



HAL
open science

Graphs with full rank 3-color matrix and few 3-colorings

Sergey Norine, Jean-Sébastien Sereni

► **To cite this version:**

Sergey Norine, Jean-Sébastien Sereni. Graphs with full rank 3-color matrix and few 3-colorings. *Journal of Combinatorial Theory, Series B*, 2008, 98 (5), pp.1115–1116. 10.1016/j.jctb.2007.12.001 . hal-00487281

HAL Id: hal-00487281

<https://hal.science/hal-00487281>

Submitted on 28 May 2010

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Graphs with full rank 3-color matrix and few 3-colorings

Serguei Norine*

Jean-Sébastien Sereni†

Abstract

We exhibit a counterexample to a conjecture of Thomassen stating that the number of distinct 3-colorings of every graph whose 3-color matrix has full column rank is superpolynomial in the number of vertices.

1 Introduction

Let G be a graph with vertices v_1, v_2, \dots, v_n . A *3-coloring* of G is a mapping that assigns to each vertex an element of the 3-element field \mathbb{GF}_3 such that adjacent vertices are given distinct elements. To each 3-coloring c of G we associate the vector $(c(v_1), c(v_2), \dots, c(v_n))$. The *3-color matrix* of G is the matrix whose rows are all these vectors. This matrix was introduced by Jensen and Thomassen [2]. They showed that this matrix has full column rank for every triangle-free planar graph, thereby strengthening a classical result by Grötzsch [1] that every triangle-free plane graph is 3-colorable. Thomassen [3] conjectured the following.

Conjecture 1. *There exists a positive number ε such that every graph on n vertices whose 3-color matrix has full column rank has at least 2^{n^ε} distinct 3-colorings.*

Thomassen [3] showed that every triangle-free planar graph has $2^{n^{1/12}/20\,000}$ distinct 3-colorings, where n is the number of vertices, verifying Conjecture 1

*School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332-0160, USA. E-mail: snorine@gmail.com. Partially supported by NSF grant 0200595.

†Institute for Theoretical Computer Science (ITI) and Department of Applied Mathematics (KAM), Faculty of Mathematics and Physics, Charles University, Malostranské náměstí 25, 118 00 Prague 1, Czech Republic. E-mail: sereni@kam.mff.cuni.cz. This author is supported by the European project IST FET AEOLUS.

for planar graphs. We disprove Conjecture 1 for general graphs in the next section, by exhibiting a family of triangle-free graphs with full column rank 3-color matrix and yet linearly many 3-colorings.

2 The counterexample

Throughout this section, we fix a positive integer k . The graph G_k has vertex set $V_k := \{v_0, v_1, \dots, v_{3k-2}\}$, and for each $j \in \{0, 1, \dots, 3k-2\}$, the vertex v_j is joined to the vertices $v_{j+k}, v_{j+k+1}, \dots, v_{j+2k-1}$ where the indices are modulo $3k-1$.

The graph G_k is triangle-free, and hence the neighborhood of each vertex is an independent set. For each vertex $v_j \in V_k$, we define c_j to be the following 3-coloring, where the indices are modulo $3k-1$.

- The vertices $v_j, v_{j+1}, \dots, v_{j+k-2}$ are colored 0;
- the vertices $v_{j+k-1}, v_{j+k}, \dots, v_{j+2k-2}$ are colored 1; and
- the vertices $v_{j+2k-1}, v_{j+2k}, \dots, v_{j+3k-2}$ are colored 2.

In the graph G_k , neighborhoods of the vertices are the only maximal independent sets. Consequently, up to a permutation of the colors, every 3-coloring of G_k is one of the colorings c_j . Therefore the graph G_k has linearly many 3-colorings, that is $6|V_k|$.

On the other hand, the 3-color matrix of G_k has full column rank. Indeed, fix a vertex $v_j \in V_k$. Let c be the 3-coloring of G_k obtained from c_{j+1} by recoloring v_j with 0. The difference of the two rows corresponding to c and c_{j+1} has a 1 on the entry corresponding to v_j , and 0 everywhere else.

We are unable to construct examples of families of graphs of girth five with full rank 3-color matrix and few 3-colorings. In fact, all the counterexamples to Conjecture 1 that we are aware of have large complete bipartite subgraphs. Thus it is natural to ask whether Conjecture 1 holds for graphs of girth five, or for graphs embeddable on a fixed surface.

References

- [1] H. Grötzsch. Zur Theorie der diskreten Gebilde. VII. Ein Dreifarbensatz für dreikreisfreie Netze auf der Kugel. *Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg. Math.-Nat. Reihe*, 8:109–120, 1958/1959.
- [2] T. R. Jensen and C. Thomassen. The color space of a graph. *J. Graph Theory*, 34(3):234–245, 2000.

- [3] C. Thomassen. Many 3-colorings of triangle-free planar graphs. *J. Combin. Theory Ser. B*, 97(3):334–349, 2007.