LVMB manifolds and simplicial spheres

Jérôme Tambour

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Abstract

LVM and LVMB manifolds are a large family of examples of non khler manifolds. For instance, Hopf manifolds and Calabi-Eckmann manifolds can be seen as LVMB manifolds. The LVM manifolds have a very natural action of the real torus and the quotient of this action is a simple polytope. This quotient allows us to relate closely LVM manifolds to the moment-angle manifolds studied (for example) by Buchstaber and Panov. Our aim is to generalize the polytopes associated to LVM manifolds to the LVMB case and study its properties. In particular, we show that it belongs to a very large class of simplicial spheres. Moreover, we show that for every sphere belonging to this class, we can construct a LMVB manifold whose associated sphere is the given sphere. We use this latter result to show that many moment-angle complexes can be endowed with a complex structure (up to product with circles).

Introduction

It is not easy to construct non khler compact complex manifolds. The simplest example is the well know Hopf's manifold ([Ho], 1948), which gives a complex structure on the product of spheres $S^{2n+1} \times S^1$ as a quotient of $\mathbb{C}^n \backslash \{0\}$ by the action of a discrete group. The Hopf's manifold has many generalizations: firstly, by Calabi and Eckmann [CE] who give a structure of complex manifold on any product of spheres (of odd dimension). Then by Santiago Lopez de Medrano, Alberto Verjovsky ([LdM] and [LdMV]) and Laurent Meersseman [M]. In these last generalizations, the authors obtain complex structures on products of spheres, and on connected sums of products of spheres, also constructed as a quotient of an open subset in $\mathbb{C}^n$ but by the action of a nondiscrete quotient this time. These manifolds are known as *LVM manifolds*.

The construction in [M] has (at least) two interesting features: on the one hand, the LVM manifolds are endowed with an action of the torus $(S^1)^n$ whose quotient is a simple convex polytope and the combinatorial type of this polytope characterizes the topology of the manifold. On the other hand, for every simple polytope P, it is possible to construct a LVM manifold whose quotient is P.

In [Bo], Frdric Bosio generalize the construction of [M] emphasing on the combinatorial aspects of LVM manifolds. This aim of this paper is to study the *LVMB manifolds* (i.e. manifolds constructed as in [Bo]) from the topological and combinatorial viewpoints. In particular, we will generalize the associated polytope of a LVM manifold to our case and prove that this generalization belongs to a large class of simplicial spheres (named here *rationally starshaped spheres*).

In the first part, we briefly recall the construction of the LVMB manifolds as a quotient of an open set in $\mathbb{C}^n$ by an holomorphic action of $\mathbb{C}^m$. In the second part, we study *fundamental sets*, the combinatorial data describing a LVMB manifold and its connection to pure simplicial complexes[1]. Mainly, we show that an important property appearing in [Bo] (the *SEU* property) is related to a well-known class of simplicial complexes: the pseudo-manifolds. In the same part, we also introduce the simplicial complex associated to a LVMB manifold and we show that this complex generalizes the associated polytope of a LVM manifold. In the third and forth parts, we are mainly interested in the properties of this complex and we show that the complex is indeed a simplicial sphere. To do that, we have to study another action whose quotient is a toric variety closely related to our LVMB manifold. This action was already studied in [MV] and [CFZ] but we need a more thorough study. Finally, in the fifth part, we make the inverse construction: starting with a rationally starshaped sphere, we construct a LVMB manifold whose associated complex is the given sphere. Using this construction, we show an important property for moment-angle complexes: up to a product of circles, every moment-angle complex arising from a starshaped sphere can be endowed with a complex structure of LVMB manifold.

To sum up, we prove the following theorems:

[1] In this paper, the expression "simplicial complex" (or even "complex") stands for "abstract simplicial complex" and a "geometric simplical complex" is a geometric complex whose cells are (straight) simplices in some Euclidean space $\mathbb{R}^n$.

Theorem 1: Let $\mathcal{N}$ be a LVMB manifold. Then its associated complex $\mathcal{P}$ is a rationally starshaped sphere. Moreover, if $\mathcal{N}$ is indeed a LVM manifold, $\mathcal{P}$ can be identified with (the dual of) its associated polytope.

Proposition: Every rationally starshaped sphere can be realized as the associated complex $\mathcal{P}$ for some LVMB manifold.

Theorem 2: Up to a product of circles, every moment-angle complex arising from a starshaped sphere can be endowed with a complex structure of LVMB manifold.

Notations

In this short section, we fix several notations which will be used throughout the text:

- We put $I_z = \{\ k \in \{1, \dots, n\}\ /\ z_k \neq 0\ \}$ for every z in $\mathbb{C}^n$.
- $\mathbb{D}$ is the closed unit disk in $\mathbb{C}$ and S^1 its boundary.
- Moreover, *exp* will be the map $\mathbb{C}^n \to (\mathbb{C}^*)^n$ defined by $exp(z) = (e^{z_1}, \dots, e^{z_n})$ (where e is the usual exponential map of $\mathbb{C}$).
- The characters of $(\mathbb{C}^*)^n$ will be noted X_n^m: $X_n^m(z) = z_1^{m_1} \dots z_n^{m_n}$.
- And the one-parameter subgroups of $(\mathbb{C}^*)^n$ will be noted λ_n^u: $\lambda_n^u(t) = (t^{u_1}, \dots, t^{u_n})$.
- $<,>$ is the usual **non hermitian** inner product on $\mathbb{C}^n$ $:< z, w >= \sum_{j=1}^n z_j w_j$
- We will identify $\mathbb{C}^m$, as a $\mathbb{R}$-vector space, to $\mathbb{R}^{2m}$ via the morphism

$$z \mapsto (Re(z_1), \dots, Re(z_n), Im(z_1), \dots, Im(z_n))$$

- We set $Re(z) = (Re(z_1), \dots, Re(z_n))$ and $Im(z) = (Im(z_1), \dots, Im(z_n))$, so

$$z = Re(z) + iIm(z) = (Re(z), Im(z))$$

- In $\mathbb{R}^n$, Conv(A) (resp. pos(A)) is the convex hull (resp. the positive hull) of a subset A.
- For every v in $\mathbb{R}^n$, we set $\widetilde{v} = (1, v) \in \mathbb{R}^{n+1}$.

1 Construction of the LVMB manifolds

In this section, we briefly recall the construction of LVMB manifolds, following the notations of [Bo]. Let M and n be two positive integers such that $n \geq M$. A *fundamental set* is a nonempty subset $\mathcal{E}$ of $\mathcal{P}(\{1, \dots, n\})$ whose elements are of cardinal M. Elements of $\mathcal{E}$ are called *fundamental subsets*.

Remark: For practical reasons, we sometimes consider fundamental sets whose elements does not belong to $\{\ 1, \dots, n\ \}$ but to another finite set (usually $\{\ 0, 1, \dots, n-1\ \}$).

Let P be a subset of $\{1, \dots, n\}$. We say that P is *acceptable* if P contains a fundamental subset. We set $\mathcal{A}$ as the set of all acceptable subsets. Finally, an element of $\{1, \dots, n\}$ will be called *indispensable* if it belongs to every fundamental subset of $\mathcal{E}$. We say that $\mathcal{E}$ is of type (M, n) (or (M, n, k) if we want to emphasize the number k of indispensable elements).

Example 1: For instance, $\mathcal{E} = \{\ \{1,2,5\}, \{1,4,5\}, \{2,3,5\}, \{3,4,5\}\ \}$ is a fundamental set of type $(3, 5, 1)$.

Two combinatorial properties (named SE and SEU^2 respectively, cf. [Bo]) will be very important in the sequel:

$$\forall\ P \in \mathcal{E},\ \ \forall\ k \in \{1, \dots, n\},\ \ \exists\ k' \in P;\ \ (P \backslash \{k'\}) \cup \{k\} \in \mathcal{E}$$

$$\forall\ P \in \mathcal{E},\ \ \forall\ k \in \{1, \dots, n\},\ \ \exists!\ k' \in P;\ \ (P \backslash \{k'\}) \cup \{k\} \in \mathcal{E}$$

[2] for Substitute's Existence and Substitute's Existence and Uniqueness

Moreover, a fundamental set $\mathcal{E}$ is *minimal* for the SEU property if it verifies this property and has no proper subset $\widetilde{\mathcal{E}}$ such that $\widetilde{\mathcal{E}}$ verifies the SEU property. Finally, we associate to $\mathcal{E}$ two open set $\mathcal{S}$ in $\mathbb{C}^n$ and $\mathcal{V}$ in $\mathbb{P}^{n-1}$ defined as follow:

$$\mathcal{S} = \{\ z \in \mathbb{C}^n / \ I_z \in \mathcal{A}\ \}$$

The open $\mathcal{S}$ is the complement of an arrangement of coordinate subspaces in $\mathbb{C}^n$. We also remark that, for all $t \in \mathbb{C}^*$, we have $I_{tz} = I_z$ so the following definition is consistant (where $\mathbb{P}^{n-1}$ is the complex projective space of dimension $n-1$):

$$\mathcal{V} = \{\ [z] \in \mathbb{P}^{n-1} / \ I_z \in \mathcal{A}\ \}$$

Example 2: In example 1, $\mathcal{E}$ verifies the SEU property and the set $\mathcal{S}$ is

$$\mathcal{S} = \{\ (z_1, z_2, z_3, z_4, z_5)\ /\ (z_1, z_3) \neq 0,\ (z_2, z_4) \neq 0,\ z_5 \neq 0\ \} \ \simeq\ \left(\mathbb{C}^2 \backslash \{0\}\right)^2 \times \mathbb{C}^*$$

Now, we suppose that $M = 2m+1$ is odd and we fix $l = (l_1, \ldots, l_n)$ a family of elements of $\mathbb{C}^m$. We can define an action (called *acceptable holomorphic action*) of $\mathbb{C}^* \times \mathbb{C}^m$ on $\mathbb{C}^n$ defined by:

$$(\alpha, T) \cdot z = (\ \alpha e^{<l_1, T>} z_1,\ \ldots,\ \alpha e^{<l_n, T>} z_n\ ) \quad \forall\ (\alpha, T, z) \in \mathbb{C}^* \times \mathbb{C}^m \times \mathbb{C}^n$$

Remark: If $m = 0$, the previous action is just the classical one defining $\mathbb{P}^{n-1}$.

In the sequel, we focus only on families l such that for every $P \in \mathcal{E}$, $(l_p)_{p \in P}$ spans $\mathbb{C}^m$ (seen as a $\mathbb{R}$-affine space). In this case, we said that $(\mathcal{E}, l)$ is a *acceptable system*. If $\alpha \in \mathbb{C}^*$, $T \in \mathbb{C}^m$ and $z \in \mathbb{C}^n$, we have $I_{(\alpha,T).z} = I_z$, so $\mathcal{S}$ is invariant for the acceptable holomorphic action. Moreover, the restriction of this action to $\mathcal{S}$ is free (cf. [Bo], p.1264).

As a consequence, we note $\mathcal{N}$ the orbit space for the action restricted to $\mathcal{S}$. Notice that we can also consider an action of $\mathbb{C}^m$ on $\mathcal{V}$ whose quotient is again $\mathcal{N}$. We call $(\mathcal{E}, l)$ a *good system* if $\mathcal{N}$ can be endowed with a complex structure such that the natural projection $\mathcal{S} \to \mathcal{N}$ is holomorphic. Such a manifold is known as a *LVMB manifold*. Since a quotient of a holomorphic manifold by a free and proper action (cf. [Hu], p.60) can be endowed with a complex structure, we only have to check the last property. Here, according to [Hu], *p.*59, we define a proper action of a Lie Group G on a topological space X as a continuous action such that the map $G \times X \to X \times X$ defined by $(g, x) \ \mapsto (g \cdot x, x)$ is proper. Notice that in our case, the group G is not discrete. If G is a discrete group which acts freely and properly on a space X, then the action is properly discontinuous.

According to [Bo], we make the following definition:

Definition: Let $(\mathcal{E}, l)$ be a acceptable system. We say that it verifies the *imbrication condition* if for every P, Q in $\mathcal{E}$, the convex hulls $Conv(l_p, p \in P)$ and $Conv(l_q, q \in Q)$ does not have disjoint interiors.

In [Bo], the following fundamental theorem is proved:

Theorem 3 ([Bo], p.1268)**:** A acceptable system is a good one if and only if $(\mathcal{E}, l)$ verifies the SE property and the imbrication condition.

Remark: In [Bo], it is also proved that a good system is minimal for the SEU property.

Finally, a *LVM manifold* is a manifold constructed as in [LdMV] or [M]. We don't explain here the whole construction of the LVM manifolds. The only thing we need here is that is a special case of LVMB. Indeed, we have the following theorem (and we use this theorem as a definition of LVM manifolds):

Theorem 4 ([Bo], p.1265)**:** Every LVM manifold is a LVMB manifold. To be more precise, let $\mathcal{O}$ be the set of points of $\mathbb{C}^m$ which are not in the convex hull of any subset of l with cardinal $2m$. Then, a good system $(\mathcal{E}, l)$ is the good system of a LVM manifold if and only if there exists an unbounded component O in $\mathcal{O}$ such that $\mathcal{E}$ is exactly the set of subsets P of $\{1, \ldots, n\}$ with $(2m+1)$ for cardinal such that O is contained in $Conv(l_p, p \in P)$.

Example 3: We come back to example 1. If we set $l_1 = l_3 = 1$, $l_2 = l_4 = i$ and $l_5 = 0$, then the imbrication condition is fulfilled and $(\mathcal{E}, l)$ is a good system. As a consequence, theorem 3 and theorem 4 imply that $\mathcal{N}$ can be endowed with a structure of a LVM manifold. In [LdMV], the LVM manifolds constructed from a good system of type $(3, n, k)$ are classified up to diffeomorphism. Here, $\mathcal{E}$ has type $(3, 5, 1)$ and we have $\mathcal{N} \approx S^3 \times S^3$. Notice that we can also use the theory of moment-angle complex (cf.the last section of this text) to do this calculation.

To conclude this section, we recall how is constructed the polytope associated to a LVM manifold $(\mathcal{E}, l)$. It is clear that the natural action of $(S^1)^n$ on $\mathbb{C}^n$ preserves $\mathcal{S}$ and that this action commutes with the holomorphic action. So, we have an induced action of $(S^1)^n$ on $\mathcal{N}$. Up to a translation of the l_j (which does not change the quotient $\mathcal{N}$), theorem 4 allow us to assume that 0 belongs to $Conv(l_1, \ldots, l_n)$ (this condition is known as *Siegel's condition*) and in this case, the quotient P of the action of $(S^1)^n$ on $\mathcal{N}$ can be identified with:

$$P = \{ \ (r_1, \ldots, r_n) \in (\mathbb{R}_+)^n \ / \ \Sigma_{i=1}^n r_j l_j = 0, \ \Sigma_{i=1}^n r_j = 1 \ \}$$

This set is clearly a polytope and it can be shown that it is simple (cf. [BM], lemma 0.12). The polytope P is called the *polytope associated* to the LVM manifold $\mathcal{N}$.

Example 4: For the previous example, we have to perform a translation on the vectors l_j with the view to respect the Siegel's condition. For example, $\mathcal{N}$ is also the LVM manifold associated with the good system $(\mathcal{E}, \lambda)$ where $\lambda_1 = \lambda_3 = \frac{3}{4} - \frac{i}{4}$, $\lambda_2 = \lambda_4 = -\frac{1}{4} + \frac{3i}{4}$ and $\lambda_5 = -\frac{1}{4} - \frac{i}{4}$. A calculation shows that the polytope P is the square

$$P = \left\{ \ \left(\frac{1}{4} - r_3, \frac{1}{4} - r_4, r_3, r_4, \frac{1}{2} \right) \ / \ r_3, r_4 \in [0, \frac{1}{4}] \ \right\}$$

2 Fundamental sets and associated complex

In this section, we will briefly study the fundamental sets introduced in [Bo] and defined above and construct a simplicial complex whose combinatorial properties reflect the geometry of a LVMB manifold. Here, $\mathcal{E}$ is a fundamental set of type (M, n). The integer M is not supposed to be odd. For the moment, our aim is to relate the above properties to some classical ones of simplicial complexes.

Let us begin with some vocabulary. *Faces*, or *simplices* are subsets of a simplicial complex. If a complex K is *pure-dimensional* (or simply *pure*), the simplices of maximal dimension d are named *facets* and the faces of dimension 0 (resp. $d-1$) are the *vertices* (resp. the *ridges*) of the complex.

The first important definition in this paper is the following:

Definition: Let $\mathcal{E}$ be a fundamental set of type (M, n). The *associated complex* of $\mathcal{E}$ is the set $\mathcal{P}$ of subsets of $\{1, \ldots, n\}$ whose complement (in $\{1, \ldots, n\}$) is acceptable.

As we will see later, this complex is the best choice for a combinatorial generalization of the associated polytope of a LVM manifold.

First properties of $\mathcal{P}$ are the following:

Proposition 2.1: Let $\mathcal{E}$ be a fundamental set of type (M, n, k). Then, its associated complex $\mathcal{P}$ is a simplicial complex on $\{1, \ldots, n\}$. Moreover, $\mathcal{P}$ is pure-dimensional of dimension $(n - M - 1)$ and has $(n-k)$ vertices. Its vertices are precisely the non-indispensable elements of $\{1, \ldots, n\}$ for $\mathcal{E}$ and its facets are exactly the complements of the subsets of $\mathcal{E}$.

Proof: $\mathcal{P}$ is obviously a simplicial complex. Moreover, the maximal simplices of $\mathcal{P}$ are exactly the complements of minimal subsets of $\mathcal{A}$, that are fundamental subsets. The latters have the same number M of elements, so every maximal simplex of $\mathcal{P}$ has $n - M$ elements. Finally, an element $j \in \{1, \ldots, n\}$ is a vertex of $\mathcal{P}$ if and only if $\{1, \ldots, n\} \backslash \{j\}$ contains a fundamental subset, that is j is not indispensable. □

Example 5: The complex $\mathcal{P}$ associated to the fundamental set of example 1 is the complex with facets $\{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 4\}\}$. So, $\mathcal{P}$ is the boundary of a square.

Remark: Conversely, every pure complex can be realized as the associated complex of a fundamental set: let $\mathcal{P}$ be a pure-dimensional simplicial complex on the set $\{1,\dots,v\}$ with dimension d and v vertices. Then, for every integer k, there exists M,n two integers and $\mathcal{E}$ a fundamental set of type (M,n,k) whose associated complex is $\mathcal{P}$. If k is fixed, this fundamental set $\mathcal{E}$ is unique.

Moreover, the *SEU* property can be expressed as a combinatorial property of $\mathcal{P}$:

Proposition 2.2: $\mathcal{E}$ verify the *SEU* property if and only if

$$\forall\ Q \in \mathcal{P}_{max},\ \forall\ k \in \{1,\dots,n\},\ \exists!\ k' \notin Q;\ (Q \cup \{k'\})\backslash\{k\} \in \mathcal{P}_{max}$$

where $\mathcal{P}_{max}$ is the set of facets of $\mathcal{P}$.

Proof: First, we assume that $\mathcal{E}$ verifies the *SEU* property. Let Q be a facet of $\mathcal{P}$ and $k \in \{1,\dots,n\}$. Then $P = Q^c$ belongs to $\mathcal{E}$. The *SEU* property implies that there is k' in P (so $k' \notin Q$) such that $P' = (P\backslash\{k'\}) \cup \{k\}$ belongs to $\mathcal{E}$. As it was claimed, P'^c is also a facet of $\mathcal{P}$. Moreover, we obviously have $P'^c = (Q \cup \{k'\})\backslash\{k\}$. Finally, if $Q'' = (Q \cup \{k''\})\backslash\{k\}$ is a facet of $\mathcal{P}$ with $k'' \neq k'$ and $k'' \notin Q$, then, we would have $(P\backslash\{k'\})\cup\{k\}$ in $\mathcal{E}$, which contradicts the *SEU* property. The proof of the converse is analogous. □

Corollary 1: Let $\mathcal{E}$ be a fundamental set. Then its associated complex $\mathcal{P}$ verify the *SEU* property if and only if every ridge of $\mathcal{P}$ is contained in exactly two facets of $\mathcal{P}$.

Proof: To begin, we assume that $\mathcal{E}$ verifies the *SEU* property. Let Q be a ridge of $\mathcal{P}$. By definition, Q is included in a facet P of $\mathcal{P}$. We put $P = Q \cup \{k\}, k \in \{1,\dots,n\}\backslash Q$. By proposition 2.2, there exists $k' \notin P$ (and so $k \neq k'$) such that $P' = (P \cup \{k'\})\backslash\{k\}$ is a facet of $\mathcal{P}$. We have $P' = Q \cup \{k'\}$ so Q is contained in at least two facets of $\mathcal{P}$. Let assume that Q is contained in a third facet $P'' = Q \cup \{k''\}$. However, in this case, we would have $P'' = (P \cup \{k''\})\backslash\{k\}$, which contradicts proposition 2.2.
Conversely, let Q be a facet of $\mathcal{P}$ and $k \in \{1,\cdots,n\}$. If $k \in Q$, then $P = Q\backslash\{k\}$ is a ridge of $\mathcal{P}$ and by hypothesis, P is contained in exactly two facets Q_1 and Q_2. One of them, say Q_1, is Q. The other is Q_2 and we have $Q_2 = P \cup \{k'\}$. Then we have $k' \notin Q$ (on the contrary, we would have $Q_2 = Q = Q_1$) and $Q_2 = (Q\cup\{k'\})\backslash\{k\}$. Moreover, if $Q_3 = (Q\cup\{k''\})\backslash\{k\}$ is another facet of $\mathcal{P}$ with $k'' \notin Q$, so Q_3 contains P and by hypothesis, $Q_3 = Q_2$ (i.e $k'' = k'$). If $k \notin Q$, we remark that the element $k' \in \{1,\dots,n\}$ such that $Q' = (Q \cup \{k'\})\backslash\{k\}$ is a facet of $\mathcal{P}$ is $k' = k$. Indeed, if $k' = k$, then $Q' = Q$ is a facet of $\mathcal{P}$. And if $k' \neq k$, then $k \notin Q \cup \{k'\}$ and, as a consequence, $Q' = Q \cup \{k'\}$ is not in $\mathcal{P}$. □

Definition: Let $\mathcal{E}$ be a fundamental set of type (M,n). We define the (unoriented) graph Γ stating that its vertices are fundamental subsets of $\mathcal{E}$ and two vertices P and Q are related by an edge if and only if there exist $k \notin P, k' \in P$ such that $Q = (P\backslash\{k'\}) \cup \{k\}$. Equivalently, we relate two subsets of $\mathcal{E}$ if and only if they differ exactly by one element. Γ is called *replacement graph* of $\mathcal{E}$.

Proposition 2.3: Let $\mathcal{E}$ be a fundamental set of type (M,n) which verifies the *SEU* property. Then, there exists an integer $p \in \mathbb{N}^*$, and fundamental sets $\mathcal{E}_j$ of type (M,n) which are minimal for the *SEU* property and such that $\mathcal{E}$ is the disjoint union $\bigsqcup_{j=1}^{p} \mathcal{E}_j$.

Proof: We proceed by induction on the cardinal of $\mathcal{E}$. If $\mathcal{E}$ is minimal for the *SEU* property, then there is nothing to do. Let assume that it is not the case: there exists a proper subset $\mathcal{E}_1$ of $\mathcal{E}$ which is minimal for the *SEU* property. We put $\overline{\mathcal{E}}$ for its complement $\mathcal{E}\backslash\mathcal{E}_1$. It is obvious that $\overline{\mathcal{E}}$ is a fundamental set (of type (M,n)). We claim that $\overline{\mathcal{E}}$ verifies the *SEU* property. Let P be an element of $\overline{\mathcal{E}}$ and $k \in \{1,\dots,n\}$. If $k \in P$, then, putting $k' = k$, we have that $(P\backslash\{k'\}) \cup \{k\} = P$ is an element of $\overline{\mathcal{E}}$. It is the only choice (for k') since P is in $\mathcal{E}$ and $\mathcal{E}$ verifies the *SEU* property. Let assume now that k is not in P. Since P is an element of $\mathcal{E}$, there exists exactly one $k' \in P$ such that $P' = (P\backslash\{k'\}) \cup \{k\}$ is an element of $\mathcal{E}$, too. We claim that P' cannot be in $\mathcal{E}_1$. Indeed, if it was the case, since $\mathcal{E}_1$ is minimal for the *SEU* property, there would exist exactly one $k'' \in P'$ such that $P'' = (P'\backslash\{k''\}) \cup \{k'\} \in \mathcal{E}_1$. But $P = (P'\backslash\{k\}) \cup \{k'\}$ is in $\mathcal{E}$ and $k \in P'$. So, $k'' = k$ and $P'' = P$. As a consequence, $\overline{\mathcal{E}}$ is a fundamental set of type (M,n) which verifies the *SEU* property with cardinal strictly smaller than $\mathcal{E}$. Applying the induction hypothesis on $\overline{\mathcal{E}}$, we have the decomposition of $\mathcal{E}$ we were looking for. □

Remark: The decomposition of the previous proposition induces a decomposition of the vertex set of Γ. In the proof, we show that an element of $\mathcal{E}_j$ is related only to others elements in $\mathcal{E}_j$. Consequently, each

set $\mathcal{E}_j$ is the vertex set of a connected component of Γ. This also implies that this decomposition is unique up to the order. We call *connected components* of $\mathcal{E}$ the sets $\mathcal{E}_j$.

Corollary 2: Let $\mathcal{E}$ be a fundamental set of type (M, n) and Γ its replacement graph. We assume that $\mathcal{E}$ verifies the SEU property. Then, the following assertions are equivalent:

1. $\mathcal{E}$ is minimal for the SEU property.
2. $\mathcal{E}$ has only one connected component.
3. Γ is connected.

Definition: Let K be a simplicial complex. K is a *pseudo-manifold* if these two properties are fulfilled:

1. every ridge of K is contained in exactly two facets.
2. for all facets σ, τ of K, there exists a chains of facets $\sigma = \sigma_1, \ldots, \sigma_n = \tau$ of K such that $\sigma_i \cap \sigma_{i+1}$ is a ridge of K for every $i \in \{0, \ldots, n-1\}$.

For instance, every simplicial sphere is a pseudo-manifold. More generally, a triangulation of a manifold (that is a simplicial complex whose realization is homeomorphic to a topological manifold) is also a pseudo-manifold. Now, the proposition below shows that the notion of pseudo-manifold is exactly the combinatorial property of $\mathcal{P}$ which characterizes the fact that $\mathcal{E}$ is minimal for the SEU property:

Proposition 2.4: Let $\mathcal{E}$ be a fundamental set with $n > M$. Then, $\mathcal{P}$ is a pseudo-manifold if and only if $\mathcal{E}$ is minimal for the SEU property.

Proof: On the one hand, let assume that $\mathcal{E}$ is minimal for the SEU property. This implies that every ridge of $\mathcal{P}$ is contained in exactly two facets (cf. corollary 1). Moreover, let σ, τ be two distinct facets of $\mathcal{P}$. So, $P = \sigma^c$ and $Q = \tau^c$ are two fundamental subsets. By minimality for the SEU property, Γ is connected (cf. corollary 2). Consequently, there exists a sequence $P_0 = P, P_1, \ldots, P_r = Q$ of fundamental subsets such that P_i and P_{i-1} differ by exactly one element. We note R_i the acceptable subset $P_{i-1} \cup P_i$ with $M+1$ elements. Its complement R_i^c is thus a face of $\mathcal{P}$ with $n - M - 1 = dim(\mathcal{P})$ elements. If we put $\sigma_i = P_i^c$, we have $R_i^c = \sigma_{i-1} \cap \sigma_i$ so $\sigma_0 = \sigma, \ldots, \sigma_r = \tau$. Consequently, $\mathcal{P}$ is a pseudo-manifold.
On the other hand, we assume that $\mathcal{P}$ is a pseudo-manifold. Then, thanks to corollary 1, $\mathcal{E}$ verifies the SEU property. Moreover, $\mathcal{E}$ will be minimal for this property if and only if Γ is connected (cf. corollary 2). Let σ, τ be two distinct elements of $\mathcal{E}$. Then σ^c and τ^c are facets of $\mathcal{P}$. Since $\mathcal{P}$ is a pseudo-manifold, there exists a sequence $\sigma_0 = \sigma^c$, $\sigma_1, \ldots, \sigma_r = \tau^c$ of facets of $\mathcal{P}$ such that for every i, σ_i and σ_{i-1} share a ridge of $\mathcal{P}$. This means that σ_i^c and σ_{i-1}^c are fundamental subsets $\mathcal{E}$ which differ only by an element, and consequently, σ_i^c and σ_{i-1}^c are related in Γ. This implies that Γ is connected and $\mathcal{E}$ is minimal for the SEU property. □

Remark: The case where $n = M$ corresponds to $\mathcal{P} = \{\emptyset\}$. It is not a pseudo-manifold since the only facet is $\emptyset$ and it doesn't contains any simplex with dimension strictly smaller.

Finally, we prove the following proposition which is the motivation for the study of the associated complex:

Proposition 2.5: Let $(\mathcal{E}, l)$ be a good system associated to a LVM manifold and P its associated polytope. Then the associated complex $\mathcal{P}$ of $\mathcal{E}$ is combinatorially equivalent to the border of the dual of P.

Proof: Since $(\mathcal{E}, l)$ is a good system associated to a LVM manifold, there exists a bounded component O in $\mathcal{O}$ such that $\mathcal{E}$ is exactly the set of subsets Q of $\{1, \ldots, n\}$ with $(2m+1)$ for cardinal such that O is included in the convex hull of $(l_q, q \in Q)$. Up to a translation, (whose effect on the action is just to introduce an automorphism of $\mathbb{C}^m \times \mathbb{C}^*$ and so does not change the action, cf. [Bo]), we can assume that $\mathcal{E}$ is exactly the set of subsets P of $\{1, \ldots, n\}$ with $(2m+1)$ for cardinal such that the convex hull of $(l_p, p \in P)$ contains 0. In this setting, according to the page 65 of [BM], the border of P is combinatorially characterized as the set of subsets I of $\{1, \ldots, n\}$ verifying

$$I \in P \Leftrightarrow 0 \in Conv(l_k, k \in I^c)$$

So, I is a subset of P if and only if I^c is acceptable, i.e. $I \in \mathcal{P}$.
As a consequence, from a viewpoint of set theory, P and $\mathcal{P}$ are the same set. We claim that the order

on these sets P and $\mathcal{P}$ are reversed. On the one hand, the order on $\mathcal{P}$ is the usual inclusion (as for every simplicial complex). On the other other, we recall the order on the face poset of P given in [BM]: every $j-$face of P is represented by a $(n-2m-1-j)-$tuple. So, facets of P are represented by a singleton and vertices by a $(n-2m-2)-$tuple. A face represented by I is contained in another face represented by J if and only if $I \supset J$. So, combinatorially speaking, the poset of P is $(\mathcal{P}, \supset)$, which prove the claim. Finally, the poset for the dual P^* is $(\mathcal{P}, \subset)$, and the proof is over. $\square$

3 Condition (K)

Consequently, $\mathcal{P}$ is exactly the object we are looking for to generalize the associated polytope of a LVM manifold to the case of LVMB manifolds. What we have to do now is to study its properties. Our main goal for the moment is to prove the following theorem:

Theorem 5: Let $(\mathcal{E}, l)$ be a good system of type $(2m+1, n)$. Then $\mathcal{P}$ is a simplicial $(n-2m-2)-$sphere.

Remark: The theorem is trivial in the LVM case since the associated complex $\mathcal{P}$ is a polytope.

To prove the previous theorem, we have to focus on good systems which verify an additional condition, called condition (K): there exists a real affine automorphism ϕ of $\mathbb{C}^m = \mathbb{R}^{2m}$ such that $\lambda_j = \phi(l_j)$ has coordinates in $\mathbb{Z}^{2m}$ for every j. For instance, if all coordinates of l_i are rationals, then $(\mathcal{E}, l)$ verify condition (K). Note that the imbrication condition is an open condition. As a consequence, it is sufficient to prove the previous theorem for good system verifying the condition (K). Indeed, since $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$, a good system $(\mathcal{E}, l)$ which does not verify the condition (K) can be replaced by a good system which verifies the condition, with the same associated complex $\mathcal{P}$.

The main interest of condition (K) stands in the fact that we can associate to our holomorphic acceptable action an algebraic action (called *algebraic acceptable action*) of $(\mathbb{C}^*)^{2m+1}$ on $\mathbb{C}^n$ (or an action of $(\mathbb{C}^*)^{2m}$ on $\mathbb{P}^{n-1}$):

Let $(\mathcal{E}, l)$ be a fundamental set of type $(2m+1, n)$ verifying condition (K). We set $l_j = a_j + ib_j, a_j, b_j \in \mathbb{Z}^m$ for every j and $a_j = (a_j^1, \ldots, a_j^m)$. We can define an action of $(\mathbb{C}^*)^{2m+1}$ on $\mathbb{C}^n$ setting:

$$(u,t,s)\cdot z = \left(u\ t_1^{a_1^1}\ldots t_m^{a_1^m}\ s_1^{b_1^1}\ldots s_m^{b_1^m}\ z_1, \ldots, u\ t_1^{a_n^1}\ldots t_m^{a_n^m}\ s_1^{b_n^1}\ldots s_m^{b_n^m}\ z_n\right)$$

$$\forall u \in \mathbb{C}^*, t, s \in (\mathbb{C}^*)^m, z \in \mathbb{C}^n$$

Using the notation $X_{2m+1}^{\widetilde{l_j}}$ for the character of $(\mathbb{C}^*)^{2m+1}$ defined by $\widetilde{l_j} = (1, l_j)$, we can resume the formula describing the acceptable algebraic action by:

$$t\cdot z = \left(X_{2m+1}^{\widetilde{l_1}}(t)z_1, \ldots, X_{2m+1}^{\widetilde{l_n}}(t)z_n\right)\ \forall t \in (\mathbb{C}^*)^{2m+1},\ z \in \mathbb{C}^n$$

It is clear that $\mathcal{S}$ is invariant by this action. So we can define X as the topological orbit space of $\mathcal{S}$ by the algebraic action. As for the holomorphic acceptable action, we can define an action of $(\mathbb{C}^*)^{2m}$ on $\mathcal{V}$ whose quotient is also X. In [CFZ], proposition 2.3, it is shown that the holomorphic acceptable action of $\mathbb{C}^m$ on $\mathcal{V}$ can be seen as the restriction of the algebraic acceptable action to a closed cocompact subgroup H of $(\mathbb{C}^*)^{2m}$. As a consequence, we can define an action of $K = (\mathbb{C}^*)^{2m}/H$ on $\mathcal{N}$ whose quotient can be homeomorphically identified with X.

The principal consequence of this result is the following:

Proposition 3.1: X is Hausdorff and compact.

Proof: Let $p : \mathcal{N} \to X$ be the canonical surjection. K is a compact Lie group so p is a closed map (cf. [Br], p.38). Consequently, X is Hausdorff. Finally, since p is continuous and $\mathcal{N}$ is compact, we can conclude that X is compact. $\square$

Another important consequence for the sequel of the article is that the algebraic action on $\mathcal{S}$ (or $\mathcal{V}$) is closed. Moreover, since every complex compact commutative Lie group is a complex compact torus (i.e.

a complex Lie group whose underlying topological space is $(S^1)^n$, cf. [L], Theorem 1.19), we can say that K is a complex compact torus.

Using an argument of [BBŚ], we show that:

Proposition 3.2: $t \in (\mathbb{C}^*)^{2m}$ is in the stabilizer of $[z]$ if and only if $\forall i, j \in I_z$, we have

$$X^{l_i}_{2m+1}(t) = X^{l_j}_{2m+1}(t)$$

Proof: Let us fix some linear order on $\mathbb{Z}^{2m}$. Up to a permutation of the homogenous coordinates of $\mathbb{P}^{n-1}$, we can assume that $l_j \leq l_{j+1}$ for every $j \in \{1, \ldots, n-1\}$ (notice that such a permutation is an equivariant automorphism of $\mathbb{P}^{n-1}$). We set $j_0 = min(I_z)$ the smallest index of nonzero coordinates of z. Then, for every t which stabilizes $[z]$, we have

$$[z] = t.[z] = [X^{l_j}_{2m+1}(t)z_j] = [0, \ldots, 0, X^{l_{j_0}}_{2m+1}(t)z_{j_0}, \ldots, X^{l_n}_{2m+1}(t)z_n]$$

So $[z] = [0, \ldots, 0, z_{j_0}, \ldots, X^{l_n - l_{j_0}}_{2m+1}(t)z_j]$. In particular, we have $X^{l_n - l_{j_0}}_{2m+1}(t)z_j = z_j$ for every j. If $j \in I_z$, then $X^{l_n - l_{j_0}}_{2m+1}(t) = 1$. □

Remark: In [BBŚ], it is shown that $[z]$ is a fixed point for the algebraic acceptable action if and only if $\forall i, j \in I_z$, we have $l_i = l_j$.

Consequently, we have:

Proposition 3.3: Every element of $\mathcal{V}$ have a finite stabilizer for the algebraic acceptable action.

Proof: First, we recall that an element of $\mathcal{V}$ has at least $(2m+1)$ nonzero coordinates and amongs this coordinates, there are $(2m+1)$ coordinates such that $(l_p)_{p \in P}$ spans $\mathbb{C}^m$ as a real affine space. Up to a permutation, we can assume that $(1, 2, \ldots, 2m+1)$ is contained in I_z and in $\mathcal{E}$. In this case, we have $X^{l_j - l_{2m+1}}_{2m+1}(t) = 1$ for every $j = 1, \ldots, 2m$ and every t in the stabilizer of $[z]$. We put $L_j = l_j - l_{2m+1}$. Writing $t_j = r_j e^{2i\pi\theta_j}$, we get that $r = (r_1, \ldots, r_{2m+1})$ and $\theta = (\theta_1, \ldots, \theta_{2m+1})$ verify the following systems:

$$M.ln(r) = 0, \ M.\theta \equiv 0 \ [1]$$

where $M = (m_{i,j})$ is the matrix defined by $m_{i,j} = L^i_j$ and $ln(r) = (ln(r_1), \ldots, ln(r_{2m+1}))$. The system being acceptable, $(l_1, \ldots, l_{2m+1})$ spans $\mathbb{C}^m$ as a real affine space, which means exactly that the real matrix M is inversible.
Consequently, $ln(r) = 0$ (i.e $|t_j| = 1$ pour tout j) and $\theta \equiv 0$ $[det(M^{-1})]$. So, $r_j e^{2i\pi\theta_j}$ can take only a finite number of values. As a conclusion, the stabilizer of $[z]$ is finite, as it was claimed. □

Remark: An analogous proof shows that the stabilizer of $z \in \mathcal{S}$ is finite, too.

4 Connection with toric varieties

In this section, our main task is to recall that X is a toric variety and to compute its fan. When the fan Σ is simplicial, we can constructed a simplicial complex K_Σ as follow: we note $\Sigma(1)$ for the set of rays of Σ and order its elements by $x_1, \ldots, x_n$. Then, the complex K_Σ is the simplicial complex on $\{1, \cdots, n\}$ defined by:

$$\forall\ J \subset \{1, \cdots, n\}, \quad (\ J \in K_\Sigma \Leftrightarrow\ pos(x_j, j \in J) \in \Sigma\ )$$

This complex K_Σ is *the underlying complex* of Σ. We recall a theorem which will be very important in the sequel:

Proposition 4.1: Let X be a normal separated toric variety and Σ its fan. We suppose that Σ is simplicial. Then, the three following assertions are equivalent:

1. X is compact.
2. Σ is complete in $\mathbb{R}^n$.
3. Σ's underlying simplicial complex is a $(n-1)$-sphere.

4.1 Toric varieties

To begin with, it is clear that $\mathcal{S}$ and $\mathcal{V}$ are separated normal toric varieties. As it is explained in [CLS], we can associate to a separated normal toric variety with group of one-parameter subgroups N a fan Σ in the real vector space $N_{\mathbb{R}} = N \otimes \mathbb{R}$ whose cones are rational with respect to the lattice N[3]. So, we can compute the fan associated to $\mathcal{S}$:

Proposition 4.2: Let $(e_i)_{i=1}^n$ be the canonical basis for $\mathbb{R}^n$. Then, the fan describing $\mathcal{S}$ in $\mathbb{R}^n$ is

$$\Sigma(\mathcal{S}) = \{\ pos(e_i, i \in I)/\ I \in \mathcal{P}\}$$

Proof: To compute the fan of a toric variety, one has to calculate limits for its one-parameter subgroups. The embedding of $(\mathbb{C}^*)^n$ in $\mathcal{S}$ is the inclusion and the one-parameter subgroups of $(\mathbb{C}^*)^n$ have the form $\lambda^m(t) = (t^{m_1}, \dots, t^{m_n})$ with $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$.
So, the limit of $\lambda^m(t)$ when t tends to 0 exists in $\mathbb{C}^n$ if and only if $m_i \geq 0$ for every i. In this case, the limit is $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ with $\epsilon_j = \delta_{m_j,0}$ (Kronecker's symbol).
Of course, the limit has to be in $\mathcal{S}$, which implies for I_ϵ to be acceptable. But $I_\epsilon = \{j/m_j = 0\}$ so the condition means exactly that $\{j/m_j > 0\}$ belongs to $\mathcal{P}$. □

Example 6: For the fundamental set $\mathcal{E} = \{\{1,2,5\},\{1,4,5\},\{2,3,5\},\{3,4,5\}\}$ of example 1, we have $\mathcal{S} = \{z/(z_1, z_3) \neq 0, (z_2, z_4) \neq 0, z_5 \neq 0\}$. As a consequence, the fan $\Sigma(\mathcal{S})$ is the fan in $\mathbb{R}^5$ whose facets are the 2−dimensional cones $pos(e_1, e_2), pos(e_1, e_4), pos(e_2, e_3)$ and $pos(e_3, e_4)$ (where $e_1, \dots, e_5$ is the canonical basis of $\mathbb{C}^5$).

Remark: We can also easily compute orbits of $\mathcal{S}$ for the action of $(\mathbb{C}^*)^n$. For $I \subset \{1, \dots, n\}$, we set $O_I = \{z \in \mathbb{C}^n / I_z = I^c\}$. Then if $z \in \mathcal{S}$, its orbit is $O_{I_z^c}$.
So, we can describe $\mathcal{S}$ as the partition: $\mathcal{S} = \bigsqcup_{I \in \mathcal{P}} O_I$

Moreover, in the orbit-cone correspondance between $\mathcal{S}$ and $\Sigma(\mathcal{S})$ (cf. [CLS] ch.3), O_I corresponds to $\sigma_I = pos(e_j / j \in I)$.

Remark: In quite the same way, we can show that the fan of $\mathcal{V}$ is $\Sigma(\mathcal{V}) = \{pos(e_i, i \in I)/I \in \mathcal{P}\}$ in $\mathbb{R}^{n-1}$, with $(e_1, \dots, e_{n-1})$ defined as the canonical basis of $\mathbb{R}^{n-1}$ and $e_n = -(e_1 + \dots + e_{n-1})$.

In [CLS], it is proven that X is an irreducible separated compact normal toric variety. In the next section, we will detail the construction with the view to identify its group of one-paramater subgroup and the structure of its fan.

Example 7: For the good system $(\mathcal{E}, l)$ with $\mathcal{E} = \{\{1,2,5\},\{1,4,5\},\{2,3,5\}\{3,4,5\}\}$ and $l_1 = l_3 = 1$, $l_2 = l_4 = i$ and $l_5 = 0$, the algebraic action is

$$(\alpha, t, s) \cdot z = (\alpha t z_1, \alpha s z_2, \alpha t z_3, \alpha s z_4, \alpha z_5)$$

Using the automorphism of $(\mathbb{C}^*)^3$ defined by $\phi(\alpha, t, s) = (\alpha, \alpha t, \alpha s)$, we can see that the quotient X of the algebraic action is also the quotient of $\mathcal{S}$ by the action defined by

$$(\alpha, t, s) \cdot z = (t z_1, s z_2, t z_3, s z_4, \alpha z_5)$$

so X is the product $\mathbb{P}^1 \times \mathbb{P}^1$ and the projection $\mathcal{N} \to X$ comes from the usual definition of $\mathbb{P}^2$ as a quotient of S^3.

To conclude this section, let $f : (\mathbb{C}^*)^{2m+1} \longrightarrow (\mathbb{C}^*)^n$ be the map defined by

$$f(u, t, s) = (X_{2m+1}^{\widetilde{l_1}}(u, t, s), \cdots, X_{2m+1}^{\widetilde{l_n}}(u, t, s))$$

The algebraic acceptable action on $\mathbb{C}^n$ is just the restriction to $Im(f)$ of the natural action of the torus $(\mathbb{C}^*)^n$ on $\mathbb{C}^n$.

[3] a cone σ is said to be rational for N if there a family S of elements of N such that $\sigma = pos(S)$. Moreover, if S is free in $N_{\mathbb{R}}$, we say that σ is simplicial

Proposition 4.3: $Ker(f)$ is finite.

Proof: Let t be in $Ker(f)$. Then $X^{\widetilde{l_j}}(t) = 1$ for every j. We set $t = (r_1 e^{2i\pi\theta_1}, \ldots, r_{2m+1} e^{2i\pi\theta_{2m+1}})$ and $l_j = (l_j^1, \ldots, l_j^{2m})$ and we have $r_1 r_2^{l_j^1} \ldots r_{2m+1}^{l_j^{2m}} = 1$ for every j. That means that we have $A.v = 0$ where $A = (a_{i,j})$ is the real matrix defined by $a_{i,j} = \widetilde{l_i}^j$ and v is the vector $v = (ln(r_j))$. Since $l_1, \ldots l_n$ affinely spans $\mathbb{R}^{2m}$, we can extract an inversible matrix $\widetilde{A}$ of order $2m+1$ from A and we can deduce that $v = 0$. This means that $|t_j| = 1$ for every j. Moreover, if $\theta = (\theta_1, \ldots, \theta_{2m+1})$, we get that $\widetilde{A}.\theta$ is an element of $\mathbb{Z}^{2m+1}$. Consequently, θ is an element of $\frac{1}{\delta}\mathbb{Z}^n$, where $\delta = det(\widetilde{A})$. Finally, we can deduce that $e^{i\theta_j}$ is a $\delta-$th root of unity for every j. Particularly, there are only a finite number of elements in the kernel of f. □

Remark: Generally, f is not injective. For instance, if we consider $\mathcal{E} = \{\{1,2,4\},\{2,3,4\}\}$ and $l_1 = 1, l_2 = i, l_3 = p, l_4 = -1 - i$, where p is a nonzero positive integer. $(\mathcal{E}, l)$ is a good system and for $p = 4$, f is not injective. Note that for $p = 3$, f is injective.

Definition: We define T_N as the quotient group $(\mathbb{C}^*)^n / Im(f)$

We recall that $(\mathbb{C}^*)^n$ is included in $\mathcal{S}$ as an dense Zariski open subset. $(\mathbb{C}^*)^n$ is invariant by the action of $(\mathbb{C}^*)^{2m+1}$. This implies that T_N can be embedded in X as an dense open subset. Moreover, the action of $(\mathbb{C}^*)^n$ on $\mathcal{S}$ commutes with the action of $(\mathbb{C}^*)^{2m+1}$, so the action of T_N on itself can be extended to an action on X. In [CLS], it is proven that X is an irreducible separated compact normal algebraic variety.

4.2 The algebraic torus T_N

We note $F : \mathbb{C}^{2m+1} \to \mathbb{C}^n$ the linear map defined by

$$F(U,T,S) = (U + < a_j, T > + < b_j, S >)_j$$

The matrix of F has $(1, l_j)$ as j-th row so F has maximal rank. So, F is injective. The family $f_j = F(e_j)$, with $(e_1, \ldots, e_{2m+1})$ the canonical basis of $\mathbb{C}^{2m+1}$ is a basis for $Im(F)$. We can notice that each f_j has integer coordinates. We complete this basis to a basis $(f_1, \ldots, f_n)$ of $\mathbb{C}^n$ with integer coordinates. Next, we define the map $G : \mathbb{C}^n \to \mathbb{C}^{n-2m-1}$ by linearity and $G(f_j) = \begin{cases} 0 & j \in \{1, \ldots, 2m+1\} \\ g_j & otherwise \end{cases}$
(with $(g_{2m+2}, \ldots, g_n)$ the canonical basis of $\mathbb{C}^{n-2m-1}$)

It is clear by construction that the following sequence is exact:

$$0 \longrightarrow \mathbb{C}^{2m+1} \xrightarrow{F} \mathbb{C}^n \xrightarrow{G} \mathbb{C}^{n-2m-1} \longrightarrow 0$$

Moreover, we have

$$F(\mathbb{Z}^{2m+1}) \subset \mathbb{Z}^n,\ G(\mathbb{Z}^n) \subset \mathbb{Z}^{n-2m-1}$$

Let $t \in (\mathbb{C}^*)^{2m+1}$ and T be some element of $\mathbb{C}^{2m+1}$ such that $t = exp(T)$. We put $g(t) = exp(G(T))$. The previous remark has for consequence that g is well defined. Moreover, we have:

Proposition 4.4: g is a group homomorphism and the following diagram is commutative:

$$\begin{array}{ccccc} (\mathbb{C}^*)^{2m+1} & \xrightarrow{f} & (\mathbb{C}^*)^n & \xrightarrow{g} & (\mathbb{C}^*)^{n-2m-1} \\ \uparrow{\scriptstyle exp} & & \uparrow{\scriptstyle exp} & & \uparrow{\scriptstyle exp} \\ \mathbb{C}^{2m+1} & \xrightarrow{F} & \mathbb{C}^n & \xrightarrow{G} & \mathbb{C}^{n-2m-1} \end{array}$$

Finally, we can prove that:

Proposition 4.5: g is surjective and $Ker(g) = Im(f)$

Proof: The surjectivity of g is clear (since G and exp are surjective).
So, we have only to show that $Ker(g) = Im(f)$. By the construction of F and G and by commutativity of the previous diagram, we have for every $t = exp(T)$, $g \circ f(t) = g \circ f \circ exp(t) = exp \circ G \circ F(T) = exp(0) = 1$

so $Im(f) \subset Ker(g)$.
Conversely, let t belong to $Ker(g)$. We put $t = exp(T)$, for some $T \in \mathbb{C}^n$ and we have $1 = g(t)$ so $exp(G(T)) = 1$.
As a consequence, $G(T) \in 2i\pi\mathbb{Z}^{n-2m-1}$, i.e.

$$G(T) = (2i\pi q_{2m+2}, \dots, 2i\pi q_n) = 2i\pi q_{2m+2} g_{2m+2} + \dots 2i\pi q_n g_n = G(2i\pi q_{2m+2} f_{2m+2} + \dots 2i\pi q_n f_n)$$

So $T - (2i\pi q_{2m+2} f_{2m+2} + \dots 2i\pi q_n f_n) \in Ker(G) = Im(F)$. We can write that

$$T = \lambda_1 f_1 + \dots \lambda_{2m+1} f_{2m+1} + 2i\pi q_{2m+2} f_{2m+2} + \dots 2i\pi q_n f_n$$

Finally, $T = F(\lambda_1 e_1 + \dots \lambda_{2m+1} e_{2m+1}) + 2i\pi(q_{2m+2} f_{2m+2} + \dots q_n f_n)$, which implies that

$$t = exp(F(\lambda_1 e_1 + \dots \lambda_{2m+1} e_{2m+1})) = f(exp(\lambda_1 e_1 + \dots \lambda_{2m+1} e_{2m+1})) \in Im(f) \ \square$$

Particularly, $T_N = (\mathbb{C}^*)^n / Im(f)$ is isomorphic to $(\mathbb{C}^*)^{n-2m-1}$ (and, as it was claimed in the previous section, X is a toric variety). We note $\overline{g}$ for the isomorphism between T_N and $(\mathbb{C}^*)^{n-2m-1}$ induced by g.

Definition: We note λ^u_T for the one-parameter subgroup of T_N defined by $\lambda^u_T = \overline{g}^{-1} \circ \lambda^u_{n-2m-1}$. Since $\overline{g}$ is an isomorphism, every one-parameter subgroup of T_N has this form.

Notation: Let $\phi : T_1 \to T_2$ be a group homomorphism between two algebraic torus T_1 and T_2. We will note ϕ^* for the morphism between the groups of one-parameter subgroup of T_1 and T_2 induced by ϕ: $\phi^*(\lambda) = \phi \circ \lambda$.

The group of one-parameter subgroup of $(\mathbb{C}^*)^n$ is $\{\lambda^u_n / u \in \mathbb{Z}^n\}$ which we will identify with $\mathbb{Z}^n$ via $u \leftrightarrow \lambda^u_n$. Via this map, the map F and G are exactly the morphisms induced by f and g (respectively):

Proposition 4.6: With the above identification, we have $F = f^*$ and $G = g^*$. Precisely:

1. For every v in $\mathbb{Z}^{2m+1}$, $f \circ \lambda^v_{2m+1}$ is the one-parameter subgroup $\lambda^{F(v)}_n$ of $(\mathbb{C}^*)^n$.
2. For every v in $\mathbb{Z}^n$, $g \circ \lambda^v_n$ is the one-parameter subgroup $\lambda^{G(v)}_{n-2m-1}$ of $(\mathbb{C}^*)^{n-2m-1}$.

Proof: Let us take some $t \in \mathbb{C}^*$.
1) We have $\lambda^v_{2m+1}(t) = (t^{v_1}, ..., t^{v_{2m+1}})$ so

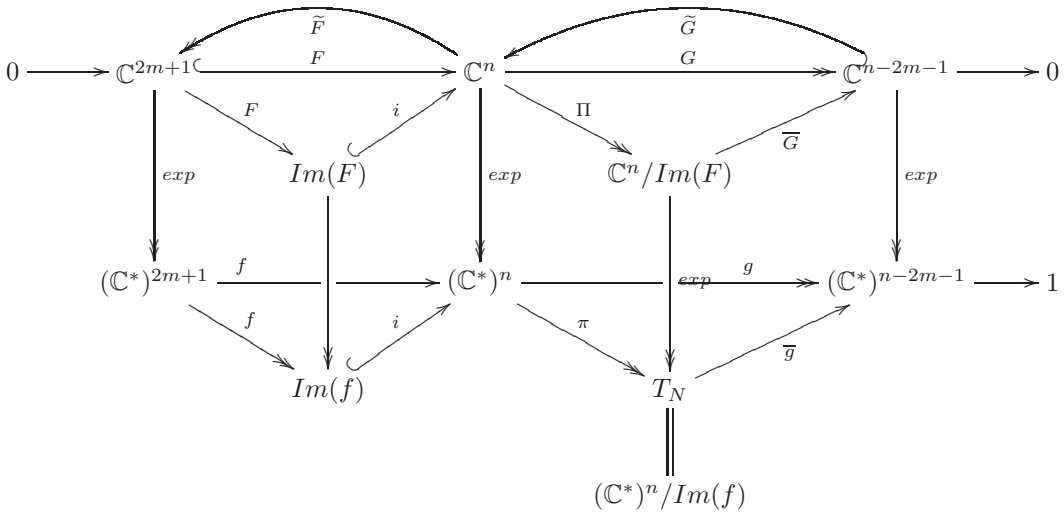
$$f \circ \lambda^v_{2m+1}(t) = (t^{v_1 + v_2 a^1_j + \dots + v_{m+1} a^m_j + \dots + v_{2m+1} a^m_j})_j = (t^{<v, \widetilde{l_j}>})_j = \lambda^{F(v)}_n(t)$$

2) Let us pick some $T \in \mathbb{C}^n$ such that $t = exp(T)$. We put $w = G(v) = (w_1, ..., w_{n-2m-1})$. If we note g_j (resp. G_j) for the coordinates functions of g (resp. G) in the canonical bases, we can easily verify that $g_j \circ exp = exp \circ G_j$ and that $w_j = G_j(v)$ for every j.
Next, we have $\lambda^v_n(t) = (t^{v_1}, \dots, t^{v_n}) = (e^{Tv_1}, \dots, e^{Tv_n})$, so $g \circ \lambda^v_n(t) = g(e^{Tv_1}, \dots, e^{Tv_n}) = exp \circ G(Tv_1, \dots, Tv_n) = exp(G_1(Tv), \dots, G_{n-2m-1}(Tv)) = (e^{G_1(Tv)}, \dots, e^{G_{n-2m-1}(Tv)})$.
Finally, $g \circ \lambda^v_n(t) = (e^{TG_1(v)}, \dots, e^{TG_{n-2m-1}(v)}) = (e^{Tw_1}, \dots, e^{Tw_{n-2m-1}}) = \lambda^w_{n-2m-1}(t)$. $\square$

We would like to identify the group N of one-parameter subgroups of T_N with some lattice in $\mathbb{C}^n / Im(F)$. A natural candidate is described as follow: Let Π be the canonical surjection $\mathbb{C}^n \to \mathbb{C}^n / Im(F)$ and $\overline{G} : \mathbb{C}^n / Im(F) \to \mathbb{C}^{n-2m-1}$ be the linear isomorphism induced by G. Notice that $\overline{G}$ is also a $\mathbb{Z}-$module isomorphism between $\mathbb{Z}^{n-2m-1}$ and $\Pi(\mathbb{Z}^n)$. If u belongs to $\mathbb{Z}^n$, we set $\lambda^{\Pi(u)}_T$ for the one-parameter subgroup $\lambda^{G(u)}_T$ (notice that this definition make sense since $Im(F) = Ker(G)$). As a consequence, we have: $N = \{\lambda^{\Pi(u)}_T / u \in \mathbb{Z}^n\}$ which can be identified with $\Pi(\mathbb{Z}^n) = \mathbb{Z}^n / Im(F)$.

Now, we can define an "exponential" map between $\mathbb{C}^n / Im(F)$ and T_N: we define $exp : \mathbb{C}^n / Im(F) \to T_N$ by setting $exp(\Pi(z)) = \pi \circ (exp(z))$ for every element z in $\mathbb{C}^n$. The fact that $Ker(g) = Im(f)$ implies that this map is well defined.(Alternatively, we can define this exponential as the map $\overline{g}^{-1} \circ exp \circ \overline{G}$). By construction, we have $\pi \circ exp = exp \circ \Pi$ and $exp \circ \overline{G} = \overline{g} \circ exp$. Moreover, for every $v \in \mathbb{Z}^n$, $\pi \circ \lambda^v_n = \lambda^{\Pi(v)}_T$, that is to say that $\Pi = \pi^*$.

To sum up, we have the following commutative diagram:



4.3 Study of the projection π

In this last section, we are in position to prove the first part of theorem 1. Let Σ denote the fan in $N_{\mathbb{R}} = N \otimes \mathbb{R}$ associated to X (this fan exists since X is separated and normal). In order to use proposition 4.1, we have to prove that Σ is simplicial. According to [CLS], we just have to prove that X is an orbifold.

As it was shown previously, the holomorphic acceptable action on $\mathcal{S}$ is free and the algebraic action on $\mathcal{S}$ has only finite stabilizers. Consequently, every stabilizer for the action of K on $\mathcal{N}$ is finite. So, X is the quotient of the compact variety $\mathcal{N}$ by the action of a compact Lie group K and every stabilizer for this action is finite.

Finally, since it is a continuous map defined on a compact set, we can claim that the map

$$\begin{array}{rccc} \phi_n : & K & \longrightarrow & \mathcal{N} \\ & h & \mapsto & h \cdot n \end{array}$$

is proper for every element n in $\mathcal{N}$. By Holmann's theorem (cf. [O], §5.1), we get that X is indeed an orbifold. Taking advantage of proposition 4.1, we have proved that K_Σ, the underlying complex of the fan Σ, is a $(n-2m-2)-$sphere.

Now, the following proposition will prove theorem 1:

Theorem 6: K_Σ and $\mathcal{P}$ are isomorphic simplicial complexes.

Proof: Fisrt of all, π is by construction a toric morphism. According to [CLS], this implies that π is equivariant (for the toric actions of $(\mathbb{C}^*)^n$ and T_N), so π sends an orbit O_I in $\mathcal{S}$ to an orbit in X. We set $\widetilde{O_I}$ to be the unique orbit in X containing $\pi(O_I)$. Moreover, the map π^* (identified with Π) from $\mathbb{R}^n$ into $N_{\mathbb{R}}$ preserves the cones.

We can easily show that $\mathcal{S}$ and X have exactly the same number of orbits, i.e that the quotient of X by its torus T_N is in bijection with the quotient of $\mathcal{S}$ by its torus $(\mathbb{C}^*)^n$. Since π is surjective, we get that the assignation $O_I \to \widetilde{O}_I$ (induced by π) is bijective. As a consequence, Π induces a bijection between the cones of $\Sigma(\mathcal{S})$ and those of Σ

If σ is a cone belonging to $\Sigma(\mathcal{S})$, we note $O(\sigma)$ the orbit in $\mathcal{S}$ associated to σ (cf [CLS], ch.3). Particularly, $O(\sigma_I) = O_I$. We will also note in the same way the orbits in X. Moreover, the image of the cone σ by Π will be noted $\widetilde{\sigma}$. So, we have $\widetilde{O(\sigma)} = O(\widetilde{\sigma})$.
Still from [CLS], π preserves the partial order of faces: if τ is a face of σ in $\Sigma(\mathcal{S})$, then $\widetilde{\tau}$ is a face of $\widetilde{\sigma}$.

On the other hand, a slight modification of the proof of the fact that $(\mathbb{C}^*)^n/Im(f)$ (i.e. $\pi(O_\emptyset)$) is isomorphic to $(\mathbb{C}^*)^{n-2m-1}$ shows that $\pi(O(\sigma))$ is isomorphic to $(\mathbb{C}^*)^{n-2m-1-dim(\sigma)}$. At the level of cones, this means that cones of $\Sigma(\mathcal{S})$ with the same dimension are sent to cones of Σ with the same dimension. In particular, Π sends rays to rays. This last property means that Π induces a bijection between vertices of $\mathcal{P}$ and vertices of K_Σ, bijection which we also note Π.
It is clear by what preceeds that this very last map is an simplicial complex isomorphism. $\square$

As a consequence, $\mathcal{P}$ is indeed a simplicial sphere. More precisely, what we have is a particular type of simplicial spheres, namely spheres which are underlying simplicial complex of some complete fan:

Definition: Let K be a d-sphere. K is said to be rationally starshaped if there exists a lattice N in $\mathbb{R}^{d+1}$, a point $p_0 \in N$ and a realization[4] $|K|$ for K in $\mathbb{R}^{d+1}$ such that every vertex of $|K|$ belongs to N and every ray emanating from p_0 intersects $|K|$ in exactly one point. The realization $|K|$ is said to be *starshaped* and we say that p_0 belongs to the *kernel* of $|K|$.

Corollary 3: Let $(\mathcal{E}, l)$ be a good system verifying (K). Then its associated complex $\mathcal{P}$ is a rationally starshaped sphere.

Proof: We have already seen that $\mathcal{P}$ is a simplicial sphere combinatorially equivalent to K_Σ. If we put $\Sigma(1) = \{\rho_1, \ldots, \rho_v\}$ for the distincts rays of Σ and $u_1, \ldots, u_v$ for generators of $\rho_1, \ldots, \rho_v$ (respectively) in N, then the geometric simplicial complex C whose simplexes are $Conv(u_i, i \in I)$ for I in $\mathcal{P}$ is obviously a realization of $\mathcal{P}$ in $\mathbb{R}^{n-2m-1}$ with rational vertices. The point 0 is in the kernel of C so $\mathcal{P}$ is rationally starshaped. □

5 Inverse construction

5.1 The construction

In this section, we give a realization theorem: for every rationally starshaped sphere, there exists a good system whose associated sphere is the given one. In [M], $p.86$, the same kind of theorem is proven for simple polytopes and LVM manifolds. One of the main interests of this theorem is that it gives us a clue to answer to an open question: does there exist a LVMB manifold who has not the same topology than a LVM manifold? The idea is to use this theorem to construct a LVMB manifold from a rationally starshaped sphere $\mathcal{P}$ which is not polytopal and that this manifold has a particular topology. For example, we expect that the LVMB manifold coming from the Brckner's sphere or the Barnette's one (the two 4-spheres with 8 vertices which are not polytopal) are good candidates.

Let $\mathcal{P}$ be a rationally starshaped d-sphere with v vertices (up to a simplicial isomorphism, we will assume that these vertices are $1, 2, \ldots, v$). So, there exists a lattice N and a realization $|\mathcal{P}|$ of $\mathcal{P}$ in $\mathbb{R}^{d+1}$ all of whose vertices belong to N. We can assume that 0 is in the kernel of $|\mathcal{P}|$ and that N is $\mathbb{Z}^{d+1}$.
To begin, we suppose that v is even and we put $v = 2m$. We note $\mathcal{E}$ the set defined by

$$\{P \in \{1, ..., v+d+1\} / P^c \; is \; a \; facet \; of \; \mathcal{P}\}$$

and $\mathcal{E}_0 = \{\{0\} \cup P / P \in \mathcal{E}\}$ We also note A the matrix whose columns are $x_1, \ldots, x_v$. We label its rows $x^1, \ldots, x^{d+1}$, and finally, we have:

Theorem 7: If $e_1, \ldots, e_v$ is the canonical basis of $\mathbb{R}^v$, then $(\mathcal{E}_0, (0, e_1, \ldots, e_v, -x^1, \ldots, -x^{d+1}))$ is a good system of type $(v+1, d+v+2)$ whose associated complex is $\mathcal{P}$.

First, since $\mathcal{P}$ is a $d-$sphere, it is a pure complex whose facets have d elements. As a consequence, every subset in $\mathcal{E}_0$ has $v+1$ elements. So, $\mathcal{E}_0$ is a fundamental set of type $(v+1, v+d+2)$ (as usual, we will note the set of its acceptable subsets $\mathcal{A}_0$). Notice that, since every facet of $\mathcal{P}$ has vertices in $\{1, \ldots, v\}$, $\mathcal{E}_0$ has $d+2$ indispensable elements. Moreover, by definition, $\mathcal{E}_0$'s associated complex is clearly $\mathcal{P}$. Since $\mathcal{P}$ is a sphere, hence a pseudo-manifold, proposition 2.4 implies that $\mathcal{E}_0$ is minimal for the SEU property. We can also notice that the same is true for $\mathcal{E}$ (with set of acceptable subsets $\mathcal{A}$).

Secondly, we have to check that the vectors $0, e_1, e_2, \ldots, e_v, -x^1, \ldots, -x^{d+1}$ "fits" with $\mathcal{E}_0$ to make a good system. We put $\rho_j = pos(x_j)$ for the ray generated by $x_j, j = 1, \ldots, v$. We also note $\Sigma(\mathcal{P})$ the fan defined by $\Sigma(\mathcal{P}) = \{pos(x_j, j \in I) / I \in \mathcal{P}\}$. Then, by definition, $\Sigma(\mathcal{P})$ is a simplicial fan whose underlying complex is $\mathcal{P}$. As a consequence, $\Sigma(\mathcal{P})$ is rational with respect to N and complete (since $\mathcal{P}$ is a sphere, cf. proposition 4.1). We set X for the compact normal toric variety associated to $\Sigma(\mathcal{P})$. Following [Ha], we will construct X as a quotient of a quasi-affine toric variety by the action of an algebraic torus.

In what follows, $(e_j)_{j=1,\ldots,N}$ is the canonical basis of $\mathbb{C}^N$ (for any N). Let $\widetilde{\Sigma}$ be the fan in $\mathbb{R}^{v+d+1}$ whose cones are $pos(e_j, j \in J)$, $J \in \mathcal{P}$. Obviously, $\widetilde{\Sigma}$ is a noncomplete simplicial fan whose underlying complex is $\mathcal{P}$, too. We note $\widetilde{X}$ the (quasi-affine) toric variety associated to this fan. Then, the open set $\mathcal{S} = \{z \in \mathbb{C}^{v+d+1} / I_z \in \mathcal{A}\}$ is exactly the set $\widetilde{X}$. Indeed, the computation of the proof of proposition 4.2

[4] The fact that a $d-$sphere has a realization in $\mathbb{R}^{d+1}$ is an open question (cf. [MW], §5).

shows that the fan of $\mathcal{S}$ is the fan in $\mathbb{R}^{v+d+1}$ whose rays are generated by the canonical basis and whose underlying complex is $\mathcal{P}$. Thus, $\mathcal{S}$ and $\widetilde{X}$ have exactly the same fan, so they agree. We can also observe that

$$\mathcal{S}_0 = \mathbb{C}^* \times \mathcal{S} = \{(z_0, z) \in \mathbb{C}^{v+d+2} / I_{z_0,z} \in \mathcal{A}_0\}$$

has the same fan, but seen in $\mathbb{R}^{v+d+2}$.

Finally, we define the map $f : (\mathbb{C}^*)^v \to (\mathbb{C}^*)^{v+d+1}$ by $f(t) = (t, X_v^{-x^1}(t), \ldots, X_v^{-x^{d+1}}(t))$. Note that $t = (X_v^{e_1}(t), \ldots, X_v^{e_v}(t))$. According to [Ha], X is the quotient of $\widetilde{X} = \mathcal{S}$ by the restriction of the toric action of $(\mathbb{C}^*)^{v+d+1}$ restricted to $Im(f)$. It's a geometric quotient because $\widetilde{\Sigma}$ is simplicial.

Considering $l_0 = 0, l_1 = e_1, \ldots, l_{2m} = e_{2m}, -x^1, \ldots, l_{v+d+2} = -x^{d+1}$ as elements of $\mathbb{C}^m$, we can define an holomorphic action of $(\mathbb{C}^*) \times \mathbb{C}^m$ on $\mathcal{S}_0$ by setting

$$(\alpha, T) \cdot (z_0, z) = (\alpha e^{<l_j, T>} z_j)_{j=0}^{v+d+2} \ \forall \alpha \in \mathbb{C}^*, t \in \mathbb{C}^m, (z_0, z) \in \mathcal{S}_0$$

It is clear that $(\mathcal{E}_0, l)$ verifies (K). Moreover, the algebraic acceptable action associated to this system has X for quotient. Indeed, a computation shows that this algebraic action is defined by

$$(\alpha, t) \cdot (z_0, z) = (\alpha z_0, f(t) \cdot z) \ \forall \alpha \in \mathbb{C}^*, t \in (\mathbb{C}^*)^{2m}, (z_0, z) \in \mathcal{S}_0$$

If we note $\mathcal{V}_0 = \{[z_0, z]/(z_0, z) \in \mathcal{S}_0\}$ the "projectivization" of $\mathcal{S}_0$, then the orbit space for the algebraic action is the quotient of $\mathcal{V}_0$ by the action defined by

$$t \cdot [z_0, z] = [z_0, f(t) \cdot z] \ \forall t \in (\mathbb{C}^*)^{2m}, [z_0, z] \in \mathcal{V}_0$$

But $\mathcal{V}_0 = \{[1, z]/z \in \mathcal{S}\}$ is homeomorphic to $\mathcal{S}$ so the claim is proved.

The only thing we have to check is that $(\mathcal{E}_0, l)$ is a good system, that is the orbit space $\mathcal{N}$ for the holomorphic action is a complex manifold. We have seen that X is a geometric quotient so the action of $(\mathbb{C}^*)^{2m}$ is proper (see, for example, [BBCM], p.28). As a consequence, the action of $\mathbb{C}^m$ is proper too, and since there is no compact subgroups in $\mathbb{C}^m$ (except $\{0\}$ of course), its action is free. Finally, the action is proper and free so $\mathcal{N}$ can be endowed with a structure of complex compact manifold.

Remark: 1) Since $\mathcal{N}$ is a complex compact manifold, the imbrication condition is fulfilled (cf. theorem 3). This gives also another proof for the fact that the SEU property is also fulfilled.
2) The transformation of $(x_1, \ldots, x_v, e_1, \ldots, e_{d+1})$ to $(e_1, \ldots, e_v, -x^1, \ldots, -x^{d+1})$ is called a linear transform of $(x_1, \ldots, x_v, e_1, \ldots, e_{d+1})$ (see [E] for example). In [M], the construction of a LVM manifold starting from a simple polytope used a special kind of linear transform called Gale (or affine) transform (see. [E])

Now, we suppose that $v = 2m + 1$ is odd. The construction is very similar. However, we have an additional step: we define an action of $(\mathbb{C}^*)^{v+1}$ on $\mathcal{S}_0$ by

$$(t_0, t) \cdot (z_0, z) = (t_0 z_0, f(t) \cdot z) = (t_0 z_0, X^{e_1}(t) z_1, \ldots, X^{-x^{d+1}}(t) z_{v+d+1})$$

where f and the action of $Im(f)$ are defined as above, and e_0 is the first vector of $\mathbb{C}^{v+1}$ with coordinates $(z_0, z_1, \ldots, z_{v+1})$. The orbit space for this last action is still X. The rest is as above: we define a fundamental set $\mathcal{E}_* = \{ \{-1\} \cup E \ / \ E \in \mathcal{E}_0 \}$ and we have:

Theorem 8: If $e_0, \ldots, e_v$ is the canonical basis of $\mathbb{R}^{v+1}$, then $(\mathcal{E}_*, (0, e_0, e_1, \ldots, e_v, (0, -x^1), \ldots, (0, -x^{d+1})))$ is a good system of type $(v + 2, d + v + 3)$ whose associated complex is $\mathcal{P}$.

5.2 LVMB manifolds and moment-angle complexes

In this section, we follow closely the definitions and notations of [BP]. We use the previous result of the above section to show that many moment-angle complexes with even dimension can be endowed with a complex structure as a LVMB manifold.

Definition: Let K be a simplicial complex on $\{1, \ldots, n\}$ with dimension $d-1$. If $\sigma \subset \{1, \ldots, n\}$, we put

$$C_\sigma = \{t \in [0, 1]^n / t_j = 1 \ \forall j \notin \sigma\}$$

and

$$B_\sigma = \{z \in \mathbb{D}^n / |z_j| = 1 \ \forall j \notin \sigma\}$$

The moment-angle complex associated to K is

$$\mathcal{Z}_{K,n} = \bigcup_{\sigma \in K} B_\sigma$$

Example 8: For instance, if K is the border of the n-simplex, $\widetilde{\mathcal{Z}}_K$ is the sphere S^{2n-1} (cf. [BP]).

In [BP], lemma 6.13, it is shown that if K is a simplicial sphere, then $\mathcal{Z}_{K,n}$ is a closed manifold. Moreover, in [Bo], to prove theorem 3 (p.1268 in [Bo]), Bosio introduces the set $\widehat{M_1'}$ defined by

$$\widehat{M_1'} = \{ \ z \in \mathbb{D}^n / \ J_z \in \mathcal{A} \ \}$$

where $J_z = \{ \ k \in \{1, \ldots, n\} / \ |z_j| = 1 \ \}$. This set is the quotient of $\mathcal{S}$ by the restriction of the holomorphic to $\mathbb{R}_+^* \times \mathbb{C}^m$ and as a consequence, $\mathcal{N}$ is the quotient of $\widehat{M_1'}$ by the *diagonal action* of S^1 defined by:

$$e^{i\theta} \cdot z = (\ e^{i\theta} z_1, \ldots, e^{i\theta} z_n \ ) \quad \forall \theta \in \mathbb{R}, \ z = (z_1, \ldots, z_n) \in \widehat{M_1'} \subset \mathbb{C}^n$$

Proposition 5.1: We have $\widehat{M_1'} = \mathcal{Z}_{\mathcal{P},n}$

Proof: Indeed, we have:

$$\begin{aligned} \widehat{M_1'} &= \bigcup_{\sigma \in \mathcal{P}} \{z \in \mathbb{D}^n / \forall j \notin \sigma, |z_j| = 1\} \\ &= \bigcup_{\tau \in \mathcal{A}} \{z \in \mathbb{D}^n / \tau \subset I_z\} \\ &= \{z \in \mathbb{D}^n / J_z \in \mathcal{A}\} \qquad \square \end{aligned}$$

Proposition 5.2: Let $(\mathcal{E}, l)$ be a good system with type $(2m+1, n, k)$ and $\mathcal{N}$ the LVMB associated to this system. If $k > 0$, then $\mathcal{N}$ is homeomorphic to a moment-angle complex.

Proof: For the clarity of the proof, we assume that n is indispensable. Let $\mathcal{P}$ be the sphere associated to $\mathcal{E}$ and $\mathcal{S}$ the open subset of $\mathbb{C}^n$ whose quotient by the holomorphic action is $\mathcal{N}$. According to proposition 5.1, the quotient $\widehat{M_1'} = \mathcal{S}/(\mathbb{R}_+^* \times \mathbb{C}^m)$ can be identified with $\mathcal{Z}_{\mathcal{P},n}$.

Let ϕ be the map defined by

$$\begin{array}{rccc} \phi & : & \mathcal{Z}_{\mathcal{P},n} & \to & \mathbb{C}^{n-1} \\ & & z & \mapsto & \left(\frac{z_1}{z_n}, \ldots, \frac{z_{n-1}}{z_n}\right) \end{array}$$

Since n is indispensable, we have $|z_n| = 1$ for every $\mathcal{Z}_{\mathcal{P},n}$ so ϕ is well defined. Moreover, ϕ is continuous and a simple calculation shows that ϕ is invariant for the diagonal action and $\phi(\mathcal{Z}_{\mathcal{P},n}) = \mathcal{Z}_{\mathcal{P},n-1}$. We claim that if $\phi(z) = \phi(w)$, then z and w belong to the same orbit for the diagonal action. Indeed, if $\phi(z) = \phi(w)$, we have

$$\left(\frac{z_1}{z_n}, \ldots, \frac{z_{n-1}}{z_n}\right) = \left(\frac{w_1}{w_n}, \ldots, \frac{w_{n-1}}{w_n}\right)$$

We have $|z_n| = |w_n| = 1$, so we put $\frac{z_n}{w_n} = e^{i\alpha}$ and we have $z = e^{i\alpha} w$.

As a consequence, ϕ induce a map $\overline{\phi} : \mathcal{N} \to \mathcal{Z}_{\mathcal{P},n-1}$ which is continuous and bijective. Actually, this is an homeomorphism since the inverse map ϕ^{-1} is the continuous map

$$\begin{array}{rccc} \phi^{-1} & : & \mathcal{Z}_{\mathcal{P},n-1} & \to & \mathcal{N} \\ & & z & \mapsto & [(z,1)] \end{array}$$

where $[(z,1)]$ denotes the equivalence class of $(z,1) \in \mathcal{Z}_{\mathcal{P},n}$ of the diagonal action.

Corollary 4: Let $\mathcal{P}$ be a rationally starshaped sphere. Then there exists $N \in \mathbb{N}^*$ such that $\mathcal{Z}_{\mathcal{P},N}$ can be endowed with a complex structure as LVMB manifold.

Proof: Since $\mathcal{P}$ is a rationally starshaped sphere, there exists a good system $(\mathcal{E}, l)$ with type $(2m+1, n, k)$ whose associated complex is $\mathcal{P}$ (cf. the previous subsection). Moreover, our construction of $(\mathcal{E}, l)$ implies that $k > 0$. So, as stated in proposition 5.2, $\mathcal{N}$ is homeomorphic to $\mathcal{Z}_{\mathcal{P},n-1}$. So, we put $N = n - 1$ and we endow $\mathcal{Z}_{\mathcal{P},N}$ with the complex structure induced by this homeomorphism.

Remark: Let N_0 be the smallest integer N as in corollary 4. Then, for every $q \in \mathbb{N}$, $\mathcal{Z}_{\mathcal{P},N_0+2q}$ can also be endowed with a complex structure and then we have $\mathcal{Z}_{\mathcal{P},N_0+2q} = \mathcal{Z}_{\mathcal{P},N_0} \times (S^1)^{2q}$. Indeed, let Λ be the matrix whose columns are the vectors of the good system $(\mathcal{E}, l)$ given in the proof of corollary 4. We put $\widetilde{\mathcal{E}} = \{\ P \cup \{N_0 + 1, N_0 + 2\} / \ P \in \mathcal{E}\}$ and we define $\lambda_1, \ldots, \lambda_{n+2}$ as the columns of the matrix

$$\begin{pmatrix} \Lambda & 0 & 0 \\ -1 \cdots -1 & 1 & 0 \\ -1 \cdots -1 & -1 & 1 \end{pmatrix}$$

Then it is easy to show that $(\widetilde{\mathcal{E}}, (\lambda_1, \ldots, \lambda_{n+2}))$ is a good system and that we have $\mathcal{Z}_{\mathcal{P},N_0+2} = \mathcal{Z}_{\mathcal{P},N_0} \times (S^1)^2$

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