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Distinguished exchangeable coalescents and generalized Fleming-Viot processes with immigration

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DRAFT VERSION

Abstract

Coalescents with multiple collisions (also called Λ-coalescents or simple exchangeable coalescents) are used as models of genealogies. We study a new class of Markovian coalescent processes connected to a population model with immigration. Imagine an infinite population with immigration labelled at each generation by $\mathbb{N} := \{1, 2, ...\}$. Some ancestral lineages cannot be followed backwards after some time because their ancestor is outside the population. The individuals with an immigrant ancestor constitute a distinguished family and we define exchangeable distinguished coalescent processes as a model for genealogy with immigration, focussing on simple distinguished coalescents, i.e such that when a coagulation occurs all the blocks involved merge as a single block. These processes are characterized by two finite measures on $[0, 1]$ denoted by $M = (\Lambda_0, \Lambda_1)$. We call them M-coalescents. We show by martingale arguments that the condition of coming down from infinity for the M-coalescent coincides with that obtained by Schweinsberg for the Λ-coalescent. In the same vein as Bertoin and Le Gall, M-coalescents are associated with some stochastic flows. The superprocess embedded can be viewed as a generalized Fleming-Viot process with immigration. The measures Λ_0 and Λ_1 specify respectively the reproduction and the immigration. The coming down from infinity of the M-coalescent will be interpreted as the initial types extinction: after a certain time, all individuals are immigrant children.

Key words. Exchangeable partitions, coalescent theory, genealogy for a population with immigration, stochastic flows, coming down from infinity.

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1 Introduction

Pitman, [23] and Sagitov, [24] defined in 1999 the class of Λ-coalescent processes, sometimes also called simple exchangeable coalescents. These coalescent processes appear as models for the genealogy of certain haploid populations with fixed size. The general motivation of this work is to define a new class of coalescent processes that may be used to describe the genealogy of a population with immigration. Heuristically, let us imagine an infinite haploid population with immigration described at each generation by $\mathbb{N} := \{1, 2, 3, ...\}$. This means that each individual has at most one parent in the population at the previous generation; indeed, immigration implies that some individuals may have parents outside this population (they are children of immigrants). Sampling n individuals in the population at some fixed generation, we group together the individuals with the same parent at the preceding generation. The individuals with an immigrant parent constitute a special family. We get a partition of $\mathbb{N}$ where each block is a family.

To give to the population a full genealogy, we may imagine a generic external ancestor, say 0, to distinguish the immigrants family. This way, all families will have an ancestor at the preceding generation. Following the ancestral lineage of an individual backwards in time, it may coalesce with some others in $\mathbb{N}$ or reach 0. In that last case the lineage is absorbed at 0. We call 0 the immigrant ancestor, and we shall therefore work with partitions of $\mathbb{N} \cup \{0\} = \mathbb{Z}_+$. We view the block containing 0 as distinguished and then we speak of distinguished partitions. As usual, a partition is identified with the sequence of its blocks in the increasing order of their smallest element. The distinguished block is thus the first.

For a population with no immigration, Kingman introduced exchangeable random partitions of $\mathbb{N}$. A random partition is exchangeable if and only if its law is invariant by the action of permutations of $\mathbb{N}$. The distinguished partitions appearing in our setting are not exchangeable on $\mathbb{Z}_+$, however their laws are invariant by the action of permutations σ of $\mathbb{Z}_+$ such that $\sigma(0) = 0$. These partitions are called exchangeable distinguished partitions. We will present an extension of Kingman's theorem that determines their structure via a paint-box construction.

This allows us to define, following the approach in Bertoin's book [4], a new class of coalescent processes, we call them exchangeable distinguished coalescents. An exchangeable distinguished coalescent is characterized in law by a measure μ^0 on the space of partitions of $\mathbb{Z}_+$, called the distinguished coagulation rate. The extension of Kingman's theorem enables us to characterize this measure, and when μ^0 is carried on the subset of simple distinguished partitions (which have only one non-trivial block), we get a representation involving two finite measures on $[0, 1]$: $M = (\Lambda_0, \Lambda_1)$. We call M-coalescents this sub-class of distinguished coalescents. The restriction of an M-coalescent to each finite subset containing 0, is a Markovian coalescent chain with the following transition rates: when the partition restricted to $\mathbb{N}$ has b blocks, two kinds of jumps are allowed : for $b \geq k \geq 2$ each k-tuple of blocks not containing 0 can merge to form a single block at rate $\int_0^1 x^{k-2}(1-x)^{b-k}\Lambda_1(dx)$, and for $b \geq k \geq 1$, each k-tuple of blocks not containing 0 can merge with the one containing 0 at rate $\int_0^1 x^{k-1}(1-x)^{b-k}\Lambda_0(dx)$. Next we consider the question of the coming down from infinity for M-coalescents. An interesting result is that the condition for M-coalescents to come down does not depend on Λ_0 and is the same for the Λ_1-coalescent found by Schweinsberg [25].

In the last section, we define some stochastic flows connected with M-coalescents. The model of continuous population embedded in the flow can be viewed as a generalized Fleming-Viot process with immigration. As in [5], the stochastic flows involved allow us to define simul-

taneously a population model forward in time and its genealogical process backward in time. A duality between M-generalized Fleming-Viot processes with immigration and M-coalescents will be studied.

In a forthcoming paper, we will define the class of generalized Fleming-Viot process with immigration by a "modified" lookdown construction a la Donnelly and Kurtz [13]. The lookdown construction will allow us to apply some ideas of Birkner *et al* in [8] to etablish a link between certain branching processes with immigration and M-generalized Fleming-Viot processes with immigration.

Outline. The paper is organized as follows. In Section 2, we recall some basic facts on random partitions, and we give some fundamental properties of exchangeable distinguished partitions (about asymptotic frequencies existence, paint-box representation). In Section 3, we define exchangeable distinguished coalescents. We etablish a characterization of their laws by an exchangeable measure μ^0 on the space of the distinguished partitions. The structure of μ^0 is entirely described what enable us to study the dust. The main reference is Chapters 2 and 3 of Bertoin's book [4]. The construction of exchangeable distinguished coalescents is very close to that for exchangeable coalescents of [4]. In Section 4, we focus on M-coalescents and study the coming down from infinity. In particular, our approach provides a new proof of Schweinsberg's result, see [25], about necessary and sufficient conditions to come down from infinity for Λ-coalescents based on martingale arguments. In Section 5, we introduce certain stochastic flows encoding M-coalescents. As in [4] and [5], these flows allow us to define a population model with immigration called M-generalized Fleming-Viot process with immigration.

2 Distinguished partitions

We begin with some general notation and properties which we will use constantly in the following sections.
For every integer $n \geq 1$, we denote by $[n]$ the set $\{1, ..., n\}$ and by $\mathcal{P}_n$ the set of its partitions. The set of partitions of $\mathbb{N}$ is denoted by $\mathcal{P}_\infty$. Let $\pi \in \mathcal{P}_\infty$, we identify the set π with the sequence $(\pi_1, \pi_2, ...)$ of the blocks of π enumerated in increasing order of their least element: for every $i \leq j$, $\min \pi_i \leq \min \pi_j$. The number of blocks of π is $\#\pi$. Let $\pi \in \mathcal{P}_\infty$ and $n \in \mathbb{N}$, $\pi_{|[n]} \in \mathcal{P}_n$ is by definition the restriction of π to $[n]$. We denote by $\mathcal{P}_{\mathbf{m}}$ the set of mass-partitions, meaning the decreasing sequences with sum less than or equal to 1:

$$\mathcal{P}_{\mathbf{m}} := \{s = (s_1, s_2, ...); \sum_{i \geq 1} s_i \leq 1, s_1 \geq s_2 \geq ... \geq 0\}.$$

Given a partition $\pi = (B_1, B_2, ...)$ and a block B of that partition, we say that B has an asymptotic frequency, denoted by $|B|$, if the following limit exists:

$$|B| := \lim_{n \to \infty} \frac{\#(B \cap [n])}{n}.$$

If each block of a partition has asymptotic frequency, this partition is said to have asymptotic frequencies. For $\pi \in \mathcal{P}_\infty$ possessing asymptotic frequencies, $|\pi|^\downarrow$ is the mass partition associated

to π that is $(|\pi|_i^{\downarrow})_{i\in\mathbb{N}}$ is the rearrangement in decreasing order of $(|\pi_i|)_{i\in\mathbb{N}}$. For every $n \in \mathbb{N}$, a permutation of $[n]$ is a bijection $\sigma : [n] \mapsto [n]$. For $n = \infty$, we call permutation of $\mathbb{N}$ any bijection σ of $\mathbb{N}$ such that $\sigma(k) = k$ when k is large enough. We define the equivalence relation $\underset{\pi}{\sim}$ by $i \underset{\pi}{\sim} j$ if i and j are in the same block of π. We denote $\sigma\pi$ the partition defined by:

$$i \underset{\sigma\pi}{\sim} j \Longleftrightarrow \sigma(i) \underset{\pi}{\sim} \sigma(j).$$

We stress that due to the ranking of the blocks, $(\sigma\pi)_i = \sigma^{-1}(\pi_{\eta(i)})$ for a certain permutation η. A random partition π of $\mathbb{N}$ is exchangeable if $\sigma\pi$ and π have the same law, for every permutation σ of $\mathbb{N}$. Kingman etablished a correspondence between exchangeable partitions laws and mass-partitions via the paint-boxes partitions. We recall briefly the construction of paint-boxes. Let s be an element of $\mathcal{P}_{\mathbf{m}}$. Let $\mathcal{V}$ be an open subset of $]0,1[$ such that the ranked sequence of lengths of its interval components is given by s. Let $U_1, ...$ be an i.i.d sequence of uniform variables on $[0,1]$. A s-paint-box is the partition induced by the equivalence relation:

$$\forall i \neq j\colon\ i \underset{\pi}{\sim} j \Longleftrightarrow U_i \text{ and } U_j \text{ belong to the same interval component of } \mathcal{V}.$$

Kingman proved that any exchangeable partition is a mixture of paint-boxes. We denote by ρ_s the law of a s-paint-box.

As explained in the Introduction, we now extend this setting by distinguishing a block, working with partitions of $\mathbb{Z}_+$.

Definition 1 *A distinguished partition π is a partition of $\mathbb{Z}_+$ where the block containing 0 is viewed as a distinguished block. Ranking the blocks in the order of their least element, the first block π_0 contains 0 and is the distinguished block of π.*

We denote by $[\,n\,]$ the set $\{0, 1, ..., n\}$, $\mathcal{P}_n^0$ is the space of distinguished partitions of $\{0, ..., n\}$. For $n = \infty$, we agree that $[\,\infty\,] = \mathbb{Z}_+ = \{0, 1, ...\}$ and then $\mathcal{P}_\infty^0$ is the space of partitions of $\mathbb{Z}_+$. A first basic property is the compactness of the space $\mathcal{P}_\infty^0$ for the distance defined by

$$d(\pi, \pi') = (1 + \max\{n \geq 0, \pi_{|[\,n\,]} = \pi'_{|[\,n\,]}\})^{-1}.$$

See [4] for a proof. Let $\pi \in \mathcal{P}_n^0$, for all $n' \in [\,\infty\,]$ such that $n' \geq n$, we define

$$\mathcal{P}_{n',\pi}^0 = \{\pi' \in \mathcal{P}_{n'}^0; \pi'_{|[\,n\,]} = \pi\}.$$

A random distinguished partition is a random element of $\mathcal{P}_\infty^0$ equipped with the σ-field generated by the finite unions of the sets $\mathcal{P}_{n,\pi}^0$ (that corresponds to the Borelian σ-field for d).

In the same way, we introduce the set of distinguished mass-partitions, meaning the sequences $s = (s_i)_{i\geq 0}$ of non-negative real numbers such that $\sum_{i\geq 0} s_i \leq 1$, ranked in decreasing order apart s_0.

$$\mathcal{P}_{\mathbf{m}}^0 := \{s = (s_0, s_1, ...); \sum_{i\geq 0} s_i \leq 1, s_0 \geq 0, s_1 \geq s_2 \geq ... \geq 0\}.$$

We identify the sets $\mathcal{P}_{\mathbf{m}}$ and $\{s \in \mathcal{P}_{\mathbf{m}}^0; s_0 = 0\}$. The dust of s is by definition the quantity $\delta := 1 - \sum_{i=0}^{\infty} s_i$. A (distinguished) mass-partition is said to be improper if the dust is positive.

For $\pi \in \mathcal{P}_\infty^0$ having asymptotic frequencies, $|\pi|^\downarrow$ is the distinguished mass-partition associated to π that is $|\pi|_0^\downarrow = |\pi_0|$ and $(|\pi|_i^\downarrow)_{i\in\mathbb{N}}$ is the rearrangement in decreasing order of $(|\pi_i|)_{i\in\mathbb{N}}$. We stress that by definition $|\pi|_0^\downarrow = |\pi_0|$.
We call permutation of $\mathbb{Z}_+$ any bijection σ of $\mathbb{Z}_+$ such that $\sigma(k) = k$ when k is large enough. Note that any permutation of $\mathbb{N}$ can be extended to a permutation of $\mathbb{Z}_+$ by deciding that $\sigma(0) = 0$.

Definition 2 *A random distinguished partition π is exchangeable if $\sigma\pi$ and π have the same law for every permutation σ of $\mathbb{Z}_+$ such that $\sigma(0) = 0$.*

It is easily seen that the restriction of an exchangeable distinguished partition to $\mathbb{N}$ is exchangeable. The converse may fail: there exist distinguished partitions which are not exchangeable though their restriction to $\mathbb{N}$ is exchangeable. We construct a counter-example: let π be a non-degenerate exchangeable random partition and π^0 obtained from π by distinguishing the block containing 1, i.e $\pi^0 = (\pi_1 \cup \{0\}, \pi_2, ...)$ with blocks enumerated in order of appearance. The restriction $\pi^0_{|\mathbb{N}} = \pi$ is exchangeable. The structure of π implies that $\mathbb{P}[\pi^0_{|[2]} = (\{0,1\},\{2\})] = \mathbb{P}[1 \underset{\pi}{\not\sim} 2] > 0$. Let σ be the permutation of $[2]$: $\sigma(0) = 0, \sigma(1) = 2, \sigma(2) = 1$. We have $\mathbb{P}[\pi^0_{|[2]} = (\{0,\sigma(1)\},\{\sigma(2)\})] = \mathbb{P}[\pi_1 = \{2\}] = 0$. We thus found a permutation such that $\mathbb{P}[\pi^0 = \sigma(\pi^0_0, ...)] \neq \mathbb{P}[\pi^0 = (\pi^0_0, ...)]$.

We define now the distinguished paint-boxes and extend the Kingman's correspondence to exchangeable distinguished partitions.

Definition 3 *A distinguished paint-box can be constructed in the following way:*
Let s be a distinguished mass partition, we denote by δ its dust. Denote by ∂, an element which does not belong to $\mathbb{Z}_+$. Let ξ be a probability on $\mathbb{Z}_+ \cup \{\partial\}$ such that for all $k \geq 0, \xi(k) = s_k$ and $\xi(\partial) = \delta$. Drawning $X_1, X_2, ...$ a sequence of i.i.d variables (ξ) and $X_0 = 0$. A s-distinguished paint box is defined by : $\forall i \neq j \geq 0$,

$$\Pi^0 : i \sim j \Longleftrightarrow X_i = X_j \neq \partial.$$

In particular, $\Pi^0_0 := \{i \geq 0; X_i = 0\} = \{i \geq 1; X_i = 0\} \cup \{0\}$.

We denote by ρ^0_s the law of an s-distinguished paint box. When $s_0 = 0$, the block Π^0_0 is the singleton $\{0\}$ and the s-distinguished paint box restricted to $\mathbb{N}$ is a classical s-paint box partition of $\mathbb{N}$. According to the previous notation for paint-boxes, we will denote its law by ρ_s.

Another way to define an s-distinguished paint-box is to consider a sub-probability α on $[0,1[$ and set $s = (\alpha(0), \alpha(x_1), \alpha(x_2), ...)$ where $x_1, x_2, ...$ are the atoms of α in $]0,1[$ ranked in decreasing order of their sizes. Let $X_1, X_2...$ be independent with law α and $X_0 = 0$, the partition Π^0 defined by: $i \neq j$: $i \sim j \Longleftrightarrow X_i = X_j$ is an s-distinguished paint-box.
Equivalently, we can work with uniform variables: an interval representation of s is a collection of disjoint intervals $(A_0, A_1, A_2, ...)$, where A_0 is $[0, s_0]$, and $(A_i)_{i\geq 1}$ such that the decreasing sequence of their lengths is $(s_1, s_2, ...)$. If we draw an infinite sequence of uniform independent variables $(U_i)_{i\geq 1}$ and fix $U_0 = 0$. The partition of $\mathbb{Z}_+$ defined by: $\pi^0 := i \sim j$ if and only if U_i and U_j fall in the same interval (if U_i falls in the dust of s then $\{i\}$ is a singleton block of π^0) is a s-distinguished paint box. We stress that its law does not depend on the choice of intervals $(A_1, A_2...)$ and then we can choose $A_i := [s_0 + ... + s_{i-1}, s_0 + ... + s_i[$ for all $i \geq 1$.

Proposition 1 *Let $s \in \mathcal{P}^0_{\boldsymbol{m}}$ and π^0 a s-distinguished paint-box.*

i) The distinguished paint-box π^0 is exchangeable.

ii) π^0 has asymptotic frequencies, and more precisely $|\pi^0|^{\downarrow} = s$.

iii) For every $i \in \mathbb{Z}_+$, if $|\pi^0_i| = 0$ then π^0_i is a singleton or empty.

iv) s is improper if and only if some blocks different from π^0_0 are singletons. In that case the set of singletons $\{i \in \mathbb{Z}_+ : i$ is a singleton of $\pi^0\}$ has an asymptotic frequency given by the dust $\delta = 1 - \sum_{i=0}^{\infty} s_i$ a.s

v) We have $\rho^0_s(0$ is singleton$)= 0$ if $s_0 > 0$ and 1 otherwise, and $\rho^0_s(1, 2, ..., q$ are singletons$)=\delta^q$, for $q \geq 1$.

Proof. If σ is a permutation of $\mathbb{Z}_+$, with $\sigma(0) = 0$, $(U_i)_{i\geq 0}$ is an exchangeable sequence, and so has the law of $(U_{\sigma(i)})_{i\geq 0}$. It comes $\mathbb{P}[i \underset{\pi^0}{\sim} j] = \mathbb{P}[\sigma(i) \underset{\pi^0}{\sim} \sigma(j)]$ and then π^0 is exchangeable. The assertion *i)* is then etablished. We stress that the block $\sigma\pi^0_0$ is given by $\sigma^{-1}(\pi^0_0)$. To show *ii)*, let $\mathcal{V} := \{[0, s_0[, A_1, A_2...\}$ be a representation of s. Let $k \geq 0$, the block π^0_k is such that $\forall i, j \in \pi^0_k$, U_i and U_j fall in the same interval. We denote this interval by $A \in \mathcal{V}$. If $k = 0$, then $A = [0, s_0[$ because $U_0 = 0$. By the strong law of large numbers, $\lim_{n\to\infty} \frac{1}{n}\#\{i \leq n; i \in \pi^0_k\} = \lim_{n\to\infty} \frac{1}{n}\#\{i \leq n; U_i \in A\} = |A|$ almost surely. We then have $|\pi^0|^{\downarrow} = (|\pi^0_0|, |\pi^0|^{\downarrow}_1, ...) = s$. The assertion *iii)* is plain: if $\mathbb{P}[1, 2 \in \pi^0_1] > 0$, then $\mathbb{P}[U_1, U_2 \in A] > 0$ for some $A \in \mathcal{V}$ and $|\pi^0_1| = |A| > 0$. Suppose now that $\mathbb{P}[0, 1 \in \pi^0_0] > 0$, the same argument shows that $s_0 > 0$. We conclude by exchangeability. The assertions *iv)* and *v)* are straigthforward. If some blocks not containing 0 are singletons then $\mathbb{P}(U_1 \in [0, 1] - \mathcal{V}) > 0$ and so $1 - \sum_{i=0}^{\infty} s_i > 0$. The assertion *v)* is trivial.□

It remains to see whether the distinguished paint-box construction of Definition 3 yields all the exchangeable distinguished partitions.

Theorem 2 *Let π^0 be a random distinguished partition. The following assertions are equivalent:*

i) π^0 is exchangeable

ii) There exists a random distinguished mass-partition $S = (S_0, S_1, ...)$ such that conditionnally given $S = s$, π^0 has the law of an s-distinguished paint box (ρ^0_s). Further $|\pi^0|^{\downarrow} = S$.

Proof. A mixture of distinguished paint-boxes is still exchangeable, this shows that *ii)* implies *i)*. Let π^0 be exchangeable. We adapt a proof of Aldous [1], see also [4] p101. We call selection map, any random function $b : \mathbb{Z}_+ \to \mathbb{Z}_+$ that maps all the points of the block π^0_0 to 0, and all the points of a block π^0_i for $i \geq 1$ to the same point of that block. Let $U_0 = 0$, $(U_i)_{i\geq 1}$ be i.i.d uniform on $[0, 1]$, independent of π^0 and of the selection map b. We define $X_n = U_{b(n)}$. The law of $(X_n, n \geq 1)$ does not depend on the choice of b. The key of the proof is the exchangeability of $(X_n)_{n\geq 1}$. Let σ be a permutation with $\sigma(0) = 0$, we have:

$$X_{\sigma(n)} = U_{b(\sigma(n))} = U'_{b'(n)},$$

where $U'_i = U_{\sigma(i)}$ and $b' = \sigma^{-1} \circ b \circ \sigma$. We verify that b' is a selection map for the partition $\sigma\pi^0$: let $i \geq 0$, and $n \in \sigma\pi^0_i$, by definition of $\sigma\pi^0$, $\sigma\pi^0_i = \sigma^{-1}(\pi^0_{\eta(i)})$ for a certain permutation η such that $\eta(0) = 0$, then there exists $k \in \pi^0_{\eta(i)}$ such that $n = \sigma^{-1}(k)$. For $i = 0$, $k \in \pi^0_0$ and then $b'(n) = \sigma^{-1}(b(k)) = \sigma^{-1}(0) = 0$ for all $n \in \sigma\pi^0_0$. For $i \geq 1$, we clearly have that $b'(n) = \sigma^{-1} \circ b(k)$ depends only on i.
The sequence $(U'_i, i \geq 1)$ has the same law as $(U_i, i \geq 1)$. By exchangeability and independence of (U_j) and π^0: $((U'_n)_{n\geq 1}, \sigma\pi^0)$ has the same law as $((U_n)_{n\geq 1}, \pi^0)$, and the sequence $(X_n, n \geq 1)$ is exchangeable. By the de Finetti theorem, we have that conditionnally on a random measure of probability $\mu := \lim \frac{1}{n}\sum_{i=1}^n \delta_{X_i}$, (X_n) is an i.i.d (μ) sequence. Moreover, by the definition of X_n: $i \underset{\pi}{\sim} j$ if and only if $X_i = X_j$. We deduce that conditionnally given $\mu = \alpha$, the distinguished partition π^0 is a $s(\alpha)$-distinguished paint box with $s(\alpha) := (\alpha(0), \alpha(x_1), ...)$. By the distinguished paint-box construction, on $\{\mu = \alpha\}$, $s(\alpha)$ is the mass-partition of π^0 and so $|\pi^0|^{\downarrow} = s(\mu)$, moreover the random sequence $S := s(\mu)$ in $\mathcal{P}^0_{\mathbf{m}}$ verifies the assertion *ii)*. To conclude π^0 has the law of a η-mixture of distinguished paint-boxes, where η is the law of $|\pi^0|^{\downarrow}$.
□

This theorem sets up a bijection between probability distributions for exchangeable distinguished partitions and probability distributions on the space of distinguished mass-partitions, $\mathcal{P}^0_{\mathbf{m}}$:

$$\mathbb{P}[\pi^0 \in .] = \int_{\mathcal{P}^0_m} \rho^0_s(.)\mathbb{P}(|\pi^0|^{\downarrow} \in ds).$$

Remark 2.1 *Let π^0 be an exchangeable distinguished partition, as for exchangeable partitions, see [21], we can show that there exists a function p such that:*

$$\mathbb{P}[\pi^0_{|[n]} = (B_0, ..., B_k)] = p(n_0, ..., n_k).$$

where $n_i = \#B_i$, $i = 0, ..., k$. Contrary to exchangeable random partitions, the function p is not totally symmetric but only invariant by permutations of the arguments $(n_1, ..., n_k)$. Indeed by exchangeability $\mathbb{P}[\pi^0 = (\sigma B_0, ..., \sigma B_k)] = p(\#B_0, ..., \#B_k) = p(\#B_{\eta(0)}, ..., \#B_{\eta(k)})$ where η is the permutation such that $\sigma\pi^0_i = \sigma^{-1}(\pi^0_{\eta(i)})$. Due to the assumption $\sigma(0) = 0$, the permutation η is such that $\eta(0) = 0$. The exchangeable distinguished partitions are thus special cases of partially exchangeable partitions in the sense of [22].
We mention that Donnelly and Joyce, [12] defined the exchangeable ordered partitions for which all the blocks are distinguished (they speak about exchangeable random ranking). They obtain a Kingman's representation for the exchangeable ordered partition structure. For every μ probability on $[0, 1]$, an "ordered paint-box" is constructed from a sequence of i.i.d. μ variables. We stress that, contrary to an exchangeable distinguished paint-box, the law of a μ-ordered paint-box depends on ***the order of the atoms*** *of μ. Exchangeable ordered partitions are also partially exchangeable but the function p has no symmetry properties. Gnedin in [15] gives a representation of exchangeable compositions which are a generalization of exchangeable random rankings. We could define distinguished partitions with several distinguished blocks. It corresponds to a population with several sources of immigration: each distinguished block gathers the progeny of an immigration source. For sake of simplicity, we distinguish here just one block.*

In the next section, we define distinguished coalescents which can be interpreted as a genealogy for a population with immigration.

3 Distinguished coalescents

Imagine an infinite haploid population with immigration. We denote by $\Pi^0(t)$ the partition of the current population into families having the same ancestor at the generation t backwards. As explained in the Introduction, individuals who have no ancestor in the population at generation t, form the distinguished block of $\Pi^0(t)$. Actually, the individual 0 can be viewed as their common ancestor. When some individuals have the same ancestor at a generation t, they have the same ancestor at any generation $t' \geq t$. In terms of partitions, all integers in the same block of $\Pi^0(t)$, are in the same block of $\Pi^0(t')$ for any $t' \geq t$. The collection of partitions $(\Pi^0(t))_{t\geq 0}$ will be a coalescent process. To define these processes and go from an exchangeable distinguished partition to another coarser, we have to introduce the coagulation operator.

3.1 Coagulation operator and distinguished coalescents

To define the distinguished exchangeable coalescents, we need to define an operator on the space of distinguished partitions.

Definition 4 *Let $\pi, \pi' \in \mathcal{P}^0_\infty$, the partition $Coag(\pi, \pi')$ is defined by $Coag(\pi, \pi')_i = \pi''_i$ where $\pi''_i = \bigcup_{j\in\pi'_i} \pi_j$. The partition $Coag(\pi, \pi')$ is exactly the one obtained by coagulating blocks of π according to blocks of π'.*

We denote by $0_{[\infty]}$ the partition into singletons $\{\{0\}, \{1\}, ...\}$, and by $1_{[\infty]}$ the trivial partition $\{\mathbb{Z}^+, \emptyset, ...\}$. Plainly, for all $n \geq 0$, $Coag(\pi, \pi')_{|[n]} = Coag(\pi_{|[n]}, \pi'_{|[n]})$ and for all $\pi \in \mathcal{P}^0_\infty, Coag(\pi, 0_{[\infty]}) = \pi$ and $Coag(\pi, 1_{[\infty]}) = 1_{[\infty]}$. Note that however, we do not have $Coag(\pi, \pi')_{|K} = Coag(\pi_{|K}, \pi'_{|K})$ for $K \subset \mathbb{N}$ in general.

Proposition 3 *Let π, π' be two independent exchangeable distinguished partitions. The distinguished partition $Coag(\pi, \pi')$ is also exchangeable.*

Proof. See the proof of Lemma 4.3 in [4].

The coagulation operator allows us to define distinguished coalescents which are Markovian processes valued in distinguished partitions of $\mathbb{N}$.

Definition 5 *A Markov process Π^0 with values in $\mathcal{P}^0_\infty$ is called a distinguished coalescent if its semi-group is given as follows. The conditional law of $\Pi^0(t + t')$ given $\Pi^0(t) = \pi^0$ is the law of $Coag(\pi^0, \pi')$ where π' is some exchangeable distinguished partition (whose law only depends on t'). A distinguished coalescent is called standard if $\Pi^0(0) = 0_{[\infty]}$.*

The properties of the coagulation operator, see [4], implies that a distinguished coalescent Π^0 fullfills the Feller property. The process has then a cdlg version and is strong Markovian. Plainly, the random partition $\Pi^0_{|\mathbb{N}}(t)$ is exchangeable for all $t \geq 0$. However, we stress that in general the process $(\Pi^0_{|\mathbb{N}}(t), t \geq 0)$ is not an exchangeable coalescent and not even Markovian. We will give an example in Section 4.2.
For every $n \geq 1$, the restriction $\Pi^0_{|[n]}$ is a continuous time Markovian chain with a semi-group given by the operator $Coag$. Let $\pi \in \mathcal{P}^0_n \setminus \{0_{[n]}\}$, we denote by q_π, the jump rate of $\Pi^0_{|[n]}$ from $0_{[n]}$ to π:

$$q_\pi := \lim_{t\to 0+} \frac{1}{t}\mathbb{P}_{0_{[n]}}[\Pi^0_{|[n]}(t) = \pi].$$

An easy adaptation of the proof of Proposition 4.4 in [4] gives the existence and uniqueness of the distinguished coagulation rate:

Definition 6 *The distinguished coagulation rate of* Π^0 *is the unique measure* μ^0 *on* $\mathcal{P}^0_\infty$*, such that* $\mu^0(\{0_{[\infty]}\}) = 0$ *and*

$$\mu^0(\mathcal{P}^0_{\infty,\pi}) = q_\pi$$

for every $n \in \mathbb{Z}_+$ *and every partition* $\pi \in \mathcal{P}^0_n$*.*
Moreover, the measure μ^0 *fullfills:*

$$\mu^0(\pi \in \mathcal{P}^0_\infty : \pi_{|[n]} \neq 0_{[n]}) < \infty \text{ and } \mu^0 \text{ is exchangeable.}$$

We construct explicitly a distinguished coalescent process with coagulation rate μ^0. Let N be a Poisson measure with intensity $dt \otimes \mu^0(d\pi)$. Let N_b be the image of N by the map $(t, \pi) \mapsto (t, \pi_{|[b]})$, its intensity, denoted by μ^0_b, is the image of μ^0 by the previous map. We denote by (t_i, π_i) the atoms of N_b and define a process $(\Pi^b(t), t \geq 0)$ by the following recursion:

For all $0 \leq t < t_1$ $\Pi^b(t) = 0_{[b]}$ and if $t_i \leq t < t_{i+1}$, $\Pi^b(t) = Coag(\Pi^b(t_{i-1}), \pi^i)$, with $t_0 = 0$.

Proposition 4 *The sequence of random partitions* $(\Pi^b(t), b \in \mathbb{N})$ *is compatible. The unique process* $(\Pi^0(t), t \geq 0)$ *such that* $\Pi^0_{|[b]}(t) = \Pi^b(t)$*, defined by* $\Pi^0_i(t) = \bigcup_{b\geq 1} \Pi^b_i(t)$ *is a distinguished coalescent with coagulation rate* μ^0*.*

Proof. Same arguments as those of Proposition 4.5 in [4] apply. □

Example 3.1 *We denote by* $K(i, j)$ *the simple distinguished partition with* i *and* j *in the same block and all the other block singleton. Let* c_0*,* c_1 *be two non-negative real numbers. The measure* $\mu^0 = c_0\mu^K_0 + c_1\mu^K_1$*, where* $\mu^K_0 := \sum_{1\leq i} \delta_{K(0,i)}$ *and* $\mu^K_1 := \sum_{1\leq i<j} \delta_{K(i,j)}$*, is a distinguished coagulation rate. The process obtained is called the Kingman's distinguished coalescent with rates* (c_0, c_1)*.*

Indeed, the measure μ^0 defined as above is plainly a distinguished coagulation rate. The Poissonian construction explains the dynamics of this process. At a constant rate c_0, a block not containing 0 merges with Π^0_0 that is a singular coagulation with the distinguished block. At a constant rate c_1, two blocks not containing 0 merges in one, that is the classic binary coagulation of Kingman's coalescent.

3.2 Characterization in law of the distinguished coalescents

The next theorem is one of the main results of this work, it claims that the law of a distinguished coalescent is characterized by two non-negative real numbers c_0, c_1 and a measure ν^0 on $\mathcal{P}^0_{\mathbf{m}}$. It should be viewed as an extension of Theorem 2 to certain infinite measures. Recall that ρ^0_s denotes the law of an exchangeable distinguished s-paint-box for $s \in \mathcal{P}^0_{\mathbf{m}}$.

Theorem 5 *Let* μ^0 *be a distinguished coagulation rate. There exists two unique real numbers* c_0, c_1 *and a unique measure* ν^0 *on* $\mathcal{P}^0_{\boldsymbol{m}}$ *which fulfills:*

$$\nu^0(0) = 0 \text{ and } \int_{\mathcal{P}^0_{\boldsymbol{m}}} (s_0 + \textstyle\sum_{i=1}^\infty s_i^2)\nu^0(ds) < \infty$$

such that :

$$\mu^0(d\pi) = c_0\mu_0^K + c_1\mu_1^K + \int_{s\in\mathcal{P}_m^0} \rho_s^0(d\pi)\nu^0(ds).$$

Conversely, let c_0, c_1 and ν^0 be two real numbers and a measure on $\mathcal{P}_{\boldsymbol{m}}^0$ verifying the previous conditions, there exists a unique (in law) distinguished coalescent with $\mu^0 = c_0\mu_0^K + c_1\mu_1^K + \rho_{\nu^0}^0$.

When ν^0 is carried on $\{s \in \mathcal{P}_{\mathbf{m}}^0; s_0 = 0\}$ (which can be identified as $\mathcal{P}_{\mathbf{m}}$), the block containing 0 is reduced to the singleton $\{0\}$ (we distinguish no block) and considering the restriction to $\mathbb{N}$, we recover the characterization of exchangeable coalescents (also called Ξ-coalescents) by Schweinsberg in [26].

Proof. Arguments used to prove this theorem are adapted from those of Theorem 4.2 in Chapter 4 of [4]. Nevertheless, we give details to highlight the fact that the condition on ν^0 differs from that of Theorem 4.2 in [4]. We denote by μ_n^0 the restriction of μ^0 to $\{\pi \in \mathcal{P}_\infty^0; \pi_{|[n]} \neq 0_{[n]}\}$. The measure $\mu_n^0 = 1_{\{\pi\in\mathcal{P}_\infty^0;\pi_{|[n]}\neq 0_{[n]}\}}\mu^0$ has a finite mass and is invariant by the permutations σ that coincide with the identity on $[n]$. We define the n-shift on distinguished partitions by the map $\pi \to \pi'$ defined by :

$$\forall i, j \geq 1: \quad i \underset{\pi'}{\sim} j \Longleftrightarrow i + n \underset{\pi}{\sim} j + n$$
$$\forall j \geq 1: \quad 0 \underset{\pi'}{\sim} j \Longleftrightarrow 0 \underset{\pi}{\sim} j + n.$$

The image of μ_n^0 by the n-shift, denoted by $\bar{\mu}_n^0$, is invariant by permutations σ of $\mathbb{Z}_+$ such that $\sigma(0) = 0$. By the Kingman's correspondence (Theorem 2):

$$\bar{\mu}_n^0(d\pi) = \int_{\mathcal{P}_{\mathbf{m}}^0} \rho_s^0(d\pi)\bar{\mu}_n^0(|\pi|^\downarrow \in ds).$$

Moreover, $\bar{\mu}_n^0$ almost every partition has asymptotic frequencies. The shift does not affect asymptotic frequencies and then μ_n^0 a.e partition has asymptotic frequencies. The measure μ^0 is the increasing limit of the μ_n^0, we deduce that μ^0 almost every partition has asymptotic frequencies.
By the distinguished paint-box representation of $\bar{\mu}_n^0$, we get :

$$\mu_n^0(n+1 \sim n+2 \not\sim 0 \text{ or } 0 \sim n+1)||\pi|^\downarrow = s) = s_0 + \textstyle\sum_{k=1}^\infty s_k^2.$$

Let ν_n^0 be the image of μ_n^0 by the map $\pi \mapsto |\pi|^\downarrow$:

$$\nu_n^0(ds) = \mu_n^0(|\pi|^\downarrow \in ds)$$

We stress that $\nu_n^0(ds) = \bar{\mu}_n^0(|\pi|^\downarrow \in ds)$ because the n-shift has no impact on asymptotic frequencies.
We have :

$$\mu_n^0(n+1 \sim n+2 \not\sim 0 \text{ or } 0 \sim n+1) \geq \textstyle\int_{\mathcal{P}_{\mathbf{m}}^0} (s_0 + \sum_{i=1}^\infty s_i^2)\nu_n^0(ds).$$

Moreover, $\mu_n^0(n+1 \sim n+2 \not\sim 0 \text{ or } 0 \sim n+1) \leq \mu^0(n+1 \sim n+2 \not\sim 0 \text{ or } 0 \sim n+1)$.

By exchangeability of μ^0:

$$\mu^0(n+1\sim n+2\not\sim 0 \text{ or } 0\sim n+1)=\mu^0(1\sim 2\not\sim 0 \text{ or } 0\sim 1)\leq \mu^0(\pi_{[2]}\neq 0_{[2]})<\infty.$$

We deduce that the finite measures ν_n^0 increase as $n\uparrow\infty$ to the measure $\nu^0:=\mu^0(|\pi|^{\downarrow}\in ds)$ and so:

$$\lim_{n\to\infty}\int_{\mathcal{P}_{\mathbf{m}}^0}(s_0+\sum_{i=1}^{\infty}s_i^2)\nu_n^0(ds)=\int_{\mathcal{P}_{\mathbf{m}}^0}(s_0+\sum_{i=1}^{\infty}s_i^2)\nu^0(ds)\leq \mu^0(\pi_{[2]}\neq 0_{[2]})<\infty.$$

Let $k\in\mathbb{N}$ and $\pi^{[k]}\in\mathcal{P}_k\backslash\{0_{[k]}\}$. The sequence of events $\pi_{|\{k+1,\dots,k+n\}}\neq 0_{\{k+1,\dots,k+n\}}$ is increasing, then we have :

$$\mu^0(\pi_{|[k]}=\pi^{[k]},|\pi|^{\downarrow}\neq 0)=\lim_{n\to\infty}\mu^0(\pi_{|[k]}=\pi^{[k]},|\pi|^{\downarrow}\neq 0,\pi_{|\{k+1,\dots,k+n\}}\neq 0_{\{k+1,\dots,k+n\}}).$$

By an obvious permutation we get :

$$\mu^0(\pi_{|[k]}=\pi^{[k]},|\pi|^{\downarrow}\neq 0,\pi_{|\{k+1,\dots,k+n\}}\neq 0_{\{k+1,\dots,k+n\}})=\bar{\mu}_n^0(\pi_{|[k]}=\pi^{[k]},|\pi|^{\downarrow}\neq 0).$$

Thus, using the distinguished paint-box representation of $\bar{\mu}_n^0$, we deduce that

$$\bar{\mu}_n^0(\pi_{|[k]}=\pi^{[k]},|\pi|^{\downarrow}\neq 0)=\int_{\mathcal{P}_{\mathbf{m}}^0}\rho_s^0(\pi_{|[k]}=\pi^{[k]},|\pi|^{\downarrow}\neq 0)\nu_n^0(ds)\underset{n\to\infty}{\longrightarrow}\int_{\mathcal{P}_{\mathbf{m}}^0}\rho_s^0(\pi_{|[k]}=\pi^{[k]},|\pi|^{\downarrow}\neq 0)\nu^0(ds).$$

As k is arbitrary, we get :

$$1_{\{|\pi|^{\downarrow}\neq 0\}}\mu^0(d\pi)=\int_{\mathcal{P}_{\mathbf{m}}^0}\rho_s^0(d\pi)\nu^0(ds).$$

It remains to study $1_{\{|\pi|^{\downarrow}=0\}}\mu^0(d\pi)$: Consider now $\tilde{\mu}^0(d\pi):=1_{\{0\sim 1,|\pi|=0\}}\mu^0(d\pi)$ which has finite mass (because $\mu_0(0\sim 1)<\infty$). We want to show that $\tilde{\mu}^0(d\pi)$ is proportional to $\delta_{K(0,1)}$, where $K(0,1)$ is the simple partition with $0\sim 1$. Let $\tilde{\mu}_2^0(d\pi)$ be the image of $\tilde{\mu}^0$ by the 2-shift. The measure $\tilde{\mu}_2^0(d\pi)$ is supporting by $\{\pi\in\mathcal{P}_\infty;|\pi|^{\downarrow}=0\}$ and is exchangeable with finite mass. By the distinguished paint-box construction, the only exchangeable partition with asymptotic frequencies $|\pi|^{\downarrow}=0$ is the partition into singletons $0_{[\infty]}$, then, $\tilde{\mu}_2^0(d\pi)=c_0\delta_{\{\{0\},\{1\},\dots\}}$. We deduce that for $\tilde{\mu}_2^0$-almost every π': $i\underset{\pi'}{\not\sim}j$, $\forall i\neq j$ and then for $\tilde{\mu}^0$-almost every π: $i+2\underset{\pi}{\not\sim}j+2$, $\forall i,j\geq 1$ and $0\underset{\pi}{\not\sim}j+2$, $\forall j\geq 1$. It implies that $\tilde{\mu}^0$-almost every distinguished partition π is simple with π_0 the only one non-trivial block.
Moreover, we get:

$$\tilde{\mu}^0(d\pi)=c_0\delta_{K(0,1)} \text{ or } \tilde{\mu_0}(0\sim 1\sim 2)>0.$$

If $\tilde{\mu}^0(0\sim 1\sim 2)>0$, then by exchangeability for all $n\geq 2$, $\tilde{\mu}^0(0\sim 1\sim n)=c_0>0$. Moreover, the collection of sets $\{\{0\sim 1\sim n\},n\geq 2\}$ is such that the intersection of two or more sets has a zero measure $\tilde{\mu}^0$ and then $\tilde{\mu}^0(\cup_{n\geq 2}\{0\sim 1\sim n\})=\sum_{n\geq 2}\tilde{\mu}^0(0\sim 1\sim n)\leq\tilde{\mu}^0(0\sim 1)$. It comes that $\tilde{\mu}^0(0\sim 1)=\infty$. This is a contradiction because $\mu^0(0\sim 1)<\infty$. We deduce that $\tilde{\mu}^0(d\pi)$ is equal to $c_0\delta_{K(0,1)}$ and then the partitions with n, 0 and 1 in the same block for some $n\geq 2$ has zero $\tilde{\mu}^0(d\pi)$ measure. By exchangeability, we have $1_{\{0\sim i,|\pi_0|=0\}}\mu^0(d\pi)=c_0\delta_{K(0,i)}$. By the same argument, the collection of sets $\{\{0\sim i\},i\geq 1\}$ is such that the intersection of two sets has a zero measure, moreover the measure $\tilde{\mu}^0$ is carried on simple partition π such that π_0 is not singleton, we then have:

$$1_{\{|\pi|=0,\pi_0\neq\{0\}\}}\mu^0(d\pi)=c_0\sum_{i\geq 1}\delta_{K(0,i)}.$$

The restriction of μ^0 to $\{\pi\in\mathcal{P}_\infty^0;\pi_0=\{0\}\}$ can be viewed as an exchangeable measure on $\mathcal{P}_\infty$, the argument to conclude is then the same as in [4] on p184. □

3.3 Dust in distinguished exchangeable coalescents

Let $(\Pi^0(t), t \geq 0)$ be a distinguished coalescent. We denote by $(D_0(t))_{t\geq 0} := (1-\sum_{i=0}^{\infty} |\Pi_i^0(t)|)_{t\geq 0}$ the process of dust of Π^0. By the distinguished paint-box structure, there is dust if and only if some blocks not containing 0 are singletons. As in [23], the Poissonian structure of distinguished coalescents allows us to find the necessary and sufficient conditions to have dust.

Proposition 6 *For all $t > 0$, the random partition $\Pi^0(t)$ has improper asymptotic frequencies with a strictly positive probability if and only if $c_1 = 0$ and $\int_{\mathcal{P}_m^0}(1-\delta)\nu^0(ds) < \infty$ where $\delta = 1 - \sum_{i=0}^{\infty} s_i$.*
In that case, the process $(\xi^0(t))_{t\geq 0} := (-ln(D_0(t)))_{t\geq 0}$ is a subordinator with Laplace exponent:

$$\phi^0(q) = c_0 q + \int_{\mathcal{P}_m^0}(1-\delta^q)\nu^0(ds).$$

Remark 3.1 *Note that the drift coefficient c_0 may be positive, which contrasts with the result of Pitman [23].*

Proof. $\Pi^0(t)$ has proper asymptotic frequencies a.s if and only if none of its block not containing 0 is singleton a.s. By exchangeability, this is equivalent to $\mathbb{P}[\Pi_1^0(t) = \{1\}] = 0$. Let N be a Poisson Point Process on $\mathcal{P}_\infty^0$ with intensity μ^0. By the Poissonian construction, the event $\{\Pi_1^0(t) = \{1\}\}$ occurs if and only if all the atoms of N on $[0,t]$ fulfill $\pi_1 = \{1\}$; we get:

$$\mathbb{P}[\Pi_1^0(t) = \{1\}] = \mathbb{P}[N([0,t] \times \{\pi \in \mathcal{P}_\infty; \pi_1 \neq \{1\}\}) = 0] = \exp(-ta)$$

where:

$$a = \mu^0(\{\pi; \pi_1 \neq \{1\}\}).$$

By Theorem 5, it comes:

$$a = \infty 1_{c_1>0} + c_0 + \rho_{\nu^0}^0(\pi_1 \neq \{1\})$$

By exchangeability,

$$\rho_{\nu_1}^0(\pi_1 \neq \{1\}) = \int_{\mathcal{P}_\mathbf{m}^0} \mathbb{P}(U_1 \text{ falls not in } [s_1, 1])\nu^0(ds) = \int_{\mathcal{P}_\mathbf{m}^0}(1-\delta)\nu^0(ds).$$

We then have:

$$a < \infty \text{ if and only if } c_1 = 0 \text{ and } \int_{\mathcal{P}_\mathbf{m}^0}(1-\delta)\nu^0(ds) < \infty.$$

The proof of the stationarity and independence of multiplicative increments of $D_0(.)$ is as in Chapter 4 of [4] (p189). We focus on the Laplace exponent of ξ_0. Let $q \geq 1$, $s \in \mathcal{P}_\mathbf{m}^0$, thanks to the paint-box structure, $\rho_s^0(1, ..., q \text{ are singletons}) = \delta^q$, and then $\rho_s^0(\{\pi; \exists 1 \leq i \leq q \text{ not singleton of } \pi\}) = 1 - \delta^q$. Thus, we compute the entire moments of $D_0(t)$:

$$\mathbb{E}[D_0(t)^q] = \mathbb{P}[1, ..., q \text{ are singletons of } \Pi^0(t)].$$

By exchangeability and Poissonian structure, the probability of the event $\{1, ..., q \text{ are singletons of } \Pi^0(t)\}$ is the probability that $N([0,t] \times \{\exists 1 \leq i \leq q \text{ not singleton of } \pi\}] = 0$. By Poissonian calculations, we get:

$$\mathbb{E}[D_0(t)^q] = \exp(-t\mu^0(\{\pi; \exists 1 \leq i \leq q \text{ which is not a singleton of } \pi\})).$$

By the previous recall on distinguished paint-boxes:

$$\mu^0(\{\pi; \exists 1 \leq i \leq q \text{ not singleton of } \pi\}) = c_0 q + \int_{\mathcal{P}^0_{\mathbf{m}}} (1-\delta^q)\nu^0(ds)$$

and then:

$$\mathbb{E}[D_0(t)^q] = \exp[-t(c_0 q + \int_{\mathcal{P}^0_{\mathbf{m}}} (1-\delta^q)\nu^0(ds))].$$

We thus have: $\mathbb{E}[D_0(t)^q] = \mathbb{E}[e^{-q\xi^0(t)}] = e^{-t\phi^0(q)}$ with ϕ^0 as in the statement. $\square$

4 The simple distinguished exchangeable coalescents : M-coalescents

In this section, we focus on simple distinguished coalescent for which the coagulation rate μ^0 is carried by the set of simple distinguished partitions. We call them, in the next, M-coalescents. These processes are the analogue of Λ-coalescents for exchangeable distinguished coalescents. Historically, the Λ-coalescent is the first exchangeable coalescent with multiple collisions to have been defined, see Pitman, [23] and Sagitov, [24]. We begin by recalling some basic facts about Λ-coalescents.

4.1 Λ-coalescents

A Λ-coalescent (also called simple exchangeable coalescent) is a process taking values in the partitions of $\mathbb{N}$ describing the genealogy of an infinite haploid population, labelled by $\mathbb{N}$ where two or more ancestral lineages merging cannot occur simultaneously. We stress that in these coalescent processes, each individual has an ancestor in the population. Immigration phenomenon is not taken into account and no block is distinguished. A simple exchangeable coalescent is a Markovian process $(\Pi(t), t \geq 0)$ on the space of partitions of $\mathbb{N}$: $\mathcal{P}_\infty$ satisfying :

i) If $n \in \mathbb{N}$, then the restriction $(\Pi_{|[n]}(t))$ is a continuous-time Markov chain valued in $\mathcal{P}_n$;

ii) For each n, $(\Pi_{|[n]}(t))$ evolves by exchangeable merging of blocks : $\Pi_{|[n]}(t) = Coag(\Pi_{|[n]}(t-), \pi')$ where π' is an independent simple exchangeable partition;

By Theorem 1 in Pitman [23], or Sagitov [24], we know that any simple exchangeable coalescent is characterized in law by a finite measure Λ on $[0,1]$. The dynamics of Π can be described as follows: whenever $\Pi_{|[n]}(t)$ is a partition with b blocks, the rate at which a k-tuple of its blocks merges is :

$$\lambda_{b,k} = \int_0^1 x^{k-2}(1-x)^{b-k}\Lambda(dx)$$

When Λ is the Dirac at 0, we recover the Kingman's coalescent. When $\Lambda(\{0\}) = 0$, the Λ-coalescent can be constructed via a Poisson Point Process on $\mathbb{R}_+ \times [0,1]$ with intensity $dt \otimes \nu(dx)$, where $\nu(dx) = x^{-2}\Lambda(dx)$:

$$N = \sum_{i \in \mathbb{N}} \delta_{(t_i, x_i)}$$

The atoms of N encode the evolution of the coalescent Π: At time $t-$, flip a coin with probability of "heads" x for each block. All blocks flipping "heads" are merged immediately. We can, also, construct a simple exchangeable partition, π' where the non trivial block is constituted by indices of "heads". Thus, we get $\Pi(t)$ by $Coag(\Pi(t-), \pi')$. In order to make this construction rigorous, one first considers the restrictions $(\Pi_{|[n]}(t))$ as in Proposition 4, since the measure $\nu(dx) := x^{-2}\Lambda(dx)$ can have an infinite mass.

4.2 M-coalescents

The distinguished exchangeable coalescents such that when a coagulation occurs all the blocks involved merge as a single block are called M-coalescents. We specify their laws by two finite measures on $[0,1]$, and study their generators in the same fashion than those of Λ-coalescents.

Definition 7 *When a distinguished coagulation rate μ^0 is carried by the set of simple distinguished partitions (with only one block non empty nor singleton), the distinguished coalescent Π^0 is said to be simple. Define the following restricted measures:*

$$\nu_0 = \nu^0 1_{\{s\in\mathcal{P}^0_m; s=(s_0,0,\ldots)\}} \text{ and } \nu_1 = \nu^0 1_{\{s\in\mathcal{P}^0_m; s=(0,s_1,0,\ldots)\}}.$$

We can write $\nu^0 = \nu_0 + \nu_1$. By a slight abuse of notation, ν_0 and ν_1 can be viewed as two measures on $[0,1]$ such that $\int_0^1 s_0\nu_0(ds_0) < \infty$ and $\int_0^1 s_1^2\nu_1(ds_1) < \infty$.

$$\mu^0 = c_0\mu_0^K + \rho^0_{\nu_0} + c_1\mu_1^K + \rho_{\nu_1}.$$

We define the finite measures $\Lambda_0(dx) = x\nu_0(dx) + c_0\delta_0$, and $\Lambda_1(dx) = x^2\nu_1(dx) + c_1\delta_0$. The law of a simple distinguished coalescent is then characterized by $M = (\Lambda_0, \Lambda_1)$, and we call this subclass the M-coalescents.

As already mentioned in Section 3, in the most cases, the restriction to $\mathbb{N}$ of a distinguished coalescent is not Markovian. Let Π^0 be a M-coalescent with for instance $\Lambda_0 = \delta_0$ and $\Lambda_1(dx) = dx$ (the Lebesgue measure). To locate the distinguished block in $\Pi^0_{|\mathbb{N}}$, we may locate a binary coagulation before time t (all other mergers involve an infinite number of blocks). The restricted process $\Pi^0_{|\mathbb{N}}$ is then not Markovian.

The explicit Poissonian construction of Proposition 4 can now be interpreted in the same way that the one of Λ-coalescent, recalled in Section 4.1. When $\Lambda_0(\{0\}) = \Lambda_1(\{0\}) = 0$, the M-coalescent associated can be constructed via two Poisson Point Processes N_0 and N_1 on $\mathbb{R}_+ \times (0,1]$ with intensities $dt \otimes \nu_0(dx)$ and $dt \otimes \nu_1(dx)$, where $\nu_0(dx) = x^{-1}\Lambda_0(dx)$ and $\nu_1(dx) = x^{-2}\Lambda_1(dx)$:

- At an atom (t_i, x_i) of N_1, flip a coin with probability of "heads" x_i for each block not containing 0. All blocks flipping "heads" are merged immediately in one block as in the Proposition 4.
- At an atom (t_i, x_i) of N_0, flip a coin with probability of "heads" x_i for each block not containing 0. All blocks flipping "heads" coagulate immediately with the distinguished block.

This construction is exactly the one obtained when we coagulate the partition at $t-$ with a simple exchangeable distinguished partition π' where the non trivial block is constituted by indexes of "heads". Thus, we construct the M-coalescent in the same way that Λ-coalescent in Section 4.1.

We investigate jump rates of a M-coalescent $(\Pi^0(t))_{t\geq 0}$. Thanks to the simple distinguished paint-box structure, we compute explicitly the jump rates of the restriction of Π^0. Let $\pi \in \mathcal{P}_n^0$ be simple, $q_\pi = \mu^0(\mathcal{P}_\infty^0, \pi)$:

- For every $2 \leq k \leq n$, if π has one block not containing 0 with k elements, then:

$$q_\pi = \lambda_{n,k} := \int_0^1 x^{k-2}(1-x)^{n-k}\Lambda_1(dx).$$

- For every $1 \leq k \leq n$, if the distinguished block of π has $k+1$ elements (counting 0), then:

$$q_\pi = r_{n,k} := \int_0^1 y^{k-1}(1-y)^{n-k}\Lambda_0(dy).$$

Let $\pi \in \mathcal{P}_p^0$ with b blocks whithout 0 and F any function defined on $\mathcal{P}_p^0$. The generator of $\Pi^0_{|[n]}$ is:

$$\mathcal{L}^*F(\pi) = \sum_{I\subset\{1,\dots,b\},|I|\geq 2} \lambda_{b,|I|}(F(c_I\pi) - F(\pi)) + \sum_{J\subset\{1,\dots,b\},|J|\geq 1} r_{b,|J|}(F(c^J\pi) - F(\pi))$$

With : $c_I\pi = Coag(\pi, \{\{0\},\{1\},...,\{I\},...\})$ and $c^J\pi = Coag(\pi, \{\{0\}\cup\{J\},\{.\},...,\{.\}\})$

4.3 Coming down from infinity for M-coalescents

Let Λ be a finite measure on $[0,1]$ and Π be a Λ-coalescent. Pitman [23] showed that if $\Lambda(\{1\}) = 0$, only the following two types of behavior are possible, either $\mathbb{P}$[For all $t > 0$, $\Pi(t)$ has infinitely many blocks] $= 1$, or $\mathbb{P}$[For all $t > 0$, $\Pi(t)$ has only finitely many blocks] $= 1$. In the second case, the process Π is said to come down from infinity. For instance, Kingman's coalescent comes down from infinity, while if $\Lambda(dx) = dx$, then the corresponding Λ-coalescent (called Bolthausen-Sznitman coalescent) does not come down from infinity. A necessary and sufficient condition for a Λ-coalescent to come down from infinity was given by Schweinsberg in [25]. Define,

$$\phi(n) = \sum_{k=2}^{n}(k-1)C_n^k\lambda_{n,k}$$

with $\lambda_{n,k} = \int_0^1 x^{k-2}(1-x)^{n-k}\Lambda(dx)$. The Λ-coalescent comes down from infinity if and only if $\sum_{n=2}^{\infty}\frac{1}{\phi(n)} < \infty$.
Define $\psi_\Lambda(q) := \int_{[0,1]}(e^{-qx} - 1 + qx)x^{-2}\Lambda(dx)$, Bertoin and Le Gall observed the following equivalence:

$$\sum_{n=2}^{\infty}\frac{1}{\phi(n)} < \infty \Longleftrightarrow \int_a^\infty \frac{dq}{\psi_\Lambda(q)} < \infty$$

where the right-hand side holds for some $a > 0$ (and then necessary for all). This equivalence is explained in a probabilistic way by Berestycki et al. in [2].

As for the Λ-coalescent, if the M-coalescent comes down from infinity, it does immediately:

Proposition 7 *Let $(\Pi^0(t))_{t\geq 0}$ be a M-coalescent, with Λ_0 and Λ_1 without mass at 1. We denote by T its time of coming down from infinity: $T = \inf\{t > 0, \#\Pi(t) < \infty\}$. We have a.s $T = 0$ or $T = \infty$.*

Proof. See Proposition 23 of [23]. □

We stress that when $\Lambda_0 + \Lambda_1$ has a mass at 1, the M-coalescent comes down from infinity. Indeed, by the Poisson construction, in an exponential time τ of parameter $(\Lambda_0 + \Lambda_1)(\{1\})$, the Poisson measure N has an atom π such that $\pi_0 = \mathbb{Z}_+$ or $\pi_1 = \mathbb{N}$. Thus for large t, the process $\Pi^0(t)$ has just one block.

It remains to focus on the case where $\Lambda_0 + \Lambda_1$ has no mass at 1. Intuitively, when the genuine Λ_1-coalescent comes down, all blocks merged in one in an almost surely finite time. On the one hand, we can think that the (Λ_0, Λ_1)-coalescent has more jumps and coagulates all its blocks faster. On the other hand, the perturbation due to the coagulation with the distinguished block on the general term of the sum, studied initially by Schweinsberg in [25], is not sufficient to induce its convergence and so the coming down. The (Λ_0, Λ_1)-coalescent comes down from infinity if and only if the Λ_1-coalescent comes down:

Theorem 8 *The (Λ_0, Λ_1)-coalescent comes down from infinity if and only if $\sum_{n=2}^{\infty} \frac{1}{\phi_1(n)} < \infty$ where $\phi_1(n) = \sum_{k=2}^{n} (k-1) C_n^k \lambda_{n,k}$ and $\lambda_{n,k}$ as in Section 4.2.*

The proof requires rather technical arguments and is given in the Appendix.

In the final section, we are interesting in a population model which has exactly a genealogy given by a M-coalescent. A powerful mean to study simultaneously the population model and its genealogy is to define some stochastic flows as Bertoin and Le Gall in [5]. A process valued in the space of *probability measures* on $[0, 1]$, $\rho_t^0, t \geq 0$ is embedded in the flow. The atoms of the random probability ρ_t^0 represent the current types frequencies in the population at time t. Moreover, ρ_t^0 has a distinguished atom at 0 representing the fraction of immigrants in the population. This process will be called M-generalized Fleming-Viot processes with immigration. Following [5], we begin by etablishing a correspondence between some stochastic flows and M-coalescent.

5 M-coalescents and generalized Fleming-Viot processes with immigration

5.1 Stochastic flows of distinguished bridges

By assumption, the population into families of the population is a distinguished exchangeable partition. Theorem 2 ensures that it has a distinguished paint-box structure. We have to study some random functions called distinguished bridges.

5.1.1 Distinguished bridges and exchangeable distinguished partitions

Considering the underlying law on $[0,1]$ associated with a s-distinguished paint-box (see definition 3), we introduce the distinguished bridges defined by:

$$b_s(r) = s_0 + \sum_{i=1}^{\infty} s_i 1_{\{V_i \leq r\}} + \delta r$$

where s is a distinguished mass-partition and $(V_i)_{i\geq 1}$ a sequence of independent uniform variables. Let $U_0 = 0$ and $(U_i)_{i\geq 1}$ be an independent sequence of i.i.d uniform variables. The partition given by $i \sim j$ iff $b_s^{-1}(U_i) = b_s^{-1}(U_j)$ is exactly the s-distinguished paint-box. When $s_0 = 0$, the bridge encodes a paint-box partition with no distinguished block.

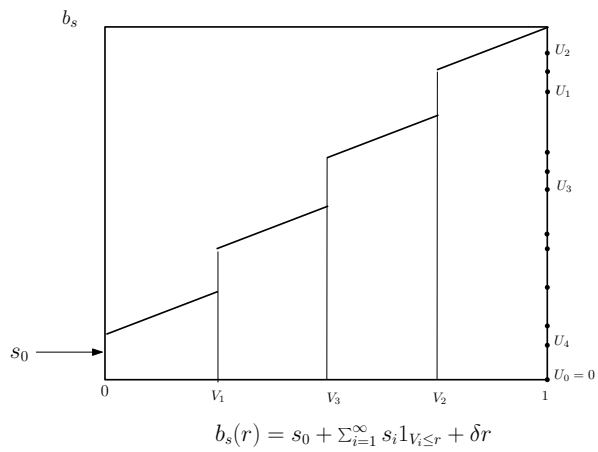

Figure 1: Distinguished bridge

Focussing on M-coalescents, we only need to focus on two types of distinguished bridges:

- Bridges with distinguished mass-partition $(0, x, 0, ...)$: $b_{0,x}(r) = x1_{V\leq r} + r(1-x)$.
- Bridges with distinguished mass-partition $(x, 0, 0, ...)$: $b_{x,0}(r) = x + r(1-x)$.

To be concise, we shall work directly with distinguished bridges of the form

$$b_{y,x}(r) = y + x1_{V\leq r} + r(1-x-y).$$

The following lemma relates the composition of distinguished bridges to the coagulation of simple distinguished partitions.

Lemma 9 *Let $x, x', y, y' \in [0,1]$ with $x+y, x'+y' \leq 1$ and $b_{y,x}$, $b_{y',x'}$ two independent distinguished bridges.*
Let $\pi : i \sim j$ iff $b_{y,x}^{-1}(U_i) = b_{y,x}^{-1}(U_j)$. We stress that π has at most two non-trivial blocks.

(i) For $i \geq 1$, we set $U_i' := b_{y,x}^{-1}(U_j)$, $\forall j \in \pi_i$. The variables $(U_i')_{i\geq 1}$ are i.i.d uniform, independent of π.

(ii) Let π' be the partition constructed from $b_{y',x'}$ and $(U_i')_{i\geq 1}$, we denote by π^0 the partition such that:
$i \sim j$ iff $b_{y,x}^{-1} \circ b_{y',x'}^{-1}(U_i) = b_{y,x}^{-1} \circ b_{y',x'}^{-1}(U_j)$. We have the identity $\pi^0 = Coag(\pi, \pi')$.

Proof. The proof is an easy adaptation of Lemma 4.8 of [4].□

This lemma is the key observation in order to associate a stochastic flow of distinguished bridges, as defined below, with M-coalescents.

Definition 8 *A flow of distinguished bridges is a collection* $(B_{s,t}, -\infty < s \leq t < \infty)$ *of distinguished bridges such that:*

- $\forall s < t < u, B_{s,u} = B_{s,t} \circ B_{t,u}$ *a.s.*
- *The law of* $B_{s,t}$ *depends only of* $t - s$*, and for any* $s_1 < ... < s_n$*,* $B_{s_1,s_2}, ..., B_{s_{n-1},s_n}$ *are independent.*
- $B_{0,0} =$*Id,* $B_{0,t} \to$ *Id in probability when* $t \to 0$*.*

5.1.2 Poissonian construction of distinguished flows encoding M-coalescent

Let M^0 and M^1 be two independent Poissonian measures on $\mathbb{R}\times[0,1]$ with intensities $dt\otimes\nu_0(dx)$ and $dt \otimes \nu_1(dx)$. We suppose $\nu_0([0,1]) + \nu_1([0,1]) < \infty$ so that $N_{s,t} := (M^0 + M^1)(]s,t] \times [0,1])$ is finite and $(N_t) = (N_{0,t})_{t\geq 0}$ is a Poisson process. Let (t_i^0, x_i^0), (t_i^1, x_i^1) be the atoms of M^0 and M^1 in $]s,t] \times [0,1]$, we define $B_{s,t} = b_{x_1} \circ ... \circ b_{x_K}$ where $K = N_{s,t}$ and where b_x denotes $b_{0,x}$ or $b_{x,0}$ according to x is an atom of M^0 or M^1. From independence of M^0 and M^1 and independence of $M^i(A)$ and $M^i(B)$, $i = 0, 1$ for A and B disjoint, $(B_{s,t})_{s\leq t}$ is a flow in the sense of definition 8.

Proposition 10 *The process* $(\Pi^0(t), t \geq 0)$ *:* $i \sim j \Leftrightarrow B_{0,t}^{-1}(U_i) = B_{0,t}^{-1}(U_j)$ *where* $B_{0,t} = b_{x_1} \circ ... \circ b_{x_{N_t}}$ *is a* M*-coalescent with* $M = (x\nu_0(dx), x^2\nu_1(dx))$*.*

Proof. Lemma 9 implies that the process $(\Pi^0(t))_{t\geq 0}$ suits to that build explicitly in Proposition 4. □
The next result defines stochastic flows for general measures ν_0 and ν_1 on $[0,1]$.

Theorem 11 *Let* (ν_0^n) *and* (ν_1^n) *be two sequences of finite measures on* $[0,1]$*. We call* $(B_{s,t}^{(n)}, -\infty < s \leq t < \infty)$ *the flow of bridges associated. Assume the weak convergences of* $\Lambda_1^n(dx) := x^2\nu_1^n(dx)$ *to* $\Lambda_1(dx) : c_1\delta_0(dx) + x^2\nu_1(dx)$ *and* $\Lambda_0^n(dy) := y\nu_0^n(dy)$ *to* $\Lambda_0(dy) := c_0\delta_0 + y\mu(dy)$*. We get:*

- $(B_{s,t}^{(n)}, \infty < s \leq t < \infty)$ *converges, in the sense of convergence of finite-dimensional distributions, to* $(B_{s,t}, \infty < s \leq t < \infty)$ *a stochastic flow.*
- *The process* $(\Pi^0(t), t \geq 0)$ *defined by :* $\Pi^0(t) : i \sim j \Leftrightarrow B_{0,t}^{-1}(U_i) = B_{0,t}^{-1}(U_j)$ *is a* M*-coalescent with rates* (Λ_0, Λ_1)*.*

Proof. We denote by $\Pi^{(n)}(s,t)$ the random partition encoded by $B_{s,t}^{(n)}$. Under the previous assumptions on ν_0 and ν_1, the jump rates $\lambda_{b,k}^{(n)} = \int_0^1 x^k(1-x)^{b-k}\nu_1^n(dx)$ and $r_{b,k}^{(n)} = \int_0^1 y^k(1-y)^{b-k}\nu_0^n(dy)$ converge:

$$\lambda_{b,k}^{(n)} \underset{n\to\infty}{\to} c_1 1_{k=2} + \int_0^1 x^k(1-x)^{b-k}\nu_1(dx) := \lambda_{b,k}.$$

$$r_{b,k}^{(n)} \underset{n\to\infty}{\to} c_0 1_{k=1} + \int_0^1 x^k(1-x)^{b-k}\nu_0(dx) := r_{b,k}.$$

The sequence of Markov chains $(\Pi^{(n)}_{|[k]}(s,t))_{t\geq s}$ converges in the sense of finite-dimensional distributions to a distinguished coalescent chain, say $\Pi^0_{|[k]}(s,t)_{t\geq s}$. By compatibility, this implies the convergence of finite dimensional distributions of $(\Pi^{(n)}(s,t))_{n\geq 1}$ to $\Pi^0(s,t)$. According to Proposition 2.9 and Lemma 4.7 in [4] (which are easily adapted in our setting), we obtain the convergence of the distinguished mass-partitions $|\Pi^{(n)}(s,t)|^{\downarrow} \underset{n\to\infty}{\to} |\Pi^0(s,t)|^{\downarrow}$ and the convergence of the bridge $B^{(n)}_{s,t}$ (which has jumps of size $|\Pi^{(n)}(s,t)|^{\downarrow}$) to a bridge $B_{s,t}$ (which has jumps of size $|\Pi^0(s,t)|^{\downarrow}$) for all $s,t\geq 0$ fixed. Thanks to the independence of $B^{(n)}_{s_1,s_2},...,B^{(n)}_{s_{k-1},s_k}$ for any $s_1<...<s_k$ and the flow property ($B^{(n)}_{s,t}\circ B^{(n)}_{t,u}=B^{(n)}_{s,u}$), the one-dimensional convergence in distribution readily extends to finite dimensional distribution. The existence of the flow B is ensured by Kolmogorov's Extension Theorem.□

Remark 5.1 *As mentioned in Section 2, we could define coalescents with several distinguished blocks. In particular, considering distinguished bridges which jump at* 0 *and* 1*, we get a flow coding a coalescent with two distinguished blocks and a population with two immigration sources.*

The composition of two distinguished bridges may be interpreted as the succession of two events (reproduction or immigration) in the population. A duality method provides a continuous population model.

5.2 The dual distinguished flow and a population model with immigration

In the same spirit of [4] and [5], we interpret the dual flow, $(\hat{B}_{s,t}) := (B_{-t,-s})$ in terms of a natural model population on $[0,1[$ with fixed size 1. We denote by $\rho^0_t(dr)$, the random Stieljes measure of $\hat{B}_{0,t}$: $\rho^0_t = d_r\hat{B}_{0,t}$, it defines a Markov process with values in the space of probability measures on $[0,1[$ (denoted by $\mathcal{M}_1$). We may think of $\rho^0_t(dr)$ and $\rho^0_t(\{0\})$ respectively as the size of the progeny at time t of the fraction dr of the initial population, and as the size at time t of the immigrants descendants.
The cocycle identity $\hat{B}_{t,u}\circ\hat{B}_{s,t}=\hat{B}_{s,u}$ ensures that (ρ^0_t) is a continuous-time Markov chain with the following dynamics, whenever the measures ν_0 and ν_1 are finite: if t is a jump time for $\rho^0_\cdot$, then the conditional law of ρ^0_t given ρ^0_{t-} is that of

- $(1-X)\rho^0_{t-}+X\delta_Z$, if t is an atom of M^1, where X is distributed as $\nu_1(.)/\nu_1([0,1])$ and Z as ρ^0_{t-}.
- $(1-Y)\rho^0_{t-}+Y\delta_0$, if t is an atom of M^0, where Y is distributed as $\nu_0(.)/\nu_0([0,1])$.

At a reproduction time (meaning an atom of M^1) an individual picked at random in the population at generation $t-$ generates a proportion X of the population at time t, as for the genuine generalized Fleming-Viot. At an immigration time (meaning an atom of M^0) the individual 0 at the time $t-$ generates a proportion Y of the population at time t. In both cases, the rest of the population at time $t-$ is reduced by a factor $1-X$ or $1-Y$ so that, at time t, the total size is still 1. We call this measure-valued process a generalized Fleming-Viot process with immigration (GFVI). The genealogy of this population (which is identified as $[0,1]$) coincides with a M-coalescent. Plainly, the generator of $(\rho^0_t, t\geq 0)$ is

$$\mathcal{L}G(\rho)=\int\nu_1(dx)\int\rho(da)[G((1-x)\rho+x\delta_a)-G(\rho)]+\int\nu_0(dy)[G((1-y)\rho+y\delta_0)-G(\rho)].$$

Thus, for any bounded function G on $\mathcal{M}_1$, the space of probability measures on $[0,1]$:

$$G(\rho_t^0) \quad - \quad \int_0^t ds \int \nu_1(dx) \int \rho_s^0(da)[G((1-x)\rho_s^0 + x\delta_a) - G(\rho_s^0)]$$
$$- \int_0^t ds \int \nu_0(dy)[G((1-y)\rho_s^0 + y\delta_0) - G(\rho_s^0)]$$

is a martingale. Considering the functions of the form:

$$G_f : \rho \in \mathcal{M}_1 \mapsto \int_{[0,1]^p} f(x_1, ..., x_p)\rho(dx_1)...\rho(dx_p) = \langle f, \rho^{\otimes p}\rangle,$$

for f a continuous function on $[0,1]^p$, we generalize in the following lemma this result for infinite measures:

Lemma 12 *Assume that ν_0 and ν_1 have infinite masses and c_0, c_1 are zero, we define the operator $\mathcal{L}$, acting on functions of the type G_f, by:*

$$\mathcal{L}G_f(\rho) = \int \nu_1(dx) \int \rho(da)[G_f((1-x)\rho + x\delta_a) - G_f(\rho)] + \int \nu_0(dy)[G_f((1-y)\rho + y\delta_0) - G_f(\rho)].$$

The process $G_f(\rho_t^0) - \int_0^t \mathcal{L}G_f(\rho_s^0)ds$ is a martingale.

Proof. Consider two sequences (ν_1^n) and (ν_0^n) of finite measures on $[0,1[$, suppose that $\Lambda_1^n(dx) := x^2\nu_1^n(dx)$ and $\Lambda_0^n(dx) := x\nu_0^n(dx)$ weakly converge to some finite measures $\Lambda_1(dx)$ and $\Lambda_0(dx)$. For $G_f(\rho) = \prod_i^p < \rho, \phi_i >$, Bertoin and Le Gall obtain in [5]:

$$\int \nu_1^n(dx) \int \rho(da)[G((1-x)\rho + x\delta_a) - G(\rho)] = \sum_{\substack{I \subset \{1,...,N\} \\ |I| \geq 2}} \lambda_{p,|I|}^n \int \rho(dx_1)...\rho(dx_p)[f(x_1^I, ..., x_p^I) - f(x_1, ..., x_p)].$$

With $(x_1^I, ..., x_p^I) = (y_1, ..., y_p)$ where for all $i \in I$, $y_i = x_{\inf I}$ and the values y_i, $i \notin I$ listed in the order of $\{1, ..., p\} \setminus I$ are the numbers $x_1, .., x_{\inf I - 1}, x_{\inf I + 1}, ..., x_{p-|I|+1}$.

The assumption on ν_1 ensures that the right side converges to

$$\sum_{I \subset \{1,...,N\}; |I| \geq 2} \lambda_{p,|I|} \int \rho(dx_1)...\rho(dx_p)[f(x_1^I, ..., x_p^I) - f(x_1, ..., x_p)].$$

It remains to study the "immigration" part, that is to etablish:

$$\int \nu_0^n(dy)[G((1-y)\rho + y\delta_0) - G(\rho)] = \sum_{J \subset \{1,...,N\}; |J| \geq 1} r_{p,|J|}^n \int \rho(dx_1)...\rho(dx_p)[f(x_1^0, ..., x_p^0) - f(x_1, ..., x_p)].$$

With $(x_1^0, ..., x_p^0) = (z_1, ..., z_p)$ where for all $i \in J$, $z_i = 0$ and the values z_i, $i \notin J$ listed in the order of $\{1, ..., p\} \setminus J$ are the numbers $x_1, .., x_{\inf J - 1}, x_{\inf J + 1}, ..., x_{p-|J|}$.

An easy calculation gives:

$$G((1-y)\rho + y\delta_0) = \prod_{i=1}^p [(1-y)\langle \rho, \phi_i\rangle + y\phi_i(0)] = \sum_{J \subset \{1,...,p\}} (1-y)^{p-|J|} y^{|J|} \prod_{j \notin J} \langle \rho, \phi_j\rangle \prod_{j \in J} \phi_j(0)$$

and then from the following identity

$$\prod_{j\notin J}\langle\rho,\phi_j\rangle\prod_{j\in J}\phi_j(0)=\int_{[0,1]^p}f(x_1^0,...,x_p^0)\rho(dx_1)...\rho(dx_p)$$

it comes:

$$\int\nu_0^n(dy)G((1-y)\rho+y\delta_0)=\sum_{J\subset\{1,...,N\};|J|\geq 1}r_{p,|J|}^n\int\rho(dx_1)...\rho(dx_p)f(x_1^0,...,x_p^0).$$

Moreover, the right hand side converges to

$$\sum_{J\subset\{1,...,N\};|J|\geq 1}r_{p,|J|}\int\rho(dx_1)...\rho(dx_p)f(x_1^0,...,x_p^0),$$

by passing to the limit in $n\to\infty$, it follows that for $f(x_1,...,x_p)=\prod_i^p\phi_i(x_i)$, the process

$$M_f(t):=G_f(\rho_t^0)-\int_0^t LG_f(\rho_s^0)ds$$

is a martingale where L is the operator defined by:

$$\begin{aligned}LG_f(\rho)=&\sum_{I\subset\{1,...,p\};|I|\geq 2}\lambda_{p,|I|}\int\rho(dx_1)...\rho(dx_p)[f(x_1^I,...,x_p^I)-f(x_1,...,x_p)]\\&+\sum_{J\subset\{1,...,p\};|J|\geq 1}r_{p,|J|}\int\rho(dx_1)...\rho(dx_p)[f(x_1^0,...,x_p^0)-f(x_1,...,x_p)].\end{aligned}$$

Since any continuous function on $[0,1]^p$ is the uniform limit of linear combinations of functions of the previous type, we easily conclude that M_f is a martingale for any continuous function on $[0,1]^p$. The statement claims that when $c_0=c_1=0$ the generator has an integral form as the one obtained for finite measures. We assume now that c_0 and c_1 are zero. Let $A_1,...,A_p$ i.i.d variables distributed as ρ, and $x,y\in[0,1]$. Let (β_j), (β_j') be two sequences of Bernoulli variables of parameters x and y. We set $I:=\{j,\beta_j=1\}$ and $J:=\{j,\beta_j'=1\}$. Let f be a continuous function on $[0,1]^p$, for $G_f(\rho)=\langle\rho^{\otimes p},f\rangle$, it is readily checked (see [8]) that :

$$\begin{aligned}&\int\rho(da)[G_f((1-x)\rho+x\delta_a)-G_f(\rho)]=\mathbb{E}[f(A_1^J...,A_p^J)]-\mathbb{E}[f(A_1,...,A_p)]\\&=\sum_{I\subset\{1,...,p\},|I|\geq 2}x^{|I|}(1-x)^{p-|I|}\int\rho(dx_1)...\rho(dx_p)(f(x_1^I,...,x_p^I)-f(x_1,...,x_p)),\end{aligned}$$

$$\begin{aligned}&G_f((1-y)\rho+y\delta_0)-G_f(\rho)=\mathbb{E}[f(A_1^0...,A_p^0)]-\mathbb{E}[f(A_1,...,A_p)]\\&=\sum_{J\subset\{1,...,p\},|J|\geq 1}y^{|J|}(1-y)^{p-|J|}\int\rho(dx_1)...\rho(dx_p)(f(x_1^0,...,x_p^0)-f(x_1,...,x_p)).\end{aligned}$$

We deduce that the process (ρ_t^0) solves the following martingale problem: for any continuous function f on $[0,1]^p$, $G_f(\rho_t^0)-\int_0^t ds\mathcal{L}G_f(\rho_s^0)$ is a martingale. $\square$

Proposition 13 *The law of the process $(\rho_t^0, t \geq 0)$ is characterized by the martingale problem of Lemma 12, and the operator $\mathcal{L}$ is an extended generator of the process $(\rho_t^0, t \geq 0)$.*

Proof. We will use the same duality argument as in Bertoin and Le Gall [5]. With their notation, we define a class of functions from $\mathcal{M}_1 \times \mathcal{P}_p^0$ to $\mathbb{R}$:

$$\Phi_f : (m, \pi) \in \mathcal{M}_1 \times \mathcal{P}_p^0 \mapsto \int \delta_0(dx_0)m(dx_1)...m(dx_{\#\pi-1})f(Y(\pi; x_1, ..., x_{\#\pi-1}))$$

with f a continuous function on $[0,1]^p$ and $Y(\pi; x_1, ..., x_{\#\pi-1}) = (y_1, ..., y_p)$ such that $y_j = x_i$ if $j \in \pi_i$ for any $i \geq 0$.
For a fixed partition π in $\mathcal{P}_n^0$, there exists a function g continuous on $[0,1]^{\#\pi-1}$ with $\mu \mapsto \Phi_f(\mu, \pi) = G_g(\mu)$ and then $\mathcal{L}\Phi_f(\mu, \pi) = \mathcal{L}G_g(\mu)$ is well-defined. We stress that for a fixed measure μ, $\Phi_f(\mu, .)$ is a function on $\mathcal{P}_p^0$. We show the following duality result:

$$\mathbb{E}[\Phi_f(\rho_0^0, \Pi^0(t))] = \mathbb{E}[\Phi_f(\rho_t^0, \Pi_0^0)].$$

By the cocycle property of the stochastic flow involved, it suffices to focus on process beginning at $\rho_0^0 = \lambda$.

$$\begin{aligned}
\mathbb{E}_\lambda[\phi_f(\rho_t^0, 0_{[p]})] &= \mathbb{E}[\int_{[0,1]^{p+1}} \delta_0(dx_0)d\hat{B}_{0,t}(x_1)...d\hat{B}_{0,t}(x_p)f(x_1, ..., x_p)] \\
&= \mathbb{E}[\int_{[0,1]^{p+1}} \delta_0(dx_0)dx_1...dx_p f(\hat{B}_{0,t}^{-1}(x_1), ..., \hat{B}_{0,t}^{-1}(x_p))] \\
&= \mathbb{E}[f(\hat{B}_{0,t}^{-1}(V_1), ..., \hat{B}_{0,t}^{-1}(V_p))]
\end{aligned}$$

With $(V_i, 0 \leq i \leq p)$ independent uniformly distributed on $[0,1]$. We define for $1 \leq i \leq \#\Pi_{|[p]}^0(t) - 1$, $V_i' := \hat{B}_{0,t}^{-1}(V_j)$ for $j \in \Pi_{i|[p]}^0(t)$. By Lemma 9, $(V_i')_{1 \leq i \leq \#\Pi_{|[p]}^0(t)-1}$ are uniform iid, independent of $\Pi^0(t)$ where Π^0 is a M-coalescent with rates $(x\nu_0(dx), x^2\nu_1(dx))$.
We get,

$$\mathbb{E}_\lambda[\phi_f(\rho_t^0, 0_{[p]})] = \mathbb{E}[\int_{[0,1]^{p+1}} \delta_0(dx_0)dx_1...dx_{\#\Pi_{|[p]}^0(t)-1} f(y_1, ...y_{\#\Pi_{|[p]}^0(t)-1})]$$

with $y_j = x_i$ if $j \in \Pi_{i|[p]}^0(t)$.

Thus, we deduce the duality result:

$$\mathbb{E}_\lambda[\Phi_f(\rho_t^0, 0_{[p]})] = \mathbb{E}[\Phi_f(\lambda, \Pi_{|[p]}^0(t))] \text{ and then } \mathcal{L}\Phi_f(\mu, \pi) = \mathcal{L}^*\Phi_f(\mu, \pi)$$

From Theorem 4.4.2 in [14], this implies uniqueness for the martingale problem, as well as strong Markov property for the solution.□

Remark 5.2 *In the case of a standard M-coalescent, ρ_0^0 is the Lebesgue measure λ, and we have:*

$$\rho_t^0(dr) = |\Pi_0^0(t)|\delta_0(dr) + \textstyle\sum_{i \geq 1} |\Pi^0(t)|_i^\downarrow \delta_{W_i}(dr) + (1 - \sum_{i \geq 0} |\Pi^0(t)|_i^\downarrow)dr$$

where $(W_i, i \geq 1)$ are independent uniform and independent of $\Pi^0(t)$.

The extinction of the initial types corresponds to the absorption of the GFVI process $(\rho_t^0, t \geq 0)$ at δ_0. Plainly this event occurs if and only if the measure Λ_0 is not the zero measure and the M-coalescent embedded is coming down from infinity. By Theorem 8, we know that the coming down from infinity depends only on the measure Λ_1. In terms of the population model, the immigration mechanism, encoded by $\Lambda_0 := c_0\delta_0 + x\nu_0(dx)$, has no impact on the extinction occurrence, provided of course that Λ_0 is not the zero measure.

6 Appendix

We recall the statement of Theorem 8 and give a proof based on martingale arguments.

Theorem 8 *The (Λ_0, Λ_1)-coalescent comes down from infinity if and only if*

$$\sum_{n=2}^{\infty} \frac{1}{\phi_1(n)} < \infty$$

where $\phi_1(n) = \sum_{k=2}^{n}(k-1)C_n^k\lambda_{n,k}$ and $\lambda_{n,k} = \int_0^1 x^{k-2}(1-x)^{n-k}\Lambda_1(dx)$.

For the Λ-coalescents, (in our setting it corresponds to have $\Lambda_0 \equiv 0$), Schweinsberg studied the mean time of "coming down from infinity" and conclude using the Kochen-Stone lemma. The proof we give here is based on martingale arguments. The sufficient part is proved using Lemma 14 which is close to the one of Proposition 4.9 page 202 in [4]. The necessary part of the proof does not follow Schweinsberg's ideas. Assuming that the coalescent comes down from infinity and the sum is infinite, we will define a supermartingale (thanks to Lemmas 6, 16 and 17) and find a contradiction (Lemma 18).

Lemma 14 *Let $(\Pi^0(t), t \geq 0)$ be a (Λ_0, Λ_1)-coalescent where $\Lambda_0 + \Lambda_1$ has no mass at 1. Let us define the fixation time*

$$\zeta := \inf\{t \geq 0, \Pi^0(t) = \{\mathbb{Z}_+, \emptyset, ...\}\}.$$

Define

$$\phi(n) = \sum_{k=2}^{n}(k-1)C_n^k\lambda_{n,k} + \Lambda_0([0,1])n,$$

then the expectation of fixation time is bounded by

$$\mathbb{E}[\zeta] \leq \sum_{n=1}^{\infty} 1/\phi(n).$$

As a consequence, if the series in the right-hand side converges, the fixation time is finite with probability one.

Proof. We shall study the process of blocks which do not contain 0: for all $t > 0$, $\Pi^*(t) := \{\Pi_1^0(t), ...\}$. This process is not partitions-valued. The jump rates of $\#\Pi^*_{|[n]}(t)$ are easily computed: for $2 \leq k \leq l+1$, $\#\Pi^*_{|[n]}$ jumps from l to $l-k+1$ with rate:

$$C_l^k \lambda_{l,k} 1_{k \leq l} + C_l^{k-1} r_{l,k-1}.$$

The first term represents coagulation of k blocks not containing 0 in one, the second represents disappearance of $k-1$ blocks (coagulation with Π_0^0). We get the infinitesimal generator of $\#\Pi^*_{|[n]}$:

$$G^{[n]} f(l) = \sum_{k=2}^{l} C_l^k \lambda_{l,k} [f(l-k+1) - f(l)] + \sum_{k=2}^{l+1} C_l^{k-1} r_{l,k-1} [f(l-k+1) - f(l)].$$

We define

$$\phi_1(n) = \textstyle\sum_{k=2}^{n} (k-1) C_n^k \lambda_{n,k} \text{ and } \phi_2(n) = \sum_{k=2}^{n+1} (k-1) C_n^{k-1} r_{n,k-1}.$$

Using the binomial formula, we get $\phi_2(n) = \Lambda_0([0,1])n$. We remark that ϕ_1 is an increasing function. Setting $\phi(n) = \phi_1(n) + \phi_2(n)$ and assuming the convergence of the sum $\sum_{n=1}^{\infty} 1/\phi(n)$, we define

$$f(l) = \sum_{k=l+1}^{\infty} 1/\phi(k).$$

The map ϕ is increasing, we thus have $f(l-k+1) - f(l) \geq \frac{k-1}{\phi(l)}$ and then

$$G^{[n]} f(l) \geq \sum_{k=2}^{l+1} (C_l^k \lambda_{l,k} + C_l^{k-1} r_{l,k-1}) \frac{k-1}{\phi(l)} = 1$$

The process $f(\#\Pi^*_{|[n]}(t)) - \int_0^t G^{[n]} f(\#\Pi^*_{|[n]}(s)) ds$ is a martingale. The quantity

$$\zeta_n := \inf\{t; \#\Pi^*_{[n]}(t) = 0\}$$

is a finite stopping time. Let $k \geq 1$, applying the optional stopping theorem to the bounded stopping time $\zeta_n \wedge k$, we get :

$$\mathbb{E}[f(\#\Pi^*_{|[n]}(\zeta_n \wedge k))] - \mathbb{E}[\int_0^{\zeta_n \wedge k} G^{[n]} f(\#\Pi^*_{|[n]}(s)) ds] = f(n)$$

With the inequality $G^{[n]} f(l) \geq 1$, we deduce that

$$\mathbb{E}[\zeta_n \wedge k] \leq \mathbb{E}[f(\#\Pi^*_{|[n]}(\zeta_n \wedge k))] - f(n).$$

By Lebesgue's theorem and monotone convergence, we have $\mathbb{E}[\zeta_n] \leq f(0) - f(n)$. Passing to the limit in n, we have $\zeta_n \uparrow \zeta_\infty := \inf\{t; \#\Pi(t) = 1\}$ and $f(n) \longrightarrow 0$, thus:

$$\mathbb{E}[\zeta_\infty] \leq f(0) = \sum_{k=1}^{\infty} 1/\phi(k).$$

Using the binomial formula, we get $\phi_1(n) = \frac{n(n-1)}{2}c + \int_{]0,1]}[(1-x)^n - 1 + nx]\nu_1(dx)$. $\square$

By simple series comparisons, we deduce the sufficient part of the theorem. Plainly $\phi(n) \geq \phi_1(n)$, and if the series $\sum_{n=1}^{\infty} 1/\phi_1(n)$ converges, then by Lemma 14 the M-coalescent comes down from infinity.

To show the necessary part of the theorem, we must look more precisely at the behavior of jumps. The next technical lemmas show that when a distinguished coalescent comes down from infinity, there is a finite number of jumps which make decrease by half or more the number of blocks. Lemma 16 will allow us to study the process of blocks number before the first of these times. Assuming that the sum $\sum_{n=2}^{\infty} \frac{1}{\phi(n)} = \infty$ is infinite, we will define a supermartingale in Lemma 17 and find a contradiction by applying the optional stopping theorem.

We have already seen that, as for the Λ-coalescent, a way to understand the dynamics of a (Λ_0, Λ_1)-coalescent, when Λ_0, Λ_1 have no mass at 0, is to imagine drawing an infinite sequence of Bernoulli variables at each jump time, with parameter x controlled by the measures $\nu_0(dx) = x^{-1}\Lambda_0(dx)$ and $\nu_1(dx) = x^{-2}\Lambda_1(dx)$. The following technical lemma allows to estimate the chance for a Bernoulli vector to have more than half terms equals to 1.

Lemma 15 *Let $(X_1, X_2, ...)$ be independent Bernoulli variables with parameter $x \in [0, 1/4[$. Defining $S_n^{(x)} = X_1 + ... + X_n$, for every n_0, there is the bound:*

$$\mathbb{P}[\exists n \geq n_0; S_n^{(x)} > \frac{n}{2}] \leq \frac{\exp(-n_0 f(x))}{1 - \exp(-f(x))}$$

with $f(x) \sim \frac{1}{2} log(1/x)$ when $x \to 0$.

Proof. By Markov inequality, for all $t > 0$,

$$\mathbb{P}[S_n^{(x)} \geq n/2] \leq e^{-nt/2}\mathbb{E}[e^{tS_n^{(x)}}] = \exp(-n[t/2 - \log(xe^t + 1 - x)]).$$

Applying this inequality for $t = \log(1/x)$, we get $\mathbb{P}[S_n^{(x)} \geq n/2] \leq e^{-nf(x)}$ where

$$f(x) = \frac{1}{2}\log(1/x) - \log(2 - x).$$

The function f is non-negative on $]0, \frac{1}{4}[$ and then we obtain the convergence of the geometric sum:

$$\mathbb{P}[\exists n \geq n_0; S_n^{(x)} > \frac{n}{2}] \leq \sum_{n \geq n_0} \exp(-nf(x)) = \frac{\exp(-n_0 f(x))}{1 - \exp(-f(x))}.$$

Moreover we have $f(x) \underset{x \to 0^+}{\sim} \frac{1}{2}\log(1/x)$.

Lemma 16 *Assume that the M-coalescent comes down from infinity. With probability one, we have:*

$$\tau := \inf\{t > 0, \#\Pi^0(t) < \frac{\#\Pi^0(t-)}{2}\} > 0.$$

Moreover, if we define $\tau_n := \inf\{t > 0, \#\Pi^0_{|\lceil n \rfloor}(t) < \frac{\#\Pi^0_{|\lceil n \rfloor}(t-)}{2}\}$, then the sequence of stopping times τ_n converges to τ almost surely.

Proof. Obvioulsy, binary coagulations play no role in the statement and we may assume that $\Lambda_0(\{0\}) = 0$ and $\Lambda_1(\{0\}) = 0$. Let N be a Poisson measure with intensity $dt \otimes (x^{-1}\Lambda_0(dx) + x^{-2}\Lambda_1(dx))$. Recall the notation in Lemma . Let $n_0 \geq 4$, we will show that:

$$N(\{(t,x); t \leq 1; \exists n \geq n_0; S_n^{(x)} \geq n/2\}) < \infty.$$

By Proposition 7, $\#\Pi^0(\epsilon) < \infty$ a.s for every $\epsilon > 0$. We will then deduce that there is a finite number of jump times before 1 where more than half blocks coagulate. By Fellerian property (and so paths regularity) of $(\Pi^0(t))_{t\geq 0}$, $\Pi^0(0+) = \Pi^0(0)$ and 0 is not a jump time, then almost surely: $\tau > 0$. Moreover, $\#\Pi^0(\tau-) < \infty$ and then for all $n \geq \#\Pi^0(\tau-)$, $\tau_n = \tau$. We deduce that $\tau_n \underset{n\to\infty}{\to} \tau$ almost surely.
By Poissonian calculations, we get:

$$\mathbb{E}[N(\{(t,x); t \leq 1; \exists n \geq n_0, S_n^{(x)} > n/2\})] = \int_0^1 (\nu_0 + \nu_1)(dx)\mathbb{P}[\exists n \geq n_0, S_n^{(x)} > n/2].$$

By Lemma 6, we get:

$$\int_0^1 (\nu_0 + \nu_1)(dx)\mathbb{P}[\exists n \geq n_0, S_n^{(x)} > n/2] \leq \int_0^{\frac{1}{4}} (\nu_0 + \nu_1)(dx)\frac{\exp(-n_0 f(x))}{1 - \exp(-f(x))} + \int_{1/4}^1 (\nu_0 + \nu_1)(dx).$$

On the one hand:

$$\int_0^{\frac{1}{4}} (\nu_0 + \nu_1)(dx)\frac{\exp(-n_0 f(x))}{1 - \exp(-f(x))} < \infty$$

because the integrand is bounded by $8x^{\frac{n_0}{2}}$ and $n_0 \geq 4$.

On the other hand:

$$\int_{1/4}^1 (\nu_0 + \nu_1)(dx) \leq \int_{1/4}^1 x^{-2}x^2(\nu_0 + \nu_1)(dx) \leq 16\int_0^1 x^2(\nu_0 + \nu_1)(dx) < \infty.$$

This completes the proof.□

Assuming that the coalescent comes down from infinity and that $\sum_{n\geq 1} \frac{1}{\Phi(n)} = \infty$, we can define a supermartingale. We will find a contradiction using the optional stopping theorem.
We define the decreasing function:

$$f(n) = \exp(-\sum_{k=1}^{n+1} \frac{1}{\phi(k)})$$

where $\phi(n) = \phi_1(n) + \phi_2(n)$ with $\phi_1(n)$ and $\phi_2(n)$ are defined as in Lemma 14.

Lemma 17 *There exists a constant $C > 0$ such that for all $n \geq 1$, $(e^{-Ct}f(\#\Pi^*_{|[n]}(t)))_{t\leq\tau_n}$ is a non-negative supermartingale.*

Proof. We recall that the generator of $(\#\Pi^*_{|[n]}(t))_{t\geq 0}$ is

$$G^{[n]}g(l) = \sum_{k=2}^{l+1}[C_l^k 1_{k\leq l}\lambda_{l,k} + C_l^{k-1}r_{l,k-1}][g(l-k+1) - g(l)].$$

Stopping the process at τ_n, the jump times where more half of blocks coagulate are ignored, and the generator of the stopped process is :

$$A^{[n]}g(l) = \sum_{k=2}^{l/2+1} [C_l^k 1_{k\leq l/2}\lambda_{l,k} + C_l^{k-1} r_{l,k-1}][g(l-k+1) - g(l)].$$

We set $\Psi(n) = \int_0^1 (e^{-nx} - 1 + nx)\nu_1(dx)$, an easy verification allows to claim the existence of $c > 0$ such that $c\Psi(q) \leq \Phi_1(q) \leq \Psi(q)$ (See remark p170 in[7]).

$$\frac{\Psi(q)}{q} = \int_0^1 (1 - e^{-qx})\nu_1([x, 1[)dx \underset{q\to\infty}{\to} \int_0^1 x\nu_1(dx) > 0.$$

Plainly, $h(q) = \Psi(q)/q$ is a concave function. So $h(q/2) \geq h(q)/2$ and $\Psi(q/2) \geq \Psi(q)/4$.

$$A^{[n]}f(l) = \sum_{k=2}^{l/2+1} [C_l^k 1_{k\leq l/2}\lambda_{l,k} + C_l^{k-1} r_{l,k-1}]f(l)[\exp(\sum_{l-k+2}^{l} \frac{1}{\Phi(j)}) - 1].$$

We have

$$\sum_{l-k+2}^{l} \frac{1}{\Phi(j)} \leq \frac{k-1}{\Phi(l-k+2)} \leq \frac{k-1}{\Phi(l/2)} \quad \forall k \leq \frac{l}{2} + 1$$

and $e^x - 1 \leq cx$ for small x, then for large l :

$$A^{[n]}f(l) \leq cf(l) \sum_{k=1}^{l/2+1} [C_l^k \lambda_{l,k} + C_l^{k-1} r_{l,k-1}] \frac{k-1}{\Phi(l/2)}.$$

Thus, $A^{[n]}f(l) \leq cf(l)\frac{\Phi(l)}{\Phi(l/2)}$.
By definition $\Phi(l) = \Phi_1(l) + \Lambda_0([0,1])l$. Moreover, $l/\Phi_1(l)$ is bounded (it converges to $(\int_{[0,1]} x\nu_1(dx))^{-1}$) then for some constant $C > 0$:

$$\frac{\Phi(l)}{\Phi(l/2)} \leq \frac{\Phi_1(l) + \Lambda([0,1])l}{\Phi_1(l/2)} = \frac{\Phi_1(l)}{\Phi_1(l/2)} + (\Lambda_0([0,1]))\frac{l}{\Phi_1(l)}\frac{\Phi_1(l)}{\Phi_1(l/2)} \leq \frac{4}{c}[1 + \Lambda_0([0,1])\frac{l}{\Phi_1(l)}] \leq C.$$

We deduce that $A^{[n]}f(l) \leq Cf(l)$ and $(e^{-Ct}f(\#\Pi^*_{|[n]}(t)))_{t\leq\tau}$ is a supermartingale. □

Lemma 18 *If $\sum_{n\geq 1} \frac{1}{\Phi(n)} = \infty$ then Π^0 does not come down from infinity.*

Proof. Assume that the M-coalescent comes down from infinity. By Proposition 7, we know that $T = 0$ a.s. Let $T_j^{(n)} := \inf\{t > 0; \#\Pi^*_{[n]}(t) \leq j\}$. We apply to the previous supermartingale, the optional stopping theorem at time $T_j^{(n)} \wedge \tau_n$ and get

$$\mathbb{E}[e^{-cT_j^{(n)}\wedge\tau_n} f(\#\Pi^*_{[n]}(\tau_n \wedge T_j^{(n)}))] \leq f(n).$$

Passing to the limit with $n \uparrow \infty$: $f(n) \to 0$, $T_j^{(n)} \uparrow T_j$ and $\tau_n \to \tau > 0$ (by Lemma 16). The time T_j is strictly positive for some j, then $\tau \wedge T_j > 0$, $\#\Pi^*(\tau \wedge T_j) < \infty$ and thus, $f(\#\Pi^*(\tau \wedge T_j)) > 0$ almost surely. We have:

$$\mathbb{E}[e^{-cT_j\wedge\tau}] = 0$$

then $T_j = \infty$ a.s, which is not possible on $T < \infty$.$\square$

It remains to etablish the necessary part of Theorem 8. When $\Lambda_0(\{0\}) = 0$, the previous lemma claims that if the M-coalescent comes down from infinity then $\sum_{n\geq 1} \frac{1}{\Phi(n)} < \infty$. It suffices to show that:

$$\sum_{n\geq 2} \frac{1}{\phi_1(n) + \Lambda_0([0,1])n} < \infty \Longrightarrow \sum_{n\geq 2} \frac{1}{\phi_1(n)} < \infty.$$

The sequence $(\frac{\Psi(n)}{n})_{n\geq 1}$ is increasing and tends to $\int_0^1 x^{-1}\Lambda_1(dx)$ (possibly infinite). From the inequality: $c\frac{\Psi(n)}{n} \leq \frac{\Phi_1(n)}{n} \leq \frac{\Psi(n)}{n}$, we get that $\frac{n}{\phi_1(n)}$ is bounded. We then have

$$\frac{1}{\phi_1(n) + \Lambda_0([0,1])n} = \frac{1}{\phi_1(n)(1 + \Lambda_0([0,1])\frac{n}{\phi_1(n)})} \geq c\frac{1}{\phi_1(n)}$$

for some constant $c > 0$.
We then get the necessary part, and combining the results, Theorem 8 is deduced.

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References

[1] D. Aldous (1985) *Exchangeability and related topics.* Ecole d't de probabilits de Saint-Flour, XIII, Lecture Notes in Math., 1117, pp1-198. Springer, Berlin.

[2] J. Berestycki, N. Berestycki and V. Limic (2008) *Interpreting Λ-coalescent speed of coming down from infinity via particle representation of superprocesses.* In preparation.

[3] N. Berestycki (2010) *Recent progress in coalescent theory* Ensaios Matemticos [Mathematical Surveys], 16. Sociedade Brasileira de Matemtica, Rio de Janeiro, 2009. 193 pp.

[4] J. Bertoin (2006) *Random Fragmentation and Coagulation Processes.* Cambridge Studies in Advanced Mathematics, 102. Cambridge University Press, Cambridge, 2006.

[5] J. Bertoin and J.-F. Le Gall (2003) *Stochastic flows associated to coalescent processes.* Probab. Theor. Rel. Fields 126, no. 2, 261-288.

[6] J. Bertoin and J.-F Le Gall (2005) *Stochastic flows associated to coalescent processes II: Stochastic differential equations* Ann. Inst. H. Poincar Probab. Statist. 41, no. 3, 307-333.

[7] J. Bertoin and J.-F. Le Gall (2006) *Stochastic flows associated to coalescent processes III: limit theorems.* Illinois J. Math. 50, no. 1-4, 147-181.

[8] M. Birkner, J. Blath, M. Capaldo, A. Etheridge, M. Mhle, J. Schweinsberg and A. Wakolbinger (2005) *Alpha-stable branching and beta-coalescents.* Electron. J. Probab. 10,no. 9, 303-325. (electronic)

[9] M.Birkner, J. Blath, M. Mhle, M. Steinrcken and J. Tams (2009) *A modified look-down construction for the Xi-Fleming-Viot process with mutation and populations with recurrent bottlenecks* ALEA Lat. Am. J. Probab. Math. Stat. 6, 25-61.

[10] D.A. Dawson, Z. Li (2010) *Stochastic equations, flows and measure-valued processes.* Preprint available via:
http://math.bnu.edu.cn/ lizh/research/pdffiles/10stflow.pdf.

[11] R. Dong, A.V. Gnedin and J. Pitman (2007) *Exchangeable partitions derived from Markovian coalescents* Ann. Appl. Probab. 17, 1172-1201.

[12] P. Donnelly, P. Joyce (1991) *Consistent ordered sampling distributions: characterization and convergence.* Adv. in Appl. Probab. 23 (1991), no. 2, 229–258.

[13] P. Donnelly and T.G Kurtz (1997) *Particle Representations for Measure-Valued Population Models* Ann. Probab. 27, 166-205.

[14] S.N. Ethier and T.G. Kurtz (1986): *Markov Processes: Characterization and Convergence.* Wiley.

[15] A. Gnedin (1997) *The representation of composition structures.* Ann. Probab. 25 , no. 3, 1437–1450.

[16] K.Kawazu and S.Watanabe (1971)*Branching Processes with immigration and related limit theorems*, Theory Probab. Appl., vol.XVI, Number 1, p37-53.

[17] J.F.C. Kingman (1978) *The representation of partition structures.* J. London Math. Soc. (2) 18, no. 2, 374–380.

[18] J. F. C. Kingman (1982) *The coalescent.* Stochastic Process. Appl. 13 235-248.

[19] A. Lageras (2007) *A population model for Λ-coalescents with neutral mutations.*, Electron. Comm. Probab. 12

[20] M. Mhle and S. Sagitov (2001) *A classification of coalescent processes for haploid exchangeable population models.* Ann. Probab. 29(4), 1547-1562.

[21] J. Pitman (2006) *Combinatorial stochastic processes.* Ecole d't de probabilits de Saint-Flour, Lect. Notes in Maths 1875, Berlin, Springer.

[22] J. Pitman (1995) *Exchangeable and partially exchangeable random partitions* Probab. Theor. Rel. Fields 102, no. 2, 145-158.

[23] J. Pitman (1999) *Coalescents with multiple collisions.* Ann. Probab. 27 1870-1902.

[24] S. Sagitov (1999) *The general coalescent with asynchronous mergers of ancestral lines.* J. Appl. Probab. 36, no. 4, 1116–1125.

[25] J. Schweinsberg (2000) *A necessary and sufficient condition for the Λ-coalescent to come down from infinity.* Electron. Comm.Probab. 5 1-11.

[26] J. Schweinsberg (2000) *Coalescents with simultaneous multiple collisions* (English summary) Electron. J. Probab. 5 , Paper no. 12, 50 pp. (electronic).