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A step towards the Bermond-Thomassen conjecture about disjoint cycles in digraphs

Nicolas Lichiardopol∗      Attila Pór†      Jean-Sébastien Sereni‡

Abstract

In 1981, Bermond and Thomassen conjectured that every digraph with minimum out-degree at least $2k - 1$ contains $k$ disjoint cycles. This conjecture is trivial for $k = 1$, and was established for $k = 2$ by Thomassen in 1983. We verify it for the next case, by proving that every digraph with minimum out-degree at least five contains three disjoint cycles. To show this, we improve Thomassen’s result by proving that every digraph whose vertices have out-degree at least three, except at most two with out-degree two, indeed contains two disjoint cycles.

1 Introduction

Our notations mainly follow that of Bang-Jensen and Gutin [2]. By cycle we mean oriented cycle, that is an oriented path starting and ending at the same vertex. A cycle of length $d$ is a $d$-cycle. A 1-cycle is a loop and a 3-cycle is also called a triangle. All digraphs contained in this paper can have loops and 2-cycles but no parallel arcs. A digraph without cycles of length at most two is an oriented graph.

Fix a digraph $D = (V, A)$. Its order is the size of the vertex-set $V$. Given a subset $X$ of $V$, the sub-digraph of $D$ induced by $X$ is the digraph

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$D[X] := (X, A')$ where $A'$ is the set of all arcs in $A$ that start and end in $X$. Two sub-digraphs $D_1$ and $D_2$ of $D$ are disjoint if their vertex-sets are. We write $v \rightarrow u$ to mean an arc from the vertex $v$ to the vertex $u$.

We let $D^*$ be the digraph obtained from $D$ by reversing the direction of every arc. For every vertex $v \in V$ let $N^+_D(v) := \{x \in V : v \rightarrow x \in A\}$ be the out-neighbourhood of $v$ in $D$, and let $d^+_D(v) := |N^+_D(v)|$ be the out-degree of $v$ in $D$. The vertices of $N^+_D(v)$ are the out-neighbours of $v$. The in-neighbourhood of $v$ in $D$ is $N^-_D(v) := N^+_D(v)^*$, and its in-degree is $d^-_D(v) := |N^-_D(v)|$. The vertices of $N^-_D(v)$ are the in-neighbours of $v$. If the context is clear, we may omit the subscript and just write $N^+(v)$ and $N^-(v)$.

Given two disjoint subsets $X, X' \subset V$, the set $X$ dominates $X'$ if $X'$ is contained in the out-neighbourhood of each vertex of $X$. If the set $X$ is composed of only one vertex $v$ we simply say that $v$ dominates $X'$. The set $X'$ is dominated if there exists a vertex dominating it. The set $X$ dominates a sub-digraph $D'$ of $D$ if it dominates its vertex-set $V(D')$.

An arc is $d$-dominated if it is dominated by a vertex of out-degree $d$.

We are interested in the following conjecture stated by Bermond and Thomassen in 1981.

**Conjecture 1** ([3]). For every positive integer $k$, every digraph with minimum out-degree at least $2k - 1$ contains $k$ disjoint cycles.

It is an obvious observation if $k$ is one, and Thomassen gave a nice and simple proof of it when $k$ is two in 1983.

**Theorem 1** ([6]). Every digraph with minimum out-degree at least three contains two disjoint cycles.

Thomassen [6] also established the existence of a finite integer $f(k)$ such that every digraph of minimum out-degree at least $f(k)$ contains $k$ disjoint cycles. As noted by Bermond and Thomassen [3], such an integer cannot be less than $2k - 1$, so the bound offered by Conjecture 1 is optimal. Alon [1] proved that for every integer $k$, the value $64k$ is suitable for $f(k)$ in 1996. Recently, Conjecture 1 has been verified for tournaments with minimum in-degree at least $2k - 1$ [4, 5].

Our main result is the following theorem, which proves Conjecture 1 when $k$ is three.

**Theorem 2.** Every digraph with minimum out-degree at least five contains three disjoint cycles.

We note that the method used by Alon [1] yields an upper bound on the order of a minimum counter-example to Conjecture 1. For instance,
when $k$ is three the order of a minimum counter example is at most 42. However, as pointed out by Alon [1], this bound is out of reach for a brute-force attack. Thus we need to develop new tools to study this conjecture and prove Theorem 2. One of them is to strengthen Theorem 1.

**Theorem 3.** Let $D$ be a digraph whose vertices have out-degree at least three, except at most two which have out-degree two. The digraph $D$ contains two disjoint cycles.

The paper is organised as follows. In the next section we slightly improve Thomassen’s result by proving Theorem 3 which is a crucial ingredient in our proof of Theorem 2. Section 3 is devoted to the proof of a property of a certain class of digraphs, which may be of independent interest. In Section 4 we establish Theorem 2. The proof proceeds by contradiction: we consider a minimum counter-example $D$—with respect to the number of vertices—to the statement of the theorem, and exhibit some of its structural properties. Then, the argument is split into two cases: in Sub-section 4.1 we suppose that $D$ does not contain a triangle while in Sub-section 4.2 we establish the result if $D$ contains a triangle.

## 2 Improving Theorem 1

As mentioned earlier, Thomassen proved that Conjecture 1 is true if $k$ is two, namely every digraph with minimum out-degree three contains two disjoint cycles. The goal of this section is to strengthen this result, by proving Theorem 3.

**Proof of Theorem 3.** Contrary to the statement, let $D = (V, A)$ be a minimum counter-example with respect to the number of vertices. We also assume that each vertex has out-degree at most three. First, observe that $D$ cannot contain a loop. If $C$ is a loop, the digraph obtained from $D$ by removing the vertex of $C$ has minimum out-degree at least one, thus it contains a cycle $C'$. The cycles $C$ and $C'$ of $D$ are disjoint, a contradiction. So the order of $D$ is at least four. We now establish two properties of $D$. Recall that a sub-digraph is 2-dominated if there exists a vertex of out-degree two dominating it.

(A) Every 2-cycle of $D$ is 2-dominated. In particular $D$ contains at most two 2-cycles.

Suppose that $C := uv$ is a 2-cycle. Let $D'$ be the digraph obtained from $D$ by removing $u$ and $v$. Then $D'$ cannot have minimum out-degree at least one, otherwise it would contain a cycle which would be disjoint from $C$, a contradiction. Therefore there exists a vertex of $D$ of out-degree two
dominating $C$, as asserted. From this fact it directly follows that $D$ does not contain more than two 2-cycles, since each vertex of out-degree two can dominate at most one 2-cycle and $D$ contains at most two vertices of out-degree two.

The next property, proved by Thomassen [6], is still valid under our weaker assumptions.

(B) Every arc of $D$ is dominated.

Suppose that $u \to v \in A$ is not dominated. By Property (A), we can assume that $v \to u$ is not an arc of $D$. Let $D'$ be the digraph obtained from $D$ by first removing all arcs out-going from $u$ except $u \to v$, and then contracting the arc $u \to v$ into a new vertex $w$. The out-degree of $w$ in $D'$ is equal to the out-degree of $v$ in $D$. Moreover, the out-degree of each other vertex of $D'$ is the same as its out-degree in $D$. Hence, by the minimality of $D$, the digraph $D'$ contains two disjoint cycles, which yield two disjoint cycles in $D$, a contradiction.

Fix a vertex $v$ and let $x$ be an in-neighbour of $v$. Note that $d_D^-(v) \geq 1$ by the minimality of $D$. As the arc $x \to v$ is dominated, there exists a vertex $y \in V$ with $\{x, v\} \subseteq N^+(y)$. Consequently the digraph $D[N^-(v)]$ has in-degree at least one and thus contains a cycle. In particular the size of the in-neighbourhood of each vertex is at least two. Observe now that if $d_D(v) \geq 3$ for every $v \in V$, then $D$ indeed contains two disjoint cycles: just apply Theorem 1 to $D^*$.

Therefore, there exists a vertex of in-degree two in $D$, and hence a 2-cycle $C_1 := uv$. By Property (A), let $z$ be a vertex of out-degree two dominating $u$ and $v$. The sub-digraph $D[N^-(z)]$ contains a cycle, which must intersect $C_1$. So we can assume that $u \to z \in A$, and we let $C_2$ be the cycle $zu$. Again by Property (A), there exists a vertex $z'$ of out-degree two that dominates $C_2$. Note that $z' \neq v$, otherwise $D$ would contain three 2-cycles, thereby contradicting Property (A). Observe also that neither $z$ nor $u$ can dominate $z'$, otherwise $D$ would contain three 2-cycles. Therefore the cycle contained in $D[N^-(z')]$ is disjoint from the 2-cycle $uz$, a contradiction. This contradiction concludes the proof.

We note that this result is optimal, since a symmetrically oriented triangle—i.e. three vertices $x_1, x_2, x_3$ with an arc from $x_i$ to $x_j$ whenever $i \neq j$—does not have two disjoint cycles. It is also optimal if we restrict ourselves to oriented graphs, since there exist oriented graphs on seven vertices with three vertices of out-degree two, four vertices of out-degree three and no two disjoint cycles. See Figure 1(a) for an example. Moreover, the oriented graph of Figure 1(b) has no two disjoint cycles, yet every vertex has out-degree three except one which has out-degree one.
3 Arc-dominated oriented graphs

We say that a digraph $D = (V, A)$ is arc-dominated if every arc of $A$ is dominated. As we will see, a minimum counter-example to Theorem 2—and more generally, to Conjecture 1—must be arc-dominated, and it must be an oriented graph—i.e. it contains neither a loop nor a 2-cycle. We put the following proposition in a dedicated section because we believe that it might be of independent interest.

**Proposition 4.** Let $D = (V, A)$ be an arc-dominated oriented graph, and let $X \subset V$ such that $D[X]$ is either acyclic or an induced cycle of $D$. There exists a cycle $C$ disjoint from $D[X]$ such that every vertex of $C$ has at least one out-neighbour in $X$.

**Proof.** We set $X' := V \setminus X$. Let $S$ be the set of vertices of $X'$ having at least one out-neighbour in $X$. Observe that it is enough to prove that $D[S]$ contains a cycle. To this end, it suffices to establish that every vertex of $S$ has at least one in-neighbour in $S$. Suppose on the contrary that there exists a vertex $v \in S$ with no in-neighbour in $S$. We set $Y := (N^-_D(v) \cup N^+_D(v)) \cap X$. By the definition of $S$, the vertex $v$ has an out-neighbour $x$ in $X$, so in particular $Y \neq \emptyset$. Since for every $y \in Y$ there is an arc between $v$ and $y$, and since $D$ is arc-dominated, there exists a vertex $z$ which dominates $\{v, y\}$. It follows that $z \in X \cap N^-_D(v) \subset Y$. In particular this proves that $D[Y]$ and hence $D[X]$ contains a cycle. This is not possible if $D[X]$ is acyclic and concludes the proof in this case. If $D[X]$ is an induced cycle $C'$ of $D$, then $D[X] = D[Y] = C'$. Consider the out-neighbour $y$ of $x$ in $C'$. By what

Figure 1: (a) An oriented graph with three vertices of out-degree two, four vertices of out-degree three and no two disjoint cycles, and (b) an oriented graph whose vertices all have out-degree three, except one which has out-degree one, and yet without two disjoint cycles. An arc from/to a box goes from/to every vertex of the box.
precedes, it is dominated by a vertex of $N_D^{-}(v) \cap X$, which must be $x$ since $C'$ is induced. This is a contradiction since $\{v, x\}$ would induce a 2-cycle in $D$.

**Corollary 5.** Let $D = (V, A)$ be an arc-dominated oriented graph. Suppose that $C$ is a cycle of $D$, and $C'$ an induced cycle disjoint from $C$. If there is no arc from a vertex of $C$ to a vertex of $C'$ then $D$ contains three disjoint cycles.

**Proof.** We apply Proposition 4 with $X$ being $V(C')$. We deduce that there exists a cycle $C_1$ disjoint from $C'$ such that every vertex of $C_1$ has an out-neighbour in $C'$. As there is no arc from $C$ to $C'$, the cycle $C_1$ is certainly disjoint from $C$. Thus, $C, C'$ and $C_1$ are three disjoint cycles of $D$. \hfill \square

## 4  Proof of Theorem 2

Our goal in this section is to establish Theorem 2. We proceed by contradiction: we suppose that the statement of the theorem is false, and consider a counter-example with the minimum number of vertices. We first establish some fundamental properties of such a digraph, which will be extensively used in the sequel. Until the end, we let $D = (V, A)$ be a counter-example to the statement of Theorem 2 with the smallest number of vertices, and subject to this with the smallest number of arcs. In particular, every vertex has out-degree exactly five. We let $n$ be the order of $D$. Note that $n \geq 5$.

**Lemma 6.** The following hold.

(i) The digraph $D$ is an oriented graph, i.e. it has no loop and no 2-cycle.

(ii) Every arc of $D$ is dominated. In particular, the in-neighbourhood of every vertex contains a cycle.

(iii) Every triangle of $D$ is dominated by three different vertices.

(iv) If a vertex $v$ dominates a cycle $C$, there exists a triangle $vuw$ with $u \in V(C)$ and $w \notin V(C)$.

**Proof.** (i) Suppose that $C$ is a cycle of $D$ of length at most two. Note that the induced sub-digraph $D'$ of $D$ obtained by removing the vertices of $C$ has minimum degree at least three. Thus, by Theorem 1, $D'$ contains two disjoint cycles, which are certainly disjoint from the cycle $C$. Hence, $D$ contains three disjoint cycles, a contradiction.
(ii) It is proved exactly as Property (B) in the proof of Theorem 3, so we do not repeat it here.

(iii) Let $C$ be a triangle of $D$, and consider the digraph $D'$ obtained from $D$ by removing the vertices of $C$. The digraph $D'$ has minimum out-degree at least two. Moreover every vertex of $D'$ that does not dominate $C$ in $D$ has out-degree at least three in $D'$. As $D'$ cannot contain two disjoint cycles—otherwise $D$ would contain three disjoint cycles—the contrapositive of Theorem 3 implies that at least three vertices of $D'$ have out-degree two, and hence these vertices dominate $C$ in $D$.

(iv) Let $C'$ be an induced cycle contained in $N^-(v)$. As $v$ dominates a cycle $C$, by (i) the cycles $C$ and $C'$ are disjoint. According to Corollary 5, there exists an arc from $C$ to $C'$, which yields the sought triangle. 

According to Item (i) of the preceding lemma, $D$ is actually an oriented graph. So, as every vertex has out-degree five, we deduce that the order $n$ of $D$ is at least 11. The proof is now split into two parts, regarding whether $D$ contains a triangle.

### 4.1 The digraph $D$ does not contain a triangle

In this sub-section, we assume that $D$ does not contain a triangle. In particular, every 4-cycle of $D$ is induced. We first establish some useful properties of $D$.

**Lemma 7.** For every vertex $v$ of $D$ the sub-digraph induced by the out-neighbours of $v$ is acyclic.

**Proof.** Since $D$ has no triangle this follows directly by Lemma 6(iv).

We define a spanning sub-digraph $D'$ of $D$ as follows. Recall that, by Lemma 6(ii), the in-neighbourhood of every vertex $u$ of $D$ contains an induced cycle $C_u$. We let $D' = (V, A')$ be the spanning sub-digraph of $D$ where $A'$ is comprised of all arcs $v \rightarrow u$ of $D$ with $v \in V(C_u)$. The obtained digraph $D'$ has some useful properties, stated in the next lemma.

**Lemma 8.** The following hold.

(i) If $v \rightarrow u$ belongs to $A'$ then $N_D^+(v) \cap N_D^-(u) \neq \emptyset$.

(ii) The digraph $D'$ is 4-regular, i.e. $d_D^+(v) = 4 = d_D^-(v)$ for every vertex $v$. In particular, $D$ contains a 4-cycle.
(iii) If the arc \( v \rightarrow u \) belongs to \( A \setminus A' \) then \( N_+^D(v) \cap N_-^D(u) = \emptyset \).

Proof. (i) Let \( v \in V(C_u) \). By the definition of \( C_u \), the out-neighbour of \( v \) in \( C_u \) dominates \( u \) in \( D' \) and belongs to \( N_+^D(v) \).

(ii) By Lemma 6(ii), for every vertex \( v \) we have \( d_{D'}^-(v) \geq 4 \) since \( D \) contains no triangle. Therefore, to prove the statement we only need to show that \( d_{D'}^+(v) \leq 4 \) for every vertex \( v \). Suppose on the contrary that \( v \) is a vertex of \( D \) with out-degree five in \( D' \). Hence, \( N_+^D(v) = N_+^{D'}(v) \). Let \( u \in N_+^D(v) \). By (i), \( N_+^D(v) \cap N_+^{D'}(u) \neq \emptyset \). So the sub-digraph of \( D' \) induced by the out-neighbours of \( v \) has minimum in-degree at least one, and hence it contains a cycle. This contradicts Lemma 7.

(iii) Suppose that \( v \rightarrow u \) is an arc of \( D \) contradicting the statement. Again, we shall prove that the out-neighbourhood of \( v \) in \( D \) contains a cycle, thereby contradicting Lemma 7. Let \( z \in N_+^D(v) \), it suffices to prove that \( z \) is dominated by a vertex of \( N_+^D(v) \). If \( z = u \) this is clear by the definition of \( v \) and \( u \), so suppose that \( z \neq u \). By (ii), the vertex \( v \) has out-degree four in \( D' \), thus \( v \rightarrow z \in A' \) and hence (i) yields the conclusion.

We prove a last preliminary lemma before turning to the proof of Theorem 2.

Lemma 9. Let \( C \) be a 4-cycle of \( D \). The following hold.

(i) There exist at least three vertices with each exactly three out-neighbours in \( C \);

(ii) at least one of the arcs of \( C \) is not in \( D' \).

Proof. (i) By Lemma 7 every vertex of \( D \) has at most three out-neighbours in \( C \). Suppose that at most two vertices of \( D \) have three out-neighbours in \( C \). Then, every vertex of the sub-digraph of \( D \) obtained by removing \( C \) has out-degree at least three, except at most two vertices that have out-degree two. By Theorem 3, it contains two disjoint cycles. These two cycles together with \( C \) yield three disjoint cycles in \( D \), a contradiction.

(ii) Suppose on the contrary that \( C := xyzt \) is a 4-cycle of \( D' \). By the preceding item, there exist three vertices \( a, b \) and \( c \) with each three out-neighbours in \( C \). Note that no vertex of \( C \) can dominate a vertex of \( \{a, b, c\} \), otherwise \( D \) would contain a triangle or a 2-cycle. As there are
We are now in position to conclude the proof, by showing that there exists a 4-cycle of $D$ included in $D'$ and thereby contradicting Lemma 8(ii). By Property (C) let $C := xyzt$ be a 4-cycle of $D$ with two consecutive arcs in $D'$. By Lemma 9(ii) at least one of the arcs of $C$ is not in $D'$. Therefore, up to renaming the vertices, we can assume that $t \rightarrow x \in A, x \rightarrow y \in A$ and $z \rightarrow t \notin A'$. Thus $C$ and $C_t := abcd$ are disjoint. By Lemma 9(ii), assume
that $a \notin C_b$. The cycles $C_t$ and $C_b$ being disjoint, $C_b$ must intersect the cycle $C$. As none of $x$, $z$ and $t$ has an out-neighbour in $C_b$, we infer that $y \in V(C_b)$. Therefore $txyb$ is a 4-cycle of $D$ which is included in $D'$, see Figure 2(b). This contradiction concludes the proof when $D$ does not contain a triangle.

4.2 The digraph $D$ contains a triangle

For every vertex $u \in V$, we let $\varphi(u)$ be the greatest integer $r$ for which there exist triangles $T_1, T_2, \ldots, T_r$ such that

- the intersection of every two triangles is the vertex $u$; and
- the in-neighbour of $u$ in $T_i$ dominates $T_{i-1}$ for every $i \in \{2, 3, \ldots, r\}$.

Thus, $\varphi(u) = 0$ if and only if $u$ is not contained in a triangle, and $1 \leq \varphi(u) \leq 5$ otherwise.

Lemma 10. Either $D$ contains two disjoint triangles, or all the triangles of $D$ share a common vertex $x$. In the latter case $\varphi(x) \geq 3$.

Proof. Let $\Phi := \max_{u \in V} \varphi(u)$. As $D$ contains a triangle, we deduce from Lemma 6(iii) and (iv) that $\Phi \geq 2$.

We suppose first that $\Phi = 2$. We shall establish that $D$ contains two disjoint triangles. Suppose on the contrary that it is not the case. Then, the following holds.

(D) Every vertex $x \in V$ such that $\varphi(x) = 2$ is dominated by a triangle.

By the definition of $\varphi$, there exist four vertices $y_1, y_2, z_1, z_2$ such that $T_1 := xy_1z_1$ and $T_2 := xy_2z_2$ are two triangles and $z_2$ dominates $T_1$. According to Lemma 6(iii), there exists a vertex $z_3 \notin \{y_1, z_1\}$ dominating $T_2$. Thus, Lemma 6(iv) implies that there exists a triangle $T_3 := z_3a_1b_1$, with $a_1 \in$
Figure 3: Configuration obtained when $\varphi(x) = 2$.

$V(T_2)$ and $b_1 \notin V(T_2)$. There are three distinct vertices that dominate $T_3$. Among the vertices so far defined, only $y_1$ and $z_1$ may dominate $T_3$. Thus, there exists $z_4 \notin \{y_1, z_1\}$ that dominates $T_3$. Moreover, there exists a triangle $T_4 := z_4a_2b_2$ with $a_2 \in V(T_3)$ and $b_2 \notin V(T_3)$. The situation is depicted in Figure 3. We set $X := \{x, y_1, z_1, y_2, z_2, z_3, z_4\}$.

If $z_1 \rightarrow z_3 \in A$, then $z_3z_2z_1$ is a triangle which dominates $x$, which would establish Property $(D)$. We thus assume in the remaining that $z_1$ does not dominate $z_3$. The vertex $b_1$ dominates $z_3$, thus either $b_1 = y_1$ or $b_1 \notin X$. We consider these two cases separately.

$b_1 \notin X$. Then $a_1$ must be $x$, otherwise $z_3a_1b_1$ and one of $T_1, T_2$ are disjoint.

Now, $T_1, T_2$ and $z_3xb_1$ show that $\varphi(x) \geq 3$, a contradiction.

$b_1 = y_1$. Consider $T_1 = z_4a_2b_2$. Note that $z_4$ dominates $b_1 = y_1$. Notice also that the vertex $b_2$ does not lie in $\{y_2, z_2\}$, otherwise $z_4z_2b_2$ and $T_1$ would be two disjoint triangles. If $b_2 = x$ then $T_1, T_2$ and $z_4z_2x$ show that $\varphi(x) \geq 3$, a contradiction. If $b_2 = z_1$, then $z_4y_1z_1$ and $T_2$ are two disjoint triangles. Thus, as $b_2 \neq b_1 = y_1$ (since $b_2 \notin V(T_3)$), we deduce that $b_2 \notin X$. As $T_4$ must intersect $T_1, T_2$ and $T_3$, we infer that $a_2 = x$.

Consequently, $z_3z_2y_1$ and $T_4$ are two disjoint triangles, a contradiction.

This establishes Property $(D)$. Note that we also have showed that $z_1$ must indeed dominate $z_3$. Hence, $\varphi(z_2) \geq 2$, by considering the triangles $T_2$ and $z_3z_2z_1$.

Now consider a vertex $x$ such that $\varphi(x) = 2$, and let $T_1$ and $T_2$ be two triangles as before. In particular, we can assume that the vertex $z_2$ satisfies $\varphi(z_2) = 2$, thus is dominated by a triangle $T$. Observe that $T_1$ and $T_2$ are two disjoint triangles, a contradiction.

In conclusion, we have proved that $D$ contains two disjoint triangles if $\Phi$ is two.

We assume now that $\Phi \geq 3$, and we let $x$ be a vertex such that $\varphi(x) = \Phi$. By contradiction, suppose that $D$ does not contain two disjoint triangles,
and yet contains a triangle $T$ not containing $x$. There exist three triangles $T_i := xy_i z_i$, $i \in \{1, 2, 3\}$, such that $V(T_i) \cap V(T_j) = \{x\}$ if $i \neq j$, and $z_i$ dominates $T_{i-1}$ if $i > 1$. As $D$ does not contain two disjoint triangles, we deduce that $T$ contains a vertex from each set $\{y_i, z_i\}$, for $i \in \{1, 2, 3\}$.

According to Lemma 6(iii), there exists a vertex $z_4$, distinct from all the vertices defined so far, that dominates the triangle $T_3$. Thus, there exists a triangle $T_4 := y_4 z_4 b$, with $a \in V(T_3)$ and $b \notin V(T_3)$. Notice that $b \neq x$. Hence, if $a \neq x$, we obtain two disjoint triangles; indeed, the triangle $T_4$ intersects at most two triangles among $T_1, T_2$ and $T_3$, because $x \notin V(T_4)$ and $z_4 \notin V(T_1) \cup V(T_2) \cup V(T_3)$. Thus, among the triangles $T_i$, $i \in \{1, 2, 3, 4\}$, at least two are disjoint, a contradiction.

Therefore, $a = x$. Let $X := \{x, y_1, z_1, y_2, z_2, y_3, z_3, z_4\}$. Note that $b$ either belongs to $\{y_2, y_1\}$ or does not belong to $X$. The latter case is not possible, since $T_4$ and $T$ would then be two disjoint triangles—because, as noted earlier, $V(T) \subset \{y_1, z_1, y_2, z_2, y_3, z_3\}$. If $b = y_2$, then $T_1$ and $z_4 z_3 y_2$ are two disjoint triangles. Therefore, we infer that $b = y_1$, so $T_4 = z_4 x y_1$. The situation is depicted in Figure 4.

As $D$ does not contain two disjoint triangles, $V(T)$ must intersect the set $\{y_1, z_4\}$. So, $y_1$ is a vertex of $T$. Now, observe that the triangles $T_2, T_3$ and $T_4$ fulfill the same conditions as do $T_1, T_2$ and $T_3$. Consequently, we deduce as previously that $y_2 \in V(T)$. So, the triangle $T$ either is $z_3 y_2 y_1$ or is comprised of the vertices $y_1, y_2$ and $y_3$. If the former case, let $u \notin X$ be a vertex dominating $T_4$. This is possible since at least three vertices dominate $T_4$. There exists a triangle $T_5$ comprised of $u$, a vertex $u_1 \in V(T_4)$ and a vertex $u_2 \notin V(T_4)$. If $u_1 \in \{y_1, z_4\}$, then $T_5$ and either $T_2$ or $T_3$ are two disjoint triangles, since $x \notin V(T_5)$. So, $u_1 = x$ and $u_2$ is either $y_2, y_3$ or a new vertex. In all cases, $T_5$ and $z_3 z_2 y_1$ are two disjoint triangles, a contradiction. Consequently, $V(T) = \{y_1, y_2, y_3\}$. Thus, none of the vertices $z_i, i \in \{1, 2, 3, 4\}$, dominates $T$. As $T$ is dominated by at least three vertices, we can choose a vertex $u$ that dominates $T$ and is different from $x$. Now,
there exists a triangle $T' := uu_1u_2$ with $u_1 \in \{y_1, y_2, y_3\}$, and $u_2 \notin V(T)$. Note that $u_2 \neq x$. Consequently, $T'$ and one triangle among $T_1, T_2$ and $T_3$ are disjoint, a contradiction. This concludes the proof.

We define now two subsets of $V$. Let $Y$ be the set of vertices contained in a triangle, and $Z$ the set of vertices dominating a triangle. We set $D_Y := D[Y]$, and $D_Z := D[Z]$. From Lemma 6(iv) we deduce that $D_Z$ is an induced subdigraph of $D_Y$. The following lemma will prove to be useful.

**Lemma 11.** The following hold.

(i) Every vertex of $Y$ has at least five in-neighbours in $D$, with at least four lying in $D_Y$;

(ii) the minimum in-degree of the digraph $D_Z$ is at least three.

**Proof.** (i) Let $T := xyz$ be a triangle containing $x$. By Lemma 6(iii), there exist three vertices $u, v$ and $w$ that dominate $T$. By the definition of $Y$, the vertices $u, v, w$ and $z$, which are all in-neighbours of $x$, belong to $Y$. Thus, it only remains to show that there exists a fifth in-neighbour of $x$ in $D$. To this end, suppose on the contrary that $d_D(x) = 4$. Consider the cycle $C_x$. Since $z$ is dominated by $\{u, v, w\}$, it cannot belong to $C_x$. Thus, $C_x$ is a triangle whose vertices are $u, v$ and $w$. In particular $T$ and $C_x$ are two disjoint cycles, and there is no arc from the triangle $T$ to the cycle $C_x$, which contradicts Corollary 5.

(ii) Let $x$ be a vertex of $D_Z$. By Lemma 6(iv) there exists a triangle $T := xyz$, along with three vertices $u, v, w$ dominating $T$. Thus, $\{u, v, w\} \subseteq N_{D_Z}(x)$, which proves the desired statement.

We finish the proof of Theorem 2 right after having established the following bound.

**Lemma 12.** Suppose that $T$ and $T'$ are two disjoint triangles of $D$. If $\ell$ is the number of arcs between $T$ and $T'$ then $n \leq 22 - \ell$.

**Proof.** Let $X := V(T) \cup V(T')$ and $X' := V \setminus X$. We shall obtain the desired inequality by counting the number $L$ of arcs from a vertex of $X$ to a vertex of $X'$. Since every vertex has out-degree five, $L$ is $4 \times 6 - \ell = 24 - \ell$. We now prove that $L \geq n + 2$, which will imply that $n + 2 \leq 24 - \ell$, and hence $n \leq 22 - \ell$. Note that every vertex of $X'$ has an in-neighbour in $X$, otherwise $D$ would contain three disjoint cycles by Lemma 6(ii). As the digraph $D[X']$ is acyclic (and of order at least $n - 6 \geq 5$), there exists a
vertex \( v \in X' \) having no in-neighbour in \( X' \), and another vertex \( w \) with at most one in-neighbour in \( X' \). All together, these two vertices have at least \( 3 + 2 = 5 \) in-neighbours in \( X \). Now, note that \( T \) and \( T' \) are two disjoint triangles in \( D_Y \). By Lemma 11(i), \( D_Y \) has minimum in-degree at least four—and so its order is at least nine. Consequently, there exists three vertices \( a, b \) and \( c \) of \( Y \setminus X \) having at least four, three and two in-neighbours in \( X \), respectively—otherwise \( D_Y \), and hence \( D \), would contain three disjoint cycles, a contradiction. According to Lemma 11(i), every vertex of \( Y \) has in-degree at least five in \( D \). If \( \{v, w\} \subset Y \), we infer from what precedes that \( L \geq 5 + 4 + 2 + n - 6 - 3 = n + 2 \). If only one of \( v, w \) lies in \( Y \), we deduce that \( L \geq 5 + 3 + 2 + 2 + n - 6 - 4 = n + 2 \), while if none of them is in \( Y \), we have \( L \geq 3 + 2 + 4 + 3 + 2 + n - 6 - 5 = n + 3 \).

We now obtain a contradiction by proving that \( D \) indeed contains three disjoint cycles. Recall that the order of \( D \) is at least 11. According to Lemma 10, either all the triangles of \( D \) share a common vertex, or \( D \) contains two disjoint triangles. We consider the two cases separately.

**Case 1:** \( D \) does not contain two disjoint triangles. In this case, all triangles of \( D \) share a common vertex, say \( x \), and we have \( \varphi(x) \geq 3 \). All the vertices of \( D_Z \) are in-neighbours of \( x \), since \( x \) is contained in every triangle. By Lemma 11(ii), the digraph \( D_Z \) has minimum in-degree at least three. We assert that \( D_Z \) has also minimum out-degree at least three. To see this, suppose the contrary, and let \( z \) be a vertex with out-degree at most two in \( D_Z \). Note that \( x \notin Z \), so \( z \neq x \). We set \( D_1 := D_Z - z \). Observe that the digraph \( D_1^* \) fulfils the hypothesis of Theorem 3, since all its vertices have out-degree at least three (by Lemma 11(ii)) except at most two vertices which have out-degree two. Thus, the digraph \( D_1^* \) contains two disjoint cycles. They yield two disjoint cycles of \( D_1 \), say \( C_1 \) and \( C_2 \). As \( z \in D_Z \), there exists a triangle \( T := zuv \) in \( D \). By the definition of \( x \), we have \( u = x \). As noticed earlier, \( Z \subseteq N_D(x) \), hence the triangle \( T \) is disjoint from both \( C_1 \) and \( C_2 \), a contradiction. Therefore \( D_Z \) has minimum out-degree at least three. Let us set \( m := |Z| \). We shall lower bound \( m \) as a function of \( n \). As \( D_Z \) has minimum out-degree three, every vertex of \( Z \) has at least four out-neighbours in \( Z \cup \{x\} \), and thus at most one in \( Z' := V \setminus (Z \cup \{x\}) \). So the following holds.

\((E)\) The number of arcs from a vertex of \( Z \) to a vertex of \( Z' \) is at most \( m \).

Furthermore, by Theorem 1, \( D \) contains two disjoint cycles \( C_1 \) and \( C_2 \) comprised of vertices of \( Z \). Observe that every vertex of \( Z' \) has at least one
in-neighbour in $Z$; otherwise, by Lemma 6(ii), $D$ would contain a cycle comprised of vertices of $Z' \cup \{x\}$, which together with $C_1$ and $C_2$ would yield three disjoint cycles, a contradiction. As $\varphi(x) \geq 3$, there exist three out-neighbours $y_1, y_2$ and $y_3$ of $x$ in $Z'$, each having at least three in-neighbours in $Z$, by Lemma 6(iii). Consequently, the following is true.

\begin{itemize}
\item[(F)] The number of arcs from a vertex of $Z$ to a vertex of $Z'$ is at least $9 + (n - 1 - m - 3) = n - m + 5$.
\end{itemize}

It follows from Properties (E) and (F) that

\begin{equation}
2m \geq n + 5. \tag{1}
\end{equation}

We now aim at bounding $|A|$, the number of arcs of $D$, in terms of $m$. Recall that $|A| = 5n$, since every vertex of $D$ has out-degree five. We partition $V$ into the sets $Z, Z'$ and $\{x\}$. Recall that every vertex has in-degree at least three, by Lemma 6(ii). As $Z \subseteq Y$, each vertex of $Z$ has at least five in-neighbours in $D$ by Lemma 11(ii). So

\begin{equation}
\sum_{v \in Z} d_D^-(v) \geq 5m. \tag{2}
\end{equation}

Recall also that $Z \subseteq N_D^-(x)$, thus

\begin{equation}
|N_D^-(x)| \geq m. \tag{3}
\end{equation}

Moreover, according to Lemma 11(i), every vertex of $Y$ has in-degree at least five in $D$, and $|Y \cap Z'| \geq 3$ since $\varphi(x) \geq 3$. In particular, $x$ has at most two out-neighbours not in $Z$. As $x$ belongs to every triangle of $D$, every vertex not in $N_D^+(x)$ has in-degree at least four in $D$, by Lemma 6(ii). Therefore we obtain

\begin{equation}
\sum_{v \in Z'} d_D^+(v) \geq 3 \times 5 + 2 \times 3 + (n - 1 - m - 5) \times 4 = 4n - 4m - 3. \tag{4}
\end{equation}

By Equations (2), (3) and (4), we infer that the number of arcs of $D$ is at least $5m + m + 4n - 4m - 3 = 4n + 2m - 3$. As $|A| = 5n$, we obtain

\begin{equation}
2m \leq n + 3. \tag{5}
\end{equation}

Equations (1) and (5) are contradictory, which concludes the first case of our proof.

**Case 2:** $D$ has two disjoint triangles. Let $T := xyz$ and $T' := x'y'z'$ be two disjoint triangles. Consider the sub-digraph $D_1$ of $D$ obtained by
removing \( T \) and \( T' \). As \( D \) does not contain three disjoint cycles, \( D_1 \) is acyclic, thus has a vertex \( u \) of out-degree zero. Hence, the vertex \( u \) has five out-neighbours among \( x, y, z, x', y' \) and \( z' \). Without loss of generality, let \( N_D^+(u) = V(T) \cup \{x', z'\} \). Necessarily, \( y' \in N_D^+(u) \), otherwise \( T, T' \) and \( C_u \) would be three disjoint cycles of \( D \), a contradiction. So \( T \) and \( T_1 := ux'y' \) are two disjoint triangles of \( D \). By Lemma 6(iii), there exists an arc from a vertex of \( T \) to a vertex of \( T_1 \). Moreover, there are at least three arcs from a vertex of \( T_1 \) to a vertex of \( T \), since the vertex \( u \) dominates \( T \). So Lemma 12 implies that \( n \leq 22 - 4 = 18 \). By Lemma 6(iii), there exist three vertices \( a_1, a_2 \) and \( a_3 \) that dominate \( T_1 \). Clearly, none of these vertices belongs to \( V(T) \cup V(T_1) \). Moreover at least one of them, say \( a_1 \), has no in-neighbour in \( \{a_1, a_2, a_3\} \), since otherwise \( T, T_1 \) and \( D[\{a_1, a_2, a_3\}] \) would be three disjoint cycles of \( D \). By Lemma 6(ii), the vertex \( a_1 \) must have an in-neighbour in \( T \), otherwise \( T, T_1 \) and \( C_{a_1} \) would be disjoint, a contradiction. Without loss of generality, we assume that \( z \in N_D^-(a_1) \). The triangle \( T_2 := uza_1 \) is dominated by three vertices \( a_4, a_5 \) and \( a_6 \). Clearly, none of these vertices belongs to \( V(T) \cup V(T_1) \cup \{a_1, a_2, a_3\} \). More precisely, among the vertices not in \( T_2 \), only \( y', a_2 \) and \( a_3 \) dominate \( u \), and none of them dominates \( a_1 \). Thus, we obtain the sub-digraph \( F \) of \( D \), depicted in Figure 5. For convenience, every vertex of \( D \) not in \( F \) is called extern.

Note that all the vertices of \( F \) belong to \( Y \), and hence have in-degree at least five in \( D \) by Lemma 11(i). As \( D \) does not contain three disjoint cycles, there exists \( i \in \{2, 3, \ldots, 6\} \) such that the vertex \( a_i \) does not have an in-neighbour in \( \{a_1, a_2, \ldots, a_6\} \). Observe that a vertex dominating the arc \( a_i \rightarrow u \) is either \( y' \) or extern, the former begin possible only if \( i \geq 4 \). We now consider two cases, regarding the value of \( i \).
\( i \in \{2, 3\} \). Without loss of generality, let \( i = 2 \). The vertex \( a_2 \) has at least one in-neighbour in \( T \), otherwise \( T, T_1 \) and \( C_{a_2} \) would be three disjoint cycles of \( D \). We consider two cases regarding whether \( z \) dominates \( a_2 \).

**z dominates \( a_2 \).** In this case, the triangle \( za_2u \) is dominated by three vertices, which must be extern. These three vertices belong to \( Y \), as do the vertices of \( F \). Thus, by Lemma 11(i), all have in-degree at least five in \( D \). Furthermore, among them the vertex \( u \) has in-degree at least \( 10 \), and \( z \) at least \( 8 \). We deduce that

\[
|A| = 5n = \sum_{v \in V} d_D(v) \geq 10 + 8 + 13 \times 5 + (n - 15) \times 3 ,
\]

which implies that \( n \) is at least \( 19 \), a contradiction.

**z does not dominate \( a_2 \).** So at least one vertex among \( x, y \) dominates \( a_2 \). By symmetry of the roles played by \( x \) and \( y \) in what follows, we assume that \( x \) dominates \( a_2 \). The triangle \( T_3 := xa_2u \) is dominated by three vertices, which must be extern. These three vertices belong to \( Z \), and hence to \( Y \). The vertices of \( F \) also belong to \( Y \), and every vertex of \( Y \) has in-degree at least five in \( D \) by Lemma 11(i). Furthermore the in-degree of \( u \) is at least \( 10 \). Thus we obtain

\[
|A| = 5n \geq 10 + 14 \times 5 + (n - 15) \times 3 ,
\]

which yields \( n \geq \frac{35}{2} \). As \( n \leq 18 \), we have \( n = 18 \). Notice that \( T \) is dominated by two vertices distinct from \( u \). So, we infer that \( d_D(x) + d_D(z) \geq 5 + 5 + 2 = 12 \). Hence, we obtain

\[
|A| = 5n \geq 10 + 12 + 12 \times 5 + (n - 15) \times 3 ,
\]

from which it follows that \( n \geq \frac{37}{2} \), a contradiction.

\( i \in \{4, 5, 6\} \). Without loss of generality, let \( i = 4 \). As \( D \) does not have three disjoint cycles, \( N_D^-(a_4) \cap \{x, y, x', y'\} \neq \emptyset \). We split this case according to the corresponding sub-cases.

**x dominates \( a_4 \).** We set \( T_3 := a_4ux \). Among the vertices of \( F \), only \( y' \) may dominate \( T_3 \). Supposing first that it is not the case, we obtain a contradiction by counting the number of arcs in \( D \). The triangle \( T_3 \) is dominated by three extern vertices. These vertices belong to \( Z \), and thus to \( Y \). Moreover, recall that all the vertices of \( F \) also belong to \( Y \), and that every vertex of \( Y \) has in-degree five, by
Lemma 11(i). Thus, there are at least 15 vertices of in-degree five and, among them, $u$ has in-degree at least 10. Also, the vertex $z$ has in-degree at least 8, because the triangle $xa_4z$ is dominated by three vertices, none of them lying in $\{y, u, a_4, a_5, a_6\}$. Therefore we obtain

$$|A| = 5n \geq 10 + 8 + 13 \times 5 + (n - 15) \times 3,$$

which yields $n \geq 19$, a contradiction.

Hence, the vertex $y'$ dominates the triangle $T_3$. We seek a contradiction by counting the number of arcs in $D$. Note that there are at least five arcs between $T$ and $T_1$, since $u$ dominates $T_1$, $y'$ dominates $x$ and there is at least one arc from $T$ to $T_1$ by Corollary 5. So, by Lemma 12, $n$ is at most 17.

We now bound the number of arcs in $D$. As $a_4$ has no in-neighbours among the other vertices $a_i$, there exist two extern vertices dominating the triangle $T_3$. Recalling that all the vertices of $F$ belong to $Y$, we obtain $|Y| \geq 14$. By Lemma 11(i), each of these vertices has in-degree at least five in $D$. Moreover, $u$ has in-degree at least 9, since it has already in-degree at least 7 in $F$. Also, the in-degree of $z$ is at least 8, because $z$ is dominated by $\{u, a_4, a_5, a_6, y\}$, and by the three vertices dominating the triangle $a_4zx$, which cannot be any of the preceding ones. Therefore we infer that

$$|A| = 5n \geq 9 + 8 + 12 \times 5 + (n - 14) \times 3,$$

and hence $n \geq \frac{35}{2}$, contradicting the conclusion of the preceding paragraph.

$y$ dominates $a_4$. Let $T_3 := a_4uy$. This triangle is dominated by three vertices. Among the vertices of $F$, only $y'$ may dominate it. Suppose first that it is not the case, i.e. $T_3$ is dominated by three extern vertices, which hence belong to $Y$. Furthermore, the triangle $T$ is dominated by two vertices different from $u$. Thus we deduce that $d_D(y) + d_D(z) \geq 5 + 5 + 2 = 12$. Note also that $u$ has in-degree at least 10. So, recalling that all the vertices of $F$ belong to $Y$, it follows that

$$|A| = 5n \geq 10 + 12 \times 5 + (n - 15) \times 3,$$

i.e. $n \geq \frac{37}{2}$, a contradiction. Consequently, we infer that $y'$ dominates $T_3$. As in the previous case, we note that there are at least five arcs between $T$ and $T_1$, and thus the Lemma 12 implies that
$n$ is at most 17. As $T_3$ is dominated by two extern vertices, notice that $u$ has in-degree at least 9 (since its in-degree in $F$ is at least 7). Moreover the triangle $a_4a_1y'$ is dominated by three vertices, and none of them belongs to \{$_{x', a_2, a_3, a_5, a_6, z}$\}. Hence, we deduce that both $a_1$ and $y'$ have in-degree at least 7 in $D$. Therefore, we obtain

$$|A| = 5n \geq 9 + 7 + 7 + 11 \times 5 + (n - 14) \times 3,$$

so $n \geq 18$, a contradiction.

$x'$ dominates $a_4$. Then the triangle $T_3 := a_4ux'$ is dominated by three extern vertices. So there are at least 15 vertices of in-degree at least five, and among them $u$ has in-degree at least 10 (since its in-degree in $F$ is at least 7), and $x'$ has in-degree at least 7 (since its in-degree in $F$ is at least 4). Therefore, we deduce that

$$|A| = 5n \geq 10 + 7 + 13 \times 5 + (n - 15) \times 3,$$

which yields $n \geq 37$, a contradiction.

None of $x, y$ and $x'$ dominates $a_4$ in $D$. In this case the vertex $y'$ must dominate $a_4$. We consider three vertices dominating the triangle $T_3 := a_4a_1y'$. Among the vertices of $F$, only $x$ and $y$ can dominate $T_3$, but none of them does since none of them is an in-neighbour of $a_4$. Thus, $T_3$ is dominated by three extern vertices. Consequently, $Y$ contains at least 15 vertices, and $u, a_1$ and $y'$ all have in-degree at least 7. It follows that

$$|A| = 5n \geq 3 \times 7 + 12 \times 5 + (n - 15) \times 3,$$

and hence, $n \geq 18$. As we know that $n \leq 18$, we have $n = 18$. In particular, there are exactly 6 extern vertices $r, s, t, r', s'$ and $t'$, with \{$_{r, s, t}$\} dominating the triangle $T_3$.

Now observe that, for every $i \in \{1, 2, 3, 4\}$, $a_i \notin N_D^+(x')$. Moreover, $V(T) \cap N_D^+(x') = \emptyset$ otherwise there would be at least five arcs between $T$ and $T'$, which would imply that $n \leq 17$ by Lemma 12, a contradiction. We assert that the in-degree of $x'$ in $D$ is at least 7. Recalling that $u, a_1$ and $y'$ also have in-degree at least 7, we would deduce that

$$|A| = 5n = 90 \geq 4 \times 7 + 11 \times 5 + 3 \times 3 = 92,$$

a contradiction. So it only remains to prove the assertion. If \{$_{a_5, a_6}$\} \cap $N_D^+(x') \neq \emptyset$, we assume without loss of generality that
$x'$ dominates $a_5$. Then, the triangle $a_5a_1x'$ is dominated by three vertices, which cannot be any of $u, a_1, a_2, a_3$. So $x'$ has at least 7 in-neighbours in $D$. If $\{a_5, a_6\} \cap N^+_D(x') = \emptyset$, the vertex $x'$ has at least four out-neighbours lying in $\{r, s, t, r', s', t'\}$. So it dominates at least one of $r, s$ and $t$, say $r$. The triangle $ra_1x'$ is dominated by three vertices, none of them lying in $\{u, a_1, a_2, a_3\}$. Thus, we again conclude that the vertex $x'$ has in-degree at least 7, which proves the assertion.

The proof of Theorem 2 is complete. \hfill \Box

References


