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► **To cite this version:**

Nicolas Auffray, Regis Bouchet, Y. Brechet. Strain gradient elastic homogenization of bidimensional cellular media. *International Journal of Solids and Structures*, 2010, 47 (13), pp.1698-1710. 10.1016/j.ijsolstr.2010.03.011 . hal-00486674

HAL Id: hal-00486674

<https://hal.science/hal-00486674>

Submitted on 26 May 2010

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Strain gradient elastic homogenization of bidimensional cellular media

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Abstract

The present paper aims at introducing an homogenization scheme for the determination of strain gradient elastic coefficients. This scheme is based on a quadratic extension of homogeneous boundary condition (HBC). It allows computing strain elastic effective tensors. This easy-to-handle computational procedure will then be used to construct overall behaviors and to verify some theoretical predictions on strain gradient elasticity.

Key words: Homogenization, Symmetry Classes, Strain Gradient Elasticity, Anisotropy

1. Introduction

Lightweight and innovative materials design is nowadays one of the most important challenge for material engineering, the goal is to reach high mechanical properties with low density materials. To achieve such contradictory objectives, scientific community focused on mesoscale structured cellular materials. Such material design requires to understand at the same time the relation between architecture and physical properties, and the explicit method to calculate those properties.

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According to a geometrical definition of a RVE (Representative Volume Elementary) a classical way to obtain the overall behavior of the cellular material is to use homogenization theory. It is well known that classical homogenization theory relies on a broad scale separation between geometric pattern and mechanical fields. If the scale separation is not broad enough, the classical theory fails to predict the overall behavior. As shown by Boutin (1996) and Forest (1998), keeping a continuum description requires to consider a generalized continuum to model the substitution material.

Especially, when designing millimetric microstructural materials to be implemented in centimetric structures (e.g. hollow spheres stacking for acoustical absorber (Gasser, 2003)) strong scale separation is not granted. Then, second order elastic effects have to be taken into account in the homogenization approach.

As the number of elastic constants in strain gradient theories dramatically increases with the order of tensors, a systematic way of identifying such coefficients is necessary. We proposed here such a method by combining recent results in extended homogenization methods and symmetry properties of higher order stiffness tensors:

- Quadratic homogenization scheme through the use of quadratic boundary conditions (Gologanu and al. (1997); Forest (1999)) ;
- Extended Voigt notations for different symmetry classes of second gradient elasticity (Auffray and al. (2009)).

Our main results are three folds:

1. First, it is shown that the circular cavity shape used in several higher order homogenization schemes (Gologanu and al. (1997); Zybell and al. (2008)) leads to a singular sixth order elasticity tensor;
2. Second, it is shown numerically that the application of the quadratic homogenization scheme provides isotropic second order effective properties for octagonal and pentagonal cell shapes, thus illustrating purely mathematical considerations of symmetry.
3. Finally, an example of chiral dependent behavior is given, illustrating an exotic property of strain gradient elasticity.

These results confirm that generalized homogenization schemes are powerful tools to estimate higher order elastic properties. Furthermore some useful operators to translate higher order moduli from the equivalent first strain gradient into the second gradient of displacement theories, according to Mindlin's formulation, are provided.

To reach those objectives, several facts on strain-gradient elasticity will be recalled in section 2. Some basic definitions and results about symmetry classes (Auffray and al. (2009)) will be summed up. In a second time, attention will be focused on the homogenization scheme. Extension of the effective modulus approach was firstly proposed by Gologanu and al. (1997) then used by Forest et al. (1998). This approach will be detailed and specified to our problem. All the needed operators will be defined. The last section will be devoted to numerical experiments on different geometrical patterns. It will be shown that isotropic homogenization proceed on a circular shape (as proposed in Gologanu and al. (1997) and Zybell and al. (2008)) leads to a degenerated isotropic tensor that should not be used in computational simulation. Another construction based on Hermann theorem consequences(Auffray (2008a)) will be proposed.

2. Mindlin's strain gradient elasticity

2.1. Constitutive law

In classical elasticity theory stress at a material point is linked to strain through the classical elasticity tensor. This relation, usually known as Hooke law, is written in tensorial fashion:

$$\sigma_{(ij)} = C_{\underline{(ij)} \underline{(lm)}} \varepsilon_{(lm)} \quad (1)$$

with $\sigma_{(ij)}$ the symmetrical-stress tensor, $\varepsilon_{(lm)}$ the strain tensor and $C_{\underline{(ij)} \underline{(lm)}}$ the tensor describing material elastic properties. The notation $()$ stands for the minor symmetries whereas $\underline{\underline{}}$ stands for the major ones.

Second grade elasticity is a kinematic enhancement of classical elasticity taking into account the second gradient of displacement in the mechanical formulation. Such a generalization could be constructed in, at least, three different, but equivalent, ways (Mindlin and Eshel, 1968). In this paper interest will be focused on the two first formulations.

In *type I* formulation, the freedom extra degrees will simply be defined as the second gradient of displacement:

$$\underline{\kappa}^I \underset{\simeq}{=} \underline{\mathbf{u}} \otimes \underline{\nabla} \otimes \underline{\nabla} \quad (2)$$

whereas in *type II* formulation the strain gradient will be considered:

$$\underline{\kappa}^{II} \underset{\simeq}{=} \underline{\varepsilon} \otimes \underline{\nabla} \quad (3)$$

These two definitions solely differ by the index symmetry of $\underline{\kappa}$. We have got $\kappa_{i(jk)}^I$ and $\kappa_{(ij)k}^{II}$ with the following relations between those two systems:

$$\kappa_{ijk}^{II} = \frac{1}{2} (\kappa_{ijk}^I + \kappa_{jik}^I) \quad (4)$$

$$\kappa_{ijk}^I = \kappa_{ijk}^{II} + \kappa_{kij}^{II} - \kappa_{jki}^{II} \quad (5)$$

As these two systems are defined up to a permutation, other properties will be introduced just for *type II* elasticity: strain gradient elasticity (SGE).

Taking into account strain-gradient effect in the mechanical formulation leads to define symmetrically the hyperstress tensor $\tau_{(ij)k}$. In each material point, the knowledge of the stress and the hyperstress tensors allows to compute the effective tensor $\eta_{(ij)}$. This tensor is defined as:

$$\eta_{(ij)} = \sigma_{(ij)} - \tau_{(ij)k,k} \quad (6)$$

It is the tensor to be considered to calculate the local equilibrium (Forest, 2004). Tensors $\sigma_{(ij)}$ and $\tau_{(ij)k}$ are related with $\varepsilon_{(lm)}$ and $\kappa_{(lm)n}$ through the following general constitutive law:

$$\sigma_{(ij)} = C_{(ij)(lm)} \varepsilon_{(lm)} + M_{(ij)(lm)n} \kappa_{(lm)n} \quad (7)$$

$$\tau_{(ij)k} = M_{(ij)k(lm)} \varepsilon_{(lm)} + A_{(ij)k(lm)n} \kappa_{(lm)n} \quad (8)$$

where the tensor $A_{(ij)k(lm)n}$ is the second order elasticity tensor and $M_{(ij)(lm)n}$ the coupling tensor between first and second order elasticity.

In a 3-D space this coupling tensor will vanish for a centro-symmetric media (Triantafyllidis and Bardenhagen, 1996). In 2-D space this tensor vanishes for any media that is even order rotational invariant (Auffray and

al., 2008b). For both cases the constitutive relation could be rewritten as follow:

$$\sigma_{(ij)} = C_{(ij) \underline{(lm)}} \varepsilon_{(lm)} \quad (9)$$

$$\tau_{(ij)k} = A_{(ij)k \underline{(lm)n}} \kappa_{(lm)n} \quad (10)$$

To switch constitutive law from one system to another, the following operators could easily be defined:

$$P_{ijklmn}^{I \rightarrow II} = \frac{1}{2} (\delta_{il} \delta_{jm} + \delta_{im} \delta_{jl}) \delta_{kn} \quad (11)$$

$$P_{ijklmn}^{II \rightarrow I} = \delta_{il} \delta_{jm} \delta_{kn} + \delta_{im} \delta_{jn} \delta_{kl} - \delta_{in} \delta_{jl} \delta_{km} \quad (12)$$

where $\underset{\approx}{P}^{I \rightarrow II}$ stands for the operator from *type I* to *type II*, and conversely.

The following relation holds true:

$$P_{ijkopq}^{I \rightarrow II} P_{opqlmn}^{II \rightarrow I} = 1_{ijklmn}^{II} \quad (13)$$

where $\underset{\approx}{1}^{II}$ should not be confused with the 6th-order identity tensor; a symmetrical relation could also be defined for $\underset{\approx}{1}^I$.

Switching between the two systems is related to the fact that most of our theoretical results are demonstrated in *type II* second-grade elasticity whereas the boundary conditions needed for the homogenization scheme are more natural in *type I* formulation. As transformations from one system to another are straightforward, it seems interesting to point out how to transfer results. Let's now introduce some results about SGE anisotropic tensors.

2.2. Anisotropic tensors

Most of the results presented here could be found and detailed in Auffray and al. (2009) and Auffray (2009). A true tensorial representation for SGE tensor will first be introduced. Then for each material symmetry group the corresponding physical group will be given including minimal number of coefficient of the associated tensor. For the hereafter studied anisotropic systems the corresponding Voigt representations will be given.

2.2.1. Voigt tensorial representation

In order to handle the second order elastic tensor, a mathematical transformation could be introduced to turn the 2-dimensional 6th-order tensor into a 6-dimensional 2nd-order tensor². This transformation allows rewriting the second order constitutive relation as³:

$$\hat{\tau}_\alpha = \hat{A}_{(\alpha\beta)} \hat{\kappa}_\beta \quad (14)$$

A rigorous way of representing the 6th-order tensor A as a 2nd-order one according to its symmetries is:

$$\begin{pmatrix} \tau_{111} \\ \tau_{221} \\ \sqrt{2}\tau_{122} \\ \tau_{222} \\ \tau_{112} \\ \sqrt{2}\tau_{121} \end{pmatrix} = \begin{pmatrix} A_{111111} & A_{111221} & \sqrt{2}A_{111122} & A_{111222} & A_{111112} & \sqrt{2}A_{111121} \\ A_{221111} & A_{221221} & \sqrt{2}A_{221122} & A_{221222} & A_{221112} & \sqrt{2}A_{221121} \\ \sqrt{2}A_{122111} & \sqrt{2}A_{122221} & 2A_{122122} & \sqrt{2}A_{122222} & \sqrt{2}A_{122112} & 2A_{122121} \\ A_{222111} & A_{222221} & \sqrt{2}A_{222122} & A_{222222} & A_{222112} & \sqrt{2}A_{222121} \\ A_{112111} & A_{112221} & \sqrt{2}A_{112122} & A_{112222} & A_{112112} & \sqrt{2}A_{112121} \\ \sqrt{2}A_{121111} & \sqrt{2}A_{121221} & 2A_{121122} & \sqrt{2}A_{121222} & \sqrt{2}A_{121112} & 2A_{121121} \end{pmatrix} \begin{pmatrix} \kappa_{111} \\ \kappa_{221} \\ \sqrt{2}\kappa_{122} \\ \kappa_{222} \\ \kappa_{112} \\ \sqrt{2}\kappa_{121} \end{pmatrix} \quad (15)$$

This is a true tensorial way of writing the constitutive relation $\tau_{(ij)k} = A_{(ij)k} \underline{(lm)n} \kappa_{(lm)n}$. Now let's consider the symmetry classes of such a linear operator.

2.2.2. Anisotropic system

In a two dimensional physical space, the group of symmetry of $A_{(ij)k} \underline{(lm)n}$ must be conjugated to an element of the following set:

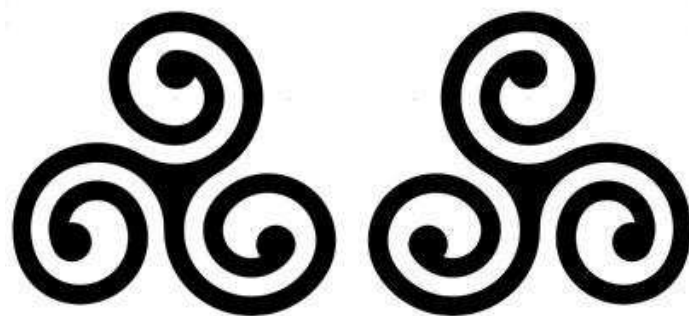
$$\Sigma_A : \{I, D_2, Z_4, D_4, Z_6, D_6, SO(2), O(2)\} \quad (16)$$

where I is the identity group. Z_n is the cyclic group of order n which is the rotation group of a chiral figure with an n -fold invariance (cf. fig.1 for an example of an Z_3 -invariant figure). Z_n is generated by an elementary rotation of $\frac{2\pi}{n}$. D_n is the dihedral group of order $2n$ which is the group of transformations that leave a regular n -gone invariant. D_n is generated by an elementary rotation of $\frac{2\pi}{n}$ and a mirror operation (cf. fig.2 for the example of an D_3 -invariant figure). $SO(2)$ is the continuous group of rotations and $O(2)$ is the 2-D orthogonal group.

Results are summed up in the following table:

²The permutation order-dimension is just a coincidence, in 3-D the same transformation would turn a 3-dimensional 6th-order tensor into a 18-dimensional 2nd-order tensor.

³The hat notation $\hat{}$ indicates a 2nd-order representation of a 6th-order tensor



(a) Levorotary triskelion (b) Dextrorotary triskelion

Figure 1: Z_3 -invariant figures

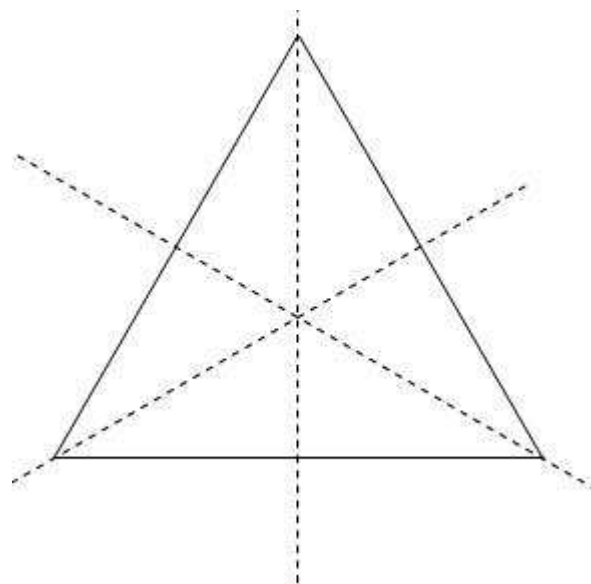


Figure 2: D_3 -invariant figure

$G_{\mathcal{M}}$	G_A	dim
I, Z_2	I	21
D_2	D_2	12
Z_4	Z_4	8
D_4	D_4	6
Z_3, Z_6	Z_6	6
D_3, D_6	D_6	5
$Z_5, Z_n, n \geq 7$	$SO(2)$	5
$D_5, D_n, n \geq 7$	$O(2)$	4

with $G_{\mathcal{M}}$ the material symmetry group; G_A the tensor symmetry group and dim the number of coefficients defining the tensor for the considered symmetry.

Voigt representation will be detailed for the following anisotropic classes: Isotropic ($O(2)$); Hexatropic (D_6); Orthotropic (D_4) and Chiral-Orthotropic (Z_4).

Isotropic.

$$\hat{\mathbb{A}}_{\sim O(2)} = \begin{pmatrix} a_1 & a_2 & \frac{a_1-a_3}{\sqrt{2}}-a_4 & 0 & 0 & 0 \\ & a_3 & a_4 & 0 & 0 & 0 \\ & & \frac{a_1+a_3}{2}-a_2 & 0 & 0 & 0 \\ & & & a_1 & a_2 & \frac{a_1-a_3}{\sqrt{2}}-a_4 \\ & & & & a_3 & a_4 \\ & & & & & \frac{a_1+a_3}{2}-a_2 \end{pmatrix} \quad (17)$$

which could be block summed up into:

$$\hat{\mathbb{A}}_{O(2)} = \begin{pmatrix} X_i & 0 \\ 0 & X_i \end{pmatrix} ; \quad X_i = \begin{pmatrix} a_1 & a_2 & \frac{a_1-a_3}{\sqrt{2}}-a_4 \\ & a_3 & a_4 \\ & & \frac{a_1+a_3}{2}-a_2 \end{pmatrix} \quad (18)$$

Hexatropic.

$$\hat{\mathbb{A}}_{\sim D_6} = \begin{pmatrix} a_1 & a_2 & \frac{a_1-a_3}{\sqrt{2}}-a_4 & 0 & 0 & 0 \\ & a_3 & a_4 & 0 & 0 & 0 \\ & & \frac{a_1+a_3}{2}-a_2 & 0 & 0 & 0 \\ & & & a_5 & a_1-a_5+a_2 & \frac{3a_1-a_3}{\sqrt{2}}-a_4-\sqrt{2}a_5 \\ & & & & a_3+a_5-a_1 & \sqrt{2}(a_5-a_1)+a_4 \\ & & & & & \frac{-3a_1+a_3}{2}-a_2+2a_5 \end{pmatrix} \quad (19)$$

So

$$\hat{\mathbb{A}}_{\sim D_6} = \hat{\mathbb{A}}_{\sim O(2)} + (a_5 - a_1)\hat{\mathbb{A}}_{\sim cor} \quad (20)$$

with $\hat{\mathbb{A}}_{\sim cor}$ an anisotropic correction which expression is:

$$\hat{\mathbb{A}}_{\sim cor} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & 1 & -1 & -\sqrt{2} \\ & & & & 1 & \sqrt{2} \\ & & & & & 2 \end{pmatrix} \quad (21)$$

Orthotropic and Chiral Orthotropic.

$$\hat{\mathbb{A}}_{\sim Z_4} = \begin{pmatrix} X_a & C_o \\ C_o^T & X_a \end{pmatrix}, \hat{\mathbb{A}}_{\sim D_4} = \begin{pmatrix} X_a & 0 \\ 0 & X_a \end{pmatrix} \quad (22)$$

$$C_a = X_a = \begin{pmatrix} a_1 & a_2 & a_5 \\ & a_3 & a_4 \\ & & a_6 \end{pmatrix}, C_o = \begin{pmatrix} 0 & a_7 & a_8 \\ -a_7 & 0 & a_8 \\ -a_8 & -a_8 & 0 \end{pmatrix} \quad (23)$$

Those are the anisotropic tensor expressions that will be checked in this paper last section through the proposed homogenization scheme.

3. Homogenization scheme

This section is organized from general comments to the specific way of computing the extra coefficients. First some remarks will be made on the FEM code used. Then the general concept of polynomial boundary condition (BC) will be discussed. It will be shown that the use of such conditions are linked with means of mechanical fields over a RVE (Representative Volume Element). The results obtained without considering any constitutive law will be detailed in the linear elastic case. This approach will lead to a concrete procedure for computing extra elastic coefficients.

3.1. Remarks on FEM-code modeling

The FEM code used for the computation is ZeBuLoN, a code designed by Onera, NorthWest Numerics and Mines ParisTech for thermomechanical computations. In the current version of this software quadratic Dirichlet

boundary conditions (QDBC) are readily available for 2-D computation. As it will be shown, QDBC are a straightforward way of computing 2nd-order coefficients in *type I* formulation. Up to the authors knowledge, such a condition extension to 2-D periodic have not been implemented in the software we use yet⁴. This is the reason why, in the following procedure and for our calculations, quadratic periodic boundary conditions are not used. This makes our numerical results stiffer than the real coefficients. But this does not change anything to the qualitative results obtained here.

3.2. Non homogeneous boundary conditions

Let's first consider the following fields: $\underline{\mathbf{u}}(x_1, x_2)$, the local displacement field defined inside the RVE, and $\underline{\mathbf{U}}$, the macroscopic field defined as $\underline{\mathbf{u}}$ mean field. Macroscopic field gradients could be defined as:

$$\underline{\mathbf{E}} = \underline{\mathbf{U}}\{\otimes\}\underline{\nabla} ; \underline{\mathbf{K}}^I = \underline{\mathbf{U}} \otimes \underline{\nabla} \otimes \underline{\nabla} \quad (24)$$

where $\{\otimes\}$ accounts for the gradient symmetrical part. As tensor $\underline{\mathbf{K}}^I$ is the displacement second gradient, we are in *type I* formulation.

To obtain higher order coefficients, Forest uses QDBC (Forest, 1999). It's a straightforward extension of homogeneous conditions used in effective modulus approach (Bornert and al., 2001).

$$\underline{\mathbf{u}}(\underline{\mathbf{x}}) = \underline{\mathbf{E}} \cdot \underline{\mathbf{x}} + \frac{1}{2} \underline{\mathbf{K}}^I : (\underline{\mathbf{x}} \otimes \underline{\mathbf{x}}), \quad \underline{\mathbf{x}} \in \partial\Omega \quad (25)$$

Tensors $\underline{\mathbf{E}}$ and $\underline{\mathbf{K}}^I$ are constant over the volume. As the spatial coordinate tensor $\underline{\mathbf{x}} \otimes \underline{\mathbf{x}}$ is completely symmetric, *type I* formulation is the most natural system to write generalized BC.

3.3. Mean relations

The coordinate system origin will be taken at the RVE gravity center and $\langle . \rangle$ will indicate a mean operator over the RVE. We got the following

⁴This remark concerns just the implementation of the FEM code used for this study. From a broader point of view, some attempts have already been made to take into account fields fluctuation in the quadratic prescription (Kouznetsova (2002); Yuan and al. (2008))

relations over the volume (Forest, 1998)

$$\langle \underline{\mathbf{u}} \otimes \underline{\nabla} \rangle = \frac{1}{V} \int_{\Omega} \underline{\mathbf{u}} \otimes \underline{\nabla} d\Omega = \underset{\sim}{\mathbf{E}} \quad (26)$$

$$\langle \underline{\mathbf{u}} \otimes \underline{\nabla} \otimes \underline{\nabla} \rangle = \frac{1}{V} \int_{\Omega} \underline{\mathbf{u}} \otimes \underline{\nabla} \otimes \underline{\nabla} d\Omega = \underset{\simeq}{\mathbf{K}}^{\text{I}} \quad (27)$$

So, setting QDBC leads to impose the mean value of a field over the considered cell.

Knowing the relations previously introduced, some other useful relations have to be defined. First of all:

$$\underset{\sim}{\mathbf{E}} = \langle \underset{\sim}{\boldsymbol{\varepsilon}} \rangle \quad (28)$$

This is true since, according to tensor symmetry:

$$\underset{\sim}{\mathbf{E}} = \langle \underline{\mathbf{u}} \otimes \underline{\nabla} \rangle \quad (29)$$

$$= \langle \underline{\nabla} \otimes \underline{\mathbf{u}} \rangle \quad (30)$$

$$= \frac{1}{2} (\langle \underline{\mathbf{u}} \otimes \underline{\nabla} \rangle + \langle \underline{\nabla} \otimes \underline{\mathbf{u}} \rangle) \quad (31)$$

$$= \langle \frac{1}{2} (\underline{\mathbf{u}} \otimes \underline{\nabla} + \underline{\nabla} \otimes \underline{\mathbf{u}}) \rangle \quad (32)$$

$$= \langle \underset{\sim}{\boldsymbol{\varepsilon}} \rangle \quad (33)$$

Considering the local strain gradient mean, we obtain:

$$\langle \underset{\sim}{\boldsymbol{\varepsilon}} \otimes \underline{\nabla} \rangle = \underset{\simeq}{\mathbf{K}}^{\text{II}} \quad (34)$$

$\underset{\simeq}{\mathbf{K}}^{\text{II}}$ is the strain gradient tensor of *type II* formulation. With little manipulations, we get

$$\langle \underset{\sim}{\boldsymbol{\varepsilon}} \otimes \underline{\nabla} \rangle = \frac{1}{2} (\langle \underline{\mathbf{u}} \otimes \underline{\nabla} \otimes \underline{\nabla} \rangle + \langle \underline{\nabla} \otimes \underline{\mathbf{u}} \otimes \underline{\nabla} \rangle) \quad (35)$$

$$= \frac{1}{2} (\langle \underline{\mathbf{u}} \otimes \underline{\nabla} \otimes \underline{\nabla} \rangle + \langle \underline{\nabla} \otimes \underline{\mathbf{u}} \otimes \underline{\nabla} \rangle) \quad (36)$$

$$= \frac{1}{2} (K_{ijk}^{\text{I}} + K_{jik}^{\text{I}}) \quad (37)$$

$$= K_{ijk}^{\text{II}} \quad (38)$$

$$= \underset{\simeq}{\mathbf{K}}^{\text{II}} \quad (39)$$

and the following relation between *type I* and *II* system is obtained:

$$\langle \underset{\sim}{\varepsilon} \otimes \underset{\sim}{\nabla} \rangle = \underset{\sim}{\mathbb{P}}^{\text{I} \rightarrow \text{II}} \therefore \underset{\sim}{\mathbb{K}}^{\text{I}} \quad (40)$$

Another important point is the Hill-Mandel lemma's extension (Gologanu and al., 1997). Let's consider a divergence-free field $\underset{\sim}{\sigma}$ and a compatible deformation field $\underset{\sim}{\varepsilon}$. Those two fields are not supposed to be linked through any constitutive law. So in *type I* formulation we have got:

$$\langle \underset{\sim}{\sigma} : \underset{\sim}{\varepsilon} \rangle = \langle \underset{\sim}{\sigma} \rangle : \underset{\sim}{\mathbb{E}} + \langle \underset{\sim}{\sigma} \{ \otimes \} \underline{x} \rangle \therefore \underset{\sim}{\mathbb{K}}^{\text{I}} \quad (41)$$

When defining

$$\underset{\sim}{\mathbb{S}} = \langle \underset{\sim}{\sigma} \rangle ; \underset{\sim}{\mathbb{T}}^{\text{I}} = \langle \underset{\sim}{\sigma} \{ \otimes \} \underline{x} \rangle \quad (42)$$

we obtain:

$$\langle \underset{\sim}{\sigma} : \underset{\sim}{\varepsilon} \rangle = \underset{\sim}{\mathbb{S}} : \underset{\sim}{\mathbb{E}} + \underset{\sim}{\mathbb{T}}^{\text{I}} \therefore \underset{\sim}{\mathbb{K}}^{\text{I}} \quad (43)$$

In a *type II* formulation

$$\langle \underset{\sim}{\sigma} : \underset{\sim}{\varepsilon} \rangle = \langle \underset{\sim}{\sigma} \rangle : \underset{\sim}{\mathbb{E}} + \langle \underset{\sim}{\sigma} \otimes \underline{x} \rangle \therefore \underset{\sim}{\mathbb{K}}^{\text{II}} \quad (44)$$

When defining

$$\underset{\sim}{\mathbb{T}}^{\text{II}} = \langle \underset{\sim}{\sigma} \otimes \underline{x} \rangle \quad (45)$$

we obtain:

$$\langle \underset{\sim}{\sigma} : \underset{\sim}{\varepsilon} \rangle = \underset{\sim}{\mathbb{S}} : \underset{\sim}{\mathbb{E}} + \underset{\sim}{\mathbb{T}}^{\text{II}} \therefore \underset{\sim}{\mathbb{K}}^{\text{II}} \quad (46)$$

The meaning of such relations is that imposing QDBC on RVE generate volume fields which mean energies are similar to the one of a strain gradient media.

To summarise, Hill-Mandel lemma expressions for a second grade media are obtained: *Type I*:

$$\langle \underset{\sim}{\sigma} : \underset{\sim}{\varepsilon} \rangle = \langle \underset{\sim}{\sigma} \rangle : \langle \underset{\sim}{\varepsilon} \rangle + \langle \underset{\sim}{\sigma} \{ \otimes \} \underline{x} \rangle \therefore \langle \underline{\mathbf{u}} \otimes \underset{\sim}{\nabla} \otimes \underset{\sim}{\nabla} \rangle \quad (47)$$

and *type II*:

$$\langle \underset{\sim}{\sigma} : \underset{\sim}{\varepsilon} \rangle = \langle \underset{\sim}{\sigma} \rangle : \langle \underset{\sim}{\varepsilon} \rangle + \langle \underset{\sim}{\sigma} \otimes \underline{x} \rangle \therefore \langle \underset{\sim}{\varepsilon} \otimes \underset{\sim}{\nabla} \rangle \quad (48)$$

All the results obtained here are independent of any constitutive relation. Let's now particularize them in the microscopic linear elastic case.

3.4. Linear elasticity

Localization tensors. The following constitutive relation is assumed

$$\underline{\sigma}(\underline{x}) = \underline{c}(\underline{x}) : \underline{\varepsilon}(\underline{x}), \quad \forall \underline{x} \in \Omega \quad (49)$$

The problem linear dependence to its BC (\underline{E} and \underline{K}^I) leads to the existence of localization tensors: $\underline{L}^1(\underline{x})$ and $\underline{L}^{I,2}(\underline{x})$. The fundamental relation is:

$$\underline{\varepsilon}(\underline{x}) = \underline{L}^1(\underline{x}) : \underline{E} + \underline{L}^{I,2}(\underline{x}) : \underline{K}^I, \quad \forall \underline{x} \in \Omega \quad (50)$$

Combination of $\langle \varepsilon \rangle = \underline{E}$ with:

$$\langle \varepsilon \rangle = \langle \underline{L}^1(\underline{x}) : \underline{E} + \underline{L}^{I,2}(\underline{x}) : \underline{K}^I \rangle \quad (51)$$

$$= \langle \underline{L}^1(\underline{x}) \rangle : \underline{E} + \langle \underline{L}^{I,2}(\underline{x}) \rangle : \underline{K}^I \quad (52)$$

leads to express the following properties of the localization tensors:

$$\langle \underline{L}^1 \rangle = \underline{1} ; \langle \underline{L}^{I,2} \rangle = \langle \underline{L}^{II,2} \rangle = 0 \quad (53)$$

The calculation of $\langle \varepsilon \otimes \nabla \rangle$ in *type II* convention leads to:

$$\langle \varepsilon \otimes \nabla \rangle = \langle \underline{L}^1(\underline{x}) \otimes \nabla \rangle : \underline{E} + \langle \underline{L}^{II,2}(\underline{x}) \otimes \nabla \rangle : \underline{K}^{II} \quad (54)$$

As $\langle \varepsilon \otimes \nabla \rangle = \underline{K}^{II}$, it can be concluded:

$$\langle \underline{L}^1 \otimes \nabla \rangle = \underline{0} ; \langle \underline{L}^{II,2} \otimes \nabla \rangle = \underline{1}^{II} \quad (55)$$

In case of *type I* formulation:

$$\langle \varepsilon \otimes \nabla \rangle = \langle \underline{L}^{I,2}(\underline{x}) \otimes \nabla \rangle : \underline{K}^I = \underline{K}^{II} \quad (56)$$

leads to

$$\langle \underline{L}^{I,2}(\underline{x}) \otimes \nabla \rangle = \langle \underline{L}^{II,2}(\underline{x}) \otimes \nabla \rangle : \underline{P}^{I \rightarrow II} \quad (57)$$

and so, as $\langle \underline{L}^{II,2}(\underline{x}) \otimes \nabla \rangle = \underline{1}^{II}$, it can be concluded that:

$$\langle \underline{L}^{I,2}(\underline{x}) \otimes \nabla \rangle = \underline{1}^{II} : \underline{P}^{I \rightarrow II} = \underline{P}^{I \rightarrow II} \quad (58)$$

where $\underset{\approx}{\underset{\approx}{\underset{\approx}{\text{P}}}}^{\text{I} \rightarrow \text{II}}$ is the 6th-order tensor previously defined such that:

$$\underset{\approx}{\underset{\approx}{\underset{\approx}{\text{P}}}}^{\text{I} \rightarrow \text{II}} \therefore \underset{\approx}{\text{K}}^{\text{I}} = \underset{\approx}{\text{K}}^{\text{II}} \quad (59)$$

Thus the following relation between localization tensors is true:

$$\underset{\approx}{\text{L}}^{\text{I},2}(\underline{x}) = \underset{\approx}{\text{L}}^{\text{II},2}(\underline{x}) \therefore \underset{\approx}{\underset{\approx}{\underset{\approx}{\text{P}}}}^{\text{I} \rightarrow \text{II}} \quad (60)$$

where $\underset{\approx}{\underset{\approx}{\underset{\approx}{\text{P}}}}^{\text{I} \rightarrow \text{II}}$ is obviously constant over the elementary cell. This relation allows us to switch from one system to another.

We can now obtain a first expression for the first and second order effective modulus.

Effective modulus. The first expression will be derived in a straightforward manner, in a second time an energetic approach will be considered. The straightforward derivation consists in writing down stress and hyperstress expressions using the relation (50). Starting with *type I* formulation, the mean stress tensor can be written:

$$\underset{\sim}{\text{S}}^{\text{I}} = \underset{\sim}{\langle} \underset{\sim}{\sigma} \underset{\sim}{\rangle} = \underset{\sim}{\langle} \underset{\approx}{\underset{\approx}{\underset{\approx}{\text{c}}}} : \underset{\approx}{\underset{\approx}{\underset{\approx}{\text{L}}}}^{\text{I}} \underset{\sim}{\rangle} + \underset{\sim}{\langle} \underset{\approx}{\underset{\approx}{\underset{\approx}{\text{c}}}} : \underset{\approx}{\underset{\approx}{\underset{\approx}{\text{L}}}}^{\text{I},2} \underset{\sim}{\rangle} \therefore \underset{\sim}{\text{K}}^{\text{I}} \quad (61)$$

and so:

$$T_{ijk}^{\text{I}} = \frac{1}{2} \langle \sigma_{ij} x_k + \sigma_{ik} x_j \rangle \quad (62)$$

$$= \langle c_{i(jlm} L_{lmno}^1 x_k \rangle E_{no} + \langle c_{i(jlm} L_{lmnop}^{\text{I},2} x_k \rangle K_{nop}^{\text{I}} \quad (63)$$

with the notation:

$$c_{i(jlm} L_{lmno}^1 x_k) = \frac{1}{2} (c_{ijlm} L_{lmno}^1 x_k + c_{iklm} L_{lmno}^1 x_j) \quad (64)$$

In a *type II* formulation:

$$\underset{\sim}{\text{S}}^{\text{II}} = \underset{\sim}{\langle} \underset{\approx}{\underset{\approx}{\underset{\approx}{\text{c}}}} : \underset{\approx}{\underset{\approx}{\underset{\approx}{\text{L}}}}^{\text{I}} \underset{\sim}{\rangle} + \underset{\sim}{\langle} \underset{\approx}{\underset{\approx}{\underset{\approx}{\text{c}}}} : \underset{\approx}{\underset{\approx}{\underset{\approx}{\text{L}}}}^{\text{II},2} \underset{\sim}{\rangle} \therefore \underset{\sim}{\text{K}}^{\text{II}} \quad (65)$$

$$\underset{\sim}{\text{T}}^{\text{II}} = \underset{\sim}{\langle} \underset{\sim}{\sigma} \otimes \underset{\sim}{\underline{x}} \underset{\sim}{\rangle} = \langle c_{ijlm} L_{lmno}^1 x_k \rangle E_{no} + \langle c_{ijlm} L_{lmnop}^{\text{II},2} x_k \rangle K_{nop}^{\text{II}} \quad (66)$$

Those results could be summed up in a matricial way:

$$\begin{pmatrix} S_{(ij)}^I \\ S_{i(jk)}^I \end{pmatrix} = \begin{pmatrix} \langle c_{ijlm} L_{lmno}^1 \rangle & \langle c_{ijlm} L_{lmnop}^{I,2} \rangle \\ \langle c_{i(jlm} L_{lmno}^1 x_k \rangle & \langle c_{i(jlm} L_{lmnop}^{I,2} x_k \rangle \end{pmatrix} \begin{pmatrix} E_{(no)} \\ K_{n(op)}^I \end{pmatrix} \quad (67)$$

$$\begin{pmatrix} S_{(ij)}^{II} \\ S_{(ij)k}^{II} \end{pmatrix} = \begin{pmatrix} \langle c_{ijlm} L_{lmno}^1 \rangle & \langle c_{ijlm} L_{lmnop}^{II,2} \rangle \\ \langle c_{ijlm} L_{lmno}^1 x_k \rangle & \langle c_{ijlm} L_{lmnop}^{II,2} x_k \rangle \end{pmatrix} \begin{pmatrix} E_{(no)} \\ K_{(no)p}^{II} \end{pmatrix} \quad (68)$$

Effective modulus could be also derived from an energetic approach. Modulus would be the same but, in some cases, related expressions could be easier to handle.

Energetic approach. Effective tensors expression could be derived from elastic energy:

$$\langle w_e \rangle = \frac{1}{2} \langle \underline{\varepsilon} : \underline{\sigma} \rangle \quad (69)$$

From expression (50) one obtains:

$$\langle \underline{\varepsilon} : \underline{\sigma} \rangle = \langle \left(\underline{L}^1(\underline{x}) : \underline{E} + \underline{L}^{I,2}(\underline{x}) : \underline{K}^I \right) : \underline{c} : \left(\underline{L}^1(\underline{x}) : \underline{E} + \underline{L}^{I,2}(\underline{x}) : \underline{K}^I \right) \rangle \quad (70)$$

this can be rewritten as:

$$\begin{aligned} \langle \underline{\varepsilon} : \underline{\sigma} \rangle = & \underline{E} : \langle \underline{L}^{T1} : \underline{c} : \underline{L}^1 \rangle : \underline{E} + 2 \underline{E} : \langle \underline{L}^{T1} : \underline{c} : \underline{L}^{I,2} \rangle : \underline{K}^I + \dots \\ & \dots \underline{K}^I : \langle \underline{L}^{TI,2} : \underline{c} : \underline{L}^{I,2} \rangle : \underline{K}^I \end{aligned} \quad (71)$$

According to Hill-Mandel lemma, the mean of microscopic energy over the elementary cell could be identified with a strain gradient macroscopic elastic energy. This energy is written:

$$W_e = \frac{1}{2} \underline{E} : \underline{C} : \underline{E} + \underline{E} : \underline{M}^I : \underline{K}^I + \frac{1}{2} \underline{K}^I : \underline{A}^I : \underline{K}^I \quad (72)$$

so, by identification,:

$$\underline{C} = \langle \underline{L}^{T1} : \underline{c} : \underline{L}^1 \rangle ; \underline{M}^I = \langle \underline{L}^{T1} : \underline{c} : \underline{L}^{I,2} \rangle ; \underline{A}^I = \langle \underline{L}^{TI,2} : \underline{c} : \underline{L}^{I,2} \rangle \quad (73)$$

And, in the matricial way,:

$$\begin{pmatrix} S^I \\ \underline{T}^I \end{pmatrix} = \begin{pmatrix} \langle \underline{L}^{T1} : \underline{c} : \underline{L}^1 \rangle & \langle \underline{L}^{T1} : \underline{c} : \underline{L}^{I,2} \rangle \\ \langle \underline{L}^{TI,2} : \underline{c} : \underline{L}^{T1} \rangle & \langle \underline{L}^{TI,2} : \underline{c} : \underline{L}^{I,2} \rangle \end{pmatrix} \begin{pmatrix} \underline{E} \\ \underline{K}^I \end{pmatrix} \quad (74)$$

Then the effective constitutive law will be written:

$$\begin{pmatrix} \underset{\approx}{\text{S}}^{\text{I}} \\ \underset{\approx}{\text{T}}^{\text{I}} \end{pmatrix} = \begin{pmatrix} \underset{\approx}{\text{C}} & \underset{\approx}{\text{M}}^{\text{I}} \\ \underset{\approx}{\text{M}}^{\text{TI}} & \underset{\approx}{\text{A}}^{\text{I}} \end{pmatrix} \begin{pmatrix} \underset{\approx}{\text{E}} \\ \underset{\approx}{\text{K}}^{\text{I}} \end{pmatrix} \quad (75)$$

Type II expressions are derived symmetrically. In Forest (1999), the straightforward and the energetic approaches are shown to be strictly equivalent.

Expressions obtained through energetic consideration are interesting because they allow to easily express transformation formulas between *type I* and *type II* formulations. Let's consider $\underset{\approx}{\text{A}}^{\text{I}}$: its two formulations are

$$\underset{\approx}{\text{A}}^{\text{I}}_{ijklmn} = \langle c_{i(jlm} L_{lmnop}^{1,2} x_k) \rangle = \langle L_{ijkop}^{\text{TI},2} c_{opqr} L_{qrlmn}^{1,2} \rangle \quad (76)$$

and so

$$\underset{\approx}{\text{A}}^{\text{II}} = \underset{\approx}{\text{P}}^{\text{TI} \rightarrow \text{I}} \cdot \underset{\approx}{\text{A}}^{\text{I}} \cdot \underset{\approx}{\text{P}}^{\text{II} \rightarrow \text{I}} \quad (77)$$

In indicial form:

$$\underset{\approx}{\text{A}}^{\text{II}}_{ijklmn} = (\delta_{io}\delta_{jp}\delta_{kq} + \delta_{ip}\delta_{jq}\delta_{ko} - \delta_{iq}\delta_{jo}\delta_{kp}) \underset{\approx}{\text{A}}^{\text{I}}_{opqrst} (\delta_{rl}\delta_{sm}\delta_{tn} + \delta_{rm}\delta_{sn}\delta_{tl} - \delta_{rn}\delta_{sl}\delta_{tm}) \quad (78)$$

Through relation (78) the conversion to *type II* formulation would easily be made.

3.5. Overall behavior computation

QDBC could be written:

$$u_i = E_{io}x_o + \frac{1}{2}K_{ipq}^{\text{I}}x_px_q, \quad \forall x \in \partial\Omega$$

and localization relation is (50)

$$\underline{\varepsilon}(\underline{x}) = \underset{\approx}{\text{L}}^1(\underline{x}) : \underset{\approx}{\text{E}} + \underset{\approx}{\text{L}}^{1,2}(\underline{x}) \cdot \underset{\approx}{\text{K}}^{\text{I}}, \quad \forall \underline{x} \in \Omega$$

The goal is to construct localization tensors without any explicit determination. Their construction will be made through elementary numerical computations.

We consider the following solicitation families

$$E_{ij}^{(ab)} = \delta_{ai}\delta_{bj} ; K_{ijk}^{(abc)} = \frac{1}{2}\delta_{ai}(\delta_{bj}\delta_{ck} + \delta_{bk}\delta_{cj}) \quad (79)$$

Cell mechanical response under those solicitations applied in a non combined way leads to the following strain fields:

$$\varepsilon_{ij}(x)_{ij}^{(ab)} = L(x)_{ijlm}^1 \delta_{al}\delta_{bm} = L(x)_{ijab}^1 \quad (80)$$

$$\varepsilon_{ij}(x)_{ij}^{(abc)} = L(x)_{ijklmn}^{I,2} \delta_{al}(\delta_{bm}\delta_{cn} + \delta_{bn}\delta_{cm}) = L(x)_{ijabc}^{I,2} \quad (81)$$

Associated stress fields are then obtained through constitutive law:

$$\sigma_{ij}(x)_{ij}^{(ab)} = c(x)_{ijkl}\varepsilon_{kl}(x)_{kl}^{(ab)} = c(x)_{ijkl}L(x)_{klab}^1 \quad (82)$$

$$\sigma_{ij}(x)_{ij}^{(abc)} = c(x)_{ijkl}\varepsilon_{ij}(x)_{ij}^{(abc)} = c(x)_{ijkl}L(x)_{klabc}^{I,2} \quad (83)$$

From effective modulus straightforward expressions we got:

$$C_{ijkl} = \langle c_{ijop}L_{opkl}^1 \rangle ; M_{ijklmn}^I = \langle c_{ijop}L_{oplmn}^{I,2} \rangle ; A_{ijklmn}^I = \langle c_{i(jop}L_{oplmn}^{I,2}x_k) \rangle$$

Those expressions could be re-expressed in terms of stress, and eventually:

$$C_{ijkl} = \langle \sigma_{ij}^{(kl)} \rangle ; M_{ijklmn}^I = \langle \sigma_{ij}^{(lmn)} \rangle ; A_{ijklmn}^I = \langle \frac{1}{2} \left(\sigma_{ij}^{(lmn)} x_k + \sigma_{ik}^{(lmn)} x_j \right) \rangle$$

Other expressions could be built-up according to energetic approach, but, from a computational point of view, they are less practical.

Now we can applied the defined homogenization procedure on several elementary cells.

4. Numerical computations

This section aims at illustrating how to apply the proposed computational procedure and to check some properties of this model. Those properties have been obtained by theoretical methods and had not been observed yet on concrete situations. The effective tensor considered here will be $A_{(ij)k(lm)n}$. The first part of this section will be devoted to a complete example of the homogenization procedure on a supposed isotropic cell. It will be shown that this case, commonly used in literature, leads to a degenerated kind of strain gradient isotropic media (Gologanu and al., 1997; Zybell and al., 2008). In a second subsection, various kind of anisotropic behaviors will be constructed. Some previously theoretically announced properties will then be checked.

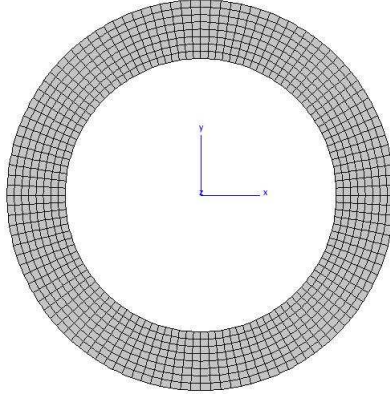


Figure 3: Elementary cell

4.1. Computation of an isotropic overall behavior

A circular cell will be considered in order to compute an isotropic strain gradient elastic overall behavior. The constitutive behavior will be isotropic linear elastic, with a Young modulus of 200 GPa and a Poisson ratio of 0.3. The external radius is of 1 mm with a thickness of 0.3 mm (see fig.3), the relative density is $\bar{\rho} = 0.51$. It is worth noting that this isotropic operator construction is physically meaningless because such a cell could not generate a dense plane tilling. On the other hand, the construction of an isotropic behavior is very simple in this case. Furthermore this mathematical trick was used before in the literature (Gologanu and al., 1997; Zybell and al., 2008). Finally, it will be shown that this construction should be avoided, because leading to a degenerated kind of isotropic medium.

The obtained displacement fields could be observed fig.4(a) and fig.4(b). As the considered media is porous, the spatial mean operator should be slightly modified (Bornert and al., 2001; Zybell and al., 2008). The proper operator is

$$\langle \underset{\sim}{\sigma} \rangle^{\star} = \frac{1}{V_T} \int_{M \sim} \sigma d\Omega_M = \bar{\rho} \langle \underset{\sim}{\sigma} \rangle_M \quad (84)$$

Degrees of freedom associated to the QDBC are, according to index symmetries,

$$\{K_{111}^I \ K_{112}^I \ K_{122}^I \ K_{211}^I \ K_{212}^I \ K_{222}^I\} \quad (85)$$

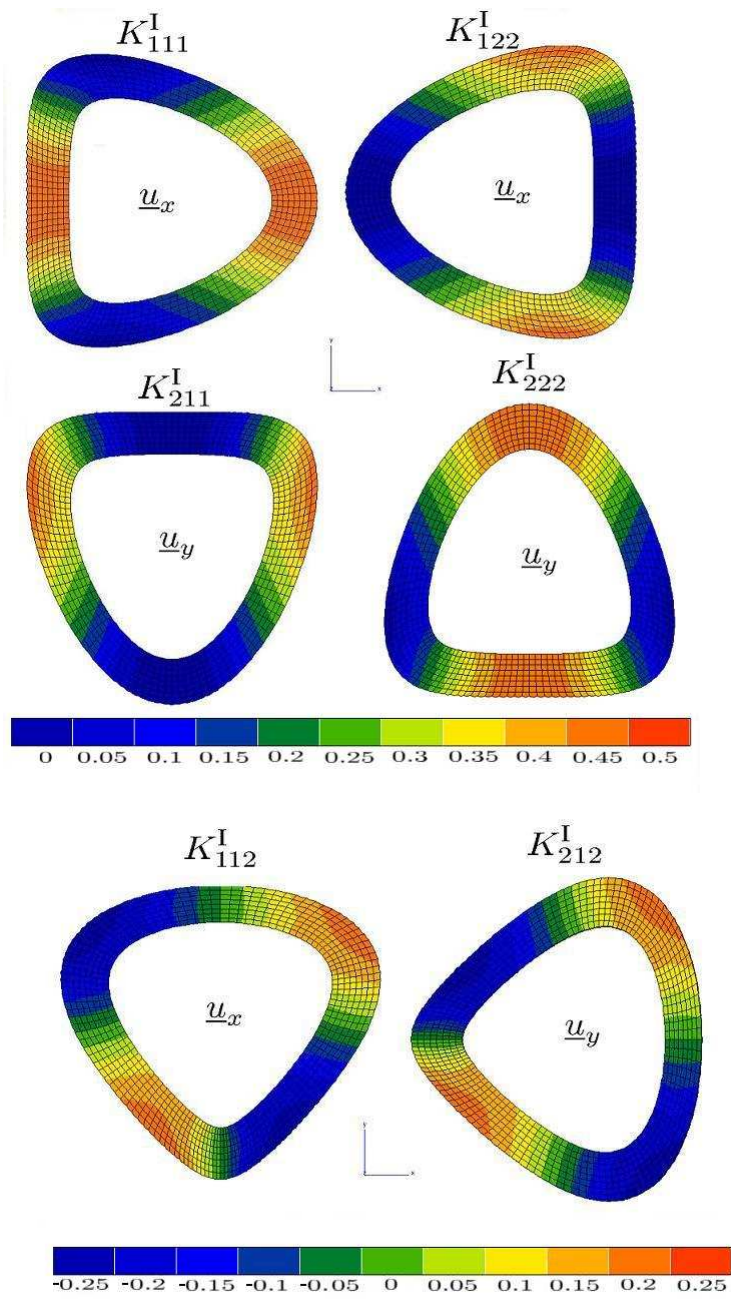


Figure 4: Displacement fields for a circular elementary cell

After 6 FEM computations, a matrix representation⁵ of $\underset{\approx}{\underset{\approx}{\mathbb{A}}^I}$ is obtained⁶:

$$\check{\mathbb{A}}^I = \begin{pmatrix} 9223 & 0 & -9223 & 0 & -2162 & 0 \\ 0 & 9223 & 0 & 2162 & 0 & -2162 \\ -9223 & 0 & 9223 & 0 & 2162 & 0 \\ 0 & 2162 & 0 & 9223 & 0 & -9223 \\ -2162 & 0 & 2162 & 0 & 9223 & 0 \\ 0 & -2162 & 0 & -9223 & 0 & 9223 \end{pmatrix} \quad (86)$$

In *type II* formulation, elementary degrees of freedom will differ:

$$\{K_{111}^{II} \ K_{112}^{II} \ K_{121}^{II} \ K_{122}^{II} \ K_{221}^{II} \ K_{222}^I\} \quad (87)$$

Both of these two systems are 6-D. Let's consider the following 8-D space:

$$\{K_{111} \ K_{112} \ K_{121} \ K_{122} \ K_{211} \ K_{212} \ K_{221} \ K_{222}\} \quad (88)$$

K^I and K^{II} could be rewritten in this system. The system switch operator is: $\underset{\approx}{\underset{\approx}{P}}^{II \rightarrow I}$:

$$P_{ijklmn}^{II \rightarrow I} = \delta_{il}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jn}\delta_{kl} - \delta_{in}\delta_{jl}\delta_{km} \quad (89)$$

could be rewritten in this 8-D space:

$$\check{P}_{\alpha\beta}^{II \rightarrow I} = \check{I}_{\alpha\beta} + \check{M}_{\alpha\beta} - \check{N}_{\alpha\beta} \quad (90)$$

with $\check{I}_{\alpha\beta}$ the 8-D identity operator. $\check{M}_{\alpha\beta}$ and $\check{N}_{\alpha\beta}$ are defined:

$$\check{M} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}; \quad \check{N} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (91)$$

⁵Notation $\check{\cdot}$ indicates a matricial, but not tensorial, representation of a tensor.

⁶Indicated values are expressed in Mpa.mm², they were calculated taking into account the cell relative density and rounded up.

$\check{A}_{\alpha\beta}^{\text{II}}$ is then straightforward to compute:

$$\check{A}_{\alpha\beta}^{\text{II}} = \check{P}_{\alpha\mu}^{\text{II}\rightarrow\text{I}} \check{A}_{\mu\nu}^{\text{II}} \check{P}_{\nu\beta}^{\text{II}\rightarrow\text{I}} \quad (92)$$

In 8-D we obtain:

$$\check{A}^{\text{II}} = \begin{pmatrix} 9223 & 0 & 0 & -9223 & 0 & -9223 & 4899 & 0 \\ 0 & 37466 & -4899 & 0 & -4899 & 0 & 0 & 4899 \\ 0 & -4899 & 9223 & 0 & 9223 & 0 & 0 & -9223 \\ -9223 & 0 & 0 & 9223 & 0 & 9223 & -4899 & 0 \\ 0 & -4899 & 9223 & 0 & 9223 & 0 & 0 & -9223 \\ -9223 & 0 & 0 & 9223 & 0 & 9223 & -4899 & 0 \\ 4899 & 0 & 0 & -4899 & 0 & -4899 & 37466 & 0 \\ 0 & 4899 & -9223 & 0 & -9223 & 0 & 0 & 9223 \end{pmatrix} \quad (93)$$

and finally in *type II* 6-D space:

$$\check{A}^{\text{II}} = \begin{pmatrix} 9223 & 0 & 0 & -9223 & 4899 & 0 \\ 0 & 37466 & -4899 & 0 & 0 & 4899 \\ 0 & -4899 & 9223 & 0 & 0 & -9223 \\ -9223 & 0 & 0 & 9223 & -4899 & 0 \\ 4899 & 0 & 0 & -4899 & 37466 & 0 \\ 0 & 4899 & -9223 & 0 & 0 & 9223 \end{pmatrix} \quad (94)$$

The obtained matrix is not yet a tensorial representation of $\underset{\approx}{A}^{\text{I}}$.

Tensorial reconstruction. Using the base introduced in Auffray and al. (2009) and presented in the first section of this paper, the following second order tensorial representation of $\underset{\approx}{A}^{\text{I}}$ is obtained:

$$\underset{\approx}{\hat{A}}^{\text{II}} = \begin{pmatrix} 9223 & 4899 & -13043 & 0 & 0 & 0 \\ 4899 & 37466 & -6928 & 0 & 0 & 0 \\ -13043 & -6928 & 18446 & 0 & 0 & 0 \\ 0 & 0 & 0 & 9223 & 4899 & -13043 \\ 0 & 0 & 0 & 4899 & 37466 & -6928 \\ 0 & 0 & 0 & -13043 & -6918 & 18446 \end{pmatrix} \quad (95)$$

In 2-D, isotropic strain gradient elasticity tensors are defined by 4 coefficients and could be blocked decomposed as:

$$\hat{\underset{\sim}{A}}_{O(2)} = \begin{pmatrix} X_i & 0 \\ 0 & X_i \end{pmatrix} ; \quad X_i = \begin{pmatrix} a_1 & a_2 & \frac{a_1-a_3}{\sqrt{2}} - a_4 \\ a_2 & a_3 & a_4 \\ \frac{a_1-a_3}{\sqrt{2}} - a_4 & a_4 & \frac{a_1+a_3}{2} - a_2 \end{pmatrix} \quad (96)$$

Considering:

$$a_1 = 9223 \text{ MPa.mm}^2 ; a_2 = 4899 \text{ MPa.mm}^2 ; a_3 = 37466 \text{ MPa.mm}^2 ; a_4 = -6928 \text{ MPa.mm}^2$$

one easily could check that the isotropic relations are verified:

$$-13043 = \frac{a_1 - a_3}{\sqrt{2}} - a_4 ; 18446 = \frac{a_1 + a_3}{2} - a_2$$

So the constructed overall behavior is truly isotropic, but the obtained operator is degenerated.

Degenerated isotropic system. The Voigt tensorial representation of $\underset{\sim}{A}^I$ is symmetric, so its eigenvalues can be determined⁷:

$$\lambda_1 = 42365,4 \text{ MPa.mm}^2 ; \lambda_2 = 22769,4 \text{ MPa.mm}^2 ; \lambda_3 = 0 \text{ MPa.mm}^2$$

This 0 eigenvalue is characteristic of the circular cell. Let's consider the matricial representation of $\underset{\sim}{A}^I$ in a *type II* convention:

$$\underset{\sim}{\check{A}}^{\text{II}} = \begin{pmatrix} 9223 & 0 & 0 & -9223 & 4899 & 0 \\ 0 & 37466 & -4899 & 0 & 0 & 4899 \\ 0 & -4899 & 9223 & 0 & 0 & -9223 \\ -9223 & 0 & 0 & 9223 & -4899 & 0 \\ 4899 & 0 & 0 & -4899 & 37466 & 0 \\ 0 & 4899 & -9223 & 0 & 0 & 9223 \end{pmatrix} \quad (97)$$

Columns represent the hyperstress component induced by an elementary QDBC. In *type II* formulation those elementary deformations are:

$$\{K_{111}^{\text{II}} \ K_{112}^{\text{II}} \ K_{121}^{\text{II}} \ K_{122}^{\text{II}} \ K_{221}^{\text{II}} \ K_{222}^{\text{I}}\}$$

⁷Their multiplicity are obviously 2

The following linear dependences could be seen : $K_{111}^{\text{II}} = -K_{122}^{\text{II}}$ and $K_{222}^{\text{II}} = -K_{211}^{\text{II}}$. These linear dependences between columns is kept while turning the former matrix into a 2nd-order tensor. So its determinant equals 0 and two eigenvalues are null.

This could formally be shown. Let's consider:

$$\underset{\sim}{K}^{\text{I}} = \frac{1}{V} \int_{\partial\Omega} \underset{\sim}{\nabla}(\underset{\sim}{u}) \cdot \underset{\sim}{n} d\partial\Omega \quad (98)$$

To avoid cumbersome notations modeling convention will be omitted here. Tensor $\underset{\sim}{K}^{(111)}$, in which the only non-null term is K_{111} , is obtained with the boundary condition

$$\underset{\sim}{u}^{(111)} = \left(u_1(x_1, x_2) = \frac{1}{2}x_1^2 ; u_2(x_1, x_2) = 0 \right) \quad (99)$$

and so as for $\underset{\sim}{K}^{(122)}$:

$$\underset{\sim}{u}^{122} = \left(u_1(x_1, x_2) = \frac{1}{2}x_2^2 ; u_2(x_1, x_2) = 0 \right) \quad (100)$$

According to the cell circular shape, we consider the change of variable:

$$x_1 = R_e \cos(\theta) ; x_2 = R_e \sin(\theta) \quad (101)$$

directly leading to:

$$u^{111} = \frac{R_e^2}{2} - u^{122} \quad (102)$$

So

$$\frac{1}{V} \int_{\partial\Omega} \underset{\sim}{\nabla}(\underset{\sim}{u}^{(111)}) \cdot \underset{\sim}{n} d\partial\Omega = - \int_{\partial\Omega} \underset{\sim}{\nabla}(\underset{\sim}{u}^{(122)}) \cdot \underset{\sim}{n} d\partial\Omega \quad (103)$$

and for a circular shape $\underset{\sim}{K}^{(111)} = -\underset{\sim}{K}^{(122)}$ and so as for $\underset{\sim}{K}^{(222)}$ and $\underset{\sim}{K}^{(211)}$. And so we get a null eigenvalue with multiplicity 2.

Isotropic second order behavior obtained through computation over a circular cell leads to a degenerated operator. The bad numerical behavior of the obtained model pointed out by the latter authors might be related to the ill conditionness of this operator. Let's turn now our interest on anisotropic geometrical cell.

4.2. Some anisotropic systems

In this subsection, different points will be considered. In a first time, some anisotropic operators will be constructed in order to check some of the properties found in Auffray and al. (2009). In a second time, isotropic operators will be built with anisotropical cells. This aims at verifying Hermann theorem and the *class jump* phenomenon. Those observations will lead to the construction of a well conditioned isotropic operator.

4.2.1. Hexagonal cell (D_6 -invariance)

Assumptions made here remain the same as the ones used for the circular case. The previously introduced method leads to the following tensorial expression:

$$\hat{\underset{\sim}{A}}^{\text{II}} = \begin{pmatrix} 12101 & 5490 & -14375 & 0 & 0 & 0 \\ 5490 & 41287 & -6263 & 0 & 0 & 0 \\ -14375 & -6263 & 21204 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10909 & 6683 & -12689 \\ 0 & 0 & 0 & 6683 & -12689 & -6263 \\ 0 & 0 & 0 & -12689 & -6263 & 18819 \end{pmatrix} \quad (104)$$

As introduced at the first section, a D_6 -invariant second order operator could be decomposed as

$$\hat{\underset{\sim}{A}}_{\sim D_6} = \hat{\underset{\sim}{A}}_{\sim O(2)} + (a_5 - a_1)\hat{\underset{\sim}{A}}_{\sim cor} \quad (105)$$

with $\hat{\underset{\sim}{A}}_{\sim O(2)}$ an isotropic tensor and $\hat{\underset{\sim}{A}}_{\sim cor}$ an anisotropic correction:

$$\hat{\underset{\sim}{A}}_{\sim cor} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & 1 & -1 & -\sqrt{2} \\ & & & & 1 & \sqrt{2} \\ & & & & & 2 \end{pmatrix} \quad (106)$$

Defining

$$a_1 = 12101 \text{ MPa.mm}^2 ; a_2 = 5490 \text{ MPa.mm}^2 ; a_3 = 40095 \text{ MPa.mm}^2$$

$$a_4 = -5621 \text{ MPa.mm}^2 ; a_5 = 10909 \text{ MPa.mm}^2$$

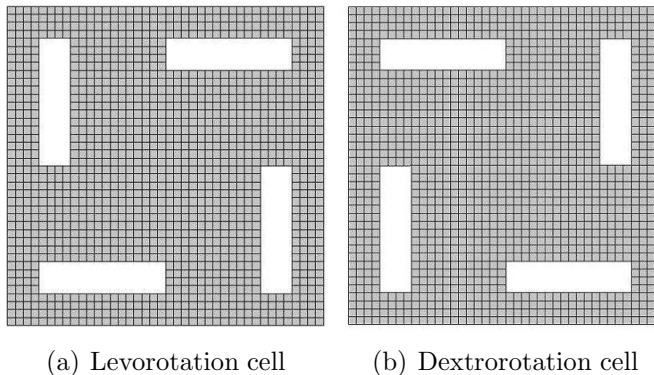


Figure 5: Z_4 -invariant cells

it is straightforward to check that the constructed operator satisfy the hexatropic conditions. It is well known that a classical overall elastic media defined over a 6-fold invariant material would be 2-D isotropic (Bornert and al., 2001). The last result is a concrete illustration that for an overall SGE media this result is no longer true. Let's turn our attention to another SGE specific behavior: chiral sensitivity.

4.2.2. Ortho-chiral cell (Z_4 -invariance)

As shown in Auffray and al. (2009) and demonstrated in Auffray (2009), SGE elasticity is chiral sensitive. An illustration of such a fact will be made considering the two elementary cells represented fig.5(a) and fig.5(b). Those square elementary cells are Z_4 -invariant but *turn* in opposite direction. Classical overall elastic media would be the same for both cells. Let's first consider the levorotation cell (fig.5(a)). Some displacement fields induced by boundary conditions are plotted⁸ in fig.6. One could readily see that the combination of cell geometry and boundary conditions induced a direct rotational displacement field. Using the tensorial block decomposition introduced §.2, the second order constitutive elastic tensor will be noted

$$\hat{\mathbb{A}}_{\sim Z_4} = \begin{pmatrix} \mathbf{X} & \mathbf{C} \\ \mathbf{C}^T & \mathbf{X} \end{pmatrix} \quad (107)$$

⁸For reading convenience plots were done in non deformed configuration.

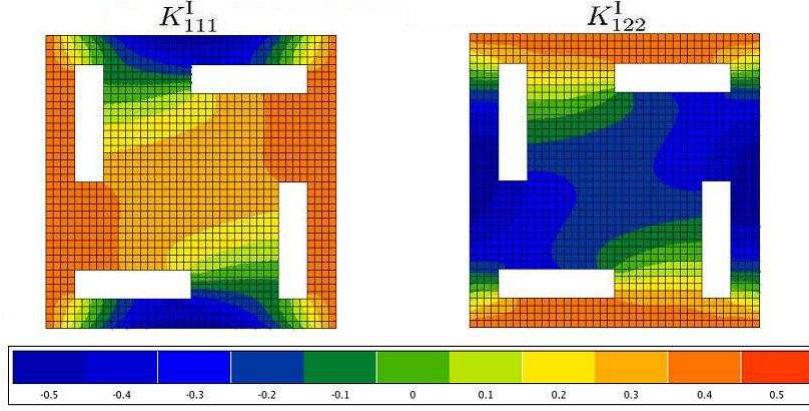


Figure 6: Displacement field along \underline{x} induced by K_{111} and K_{122}

with

$$\mathbf{X} = \begin{pmatrix} 20630 & 6073 & -13628 \\ 6073 & 56189 & -3831 \\ -13628 & -3831 & 20748 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 0 & 3334 & 1351 \\ -3334 & 0 & 5848 \\ -1351 & -5848 & 0 \end{pmatrix} \quad (108)$$

Square cell symmetries are found completed by a coupling mechanism between \underline{x} and \underline{y} directions. As theoretically determined this coupling mechanism is represented by an antisymmetrical tensor. Computed in this way the Z_4 -invariant operator seems to depend on 9 coefficients. Nevertheless, as shown in Auffray and al. (2009), one can find a rotation lowering the number of coefficients to 8. If we write down:

$$\mathbf{X} = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_2 & a_4 & a_5 \\ a_3 & a_5 & a_6 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 0 & a_7 & a_8 \\ -a_7 & 0 & a_9 \\ -a_8 & -a_9 & 0 \end{pmatrix} \quad (109)$$

The rotation angle solves the following equation:

$$\tan(4\theta) = \frac{2\sqrt{2}(a_8 - a_9)}{a_1 + a_4 - 2(a_2 + a_6)} \quad (110)$$

leading in our case to $\theta = -0,129615$ rad. Thus, after transformation:

$$\hat{\mathbf{A}}_{\sim Z_4} = \begin{pmatrix} \mathbf{X} & \mathbf{C}_o \\ \mathbf{C}_o^T & \mathbf{X} \end{pmatrix} \quad (111)$$

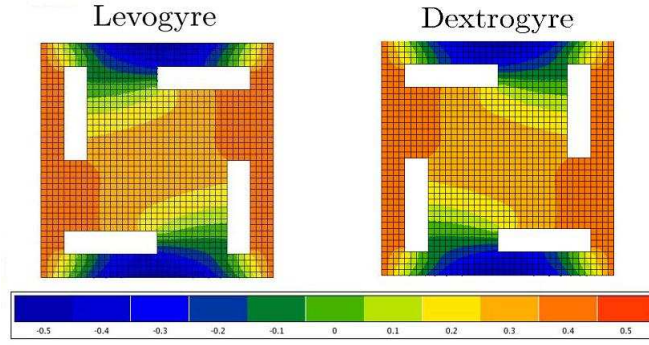


Figure 7: Displacement field along \underline{x} induced by K_{111}

with

$$\mathbf{X} = \begin{pmatrix} 19321 & 5651 & -12404 \\ 5651 & 58341 & -2607 \\ -12404 & -2607 & 20347 \end{pmatrix}, \quad \mathbf{C}_o = \begin{pmatrix} 0 & 1434 & 2256 \\ -1434 & 0 & 2256 \\ -2256 & -2256 & 0 \end{pmatrix} \quad (112)$$

In agreement with the announced results.

Now considering a dextrorotation cell (fig.5(b)), the second order constitutive elastic tensor will just differ according to:

$$\mathbf{C}_{o \text{ dex}} = \begin{pmatrix} 0 & -1434 & -2256 \\ 1434 & 0 & -2256 \\ 2256 & 2256 & 0 \end{pmatrix} = -\mathbf{C}_{o \text{ lev}} \quad (113)$$

The rotation displacement field turns in this case indirectly (see fig.7).

4.3. Hermann theorem and class-jump phenomenon

This last subsection aims at verifying some consequences of Hermann theorem in 2-D. The demonstration of this theorem can be found in Auffray (2008a) and a study of its reciprocity was studied in Auffray and al. (2008b).

This theorem states that an n th-order tensorial property will be 2-D hemitropic if the material is p -fold invariant, p greater than n . Furthermore if the material is achiral, the tensorial property is isotropic. In our case, the second grade elasticity is modeled by a 6th-order tensor, so the theorem can be checked for an heptagon. But, for some reasons that will be detailed hereafter, a octagonal cell will first be considered.

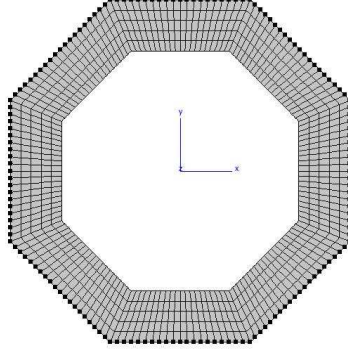


Figure 8: Octagonal cell

4.3.1. Octagonal cell (D_8 invariance)

Let's consider an octagonal cell (see fig.8): As expected the constructed tensor

$$\hat{\underset{\sim}{A}}^{\text{II}} = \begin{pmatrix} 10345 & 5426 & -13482 & 0 & 0 & 0 \\ -5426 & 39464 & -5377 & 0 & 0 & 0 \\ -13482 & -5377 & 19479 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10345 & 5426 & -13482 \\ 0 & 0 & 0 & -5426 & 39464 & -5377 \\ 0 & 0 & 0 & -13482 & -5377 & 19479 \end{pmatrix} \quad (114)$$

verifies the previously introduced isotropic relation. As in the circular case, the plane can not be densely tilled with an octagonal elementary cell. So as previously explained this construction has no physical meaning. But if the strict periodicity condition of the tilling is dropped down, some figures are, in a certain sens, D_8 invariant, such as the Ammann-Beenker tilling (see fig.9). Such kind of tilling belongs to the class of quasiperiodic structures.

Another point is to consider this tensor's eigenvalues :

$$\lambda_1 = 44626 \text{ MPa.mm}^2; \lambda_2 = 23985 \text{ MPa.mm}^2; \lambda_3 = 675 \text{ MPa.mm}^2$$

So this time the isotropic tensor is not degenerated. Its numerical properties should be far better than the ones computed on a circular cell.

In 3-D, Hermann theorem gives necessary and sufficient condition for transverse isotropy, but in a 2-D space this condition is no longer necessary.

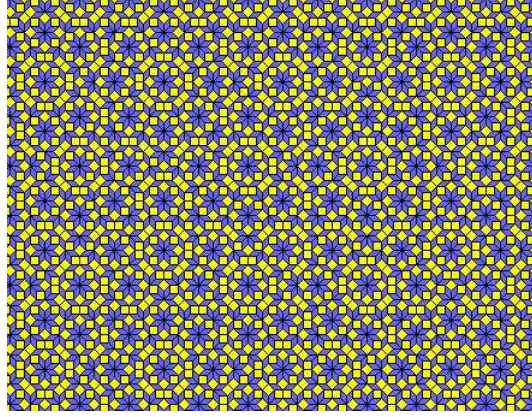


Figure 9: Ammann-Beenker tiling

In 2-D, an odd material rotational invariance leads to a twice order physical invariance. As a consequence *class jump* phenomenons could occur. So, according to this result, strain gradient elastic tensors defined on a 5-fold invariant material should be isotropic.

4.3.2. Pentagonal cell (D_5 invariance)

Let's consider a pentagonal cell (fig.10). As expected, the constructed tensor

$$\hat{\underset{\sim}{A}}^{\Pi} = \begin{pmatrix} 13102 & 6887 & -13559 & 0 & 0 & 0 \\ 6887 & 42168 & -6994 & 0 & 0 & 0 \\ -13559 & -6994 & 20748 & 0 & 0 & 0 \\ 0 & 0 & 0 & 13101 & 6887 & -13559 \\ 0 & 0 & 0 & 6887 & 42168 & -6994 \\ 0 & 0 & 0 & -13559 & -6994 & 20748 \end{pmatrix} \quad (115)$$

verifies the previously introduced isotropic relation⁹. Furthermore, this tensor eigenvalues are:

$$\lambda_1 = 47835 \text{ MPa.mm}^2; \lambda_2 = 25389 \text{ MPa.mm}^2 ; \lambda_3 = 2792 \text{ MPa.mm}^2$$

⁹In such a case the strain gradient elastic tensor is isotropic, but the whole behavior is not isotropic. As an odd order invariance is considered, the fifth-order tensor coupling strain and strain gradient effects has to be taken into account.

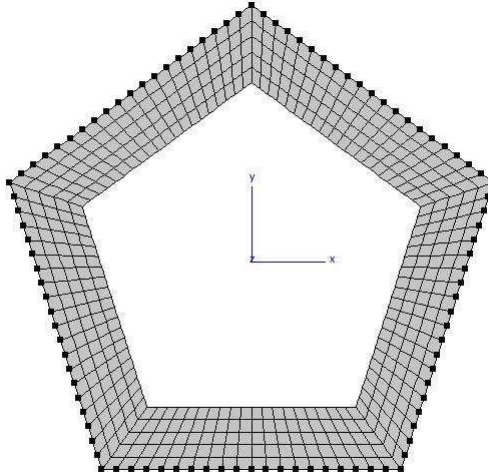


Figure 10: Pentagonal cell

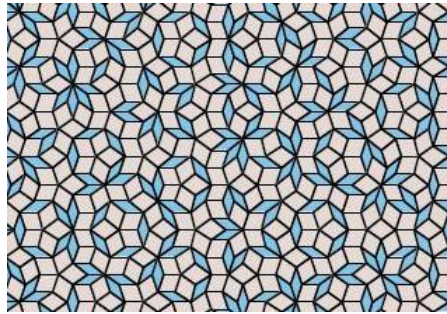


Figure 11: Penrose tiling

The obtained second order isotropic tensor is not degenerated. As before the plane couldn't be densely tilled with a pentagonal elementary cell. Again, if the strict periodicity condition is dropped down, some tilling are D_5 invariant: the Penrose tilling for example (fig.11).

This last example shows that the predicted *class jump* phenomenons could be observed. An interesting consequence is that the use of Hermann theorem allows to define non-singular isotropic second order operators. This point is interesting for numerical applications. As it does not exist any periodic tillings for which strain gradient elasticity is isotropic, this procedure might

seems to be artificial. So, to be correct, such a construction should be understood in some *quasiperiodic* ways. This way to construct an isotropic behavior is thus more physical and better conditioned than the ones found in the literature.

5. Conclusion

In this paper a straightforward and easy-to-use method was proposed to obtain strain gradient overall behavior of bidimensional materials. The combination of extended Voigt notations for strain-gradient elasticity with quadratic homogenization scheme sheds some new lights on the overall operators.

Our main results are three folds :

1. First, it is shown that the circular cavity shape used in several higher order homogenization schemes (Gologanu and al. (1997); Zybell and al. (2008)), leads to a singular sixth order elasticity tensor.
2. Second, it is shown numerically that the application of the quadratic homogenization scheme provides isotropic second order effective properties for octagonal and pentagonal cell shapes, thus illustrating purely mathematical considerations of symmetry.
3. Finally, an example of chiral dependent behavior is given illustrating an exotic property of strain gradient elasticity.

These results confirm that generalized homogenization schemes are powerful tools to estimate higher order elastic properties. Furthermore some useful operators to translate higher order moduli from the equivalent first strain gradient into the second gradient of displacement theories, according to Mindlin's formulation, are provided. Following the scheme presented here, extension of the method to 3-D situation is straightforward.

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