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COMBINATORIAL CHARACTERIZATION OF THE LANGUAGE RECOGNIZED BY FACTOR AND SUFFIX ORACLES

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ABSTRACT

Sequence Analysis requires to elaborate data structures, which allow both an efficient storage and use. A new one was introduced in 1999 by Cyril ALLAUZEN, Maxime CROCHEMORE and Mathieu RAFFINOT. This structure is linear on the size of the represented word both in time and space. It has the smallest number of states and it accepts at least all substrings of the represented word. This structure is called *Factor Oracle*. Authors developed another structure on the basis of Factor Oracle, which has the same properties except it accepts at least all suffixes instead of all factors of the represented word. This structure is then called *Suffix Oracle*. The characterization of the language recognized by the Factor/Suffix Oracle of a given word is an open problem, for which we provide a solution. Using this result, we show that these structures may accept an exponential number of words, which are not factors/suffixes of the given word.

Keywords: Factor Oracle, Suffix Oracle, automata, language, characterization.

1. Introduction

Several structures have been developed in text indexation: we can cite Tries [1], Suffix Automata [1, 2], Suffix Trees [1, 3]... Their objective is to represent a text or a word s (*i.e.* a succession of symbols taken in an arbitrary alphabet denoted by Σ), in order to "quickly" determine whether this word contains some specific sub-word. This sub-word is then called a factor of s.

ALLAUZEN & al. [4, 5, 6] described a method allowing to build an acyclic automaton, which accepts at **least** all factors of s, which have as few states as possible (|s|+1) and which is linear in the number of transitions (2|s|-1). When each state is final in this automaton, the structure is called a Factor Oracle. By keeping only "particular" states as final, this automaton becomes a Suffix Oracle.

Several advantages have been described for this structure. The construction algorithm is easy to understand and implement; such advantage doesn't exist in the most efficient algorithm to build Suffix Trees [3]. Oracles are homogeneous automata, *i.e.* all transitions ingoing to a same state are labeled with a same symbol. Thus, it is not necessary to label edges anymore. Therefore this structure requires less memory than Suffix Trees or Tries. LEFEBVRE & al. [7, 8, 9] used it for repeated motifs discovery over large genomic data and obtained in a few seconds similar results to the ones obtained by using thousands of BLASTN requests. Authors also used the Factor Oracle for text compression [10].

However, at least two open questions are linked to these Oracles: the first one is about the characterization of the language recognized by Oracles; and the second question concerns the existence of a linear algorithm in time and space to build an automaton, which accepts all factors/suffixes of a word s and which is minimal in number of transitions.

When using these Oracles, the main difficulty is to distinguish true and false positives. Therefore we will provide in the next section several definitions related to the construction of Oracles. We will characterize the language recognized by this structure in section 3. Finally, some results using Oracles will also be described.

2. Definitions

In the following sections, we call Fact(s) (resp. Suff(s) and Pref(s)) the set of factors (resp. suffixes and prefixes) of $s \in \Sigma^+$. We call $Pref_s(i)$ the prefix of s, which has length $i \ge 0$. Given $x \in Fact(s)$, $Nb_s(x)$ is the number of occurrences of x in s and x is repeated if and only if $Nb_s(x) \ge 2$.

Definition 1 Given a word $s \in \Sigma^+$ and x a factor of s, we define the function <u>Pos</u> as the position of the first occurrence of x in s = uxv $(u, v \in \Sigma^*)$: $Pos_s(x) = |u| + 1$. We also define the function <u>poccur</u> such that $poccur_s(x) = |ux| = Pos_s(x) + |x| - 1$.

2.1. Oracles

The Oracle construction is defined by the algorithm of ALLAUZEN & al. [4] (see algorithm 1). Authors gave another algorithm to build the same automaton in linear time on the size of s. However, since we are only interested in properties of the Oracle, we do not report it in this paper.

Definition 2 [4] Given a word $s \in \Sigma^*$, we define the Factor Oracle of s as the automaton obtained by the algorithm 1, where all states are final. It is denoted by FO(s). We define the Suffix Oracle of s as the automaton obtained by the same algorithm, where a state e_i $(0 \le i \le |s|)$ is final if and only if there exists a path labeled by a suffix of s from the initial state to the state e_i . It is denoted by SO(s).

We use the term <u>Oracle</u> to equally designate the Factor or the Suffix Oracle of a word s and we denote it by O(s). We define a relation of order between states in these Oracles. Indeed, if we have two states e_i and e_j such that $i \leq j$, then $e_i \leq e_j$.

^a As mentioned in [11], the term -|u| (line 17) is unfortunately missing in the original algorithm.

²

Algorithm 1: Construction of the Factor Oracle of a word^a

```
Begin
       \tilde{C} reate the initial state labeled by e_0
       For i from 1 to |s| Do
          Greate a state labeled by e_i
Build a transition from the state e_{i-1} to the state e_i labeled by s[i]
10
11
       End For
12
13
       For i from 0 to |s| - 1 Do
          Let u be a word of minimal length recognized at state e_i
14
         For All \alpha \in \Sigma \setminus \{s[i+1]\} Do
If u\alpha \in Fact(s[i-|u|+1...|s|]) Then
15
16
                \begin{array}{l} j \leftarrow poccur_{s[i-|u|+1..|s|]}(u\alpha) - |u| \\ \text{Build a transition from the state } e_i \text{ to } e_{i+j} \text{ labeled by } \alpha \end{array}
17
18
19
            End If
20
         End For All
21
      End For
   End
22
```

Definition 3 Given a word $s \in \Sigma^*$ and a word x accepted at the state $e_i (0 \le i \le |s|)$ in the Oracle of s, we define the function <u>State</u> as $State(x) = e_i$.

Lemma 1 [4] Given a word $s \in \Sigma^*$, a unique word with minimal length is accepted at each state e_i $(0 \le i \le |s|)$ in the Oracle of s. It is denoted by $min(e_i)$.

Lemma 2 [4] Given a word $s \in \Sigma^*$, its Oracle and an integer $i \ (0 \le i \le |s|)$, then $min(e_i) \in Fact(s)$ and $i = poccur_s(min(e_i))$.

Notation 1 Given a word $s \in \Sigma^*$, let $\#_{in}(e_i)$ and $\#_{out}(e_i)$ denote the number of ingoing/outgoing transitions to/from state e_i $(0 \le i \le |s|)$ in the Oracle of s.

2.2. Canonical Factors & Contraction Operation

In this section, we will define particular factors from a given word and an operation needed to characterize the language. Then we will define the sets of words obtained with this operation.

Definition 4 Given a word $s \in \Sigma^*$ and its Oracle, we define the set of <u>Canonical</u> <u>Factors</u> of s as $\mathcal{F}_s = \{\min(e_i) \mid 1 \leq i \leq |s| \land (\#_{out}(e_i) > 1 \lor \#_{in}(e_i) > 1)\}$. Given a suffix t of s and a Canonical Factor f of s, f is a <u>conserved</u> Canonical Factor of sin t if the first occurrence of f in s appears in t. The set of the conserved Canonical Factors of s in t is denoted by $\mathcal{F}_{s,t}$ (thus $\mathcal{F}_{s,t} \subseteq \mathcal{F}_s$).

Definition 5 Given a word $s \in \Sigma^*$ and a repeated Canonical Factor f of s such that:

$$\begin{cases} s = ufv & (u, v \in \Sigma^*) \\ fv = wfx & (w \in \Sigma^+, x \in \Sigma^*) \\ Pos_s(f) = |u| + 1 \end{cases}$$

then the pair (|u| + 1, |uw| + 1) is a <u>contraction</u> of s by f.

Notation 2 Given a word $s \in \Sigma^*$ and a Canonical Factor $f \in \mathcal{F}_s$, \mathcal{C}_s^f is the set of contractions of s by f and \mathcal{C}_s^* ($\equiv \bigcup_{f \in \mathcal{F}_s} \mathcal{C}_s^f$) is the set of all the contractions we can apply to s. Given a suffix t of s = t't ($t, t' \in \Sigma^*$), then $\mathcal{C}_{s,t}^*$ is the subset of \mathcal{C}_s^* such

that $C_{s,t}^* = \{(p,q) \mid (p,q) \in C_s^* \land p > |t'|\}.$ In the following, we will use sets of contractions to produce new words from a

given one. Then we will use these words to characterize the language recognized by Oracles.

Definition 6 A set C of contractions is <u>coherent</u> if and only if it does not contain two contractions (i_1, j_1) , (i_2, j_2) such that: $i_1 < i_2 < j_1 < j_2$. Furthermore, C is <u>minimal</u> if and only if it does not contain two contractions (i_1, j_1) and (i_2, j_2) such that $i_1 \leq i_2 < j_2 \leq j_1$ or such that $i_1 < j_1 = i_2 < j_2$.

Definition 7 Given a word $s \in \Sigma^*$ and a coherent and minimal set of contractions $C = \{(p_1, q_1), \ldots, (p_k, q_k)\}$ (associated to the set of canonical factors $\{f_1, \ldots, f_k\}$), then we define the function Word as following:

$$Word(s, C) = s[1..p_1 - 1] s[q_1..p_2 - 1] \dots s[q_{k-1}..p_k - 1] s[q_k..|s|].$$



Fig. 1. Words obtained using the contraction operation (see Definition 5).

We are only interested by words obtained by the contraction operation. So we will only consider coherent and minimal sets of contractions without loss of generality. Note that the word remains the same whatever the order of contractions (see figure 1).

Definition 8 We define $\mathcal{E}(s) = \bigcup_{\mathcal{C} \subseteq \mathcal{C}_s^*} Word(s, \mathcal{C})$, as the <u>closure</u> of s.

Example: Consider the word *gaccattctc* (see figure 2). Its set of Canonical Factors is $\mathcal{F}_{gaccattctc} = \{a, c, ca, t, tc, ct\}$ and then $\mathcal{C}^*_{gaccattctc} = \{(2, 5), (3, 4), (3, 8), (3, 10), (6, 7), (6, 9), (7, 9)\}$. Now, consider the set $\mathcal{C} = \{(2, 5), (7, 9)\}$ ($\mathcal{C} \subseteq \mathcal{C}^*_{gaccattctc}$), then $Word(gaccattctc, \mathcal{C}) = gaccattctc + c$. The closure of gaccattctc is:

 $\mathcal{E}(gaccattctc) = \left\{ \begin{array}{l} gac, gacatc, gacatctc, gacattc, gacattcc, gaccatc, gaccatctc, \\ gaccattc, gaccattcc, gactc, gatc, gatctc, gattc, gattctc \end{array} \right\}.$



Fig. 2. Suffix Oracle of gaccattete. In the Factor Oracle, all states are final.

3. Characterization of the language recognized by Oracles

Given a word $s \in \Sigma^*$, we saw how to build the corresponding Factor (resp. Suffix) Oracle. This Oracle allows to recognize the factors (resp. suffixes) of s, but it also accepts additional words. For example the word *atc* is accepted by the Factor (resp. Suffix) Oracle of *gaccattctc* (see figure 2), whereas it is neither a factor nor a suffix of this sequence. We will see that the Suffix Oracle of s exactly recognizes all suffixes of words from $\mathcal{E}(s)$ and we will use this result to prove that the Factor Oracle of s exactly recognizes all factors of words from $\mathcal{E}(s)$.

We use following Lemmas to prove the result concerning Suffix Oracles. These Lemmas have been proved in [4].

Lemma 3 [4] Given a word $s \in \Sigma^*$ and an integer $i \ (0 \le i \le |s|)$, then $min(e_i)$ is suffix of all words recognized at state e_i in the Oracle of s.

Lemma 4 [4] Given a word $s \in \Sigma^*$ and a factor w of s, then w is recognized at state e_i $(1 \le i \le poccur_s(w))$ in the Oracle of s.

Lemma 5 [4] Given a word $s \in \Sigma^*$ and an integer $i \ (0 \le i \le |s|)$, then every path ending by $min(e_i)$ in the Oracle of s leads to a state e_j such that $j \ge i$.

Lemma 6 [4] Given a word $s \in \Sigma^*$ and $w \in \Sigma^*$ a word accepted at state e_i $(0 \le i \le |s|)$ in the Oracle of s, then every suffix of w is also recognized by the Oracle at state e_j such that $j \le i$.

The proof of Lemma 6 is only given in [4] for Factor Oracles. We need this result to be true for Suffix Oracles.

Proof. The original Lemma gives us that if x is a suffix of w, then $State(x) \leq State(w)$. We need to prove that if State(w) is final, then State(x) is final. Therefore, we have to consider two cases. When $|x| \geq |min(e_i)|$, we have $min(e_i) \in Suff(x)$ and thus, according to Lemma 5, $State(x) \geq State(min(e_i))$. Since $State(min(e_i)) = e_i = State(w)$, we conclude that State(x) = State(w). When we have $|x| < |min(e_i)|$, since the state e_i is final, then there exists a suffix t of s such that $State(t) = e_i$. According to Lemma 3, we conclude that $min(e_i) \in$ $Suff(t) \subseteq Suff(s)$. Since x and $min(e_i)$ are suffixes of w, then $|x| < |min(e_i)| \Rightarrow$ $x \in Suff(min(e_i))$. So x is also suffix of s and according to Definition of the Suffix Oracle State(x) is final.

We use these previous results to show that the two following Lemmas.

Lemma 7 Given a word $s \in \Sigma^*$, a Canonical Factor f of $s = ufv (u, v \in \Sigma^*)$ such that f is not repeated in uf and a set C of contractions ($C \in C_s^*$) such that Word(uf, C) = wf, then the Oracle of s accepts wf and f at the same state.

Proof. We denote by $C_i \subseteq C_s^*$ a set of contractions, which has cardinality *i*. In the same way, we denote by $w_i f$ the word obtained when we apply contractions C_i to uf (warning: $w_i f = Word(uf, C_i) \Rightarrow w_i = Word(u, C_i)$). By induction on the size of C_i , we show that $State(Word(uf, C_i)) = State(f)$ for all $C_i \in C_s^*$. Let $e_x = State(f)$ $(f = min(e_x)$ by Definition of f) and $e_{x'_i} = State(Word(uf, C_i))$.

If we consider C_0 , then $Word(uf, C_0) = uf$. According to Lemma 5, $x'_0 \ge x$. Furthermore, according to Lemma 4 applied to uf, we have $x'_0 \le poccur_s(uf)$.

However by Definition of f, $poccur_s(f) = |uf| = poccur_s(uf)$. Therefore we have $x'_0 \leq x$ and $x'_0 = x$.

If this lemma is true for a set $C_i \subset C_s^*$ of contractions, then it is true for a set $C_{i+1} = C_i \cup \{(p,q)\}$. We assume without loss of generality (see figure 1) that (p,q) is the last contraction in C_{i+1} (by ascending order over positions). Let b be the Canonical Factor used by this contraction. We can write uf = s[1..p-1]s[p..q-1]s[q..|uf|]. Since (p,q) is chosen as the last contraction, all contractions in C_i are applicable to s[1..p-1]. So we define $a, c \in \Sigma^*$ such that $w_i f = as[p..|uf|] = abc$ and $d \in \Sigma^*$ such that $w_{i+1}f = as[q..|uf|] = abd$. Also $ab = Word(s[1..p-1]b, C_i)$: the opposite would mean that contraction (p,q) can't be operated from b and according to the induction hypothesis, State(ab) = State(b). From this, we conclude that State(abc) = State(bc) and State(abd) = State(bd). We know that bd (= s[q..|uf|]) is a suffix of bc (= s[p..|uf|]) and according to Lemma 6:

$$\begin{array}{rcl} State(bd) &\leq State(bc) \\ \Leftrightarrow & State(abd) &\leq State(abc) \\ \Leftrightarrow & State(w_{i+1}f) &\leq State(w_if) \\ \Leftrightarrow & State(w_{i+1}f) &\leq State(f) \end{array}$$

However, according to Lemma 5, we have $State(w_{i+1}f) \ge State(f)$ and therefore $State(w_{i+1}f) = State(f)$. This lemma is then true for all $C_i \subseteq C_s^*$.

Fig. 3. Illustration of Lemma 7.

Now, we show how to obtain a contraction, starting with transition of type $e_i \rightarrow e_j$ with j > i + 1 (that we call an <u>external transition</u> in the subsequent).

Lemma 8 Given a word $s \in \Sigma^*$ and an integer $i \ (0 \le i < |s|)$ such that $\#_{out}(e_i) > 1$, let p be a external transition from state e_i to state $e_{i+j} \ (j > 1)$ labeled by α and $u = min(e_i)$. Then an occurrence of $u \alpha$ exists at position (i + j - |u|) of s (see figure 5 – page 8). Moreover, the contraction (i - |u| + 1, i - |u| + j) of s by u exists too.

Proof. According to the construction algorithm (see algorithm 1), the transition p is added from e_i to e_{i+j} because a position j in s[i - |u| + 1..|s|] is such that: $j = poccur_{s[i-|u|+1..|s|]}(u\alpha) - |u|$. We also have $u\alpha \in Fact(s)$ because $u\alpha \in Fact(s[i - |u|+1..|s|])$. CLEOPHAS & al. [11] proved that since $u = min(e_i)$ and $u\alpha \in Fact(s)$, then $i - |u| + poccur_{s[i-|u|+1..|s|]}(u\alpha) = poccur_s(u\alpha)$. We have $i + j = poccur_s(u\alpha)$ and finally $s[i + j - |u|, i + j] = u\alpha$.

We use algorithm <u>Contractor</u> (see algorithm 2) to give a characterization of the language accepted by the Oracle of a word s. Given a word $s \in \Sigma^*$ and its Suffix Oracle SO(s), initial inputs of <u>Contractor</u> are a word w accepted by SO(s) and t, the maximal suffix of s beginning with w[1]. This algorithm outputs a set $\mathcal{C} \in \mathcal{C}_s^*$ of contractions such that $w = Word(t, \mathcal{C})$.

Algorithm 2: Contractions needed to transform s = t't $(t, t' \in \Sigma^*)$ into t'w

1	Initialization: $S_0 = t$, $S_0^w = w$, $\mathcal{C}_0 = \emptyset$, $sdec = s - t $	$\% \ t$ is a suffix of s $\%$
2 3 4	$\begin{array}{llllllllllllllllllllllllllllllllllll$	"contracted" %
5	C _i % Set of contractions %	
6	Output: a set of contractions	
7 8	Begin	
9	$p_i \leftarrow \text{longest common prefix between } S_i \text{ and } S_i^-$	(Claim 1, item 1)
10	$e_{q_i} \leftarrow State(p_i)$	(Claim 1, item 2)
11	$f_i \leftarrow min(e_{q_i})$	
12	If $(p_i < S_i^-)$ Then	
13	$e_{r_i} \leftarrow \underline{\operatorname{Transition}}(e_{q_i}, S_i^{\omega}[p_i +1])$	(Claim 1, item 4)
14	$C_{i+1} \leftarrow C_i \cup \{c_i\}, \ c_i = (q_i - f_i + 1, r_i - f_i)$	(Claim 2, item 2)
15	$S_{i+1}^{\omega} \leftarrow S_i^{\omega}[p_i - f_i + 1 S_i^{\omega}]$	(Claim 1, item 3)
16	$S_{i+1} \leftarrow t[r_i - f_i - sdec t]$	(Claim 1, item 3)
17	Return Contractor $(S_{i+1}, S_{i+1}^{\omega}, \mathcal{C}_{i+1})$	
18	Else	
19	$ \mathbf{f}_{i}(S_{i} > S_{i}^{w}) \text{ Then }$	
20	$C_{i+1} \leftarrow C_i \cup \{c_i\}, \ c_i = (q_i - f_i + 1, s - f_i + 1)$	(Claim 3)
21	Else	
22	$\mathcal{C}_{i+1} \leftarrow \mathcal{C}_i$	(Claim 3)
23	End If	
24	Return C_{i+1}	
25	End If	
26	End	

By Definition 7, given a word $s \in \Sigma^*$ such that s = t't $(t, t' \in \Sigma^*)$ and a set $C \in \mathcal{C}^*_{s,t}$ of contractions, t'w = Word(s, C) is then a concatenation of substrings of s. These substrings can be assimilated as prefixes of suffixes of s. A contraction is then a "jump" from one substring to the next one. The main idea of this algorithm is to read from left to right the sequences t and w, in order to compute the longest common prefixes between given suffixes of t and w. After each stage, the algorithm computes the length of the "jump" to go to the next suffix of t to consider.

Inputs are words S_i , S_i^w $(i \ge 0)$ and a set C_i of contractions. We initialize $S_0 = t$, $S_0^w = w$, $C_0 = \emptyset$ and p_i (line 9) as the longest prefix of S_i and S_i^w . So:

$$S_i = p_i S'_i \quad \text{and} \quad S^w_i = p_i S'^w_i. \tag{1}$$

Let $e_{q_i} = State(p_i)$ and $f_i = min(e_{q_i})$ ((lines 10 and 11), Lemma 3 provides:

$$p_i = p'_i f_i \quad (p'_i \in \Sigma^*).$$

$$\tag{2}$$

The variable e_{r_i} (line 13) is the state reached by the transition from e_{q_i} and labeled by $\alpha = S_i^w[|p_i| + 1] = S_i'^w[1]$. The set \mathcal{C}_{i+1} has cardinality i + 1. The variable sdec = |s| - |t| is necessary to compute S_{i+1} (line 15), because each state e_i is linked to the i^{th} character of s, not to the character (i - |s| + |t|) of t.

Claim 1 Following assertions are true for all $i \ge 0$ (see figure 4):

- 1. $f_i \alpha \in Pref(p_{i+1})$.
- 2. $S_i = t[q_i |p_i| + 1 sdec..|t|].$
- 3. S_{i+1} and S_{i+1}^w are respectively suffixes of S_i and S_i^w ; S_i and S_i^w $(i \ge 0)$ are respectively suffixes of t and w.
- 4. Transition from e_{q_i} to e_{r_i} labeled by α always exists.
 - 7



Fig. 4. Visualization of Contractor on S_i and S_i^w .

Proof.

1. Since $f_i = min(e_{q_i})$ and according to Lemma 8, we have $s[r_i - |f_i|..r_i] = t[r_i - |f_i| - sdec..r_i - sdec] = f_i \alpha$. So S_{i+1} and S_{i+1}^w begin with $f_i \alpha$ (line 15).

2. For i = 0 (initialization case), $S_0 = t$ and t is the longest suffix of s beginning by w[1]. If $e_x = State(S_0[1])$ (x > 0), then $t[x - sdec..|t|] = S_0$ and $State(p_0) = x + |p_0| - 1 = e_{q_i}$. Thus we conclude that $S_0 = s[q_0 - |p_0| + 1 - sdec..|s|]$. At iteration i, we have $S_{i+1} = t[r_i - |f_i| - sdec..|t|]$ (line 16). Since S_{i+1}^w begins by $f_i \alpha$ (item 1), we conclude that $q_{i+1} = r_i + |p_{i+1}| - |f_i| - 1$ and finally we have $S_{i+1} = t[r_i - |f_i| - sdec..|t|] = t[q_{i+1} - |p_{i+1}| + 1 - sdec..|t|]$.

3. This assertion is true for S_i^w because S_{i+1}^w is suffix of S_i^w by construction (line 15) and $S_0^w = w$. Concerning S_i , we have $S_0 = t$ thus the property is true for i = 0. Suppose that S_i is suffix of t, from item 2, we have $S_i = t[q_i - |p_i| + 1 - sdec..|t|]$. We also have $S_{i+1} = t[r_i - |f_i| - sdec..|t|]$ (line 16). So we conclude using Eq. (2) that $q_i - |p_i| = q_i - |p'_i| - |f_i|$. Since $|p'_i| \ge 0$, we have $q_i - |f_i| \ge q_i - |p_i|$ and $r_i > q_i$. Finally, we conclude that $r_i - |f_i| > q_i - |p_i|$ and that S_{i+1} is a suffix of S_i .

4. According to item 3, S_i^w is suffix of w. Then S_i^w is accepted by O(s)(Lemma 6). Using Eq. (1) with $S_i'^w[1] = \alpha$ the transition must exist and implies $\#_{out}(e_{q_i}) \geq 2$. So $f_i = min(e_{q_i}) \in \mathcal{F}_s$ by Definition of Canonical Factors. \Box

From Eq. (1) and Claim 1 (item 4):

$$\begin{cases} t = t'_i S_i & (t'_i \in \Sigma^*) \\ w = w'_i S^w_i = w'_i p'_i S^w_{i+1} & (w'_i \in \Sigma^*). \end{cases}$$
(3)



Fig. 5. Illustration of a step in the algorithm Contractor $(\alpha = S_i^w[|p_i| + 1])$.

Claim 2 For all $i \ge 0$ (see figure 5):

1. $State(w'_i p_i) = State(t'_i p_i) = State(p_i) = e_{q_i}$.

2. c_i is a contraction of t'_iS_i (resp. w'_iS_i) provided by f_i . Result of this contraction is $t'_ip'_iS_{i+1}$ (resp. $w'_ip'_iS_{i+1} = w'_{i+1}S_{i+1}$).

Proof.

1. For i = 0, $t'_i = w'_i = \epsilon$. Let us suppose that the property is true for i and therefore for i+1. From Claim 1 (item 2), we conclude that S_i labeled a path using only "main" transitions (*i.e.* transitions of type $e_j \to e_{j+1}$) from $e_{q_i-|p_i|}$ to $e_{|s|}$ in O(s). According to Claim 1 (item 3) we conclude that:

$$S_i = uS_{i+1} \quad (u \in \Sigma^*). \tag{4}$$

So, the state e_x $(x > q_i - |p_i|)$ is such that S_{i+1} labeled a path using only "main" transitions from e_x to $e_{|s|}$ in O(s) and more precisely $x = r_i - |f_i| - 1$. We have $t'_{i+1} = t'_i u$ (Eq. (3) and (4)) and $State(t'_i u) = e_x$. Since $f_i = min(e_{q_i})$ and there is a transition from e_{q_i} to e_{r_i} labeled by α (Claim 1, item 4), $State(t'_{i+1}f_i\alpha) = State(t'_i u f_i \alpha) = State(f_i \alpha) = e_{r_i}$ and furthermore, $p_{i+1} = f_i \alpha v$ ($v \in \Sigma^*$). So we have $State(t'_{i+1}f_i \alpha v) = State(t'_{i+1}p_{i+1}) = State(p_{i+1})$.

2. From Eq. (1), (2) and (3), we conclude that:

$$t = t_i' S_i = t_i' p_i' f_i S_i'.$$

$$\tag{5}$$

Since $S_{i+1} \in Suff(S_i)$, we have $S_i = uS_{i+1}$ $(u \in \Sigma^+)$ and from Eq. (5), $t'_i p'_i f_i S'_i = t'_i uS_{i+1}$. According to Claim 1 (item 1), we have $t'_i p'_i f_i S'_i = t'_i uf_i \alpha u'$ $(u' \in \Sigma^*)$. Since we have $State(t'_i p'_i f_i) = State(f_i)$ and $|u| > |p'_i|$ (because $S'_i[1] \neq \alpha$), we can contract $t'_i S_i$ by f_i and $t'_i p'_i f_i \alpha u' = t'_i p'_i S_{i+1}$. We can conclude that $w'_i S_i = w'_i p_i S'_i$ is contracted by f_i in $w'_i p'_i S_{i+1}$ because $State(w'_i p_i) = State(t'_i p_i)$. According to Eq. (3), we conclude that $w'_{i+1} = w'_i p'_i$ and $w'_i p'_i S_{i+1} = w'_{i+1} S_{i+1}$.

Claim 2 shows that a contraction c_i of $t'_i S_i$ by f_i is also a contraction for t and for $w'_i S_i$. Consider the i^{th} iteration, we have $|S^w_i| > |S^w_{i+1}|$ or $|S^w_i| = |S^w_{i+1}|$ and $|p_{i+1}| > |p_i|$ (if $f_i = p_i$). Since $p_i > 0$, we ensure that recursion stops at iteration j > i with $p_j = S^w_j$ (j > i).

Claim 3 Given an integer $i \ge 0$ such that $p_i = S_i^w$, then t needs a last contraction if and only if $|S_i^w| \ne |S_i|$.

Proof. The word obtained with the set C_i of contractions is $w'_i p_i S'_i$ (Claim 2, item 2). If $S_i^w = S_i$, then we have $S'_i = \epsilon$ and we conclude that C_i is complete (line 20). If $S_i^w \neq S_i$, according to Claim 2 (item 1), we have $State(w'_i p_i) = e_{q_i}$. By Definition of the final state in a Suffix Oracle, $min(e_{q_i}) \in Suff(w)$ (Lemma 3) and $min(e_{q_i}) \in Suff(t)$. A last contraction is needed to complete the set of contractions (line 22).

Considering the i^{th} call of <u>Contractor</u>, inputs are $S_i = p_i S'_i$, $S^w_i = p_i S'^w_i$ and C_i . This set is the set of contractions necessary to transform $t'_i \in Pref(t)$ into $w'_i \in Pref(w)$. The variable p_i refers to the longest common prefix of S_i and S^w_i $(p_i$ is both a factor of t and w, Claim 1). Two cases may occur for this call (line 12. When $|p_i| = |S^w_i|$, the recursion ends. Otherwise, $|p_i| \neq |S^w_i|$ and at least another contraction is necessary until $|p_j| = |S^w_j|$ (j > i). From Claim 2, the i^{th} contraction allows to continue with the suffix S_{i+1} . At the end of the process (*i.e.* the end of w), we return the set $\mathcal{C} = \mathcal{C}_j$ and Lemma 9 ensures that $w = Word(t, \mathcal{C})$.

Lemma 9 Given a word $s \in \Sigma^*$, its Suffix Oracle, a word $w \in \Sigma^*$ accepted by SO(s) and the longest suffix t of s such that w[1] = t[1], then <u>Contractor(t, w, \emptyset)</u> outputs a set C, which is such that w = Word(t, C).

Proof. Let $i \ge 0$ such that $S_{i+1}^w = p_{i+1}$. According to Claim 2, we conclude that C_{i+1} is a coherent set of contractions of t. Since p_{i+1} is prefix of S_{i+1} we have:

$$Word(t, \mathcal{C}_{i+1}) = w'_{i+1}S_{i+1} = w'_{i+1}S_{i+1}^w u = w'_{i+1}p_{i+1}u \quad (u \in \Sigma^*)$$

If $u = \epsilon$, we have $Word(t, \mathcal{C}_{i+1}) = w$ (Eq. (3)). Else (Claim 3) a ultimate contraction c_{i+1} by f_{i+1} transforms $w'_{i+1}S^w_{i+1}u$ into $w'_{i+1}S^w_{i+1} = w = Word(t, \mathcal{C}_{i+1} \cup \{c_{i+1}\})$. Finally <u>Contractor</u> provides a set \mathcal{C} such that $w = Word(t, \mathcal{C})$.

We can notice that:

1. C is not always minimal.

2. *C* is coherent. Let (a, b) and (c, d) be two contractions successively added to *C*. We have a < b and c < d because $r_i > q_i$ and $|s| > q_i$ (lines 14 and 20). If $e_{q_{i+1}} = State(p_{i+1}) = e_{r_i}$, then b = c, else $p_{i+1} = f_i \alpha v$ ($\alpha = S_{[}^w |p_i| + 1$], $v \neq \epsilon$) and $e_{q_{i+1}} > e_{r_i}$, therefore b < c.

Following theorems are the main purpose of this paper:

Theorem 1 Exactly all suffixes of words from $\mathcal{E}(s)$ are recognized by the Suffix Oracle of s.

Proof.

'⇒': Each suffix of words from $\mathcal{E}(s)$ is recognized by the Suffix Oracle of s. According to Lemma 6, if w is accepted by SO(s), each suffix of w is also accepted by SO(s). So we only need to prove that each word from $\mathcal{E}(s)$ is accepted by SO(s). Let $\mathcal{C} \in \mathcal{C}_s^*$ be a set of contractions applicable to s and $w = Word(s, \mathcal{C})$. The set \mathcal{C}_i is the set of the first i contractions of \mathcal{C} (chosen without loss of generality by ascending order over positions – see figure 1), (x_j, y_j) is the j^{th} contraction, which use the Canonical Factor f_j (1 ≤ $j \le i$) and $w_j = Word(s, \mathcal{C}_j)$. The property (P) to prove is that if w_i (0 ≤ $i < |\mathcal{C}|$) is accepted by SO(s), then w_{i+1} is accepted too. We have:

$$\begin{cases} w_i &= s[1..x_1 - 1] s[y_1..x_2 - 1] \dots s[y_i..|s|] \\ s[y_i..y_i + |f_i| - 1] &= f_i \end{cases}$$

By Definition of the Canonical Factors, f_{i+1} does not occur in s before position x_{i+1} ($x_{i+1} > y_i$). We have $w_i = v'f_{i+1}u$ and $w_{i+1} = v'f_{i+1}u'$ with $v' = s[1..x_1 - 1]s[y_1..x_2 - 1]...s[y_i..x_{i+1} - 1]$ and $f_{i+1}u = u''f_{i+1}u'$ ($u'' \in \Sigma^+$).

Considering the contraction (x_{i+1}, y_{i+1}) , we have $|s| - |f_{i+1}u| + 1 = x_{i+1}$ and $|s| - |f_{i+1}u'| + 1 = y_{i+1}$ (because the contractions are in ascending order). The word s is then not yet modified after positions x_{i+1} , so $f_{i+1}u$ and $f_{i+1}u'$ are suffixes of s. Consider the state $q = State(f_{i+1})$ of SO(s), according to Lemma 7:

$$State(v'f_{i+1}) = q. (6)$$

The suffix $f_{i+1}u'$ of s is necessarily recognized by SO(s). So the path accepting this suffix in SO(s) go through the state q. Starting from q, we can read u' and

reach a final state and therefore, according to Eq. (6), SO(s) also accepts the word $w_{i+1} = v'f_{i+1}u'$. Finally, the property (P) is true for all i ($0 \le i < |\mathcal{C}|$) and since $w_0 = s$, we can prove by induction on i that the Suffix Oracle of s recognizes all words w_i ($0 \le i \le |\mathcal{C}|$). Lemma 6 allows to conclude that each suffix of words from $\mathcal{E}(s)$ is recognized by SO(s).

'⇐': Each word recognized by the Suffix Oracle of s is suffix of a word from $\mathcal{E}(s)$. Let w be a word accepted by the Suffix Oracle of s and t be the longest suffix of s = t't $(t, t' \in \Sigma^*)$ beginning with w[1]. Then a set C of contractions such that t'w = Word(t't, C) exists (Lemma 9). Since the word w is suffix of t'w and $t'w \in \mathcal{E}(s)$, each word accepted by SO(s) is a suffix of a word from $\mathcal{E}(s)$.

On the basis of Theorem 1 we give a similar result, which is available for Factor Oracles instead of Suffix Oracles.

Theorem 2 Exactly all factors of words from $\mathcal{E}(s)$ are recognized by the Factor Oracle of s.

Proof.

 \Rightarrow : Each factor of words from $\mathcal{E}(s)$ is recognized by the Factor Oracle of s.

Let SO(s) be the Suffix Oracle of $s, u \in \mathcal{E}(s)$ and m a factor (i.e. a prefix of a suffix) of u = mv ($v \in \Sigma^*$). Then mv is accepted by SO(s) (Theorem 1). Thus, a path ($e_0 \to e_{x_1} \to \ldots \to e_{x_{|mv|}}$) exists in SO(s), which recognizes mv. Therefore, we conclude that m labeled a path ($e_0 \to e_{x_1} \to \ldots \to e_{x_{|m|}}$) (with $e_{x_{|m|}}$ final).

'⇐': Each word recognized by the Factor Oracle of s is factor of a word from $\mathcal{E}(s)$. Let SO(s) be the Suffix Oracle of s and m a word accepted by FO(s). If SO(s) recognizes m then m is a suffix of a word from $\mathcal{E}(s)$ (Theorem 1). Suppose that SO(s) does not recognize m. Then FO(s) recognizes m at state $e_{x_{|m|}}$ (not final in SO(s)). Furthermore, the path $(e_0 \rightarrow e_1 \rightarrow \ldots \rightarrow e_{|s|})$ exists in O(s) and $e_{|s|}$ is final in SO(s). We conclude that a path from $e_{x_{|m|}}$ to $e_{|s|}$ exists in SO(s). Then, the word m is prefix of a word recognized by SO(s) and therefore m is prefix of a suffix of some $u \in \mathcal{E}(s)$. Thus, m is a factor of a word of $\mathcal{E}(s)$.

4. Properties upon Oracles & Future Works

According to CLEOPHAS & al. [11], the Oracle is not minimal in number of transitions among the set of homogeneous automata. Furthermore, if we consider the set of homogeneous automata, which recognize at least all factors (resp. suffixes) of s and which have the same number of states and at most the same number of transitions than the Factor (resp. Suffix) Oracle, we show that the Oracle is not minimal on the number of accepted words. The Oracle of axttyabcdeatzattwu (see figure 6) has 35 transitions, the Factor Oracle accepts 247 words and the Suffix Oracle accepts 39 words. Though another homogeneous automaton (see figure 7), which recognizes at least all factors (resp. suffixes) of axttyabcdeatzattwu and which has only 34 transitions exists. The "Factor" version of this automaton recognizes only 236 words and its "Suffix" version accepts only 30 words. Moreover, we provide an example which has both less transitions, and less accepted words than the corresponding Oracle.



Fig. 6. Factor Oracle of the word axttyabcdeatzattwu.



Fig. 7. This automaton (considering only the continuous lines) accepts all factors of the word *axttyabcdeatzattwu*. The bold transition (from e_1 to e_3) is the only one, which is not present in the Factor Oracle of this word (see figure 6) though the two dotted ones (from e_1 to e_{12} and from e_{12} to e_{16}) are present in the Factor Oracle, but not in this automaton.

In some cases, we observe that the number of words accepted by Oracles does not allow confidence to this structure when it is used to detect factors or suffixes of words. Even if the number of false positive can sometimes be equal to 0 (e.g. aaaaaa...), it can also be exponential. Indeed, we can build a word s such that each subset of C_s^* is coherent and minimal. For example: s = aabbccddee..., the set C_s^* of contractions, which are available on such a word, is $\{(1,2), (3,4), \ldots, (|s| 1, |s|)\}$. If we consider any non-empty subset $C \subseteq (C_s^* \setminus \{(1,2)\})$ of contractions, it is easy to notice that $Word(s, C) \notin Fact(s)$. Besides, all words obtained from such subsets are pairwise different.

Number of subsets is:
$$\sum_{i=1}^{|\mathcal{C}_s^*|-1} \binom{|\mathcal{C}_s^*|-1}{i} = \sum_{i=1}^{\frac{|s|}{2}-1} \binom{|s|}{2} = 1 = 2^{\frac{|s|}{2}-1} - 1.$$
 Then,

the number of words, which are accepted by the Oracles and are not factor/suffix of s, is $\mathcal{O}(2^{|s|})$.

To better use this structure, we need to improve or to slightly modify it. However, better knowledge about the Oracle structure would be useful for future works. Indeed, it could be interesting to have an empirical or a statistical estimation of the accuracy of the Oracle (time and quality of the results), when it is substituted to Tries or Suffix Trees in algorithms.

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