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# A New Interior Point Method for Linear Complementarity Problem

Youssef ELFOUTAYENI<sup>(1,2)</sup> and Mohamed KHALADI<sup>(2)</sup>

<sup>1</sup>Computer Sciences Department

School of Engineering and Innovation of Marrakech

youssef.elfoutayeni@campusmarrakech.com

<sup>2</sup>Department of Mathematical

Faculty Semlalia, University CADI AYYAD

khaladi@ucam.ac.ma

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## Abstract

For a given  $n$ -vector  $q$  and a real square matrix  $M \in \mathbb{R}^{n \times n}$ , the linear complementarity problem, denoted  $LCP(M, q)$ , is that of finding nonnegative vector  $z \in \mathbb{R}^n$  such that  $z^T(Mz + q) = 0$  and  $Mz + q \geq 0$ . In this paper we suppose that the matrix  $M$  must be a symmetric and positive definite and the set

$$S = \{z \in \mathbb{R}^n / z > 0 \text{ and } Mz + q > 0\};$$

named interior points set of the  $LCP(M, q)$  must be nonempty.

The aim of this paper is to show that the  $LCP(M, q)$  is completely equivalent to a convex quadratic programming problem ( $CQPP$ ) under linear constraints. To solve the second problem, we propose an iterative method of interior points which converge in polynomial time to the exact solution; this convergence requires at most  $o(n^{0,5}L)$  iterations, where  $n$  is the number of the variables and  $L$  is the length of a binary coding of the input; furthermore, the algorithm does not exceed  $o(n^{3,5}L)$  arithmetic operations until its convergence and in the end, we close our paper with some numerical examples which illustrate our theoretical results.

Key words and phrases: Linear Complementarity Problem, *Convex Quadratic Programming with Equilibrium Constraints*, *Matrix symmetric and positive definite*, *Interior Point Algorithm*.

## 1. Introduction

We consider the linear complementarity problem  $LCP(M, q)$  that is, given a real square matrix  $M \in \mathbb{R}^{n \times n}$  and  $q$  an element of  $\mathbb{R}^n$ , find vectors  $z \in \mathbb{R}^n$  such that

$$\begin{cases} \langle z, Mz + q \rangle = 0 \\ Mz + q \geq 0 \\ z \geq 0 \end{cases}$$

This problem has important applications in game theory, operational research, and some other areas of engineering (see [3], [4], [5], [6], [7], [9], [10], [13], [14], [15], [16] and [17]). For solving this problem, many results exist, for instance Lemke[11] have developed an algorithm for solving a linear complementarity problem which is based on pivot steps. Mukherjee[12] gave an iterative method for finding a solution to a linear and quasi-linear complementarity problem. Kojima[8] and Achache[1] have showed that the linear complementarity problem is completely equivalent to solving quadratic convex problem (*QCP*); each of them gave a different iterative method to solve the second problem.

Our notation in this paper is the usual one. In particular,  $\mathbb{R}^n$  denotes the space of real  $n$ -dimensional vectors,

$\mathbb{R}_+^n := \{x \in \mathbb{R}^n : x_i \geq 0, i = 1..n\}$  is the nonnegative orthant and its interior is  $\mathbb{R}_{++}^n := \{x \in \mathbb{R}^n : x_i > 0, i = 1..n\}$ .

With  $x \in \mathbb{R}^n$  we define  $|x| = (|x_1|, \dots, |x_n|)^T \in \mathbb{R}^n$ .

We denote by  $I$  the identity matrix.

Let  $x, y \in \mathbb{R}^n$ ,  $x^T y$  or  $\langle x, y \rangle$  is the inner product of the  $x$  and  $y$ ;  $\|x\|$  is the Euclidean norm.

For  $x \in \mathbb{R}^n$  and  $k$  a nonnegative integer,  $x^{(k)}$  refers to the vector obtained after  $k$  iterations; for  $1 \leq i \leq n$ ,  $x_i$  refers to the  $i^{th}$  element of  $x$ , and  $x_i^{(k)}$  refers to the  $i^{th}$  element of the vector obtained after  $k$  iterations.

For  $A \in \mathbb{R}^{n \times n}$  and  $k$  a nonnegative integer,  $A^{(k)}$  refers to the matrix obtained after  $k$  iterations; for  $p$  a nonnegative integer,  $A^{p,(k)}$  refers to the matrix at puissance  $p$  obtained after  $k$  iterations and  $A^{-p,(k)}$  denotes the inverse of the matrix  $A^{p,(k)}$ .

Let  $x, y \in \mathbb{R}^n$ , the expression  $x \leq y$  (respectively  $x < y$ ) meaning that  $x_i \leq y_i$  (respectively  $x_i < y_i$ ) for each  $i = 1..n$ .

Given a vector  $x$  in  $\mathbb{R}^n$ ,  $X = \text{diag}(x)$  is the  $n \times n$  diagonal matrix with  $X_{ii} = x_i$  for all  $i$  and  $X_{ij} = 0$  for all  $i \neq j$ .

The transpose of a vector (respectively matrix) is denoted by super script  $T$ , such as the transpose of the vector  $x$  (respectively the matrix  $A$ ) is given by  $x^T$  (respectively  $A^T$ ).

Remember that the spectrum  $\sigma(A)$  of the matrix  $A$  is the set of its eigenvalues and its spectral radius  $\rho$  is given by  $\rho(A) := \sup\{|\lambda| \text{ such that } \lambda \in \sigma(A)\}$ .

We recall that a matrix  $M$  is called symmetric and positive definite matrix if and only if

$$x^t M x > 0, \forall x \neq 0.$$

and a matrix  $M$  is called symmetric and positive matrix if and only if

$$x^t Mx \geq 0, \forall x.$$

The paper is organized as follows. In the next section, we show that solving linear complementarity problem associated with a matrix  $M$  and a vector  $q$  is completely equivalent to finding the minimum of a convex quadratic programming problem ( $CQPP$ ); for solving the second problem we propose to construct a sequence of vectors  $\{z^{(k)}\}_{k=0,1,\dots}$  which converges to a vector  $z^*$  (the exact solution of linear complementarity problem  $LCP$ ). In the third section, we show that the convergence of this method requires  $o(\sqrt{n}L)$  number by iteration where  $L$  is the length of a binary coding of the input data of the problem and in the end, we close our paper with some numerical examples which illustrate our theoretical results.

## 2. Equivalent reformulation of the problem

In this section, we show that solving a linear complementarity problem ( $LCP$ ) is equivalent to finding the minimum of a convex quadratic programming problem ( $CQPP$ ) under linear constraints.

Consider the linear complementarity problem as follows

Find  $z \in \mathbb{R}^n$  such that:

$$\begin{cases} \langle z, Mz + q \rangle = 0 \\ Mz + q \geq 0 \\ z \geq 0 \end{cases} \quad (1)$$

where  $M \in \mathbb{R}^{n \times n}$  and  $q \in \mathbb{R}^n$  are given data.

and let's consider

$$S = \{z \in \mathbb{R}^n / z > 0 \text{ and } Mz + q > 0\};$$

named interior points set of the  $LCP(M, q)$ .

**Theorem 1** : *If  $M$  is symmetric positive matrix and the interior points set of the linear complementarity problem*

$$S := \{z \in \mathbb{R}^n / z > 0 \text{ and } Mz + q > 0\}$$

*is nonempty then the problem  $LCP(M, q)$  has one and only one solution.*

**Proof.** : For a proof of the above theorem we refer to [1]. ■

Consider the minimization problem under linear constraints ( $CQPP$ ) as follows

$$\begin{cases} \min f(z) := \langle z, Mz + q \rangle \\ \text{Subject to:} \\ Mz + q \geq 0 \\ z \geq 0 \end{cases} \quad (2)$$

We note that if  $z^*$  is a solution of the linear complementarity problem  $LCP(M, q)$  then  $zero$  is the global minimum of the problem(2) (see [18]). Inversely, if  $z^*$  is the minimum of the problem(2) then we have two situations:

- If  $f(z^*) = 0$ , then  $z^*$  is a solution of the  $LCP(M, q)$ .
- Otherwise (ie  $f(z^*) > 0$ ), then the linear complementarity problem  $LCP(M, q)$  admits no solution.

That is to say the problem(1) and the problem(2) are equivalent.

The Lagrange function associated with the problem(2) is given by

$$L(z, \lambda_1, \lambda_2) := f(z) - \langle \lambda_1, Mz + q \rangle - \langle \lambda_2, z \rangle$$

where  $\lambda_1$  and  $\lambda_2$  are the Lagrange multipliers associated with the problem(2).

The KKT conditions applied to the problem(2) imply that if  $z$  is a stationary point, then there exists  $\lambda_1 \in IR^n$  and  $\lambda_2 \in IR^n$  such that

$$\begin{cases} 2Mz + q - M\lambda_1 - \lambda_2 = 0 \\ \lambda_1^T (Mz + q) = 0 \\ \lambda_2^T z = 0 \\ Mz + q \geq 0 \\ z \geq 0 \\ \lambda_1 \geq 0 \\ \lambda_2 \geq 0. \end{cases} \quad (3)$$

Let's consider

$$T := \{(z, \lambda_1, \lambda_2) / 2Mz + q - M\lambda_1 - \lambda_2 = 0 ; z, Mz + q, \lambda_1, \lambda_2 \geq 0\}.$$

In this paper we propose to construct a sequence of vectors

$$\{(z^{(k)}, \lambda_1^{(k)}, \lambda_2^{(k)})\}_{k=0..} \in T$$

such that

$$\lim_{k \rightarrow +\infty} (\langle \lambda_1^{(k)}, Mz^{(k)} + q \rangle + \langle \lambda_2^{(k)}, z^{(k)} \rangle) = 0.$$

To achieve this objective we consider the iterative method which starts from an arbitrary point  $(z^{(0)}, \lambda_1^{(0)}, \lambda_2^{(0)}) \in T$  and generates successively points

$$z^{(k+1)} := z^{(k)} + d^{(k)}$$

where the vectors  $d^{(k)}$  are the directions chosen to generate the vectors  $z^{(k+1)}$ , they are defined by the Newton directions associated with the following penalized problem (see [2])

$$\begin{cases} \min f_\mu(z) \\ \text{Subjects to:} \\ Mz + q > 0 \\ z > 0 \end{cases}$$

where

$$f_\mu(z) := \langle z, Mz \rangle + \langle q, z \rangle - \mu \sum_{i=1}^n \log(z_i) - \mu \sum_{i=1}^n \log(w_i)$$

and

$$w = Mz + q.$$

The Newton direction at  $z^{(k)}$  is the optimal solution of the following quadratic problem

$$\min g_{\mu, z^{(k)}}(d^{(k)}) \quad (4)$$

where

$$g_\mu(z^{(k)}) := \frac{1}{2} \langle d^{(k)}, \nabla^2 f_{\mu_k}(z^{(k)}) d^{(k)} \rangle + \langle \nabla f_{\mu_k}(z^{(k)}), d^{(k)} \rangle.$$

Now we show that

**Theorem 2 :** *The problem(4) admits one and only one solution; this solution is given by*

$$d^{(k)} = -H^{-1, (k)} \nabla f_{\mu_k}(z^{(k)})$$

where

$$\left\{ \begin{array}{l} H^{(k)} := 2M + \mu_k(MW^{-2, (k)}M + Z^{-2, (k)}) \\ Z := \text{diag}(z) \\ W := \text{diag}(w) \\ w = Mz + q. \end{array} \right.$$

**Proof. :** The Hessian of the penalty function  $f_\mu$  at  $z^{(k)}$  noted by  $H^{(k)}$  is given by

$$H^{(k)} := 2M + \mu_k(MW^{-2, (k)}M + Z^{-2, (k)}).$$

Since the matrix  $M$  is symmetric and positive definite, then the matrix  $H^{(k)}$  is symmetric definite positive.

Therefore, the problem(4) admits one and only one solution.

This solution is given by

$$H^{(k)} d^{(k)} + \nabla f_{\mu_k}(z^{(k)}) = 0$$

this implies

$$d^{(k)} = -H^{-1, (k)} \nabla f_{\mu_k}(z^{(k)}).$$

■

After calculating the direction  $d^{(k)}$ , we can write

$$\left\{ \begin{array}{l} z^{(k+1)} := z^{(k)} + d^{(k)} \\ \lambda_1^{(k+1)} := \mu_k W^{-1, (k)} (e - W^{-1, (k)} M d^{(k)}) \\ \lambda_2^{(k+1)} := \mu_k Z^{-1, (k)} (e - Z^{-1, (k)} d^{(k)}) \end{array} \right.$$

Now we give the following algorithm for solving our problem

**Algorithm:**

**(Initialization):**

$$k = 0$$

$\mu_0 > 0$ : Parameter penalty.

$\epsilon > 0$ : Tolerance wanted.

**(Calculation of the vector  $z^{(k+1)}, \lambda_1^{(k+1)}, \lambda_2^{(k+1)}$ )**

$$\begin{cases} d^{(k)} = -H^{-1,(k)} \nabla f_{\mu_k}(z^{(k)}) \\ z^{(k+1)} := z^{(k)} + d^{(k)} \\ w^{(k+1)} := Mz^{(k+1)} + q \\ \lambda_1^{(k+1)} := \mu_k W^{-1,(k)}(e - W^{-1,(k)}Md^{(k)}) \\ \lambda_2^{(k+1)} := \mu_k Z^{-1,(k)}(e - Z^{-1,(k)}d^{(k)}) \\ \mu_{k+1} := 2\mu_k \frac{\delta^2 + \sqrt{n}}{\delta + 2\sqrt{n}} \end{cases}$$

**(Stopping criterion):**

if  $\langle \lambda_1^{(k+1)}, w^{(k+1)} \rangle + \langle \lambda_2^{(k+1)}, z^{(k+1)} \rangle < \epsilon$  then STOP.

otherwise:

$$k := k + 1.$$

GO TO the previous step.

### 3. Convergence

In this section, in the one hand, we show that all points generated by this algorithm are in the set  $T$  and  $(\langle \lambda_1^{(k)}, w^{(k)} \rangle + \langle \lambda_2^{(k)}, z^{(k)} \rangle)$  converges to zero when  $k \rightarrow +\infty$ ; on the other hand, we prove that our algorithm has  $o(\sqrt{n}L)$  iteration complexity; more precisely we will show that the algorithm does not exceed  $O(n^{3.5}L)$  arithmetic operations until its convergence. To so do, we suppose that there exist  $z^{(0)} > 0$  and  $0 < \delta < 1/2$  such that:  $\|Z^{-1,(0)}d^{(0)}\| \leq \delta$  and  $\|W^{-1,(0)}Md^{(0)}\| \leq \delta$  and we prove that if  $k$  is positive integer then we have

$$\|Z^{-1,(k)}d^{(k)}\| \leq \delta$$

and

$$\|W^{-1,(k)}Md^{(k)}\| \leq \delta$$

to achieve this goal, we will need the following four lemmas

**Lemma 3** : Let  $k$  be a positive integer, if

$$\|Z^{-1,(k)}d^{(k)}\| \leq \delta$$

then we have

$$\|Z^{(k+1)}Z^{-1,(k)} - I\| \leq \delta.$$

where  $I$  is the identity matrix.

**Proof.** : For each  $i = 1, \dots, n$  we have

$$\begin{aligned} ((Z^{(k+1)}Z^{-1,(k)} - I)e)_i &= \frac{z_i^{(k+1)}}{z_i^{(k)}} - 1 \\ &= \frac{d_i^{(k)}}{z_i^{(k)}} \\ &= (Z^{-1,(k)}d^{(k)})_i. \end{aligned}$$

■

**Lemma 4** : Let  $k$  be a positive integer, if

$$\|W^{-1,(k)}Md^{(k)}\| \leq \delta$$

then we have

$$\|W^{(k+1)}W^{-1,(k)} - I\| \leq \delta.$$

**Proof.** : For each  $i = 1, \dots, n$  we have

$$\begin{aligned} ((W^{(k+1)}W^{-1,(k)} - I)e)_i &= \frac{w_i^{(k+1)}}{w_i^{(k)}} - 1 & (5) \\ &= \frac{\sum_{j=1}^n m_{ij}d_j^{(k)}}{w_i^{(k)}} \\ &= (W^{-1,(k)}Md^{(k)})_i. \end{aligned}$$

■

**Lemma 5** : Let  $k$  be a positive integer, if

$$\|Z^{-1,(k)}d^{(k)}\| \leq \delta$$

then we have

$$\|Z^{(k+1)}Z^{-2,(k)}d^{(k)} + \frac{\mu_{k+1}}{\mu_k}e - Z^{(k+1)}Z^{-1,(k)}e\| \leq \delta \frac{\delta^2 + \sqrt{n}}{\delta + 2\sqrt{n}}.$$

where

$$\mu_{k+1} := 2\mu_k \frac{\delta^2 + \sqrt{n}}{\delta + 2\sqrt{n}} \quad (6)$$

**Proof.** To show that we use the definition

$$\frac{\mu_{k+1}}{\mu_k}e := e - \frac{\delta - \delta^2}{\delta + 2\sqrt{n}}e; \quad (11)$$

and the relation

$$Z^{(k+1)} = Z^{(k)} + D^{(k)}$$

where  $Z^{(k)}$ ,  $D^{(k)}$  are respectively the diagonal matrix of the vector  $z^{(k)}$  and  $d^{(k)}$ .

By multiplying the last relation by the matrix  $Z^{-1,(k)}$  we get

$$\begin{aligned} Z^{(k+1)} Z^{-1,(k)} &= I + D^{(k)} Z^{-1,(k)} \\ &= I + Z^{-1,(k)} D^{(k)} \end{aligned}$$

this implies

$$\begin{aligned} (Z^{(k+1)} Z^{-1,(k)} - I)e &= Z^{-1,(k)} D^{(k)} e \\ &= Z^{-1,(k)} d^{(k)} \end{aligned} \quad (2)$$

if we note that

$$\Delta_1 := \|Z^{(k+1)} Z^{-2,(k)} d^{(k)} + \frac{\mu_{k+1}}{\mu_k} e - Z^{(k+1)} Z^{-1,(k)} e\|$$

then we have

$$\begin{aligned} \Delta_1 &= \|(Z^{(k+1)} Z^{-1,(k)} - I) Z^{-1,(k)} d^{(k)} - \frac{\delta - 2\delta^2}{\delta + 2\sqrt{n}} e\| \\ &\leq \|Z^{(k+1)} Z^{-1,(k)} - I\| \|Z^{-1,(k)} d^{(k)}\| + \sqrt{n} \frac{\delta - 2\delta^2}{\delta + 2\sqrt{n}} \\ &\leq \delta \cdot \delta + \sqrt{n} \frac{\delta - 2\delta^2}{\delta + 2\sqrt{n}} \\ &= \delta \frac{\delta^2 + \sqrt{n}}{\delta + 2\sqrt{n}}. \end{aligned}$$

■

**Lemma 6** : Let  $k$  be a positive integer, if

$$\|W^{-1,(k)} M d^{(k)}\| \leq \delta$$

then we have

$$\|W^{(k+1)} W^{-2,(k)} M d^{(k)} + \frac{\mu_{k+1}}{\mu_k} e - W^{(k+1)} W^{-1,(k)} e\| \leq \delta \frac{\delta^2 + \sqrt{n}}{\delta + 2\sqrt{n}}.$$

**Proof.** : In the one hand, if we use (5) we can write

$$(I - W^{(k+1)} W^{-1,(k)})e = -W^{-1,(k)} M d^{(k)}$$

and in the other hand, if we note that

$$\Delta_2 := \|W^{(k+1)} W^{-2,(k)} M d^{(k)} + \frac{\mu_{k+1}}{\mu_k} e - W^{(k+1)} W^{-1,(k)} e\|$$

then we get

$$\begin{aligned}
 \Delta_2 &= \|W^{(k+1)}W^{-2,(k)}Md^{(k)} + e - W^{(k+1)}W^{-1,(k)}e - \frac{\delta - 2\delta^2}{\delta + 2\sqrt{n}}e\| \\
 &= \|W^{(k+1)}W^{-2,(k)}Md^{(k)} + (I - W^{(k+1)}W^{-1,(k)})e - \frac{\delta - 2\delta^2}{\delta + 2\sqrt{n}}e\| \\
 &\leq \|W^{(k+1)}W^{-2,(k)}Md^{(k)} - W^{-1,(k)}Md^{(k)} - \frac{\delta - 2\delta^2}{\delta + 2\sqrt{n}}e\| \\
 &\leq \|(W^{(k+1)}W^{-1,(k)} - I)W^{-1,(k)}Md^{(k)} - \frac{\delta - 2\delta^2}{\delta + 2\sqrt{n}}e\| \\
 &\leq \|W^{(k+1)}W^{-1,(k)} - I\| \|W^{-1,(k)}Md^{(k)}\| + \sqrt{n} \frac{\delta - 2\delta^2}{\delta + 2\sqrt{n}}e\| \\
 &\leq \delta \cdot \delta + \sqrt{n} \frac{\delta - 2\delta^2}{\delta + 2\sqrt{n}} \\
 &= \delta \frac{\delta^2 + \sqrt{n}}{\delta + 2\sqrt{n}}.
 \end{aligned}$$

■

Now we show that

**Theorem 7** : *If we suppose that*

$$\begin{cases} \|Z^{-1,(k)}d^{(k)}\| \leq \delta \\ \|W^{-1,(k)}Md^{(k)}\| \leq \delta \end{cases}$$

then we have

$$\begin{cases} \|Z^{-1,(k+1)}d^{(k+1)}\| \leq \delta \\ \|W^{-1,(k+1)}Md^{(k+1)}\| \leq \delta \end{cases}$$

**Proof.** : Since

$$H^{(k)}d^{(k)} + \nabla f_{\mu_k}(z^{(k)}) = 0$$

we have

$$\begin{cases} H^{(k)}d^{(k)} + \nabla f_{\mu_k}(z^{(k)}) = 0 \\ H^{(k+1)}d^{(k+1)} + \nabla f_{\mu_{k+1}}(z^{(k+1)}) = 0 \end{cases}$$

By multiplying the two equations by  $d^{(k+1)}$  we get

$$\begin{cases} \langle d^{(k+1)}, (2M + \mu_k(MW^{-2,(k)}M + Z^{-2,(k)}))d^{(k)} \rangle + \\ \quad \langle d^{(k+1)}, 2Mz^{(k)} + q - \mu_k(MW^{-1,(k)} + Z^{-1,(k)})e \rangle = 0 \\ \langle d^{(k+1)}, (2M + \mu_{k+1}(MW^{-2,(k+1)}M + Z^{-2,(k+1)}))d^{(k+1)} \rangle + \\ \quad \langle d^{(k+1)}, 2Mz^{(k+1)} + q - \mu_{k+1}(MW^{-1,(k+1)} + Z^{-1,(k+1)})e \rangle = 0 \end{cases}$$

if we note

$$\Delta_3 := \mu_{k+1}(\|W^{-1,(k+1)}Md^{(k+1)}\|^2 + \|Z^{-1,(k+1)}d^{(k+1)}\|^2)$$

then we have

$$\begin{aligned}
\Delta_3 &\leq \mu_{k+1} \langle d^{(k+1)}, (MW^{-1,(k+1)} + Z^{-1,(k+1)})e \rangle - \langle d^{(k+1)}, 2Mz^{(k+1)} + q \rangle \\
&= \mu_{k+1} \langle W^{-1,(k+1)}Md^{(k+1)}, e \rangle + \mu_{k+1} \langle Z^{-1,(k+1)}d^{(k+1)}, e \rangle \\
&\quad - (\mu_k \langle d^{(k+1)}, (MW^{-1,(k)} + Z^{-1,(k)})e \rangle \\
&\quad - \mu_k \langle d^{(k+1)}, (MW^{-2,(k)}M + Z^{-2,(k)})d^{(k)} \rangle) \\
&= \mu_k \langle d^{(k+1)}, \frac{\mu_{k+1}}{\mu_k} MW^{-1,(k+1)}e - MW^{-1,(k)}e + MW^{-2,(k)}Md^{(k)} \rangle \\
&\quad + \mu_k \langle d^{(k+1)}, \frac{\mu_{k+1}}{\mu_k} Z^{-1,(k+1)}e - Z^{-1,(k)}e + Z^{-2,(k)}d^{(k)} \rangle \\
&= \mu_k \langle d^{(k+1)}, MW^{-1,(k+1)}(W^{(k+1)}W^{-2,(k)}Md^{(k)} + \frac{\mu_{k+1}}{\mu_k}e - W^{(k+1)}W^{-1,(k)}e) \rangle \\
&\quad + \mu_k \langle d^{(k+1)}, Z^{-1,(k+1)}(Z^{(k+1)}Z^{-2,(k)}d^{(k)} + \frac{\mu_{k+1}}{\mu_k}e - Z^{(k+1)}Z^{-1,(k)}e) \rangle \\
&\leq \mu_k \delta \frac{\delta^2 + \sqrt{n}}{\delta^2 + 2\sqrt{n}} (\|W^{-1,(k+1)}Md^{(k+1)}\| + \|Z^{-1,(k+1)}d^{(k+1)}\|)
\end{aligned}$$

so

$$\mu_{k+1} (\|W^{-1,(k+1)}Md^{(k+1)}\| + \|Z^{-1,(k+1)}d^{(k+1)}\|) \leq 2\mu_k \delta \frac{\delta^2 + \sqrt{n}}{\delta^2 + 2\sqrt{n}}$$

finally

$$\|W^{-1,(k+1)}Md^{(k+1)}\| + \|Z^{-1,(k+1)}d^{(k+1)}\| \leq \delta$$

hence the result. ■

Now we show that all points generated by the algorithm  $(z^{(k)}, \lambda_1^{(k)}, \lambda_2^{(k)})$  remain in the set  $T$ .

For that, let's assume that:

$$(z^{(k)}, \lambda_1^{(k)}, \lambda_2^{(k)}) \in T$$

then we have

•

$$\begin{aligned}
z^{(k+1)} &= z^{(k)} + d^{(k)} \\
&= Z^{(k)}(e + Z^{-1,(k)}d^{(k)}) \\
&> 0.
\end{aligned}$$

•

$$\begin{aligned}
w^{(k+1)} &= Mz^{(k+1)} + q \\
&= w^{(k)} + Md^{(k)} \\
&= W^{(k)}(e + W^{-1,(k)}Md^{(k)}) \\
&> 0.
\end{aligned}$$

•

$$\begin{aligned}\lambda_1^{(k+1)} &:= \mu_k W^{-1,(k)}(e - W^{-1,(k)} M d^{(k)}) \\ &> 0.\end{aligned}$$

•

$$\begin{aligned}\lambda_2^{(k+1)} &:= \mu_k Z^{-1,(k)}(e - Z^{-1,(k)} d^{(k)}) \\ &> 0.\end{aligned}$$

•

$$\begin{aligned}\lambda_2^{(k+1)} &= \mu_k Z^{-1,(k)}(e - Z^{-1,(k)} d^{(k)}) \\ &= \mu_k Z^{-1,(k)} e - \mu_k Z^{-2,(k)} d^{(k)} \\ &= \mu_k Z^{-1,(k)} e - (H^{(k)} - 2M - \mu_k M W^{-2,(k)} M) d^{(k)} \\ &= \mu_k Z^{-1,(k)} e - H^{(k)} d^{(k)} + (2M z^{(k+1)} + q) - (2M z^{(k)} + q) \\ &\quad + \mu_k M W^{-2,(k)} M d^{(k)} \\ &= \mu_k Z^{-1,(k)} e + (2M z^{(k)} + q - \mu_k M W^{-1,(k)} e - \mu_k Z^{-1,(k)} e) \\ &\quad + (2M z^{(k+1)} + q) - (2M z^{(k)} + q) + \mu_k M W^{-2,(k)} M d^{(k)} \\ &= -\mu_k M W^{-1,(k)} e + (2M z^{(k+1)} + q) + \mu_k M W^{-2,(k)} M d^{(k)} \\ &= (2M z^{(k+1)} + q) - \mu_k M W^{-1,(k)}(e - W^{-1,(k)} M d^{(k)}) \\ &= (2M z^{(k+1)} + q) - M \lambda_1^{(k+1)}.\end{aligned}$$

Now we give the following proposition

**Proposition 8** : For any positive integer  $k$ , we have

$$\langle w^{(k)}, \lambda_1^{(k)} \rangle + \langle z^{(k)}, \lambda_2^{(k)} \rangle \leq 2\mu_{k-1}(\delta + \sqrt{n})^2.$$

**Proof.** : From the above we can write

$$\begin{aligned}&\langle w^{(k+1)}, \lambda_1^{(k+1)} \rangle + \langle z^{(k+1)}, \lambda_2^{(k+1)} \rangle \\ &= \langle W^{(k)}(e + W^{-1,(k)} M d^{(k)}), \mu_k W^{-1,(k)}(e - W^{-1,(k)} M d^{(k)}) \rangle + \\ &\quad \langle Z^{(k)}(e + Z^{-1,(k)} d^{(k)}), \mu_k Z^{-1,(k)}(e - Z^{-1,(k)} d^{(k)}) \rangle \\ &= \mu_k (\langle e + W^{-1,(k)} M d^{(k)}, e - W^{-1,(k)} M d^{(k)} \rangle + \\ &\quad \langle e + Z^{-1,(k)} d^{(k)}, e - Z^{-1,(k)} d^{(k)} \rangle) \\ &\leq 2\mu_k(\delta + \sqrt{n})^2.\end{aligned}$$

■

Hence the quantity  $(\langle w^{(k)}, \lambda_1^{(k)} \rangle + \langle z^{(k)}, \lambda_2^{(k)} \rangle)$  converges to zero. Now we prove that our algorithm has  $o(\sqrt{n}L)$  iteration complexity; more

precisely we will show that the algorithm does not exceed  $o(n^{3.5}L)$  arithmetic operations until its convergence, and in the end we will establish the total number of iterations performed by the algorithm so that its convergence is reached.

**Proposition 9** : *The total number of iterations performed by the algorithm is less than or equal to*

$$k^* = \frac{\delta + 2\sqrt{n}}{\delta - \delta^2 + \sqrt{n}} \ln \frac{2(\delta + \sqrt{n})^2 \mu_0}{\epsilon}$$

where  $\epsilon$  denotes the tolerance of the problem and  $\mu_0$  is the parameter of the initial penalty.

**Proof.** : The algorithm ends when

$$2(\delta + \sqrt{n})^2 \mu_k \leq \epsilon$$

thus, it is enough to show that  $k^*$  satisfies this inequality.

From the definition of  $k^*$  we can write

$$\begin{aligned} \ln(\epsilon) &= -\frac{\delta - \delta^2 + \sqrt{n}}{\delta + 2\sqrt{n}} k^* + \ln[2(\delta + \sqrt{n})^2 \mu_0] \\ &\geq k^* \ln\left(\frac{\delta^2 + \sqrt{n}}{\delta + 2\sqrt{n}}\right) + \ln[2(\delta + \sqrt{n})^2 \mu_0] \\ &= \ln\left(\frac{\delta^2 + \sqrt{n}}{\delta + 2\sqrt{n}}\right)^{k^*} + \ln[2(\delta + \sqrt{n})^2 \mu_0] \\ &= \ln[2(\delta + \sqrt{n})^2 \mu_0 \left(\frac{\delta^2 + \sqrt{n}}{\delta + 2\sqrt{n}}\right)^{k^*}] \\ &= \ln[2(\delta + \sqrt{n})^2 \mu_{k^*}] \end{aligned}$$

this implies

$$2(\delta + \sqrt{n})^2 \mu_{k^*} \leq \epsilon$$

■

If we denote by  $L$  the length of a binary coding of the input data of the problem  $LCP(M, q)$ , then we have

**Corollary 10** : *If  $\text{Log}_2(\mu_0) = o(L)$  and  $\text{Log}_2(\epsilon) = -o(L)$  then the algorithm stops in at most  $o(\sqrt{n}L)$  iterations with  $o(n^{3.5}L)$  arithmetic operations.*

**Proof.** : From this assumption we have

$$\begin{cases} \text{Log}_2(\mu_0) = o(L) \\ \text{and} \\ \text{Log}_2(\epsilon) = -o(L) \end{cases}$$

then

$$k^* = o(\sqrt{n}L)$$

But to solve a linear system we need  $o(n^3)$  arithmetic operations, then the algorithm converges in at most  $o(n^{3.5}L)$  arithmetic operations. ■

### 1. Numerical example

In this part, we consider the following example to test our method. Consider the following linear complementarity problem

**Example 11** Find vector  $z$  in  $\mathbb{R}^n$  satisfying  $z^T(Mz+q) = 0$ ,  $Mz+q \geq 0$ ,  $z \geq 0$ ,

$$\text{where } M = \begin{bmatrix} 100 & -2 & -3 & -4 \\ -2 & 50 & -6 & -7 \\ -3 & -6 & 100 & -11 \\ -4 & -7 & -11 & 200 \end{bmatrix} \text{ and } q = \begin{bmatrix} 1 \\ -2 \\ 3 \\ -4 \end{bmatrix}.$$

The exact solution is  $z^* = (0, \frac{4}{93}, 0, \frac{2}{93})^T$ .

The solution of this problem with six significant digits is presented in following Table.

Iteration	$x_1$	$x_2$	$x_3$	$x_4$
$k=01$	1,217700	4,030283	10,748913	6,834140
$k=05$	1,094700	3,627983	9,659913	6,147865
$k=10$	0,971700	3,225683	8,570913	5,461590
$k=15$	0,848700	2,823383	7,481913	4,775315
$k=20$	0,725700	2,421083	6,392913	4,089040
$k=25$	0,602700	2,018783	5,303913	3,402765
$k=30$	0,479700	1,616483	4,214913	2,716490
$k=35$	0,356700	1,214183	3,125913	2,030215
$k=40$	0,233700	0,811883	2,036913	1,343940
$k=45$	0,110700	0,409583	0,947913	0,657665
$k=50$	0,073800	0,288893	0,621213	0,451783
$k=52$	0,036900	0,168203	0,294513	0,245900
$k=53$	0,012300	0,096750	0,012340	0,145670
$k=54$	0,000000	0,043010	0,000000	0,021505

### Conclusion:

In this paper we have showed that, on the one hand, linear complementarity problem can be written as a quadratic convex programming problem; on the other, we have built a method to solve it; the convergence of this method requires  $o(\sqrt{n}L)$  number of iterations where  $L$  is the length of a binary coding of the input data of the problem  $LCP(M, q)$ .

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