chi^*(v) \geq 5 - \frac{1}{2} - 7 \cdot \frac{1}{2} - 2 \cdot \frac{1}{3} = \frac{1}{3} > 0$.

This establishes that the final charge of every vertex is nonnegative; therefore, the proof of Theorem 1 is now complete.

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