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# The lower central and derived series of the braid groups of the projective plane

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## Abstract

*In this paper, we determine the lower central and derived series for the braid groups of the projective plane. We are motivated in part by the study of Fadell-Neuwirth short exact sequences, but the problem is interesting in its own right.*

*The  $n$ -string braid groups  $B_n(\mathbb{R}P^2)$  of the projective plane  $\mathbb{R}P^2$  were originally studied by Van Buskirk during the 1960's, and are of particular interest due to the fact that they have torsion. The group  $B_1(\mathbb{R}P^2)$  (resp.  $B_2(\mathbb{R}P^2)$ ) is isomorphic to the cyclic group  $\mathbb{Z}_2$  of order 2 (resp. the generalised quaternion group of order 16) and hence their lower central and derived series are known. If  $n > 2$ , we first prove that the lower central series of  $B_n(\mathbb{R}P^2)$  is constant from the commutator subgroup onwards. We observe that  $\Gamma_2(B_3(\mathbb{R}P^2))$  is isomorphic to  $(\mathbb{F}_3 \rtimes \mathcal{Q}_8) \rtimes \mathbb{Z}_3$ , where  $\mathbb{F}_k$  denotes the free group of rank  $k$ , and  $\mathcal{Q}_8$  denotes the quaternion group of order 8, and that  $\Gamma_2(B_4(\mathbb{R}P^2))$  is an extension of an index 2 subgroup  $K$  of  $P_4(\mathbb{R}P^2)$  by  $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ . As for the derived series of  $B_n(\mathbb{R}P^2)$ , we show that for all  $n \geq 5$ , it is constant from the derived subgroup onwards. The group  $B_n(\mathbb{R}P^2)$  being finite and soluble for  $n \leq 2$ , the critical cases are  $n = 3, 4$ . We are able to determine completely the derived series of  $B_3(\mathbb{R}P^2)$ . The subgroups  $(B_3(\mathbb{R}P^2))^{(1)}$ ,  $(B_3(\mathbb{R}P^2))^{(2)}$  and  $(B_3(\mathbb{R}P^2))^{(3)}$  are isomorphic respectively to  $(\mathbb{F}_3 \rtimes \mathcal{Q}_8) \rtimes \mathbb{Z}_3$ ,  $\mathbb{F}_3 \rtimes \mathcal{Q}_8$  and  $\mathbb{F}_9 \times \mathbb{Z}_2$ , and we compute the derived series quotients of these groups. From  $(B_3(\mathbb{R}P^2))^{(4)}$  onwards, the derived series of  $B_3(\mathbb{R}P^2)$ , as well as its successive derived series quotients, coincide with those of  $\mathbb{F}_9$ . We analyse the derived series of  $B_4(\mathbb{R}P^2)$  and its quotients up to  $(B_4(\mathbb{R}P^2))^{(4)}$ , and we show that  $(B_4(\mathbb{R}P^2))^{(4)}$  is a semi-direct of  $\mathbb{F}_{129}$  by  $\mathbb{F}_{17}$ . Finally, we give a presentation of  $\Gamma_2(B_n(\mathbb{R}P^2))$ .*

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# 1 Introduction

## 1.1 Generalities and definitions

Let  $n \in \mathbb{N}$ . The braid groups of the plane  $\mathbb{E}^2$ , denoted by  $B_n$ , and known as *Artin braid groups*, were introduced by E. Artin in 1925 [A1, A2, A3], and admit the following well-known presentation:  $B_n$  is generated by elements  $\sigma_1, \dots, \sigma_{n-1}$ , subject to the classical *Artin relations*:

$$\begin{cases} \sigma_i \sigma_j = \sigma_j \sigma_i & \text{if } |i - j| \geq 2 \text{ and } 1 \leq i, j \leq n - 1 \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & \text{for all } 1 \leq i \leq n - 2. \end{cases}$$

A natural generalisation to braid groups of arbitrary topological spaces was made at the beginning of the 1960's by Fox (using the notion of configuration space) [FoN]. The braid groups of compact, connected surfaces have been widely studied; (finite) presentations were obtained in [Z1, Z2, Bi1, Sc]. As well as being interesting in their own right, braid groups have played an important rôle in many branches of mathematics, for example in topology, geometry, algebra and dynamical systems, and notably in the study of knots and links [BZ], of the mapping class groups [Bi2, Bi3], and of configuration spaces [CG, FH]. The reader may consult [Bi2, Han, R] for some general references on the theory of braid groups.

Let  $M$  be a connected manifold of dimension 2 (or *surface*), perhaps with boundary. Further, we shall suppose that  $M$  is homeomorphic to a compact 2-manifold with a finite (possibly zero) number of points removed from its interior. We recall two (equivalent) definitions of surface braid groups. The first is that due to Fox. Let  $F_n(M)$  denote the  $n^{\text{th}}$  *configuration space* of  $M$ , namely the set of all ordered  $n$ -tuples of distinct points of  $M$ :

$$F_n(M) = \{(x_1, \dots, x_n) \mid x_i \in M \text{ and } x_i \neq x_j \text{ if } i \neq j\}.$$

Since  $F_n(M)$  is a subspace of the  $n$ -fold Cartesian product of  $M$  with itself, the topology on  $M$  induces a topology on  $F_n(M)$ . Then we define the  $n$ -*string pure braid group*  $P_n(M)$  of  $M$  to be  $P_n(M) = \pi_1(F_n(M))$ . There is a natural action of the symmetric group  $S_n$  on  $F_n(M)$  by permutation of coordinates, and the resulting orbit space  $F_n(M)/S_n$  shall be denoted by  $D_n(M)$ . The fundamental group  $\pi_1(D_n(M))$  is called the  $n$ -*string (full) braid group* of  $M$ , and shall be denoted by  $B_n(M)$ . Notice that the projection  $F_n(M) \rightarrow D_n(M)$  is a regular  $n!$ -fold covering map. It is well known that  $B_n$  is isomorphic to  $B_n(\mathbb{D}^2)$ , and that the subgroup  $P_n$  of pure braids of  $B_n$  is isomorphic to  $P_n(\mathbb{D}^2)$ , where  $\mathbb{D}^2$  is the closed 2-disc.

The second definition of surface braid groups is geometric. Let  $\mathcal{P} = \{p_1, \dots, p_n\}$  be a set of  $n$  distinct points of  $M$ . A *geometric braid* of  $M$  with basepoint  $\mathcal{P}$  is a collection  $\beta = (\beta_1, \dots, \beta_n)$  of  $n$  paths  $\beta: [0, 1] \rightarrow M$  such that:

- (a) for all  $i = 1, \dots, n$ ,  $\beta_i(0) = p_i$  and  $\beta_i(1) \in \mathcal{P}$ .
- (b) for all  $i, j = 1, \dots, n$  and  $i \neq j$ , and for all  $t \in [0, 1]$ ,  $\beta_i(t) \neq \beta_j(t)$ .

Two geometric braids are said to be *equivalent* if there exists a homotopy between them through geometric braids. The usual concatenation of paths induces a group operation on the set of equivalence classes of geometric braids. This group is isomorphic to  $B_n(M)$ , and does not depend on the choice of  $\mathcal{P}$ . The subgroup of *pure* braids, satisfying additionally  $\beta_i(1) = p_i$  for all  $i = 1, \dots, n$ , is isomorphic to  $P_n(M)$ . There is a natural surjective homomorphism  $\tau: B_n(M) \rightarrow S_n$  which to a geometric braid  $\beta$  associates the permutation  $\tau(\beta)$  defined by  $\beta_i(1) = p_{\tau(\beta)(i)}$ . The kernel is precisely  $P_n(M)$ ,

and we thus obtain the following short exact sequence:

$$1 \longrightarrow P_n(M) \longrightarrow B_n(M) \xrightarrow{\tau} S_n \longrightarrow 1. \quad (1)$$

In this paper, we shall be primarily interested in the braid groups of the real projective plane  $\mathbb{R}P^2$ . Along with the braid groups of the 2-sphere, they are of particular interest, notably because they have non-trivial centre (which is also the case for the Artin braid groups), and torsion elements (which were characterised by Murasugi [Mu], see also [GG9]). We recall briefly some of their properties. If  $\mathbb{D}^2 \subseteq \mathbb{R}P^2$  is a topological disc, there is a group homomorphism  $\iota: B_n(\mathbb{D}^2) \longrightarrow B_n(\mathbb{R}P^2)$  induced by the inclusion. If  $\beta \in B_n(\mathbb{D}^2)$  then its image  $\iota(\beta)$  shall be denoted simply by  $\beta$ . A presentation of  $B_n(\mathbb{R}P^2)$  was given in [VB] (see Proposition 4); in [GG4], we obtained a presentation of  $P_n(\mathbb{R}P^2)$ . The first two braid groups of  $\mathbb{R}P^2$  are finite:  $B_1(\mathbb{R}P^2)$  and  $B_2(\mathbb{R}P^2)$  are isomorphic to  $\mathbb{Z}_2$  and  $Q_{16}$  respectively, where for  $m \geq 2$ ,  $Q_{4m}$  denotes the generalised quaternion group of order  $4m$  [VB]. If  $n \geq 3$  then  $B_n(\mathbb{R}P^2)$  is infinite. For  $n = 3$ , the Fadell-Neuwirth short exact sequence of pure braid groups yields the fact that  $P_3(\mathbb{R}P^2)$  is isomorphic to a semi-direct product of a free group of rank two by  $Q_8$ . If  $n \geq 2$ , the so-called ‘full twist’ braid  $\Delta_n^2 = (\sigma_1 \cdots \sigma_{n-1})^n$  generates the centre  $Z(B_n(\mathbb{R}P^2))$  of  $B_n(\mathbb{R}P^2)$ , and is the unique element of  $B_n(\mathbb{R}P^2)$  of order 2. Here  $\Delta_n$  denotes the Garside (or ‘half twist’) element of  $B_n(\mathbb{R}P^2)$ , defined by

$$\Delta_n = (\sigma_1 \cdots \sigma_{n-1})(\sigma_1 \cdots \sigma_{n-2}) \cdots (\sigma_1 \sigma_2) \sigma_1.$$

Further, the torsion of  $B_n(\mathbb{R}P^2)$  is  $4n$  and  $4(n-1)$ , and that of  $P_n(\mathbb{R}P^2)$  is 2 and 4 [GG3]. In [GG8], we classified the virtually cyclic subgroups of  $B_n(\mathbb{R}P^2)$  for all  $n \in \mathbb{N}$ , and in [GG9], we characterised the finite subgroups of  $B_n(\mathbb{R}P^2)$ .

Our aim in this paper is to study the lower central and derived series of the braid groups of  $\mathbb{R}P^2$ . We recall some definitions and notation concerning these series. If  $G$  is a group, then its *lower central series*  $\{\Gamma_i(G)\}_{i \in \mathbb{N}}$  is defined inductively by  $\Gamma_1(G) = G$ , and  $\Gamma_{i+1}(G) = [G, \Gamma_i(G)]$  for all  $i \in \mathbb{N}$ , and its *derived series*  $\{G^{(i)}\}_{i \in \mathbb{N} \cup \{0\}}$  is defined inductively by  $G^{(0)} = G$ , and  $G^{(i)} = [G^{(i-1)}, G^{(i-1)}]$  for all  $i \in \mathbb{N}$ . One may check easily that  $\Gamma_i(G) \supseteq \Gamma_{i+1}(G)$  and  $G^{(i-1)} \supseteq G^{(i)}$  for all  $i \in \mathbb{N}$ , and for all  $j \in \mathbb{N}$ ,  $j > i$ ,  $\Gamma_j(G)$  (resp.  $G^{(j)}$ ) is a normal subgroup of  $\Gamma_i(G)$  (resp.  $G^{(i)}$ ). Notice that  $\Gamma_2(G) = G^{(1)}$  is the *commutator subgroup* of  $G$ . The *Abelianisation* of the group  $G$ , denoted by  $G^{\text{Ab}}$  is the quotient  $G/\Gamma_2(G)$ ; the *Abelianisation* of an element  $g \in G$  is its  $\Gamma_2(G)$ -coset in  $G^{\text{Ab}}$ . The group  $G$  is said to be *perfect* if  $G = G^{(1)}$ , or equivalently if  $G^{\text{Ab}} = \{1\}$ . Following P. Hall, for any group-theoretic property  $\mathcal{P}$ , a group  $G$  is said to be *residually*  $\mathcal{P}$  if for any (non-trivial) element  $x \in G$ , there exists a group  $H$  with the property  $\mathcal{P}$  and a surjective homomorphism  $\varphi: G \longrightarrow H$  such that  $\varphi(x) \neq 1$ . It is well known that a group  $G$  is *residually nilpotent* (respectively *residually soluble*) if and only if  $\bigcap_{i \geq 1} \Gamma_i(G) = \{1\}$  (respectively  $\bigcap_{i \geq 0} G^{(i)} = \{1\}$ ). If  $g, h \in G$  then  $[g, h] = ghg^{-1}h^{-1}$  will denote their commutator.

The lower central series of groups and their successive quotients  $\Gamma_i/\Gamma_{i+1}$  are isomorphism invariants, and have been widely studied using commutator calculus, in particular for free groups of finite rank [Hal, MKS]. Falk and Randell, and independently Kohno investigated the lower central series of the pure braid group  $P_n$ , and were able to conclude that  $P_n$  is residually nilpotent [FR1, Ko]. Falk and Randell also studied the lower central series of generalised pure braid groups [FR2, FR3]. Using the

Reidemeister-Schreier rewriting process, Gorin and Lin obtained a presentation of the commutator subgroup of  $B_n$  for  $n \geq 3$  [GL]. For  $n \geq 5$ , they were able to infer that  $(B_n)^{(1)} = (B_n)^{(2)}$ , and so  $(B_n)^{(1)}$  is perfect. From this it follows that  $\Gamma_2(B_n) = \Gamma_3(B_n)$ , hence  $B_n$  is not residually nilpotent. If  $n = 3$  then they showed that  $(B_3)^{(1)}$  is a free group of rank 2, while if  $n = 4$ , they proved that  $(B_4)^{(1)}$  is a semi-direct product of two free groups of rank 2. By considering the action, one may see that  $(B_4)^{(1)} \not\cong (B_4)^{(2)}$ . The work of Gorin and Lin on these series was motivated by the study of almost periodic solutions of algebraic equations with almost periodic coefficients. In [GG5, GG6] we studied the lower central and derived series of the 2-sphere  $\mathbb{S}^2$  and the finitely-punctured 2-sphere. For  $\mathbb{S}^2$ , the case  $n = 4$  is critical, in the sense that if  $n \neq 4$ ,  $B_n(\mathbb{S}^2)$  is residually soluble if and only if  $n < 4$ . It is an open question as to whether  $B_4(\mathbb{S}^2)$  is residually soluble.

The above comments indicate that the study of the lower central and derived series of the braid groups of  $\mathbb{R}P^2$  is an important problem in its own right, and it helps us to understand better the structure of such groups. But we are also motivated by the interesting question of the existence of a section (the ‘splitting problem’) for the following two short exact sequences of braid groups (notably for the case  $M = \mathbb{R}P^2$ ) obtained by considering the long exact sequences in homotopy of fibrations of the corresponding configuration spaces:

(a) let  $m, n \in \mathbb{N}$  and  $m > n$ . Then we have the *Fadell-Neuwirth short exact sequence of pure braid groups* [FaN]:

$$1 \longrightarrow P_n(M \setminus \{x_1, \dots, x_m\}) \xrightarrow{i_*} P_{m+n}(\mathbb{R}P^2) \xrightarrow{p_*} P_m(\mathbb{R}P^2) \longrightarrow 1, \quad (2)$$

where  $m \geq 3$  if  $M = \mathbb{S}^2$  [Fa, FVB],  $m \geq 2$  if  $M = \mathbb{R}P^2$  [VB], and  $m \geq 1$  otherwise [FaN], and where  $p_*$  is the group homomorphism which geometrically corresponds to forgetting the last  $n$  strings, and  $i_*$  is inclusion (we consider  $P_n(M \setminus \{x_1, \dots, x_m\})$  to be the subgroup of  $P_{m+n}(\mathbb{R}P^2)$  of pure braids whose last  $m$  strings are vertical). This short exact sequence plays a central rôle in the study of surface braid groups. It was used by [PR] to study mapping class groups, in the work of [GMP] on Vassiliev invariants for braid groups, as well as to obtain presentations for surface pure braid groups [Bi1, Sc, GG1, GG4, GG6].

(b) let  $m, n \in \mathbb{N}$ . Consider the group homomorphism  $\tau: B_{m+n}(M) \longrightarrow S_{m+n}$ , and let  $B_{m,n}(M) = \tau^{-1}(S_m \times S_n)$  be the inverse image of the subgroup  $S_m \times S_n$  of  $S_{m+n}$ . As in the pure braid group case, we obtain a generalisation of the Fadell-Neuwirth short exact sequence [GG2]:

$$1 \longrightarrow B_n(M \setminus \{x_1, \dots, x_m\}) \longrightarrow B_{m,n}(M) \xrightarrow{p_*} B_m(M) \longrightarrow 1, \quad (3)$$

where we take  $m \geq 3$  if  $M = \mathbb{S}^2$ ,  $m \geq 2$  if  $M = \mathbb{R}P^2$  and  $m \geq 1$  otherwise. Once more,  $p_*$  corresponds geometrically to forgetting the last  $n$  strings.

We remark that if the above conditions on  $n$  and  $m$  are satisfied then the existence of a section for  $p_*$  is equivalent to that of a geometric section for the corresponding configuration spaces (cf. [GG3, GG4]). The authors have recently solved the splitting problem for the short exact sequence (2) for all surfaces [GG7]. In [GG4], we studied the short exact sequence (3) in the case  $M = \mathbb{S}^2$  of the sphere, and showed that if  $m = 3$  then (3) splits if and only if  $n \equiv 0, 2 \pmod{3}$ . Further, if  $m \geq 4$  and (3) splits then there exist  $\varepsilon_1, \varepsilon_2 \in \{0, 1\}$  such that  $n \equiv \varepsilon_1(m-1)(m-2) - \varepsilon_2 m(m-2) \pmod{m(m-1)(m-2)}$ . An open question is whether this condition is also sufficient.

Our main aim in this paper is to study the lower central and derived series of the braid groups of  $\mathbb{R}P^2$ . This was motivated in part by the study of the problem of the existence of a section for the short exact sequences (2) and (3). To obtain a positive answer, it suffices of course to exhibit an explicit section (although this may be easier said than done!). However, and in spite of the fact that we possess presentations of surface braid groups, in general it is very difficult to prove directly that such an extension does not split. One of the main methods that we used to prove the non-splitting of (2) for  $n \geq 2$  and of (3) for  $m \geq 4$  was based on the following observation: let  $1 \rightarrow K \rightarrow G \rightarrow Q \rightarrow 1$  be a split extension of groups, where  $K$  is a normal subgroup of  $G$ , and let  $H$  be a normal subgroup of  $G$  contained in  $K$ . Then the extension  $1 \rightarrow K/H \rightarrow G/H \rightarrow Q \rightarrow 1$  also splits. The condition on  $H$  is satisfied for example if  $H$  is an element of either the lower central series or the derived series of  $K$ . In [GG1], considering the extension (2) with  $n \geq 3$ , we showed that it was sufficient to take  $H = \Gamma_2(K)$  to prove the non-splitting of the quotiented extension, and hence that of the full extension. In this case, the kernel  $K/\Gamma_2(K)$  is Abelian, which simplifies somewhat the calculations in  $G/H$ . This was also the case in [GG4] for the extension (3) with  $m \geq 4$ . However, for the extension (2) with  $n = 2$ , it was necessary to go a stage further in the lower central series, and take  $H = \Gamma_3(K)$ . From the point of view of the splitting problem, it is thus helpful to know the lower central and derived series of the braid groups occurring in these group extensions. But as we indicated earlier, these series are of course interesting in their own right, and help us to understand better the structure of surface braid groups.

## 1.2 Statement of the main results

This paper is organised as follows. In Section 2, we recall some general results concerning the splitting of the short exact sequence  $1 \rightarrow \Gamma_2(B_n(\mathbb{R}P^2)) \rightarrow B_n(\mathbb{R}P^2) \rightarrow (B_n(\mathbb{R}P^2))^{\text{Ab}} \rightarrow 1$ , where  $(B_n(\mathbb{R}P^2))^{\text{Ab}}$  is the Abelianisation of  $B_n(\mathbb{R}P^2)$ , as well as homological conditions for the stabilisation of the lower central series of a group (Lemma 7). We then go on to study the lower central series of  $B_n(\mathbb{R}P^2)$ , and we prove the following result.

**Theorem 1.** *The lower central series of  $B_n(\mathbb{R}P^2)$  is as follows.*

- (a) *If  $n = 1$  then  $B_1(\mathbb{R}P^2) = P_1(\mathbb{R}P^2) \cong \mathbb{Z}_2$ , and  $\Gamma_i(B_1(\mathbb{R}P^2)) = \{1\}$  for all  $i \geq 2$ .*
  - (b) *If  $n = 2$  then  $B_2(\mathbb{R}P^2)$  is isomorphic to the generalised quaternion group  $\mathcal{Q}_{16}$  of order 16. Its lower central series is given by  $\Gamma_2(B_2(\mathbb{R}P^2)) \cong \mathbb{Z}_4$ ,  $\Gamma_3(B_2(\mathbb{R}P^2)) \cong \mathbb{Z}_2$  and  $\Gamma_i(B_2(\mathbb{R}P^2)) = \{1\}$  for all  $i \geq 4$ .*
  - (c) *For all  $n \geq 3$ , the lower central series of  $B_n(\mathbb{R}P^2)$  is constant from the commutator subgroup onwards:  $\Gamma_m(B_n(\mathbb{R}P^2)) = \Gamma_2(B_n(\mathbb{R}P^2))$  for all  $m \geq 2$ .*
- Further, a presentation of  $\Gamma_2(B_n(\mathbb{R}P^2))$  is given in Proposition 12.*

The lower central series of  $B_n(\mathbb{R}P^2)$  is thus completely determined. In particular, for all  $n \neq 2$ , the lower central series of  $B_n(\mathbb{R}P^2)$  is constant from the commutator subgroup onwards, and  $B_n(\mathbb{R}P^2)$  is residually nilpotent if and only if  $n \leq 3$ . A presentation of  $\Gamma_2(B_n(\mathbb{R}P^2))$  is given in Proposition 12 in Section 4. The case  $n = 3$  is particularly interesting: as we shall see in Proposition 8,  $\Gamma_2(B_3(\mathbb{R}P^2))$  is a semi-direct of the form  $(\mathbb{F}_3 \rtimes \mathcal{Q}_8) \rtimes \mathbb{Z}_3$ . This may be compared with Gorin and Lin's results for  $\Gamma_2(B_3)$  and  $\Gamma_2(B_4)$  [GL] and with our result for  $B_4(\mathbb{S}^2)$  [GG5].

In Section 3, we study the derived series of  $B_n(\mathbb{R}P^2)$ . As in the case of  $B_n$  and  $B_n(\mathbb{S}^2)$  [GL, GG5],  $(B_n(\mathbb{R}P^2))^{(1)}$  is perfect if  $n \geq 5$ , in other words, the derived series of  $B_n(\mathbb{R}P^2)$  is constant from  $(B_n(\mathbb{R}P^2))^{(1)}$  onwards. The cases  $n = 1, 2$  are straightforward, and the groups  $B_n(\mathbb{R}P^2)$  are finite and soluble. In the case  $n = 3$ , we make use of the semi-direct product decomposition of  $(B_3(\mathbb{R}P^2))^{(1)}$  of Proposition 8.

**Theorem 2.** *Let  $n \in \mathbb{N}$ ,  $n \neq 4$ . The derived series of  $B_n(\mathbb{R}P^2)$  is as follows.*

- (a) *If  $n = 1$  then  $(B_1(\mathbb{R}P^2))^{(1)} = \{1\}$ .*
- (b) *If  $n = 2$  then  $(B_2(\mathbb{R}P^2))^{(1)} \cong \mathbb{Z}_4$  and  $(B_2(\mathbb{R}P^2))^{(2)} = \{1\}$ .*
- (c) *Suppose that  $n = 3$ . Then*
  - (i)  *$(B_3(\mathbb{R}P^2))^{(1)} = \Gamma_2(B_3(\mathbb{R}P^2))$  fits into the short exact sequence*

$$1 \longrightarrow K \longrightarrow (B_3(\mathbb{R}P^2))^{(1)} \longrightarrow \mathbb{Z}_3 \longrightarrow 1,$$

*where  $K$  is an index 2 subgroup of  $P_3(\mathbb{R}P^2)$ .*

(ii) *This short exact sequence splits; a section is given by associating  $(\rho_3\sigma_2\sigma_1)^4 \in (B_3(\mathbb{R}P^2))^{(1)}$  to a generator of  $\mathbb{Z}_3$ . The commutator subgroup  $(B_3(\mathbb{R}P^2))^{(1)}$  is isomorphic to  $(\mathbb{F}_3 \rtimes \mathcal{Q}_8) \rtimes \mathbb{Z}_3$ , where the actions are given by Proposition 8.*

(iii) *We have  $(B_3(\mathbb{R}P^2))^{(2)} \cong \mathbb{F}_3 \rtimes \mathcal{Q}_8$ , where the action is given by Proposition 8. The quotient  $(B_3(\mathbb{R}P^2))^{(1)}/(B_3(\mathbb{R}P^2))^{(2)} \cong \mathbb{Z}_3$ , and there is a short exact sequence*

$$1 \longrightarrow (B_3(\mathbb{R}P^2))^{(1)}/(B_3(\mathbb{R}P^2))^{(2)} \longrightarrow B_3(\mathbb{R}P^2)/(B_3(\mathbb{R}P^2))^{(2)} \longrightarrow B_3(\mathbb{R}P^2)/(B_3(\mathbb{R}P^2))^{(1)} \longrightarrow 1,$$

*where the extension  $B_3(\mathbb{R}P^2)/(B_3(\mathbb{R}P^2))^{(2)}$  is isomorphic to the dihedral group  $Dih_{12}$  of order 12. Moreover,  $(B_3(\mathbb{R}P^2))^{(2)}/(B_3(\mathbb{R}P^2))^{(3)} \cong \mathbb{Z}_2^4$ , and  $B_3(\mathbb{R}P^2)/(B_3(\mathbb{R}P^2))^{(3)}$  is an extension of  $\mathbb{Z}_2^4$  by  $Dih_{12}$ , so is of order 192.*

(iv) *We have  $(B_3(\mathbb{R}P^2))^{(3)} \cong \mathbb{F}_9 \oplus \mathbb{Z}_2$  and  $(B_3(\mathbb{R}P^2))^{(3)}/(B_3(\mathbb{R}P^2))^{(4)} \cong \mathbb{Z}^9 \oplus \mathbb{Z}_2$ . Further,  $B_3(\mathbb{R}P^2)/(B_3(\mathbb{R}P^2))^{(4)}$  is an extension of  $\mathbb{Z}^9 \oplus \mathbb{Z}_2$  by  $B_3(\mathbb{R}P^2)/(B_3(\mathbb{R}P^2))^{(3)}$ , so is infinite, and for all  $i \geq 4$ ,  $(B_3(\mathbb{R}P^2))^{(i)} \cong (\mathbb{F}_9)^{(i-3)}$ .*

(d) *If  $n \geq 5$  then  $(B_n(\mathbb{R}P^2))^{(2)} = (B_n(\mathbb{R}P^2))^{(1)}$ , so  $(B_n(\mathbb{R}P^2))^{(1)}$  is perfect. A presentation of  $(B_n(\mathbb{R}P^2))^{(1)}$  is given in Proposition 12.*

So if  $n \neq 4$ , the derived series of  $B_n(\mathbb{R}P^2)$  is thus completely determined (up to knowing the derived series of the free group  $\mathbb{F}_9$  of rank 9). In particular, if  $n \neq 4$ ,  $B_n(\mathbb{R}P^2)$  is residually soluble if and only if  $n < 4$  (Corollary 10). We remark that part (c) of Theorem 1 and the first statement of part (d) of Theorem 2 appeared in [BM] where the authors asserted that the results may be proved along the lines of our proof in the case of the sphere [GG5]. We give the details of the proofs. As for  $B_n$  and  $B_n(\mathbb{S}^2)$  [GL, GG5, GG6], the case  $n = 4$  is somewhat delicate. We are able to determine some of the terms and quotients of the derived series of  $B_4(\mathbb{R}P^2)$ .

**Theorem 3.** *Suppose that  $n = 4$ .*

(a) *The group  $(B_4(\mathbb{R}P^2))^{(1)} = \Gamma_2(B_4(\mathbb{R}P^2))$  is given by an extension*

$$1 \longrightarrow K \longrightarrow (B_4(\mathbb{R}P^2))^{(1)} \longrightarrow A_4 \longrightarrow 1$$

*where  $K$  is a subgroup of  $P_4(\mathbb{R}P^2)$  of index two.*

(b) (i) We have the following isomorphism:

$$(B_4(\mathbb{R}P^2))^{(1)} \cong (B_4(\mathbb{R}P^2))^{(2)} \rtimes \mathbb{Z}_3,$$

where the action on  $(B_4(\mathbb{R}P^2))^{(2)}$  is given by conjugation by  $(\rho_3\sigma_2\sigma_1)^4$ , and

$$(B_4(\mathbb{R}P^2))^{(1)}/(B_4(\mathbb{R}P^2))^{(2)} \cong \mathbb{Z}_3.$$

(ii) We have a short exact sequence

$$1 \longrightarrow (B_4(\mathbb{R}P^2))^{(1)}/(B_4(\mathbb{R}P^2))^{(2)} \longrightarrow B_4(\mathbb{R}P^2)/(B_4(\mathbb{R}P^2))^{(2)} \longrightarrow B_4(\mathbb{R}P^2)/(B_4(\mathbb{R}P^2))^{(1)} \longrightarrow 1,$$

where  $B_4(\mathbb{R}P^2)/(B_4(\mathbb{R}P^2))^{(2)}$  is isomorphic to the dihedral group  $Dih_{12}$  of order 12.

(iii) The group  $(B_4(\mathbb{R}P^2))^{(2)}$  is given by an extension

$$1 \longrightarrow K \longrightarrow (B_4(\mathbb{R}P^2))^{(2)} \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 \longrightarrow 1.$$

(c)  $(B_4(\mathbb{R}P^2))^{(2)}/(B_4(\mathbb{R}P^2))^{(3)} \cong \mathbb{Z}_2^4$ , and  $(B_4(\mathbb{R}P^2))^{(1)}/(B_4(\mathbb{R}P^2))^{(3)} \cong \mathbb{Z}_2^4 \rtimes \mathbb{Z}_3$ , where the action of  $\mathbb{Z}_3$  permutes cyclically the three non-trivial elements of the first and second (resp. the third and fourth) copies of  $\mathbb{Z}_2$ .

(d) The group  $(B_4(\mathbb{R}P^2))^{(3)}$  is a subgroup of  $K$  of index four. Further,

$$(B_4(\mathbb{R}P^2))^{(3)} \cong (\mathbb{F}_5 \rtimes \mathbb{F}_3) \rtimes \mathbb{Z}_4,$$

where the action is described by equations (128)–(131). Moreover,

$$(B_4(\mathbb{R}P^2))^{(3)}/(B_4(\mathbb{R}P^2))^{(4)} \cong \mathbb{Z}_2^8 \oplus \mathbb{Z}_4,$$

and  $(B_4(\mathbb{R}P^2))^{(4)}$  is a semi-direct product of the form  $\mathbb{F}_{129} \rtimes \mathbb{F}_{17}$  where the action is that induced by  $\mathbb{F}_3$  on  $\mathbb{F}_5$ . From  $i = 4$  onwards, we have  $(B_4(\mathbb{R}P^2))^{(i+4)} \cong (\mathbb{F}_{129} \rtimes \mathbb{F}_{17})^{(i)}$  for all  $i \geq 0$ .

A presentation of  $(B_4(\mathbb{R}P^2))^{(1)}$  derived from that of Proposition 12 is given during the proof of Theorem 3. As in the case of  $B_4(\mathbb{S}^2)$ , it is an open question as to whether  $B_4(\mathbb{R}P^2)$  is residually soluble or not.

In [BGG], the lower central series of braid groups of orientable surfaces of genus  $g \geq 1$ , with and without boundary, was analysed. The study of the lower central series of non-orientable surfaces of genus at least two is the subject of work in progress.

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## 2 The lower central series of $B_n(\mathbb{R}P^2)$

The main aim of this section is to prove Theorem 1, which describes the lower central series of  $B_n(\mathbb{R}P^2)$ . Before doing so, we state some general results concerning  $B_n(\mathbb{R}P^2)$ , as well as some general homological conditions for the stabilisation of the lower central series of a group (Lemma 7). We start by recalling Van Buskirk's presentation of  $B_n(\mathbb{R}P^2)$ .

**Proposition 4** (Van Buskirk [VB]). *Let  $n \in \mathbb{N}$ . The following constitutes a presentation of the group  $B_n(\mathbb{R}P^2)$ :*

**generators:**  $\sigma_1, \dots, \sigma_{n-1}, \rho_1, \dots, \rho_n$ .

**relations:**

$$\begin{aligned} \sigma_i \sigma_j &= \sigma_j \sigma_i \quad \text{if } |i - j| \geq 2, \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{for } 1 \leq i \leq n - 2, \\ \sigma_i \rho_j &= \rho_j \sigma_i \quad \text{for } j \neq i, i + 1, \\ \rho_{i+1} &= \sigma_i^{-1} \rho_i \sigma_i^{-1} \quad \text{for } 1 \leq i \leq n - 1, \\ \rho_{i+1}^{-1} \rho_i^{-1} \rho_{i+1} \rho_i &= \sigma_i^2 \quad \text{for } 1 \leq i \leq n - 1, \\ \rho_1^2 &= \sigma_1 \sigma_2 \cdots \sigma_{n-2} \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_2 \sigma_1. \end{aligned}$$

*Remark 5.* Let  $n \in \mathbb{N}$ . It is well known that  $\{B_{i,j}, \rho_k \mid 1 \leq i < j \leq n, 1 \leq k \leq n\}$  is a generating set for  $P_n(\mathbb{R}P^2)$ , where

$$B_{i,j} = \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}.$$

Let  $n \in \mathbb{N}$ , let  $(B_n(\mathbb{R}P^2))^{\text{Ab}} = B_n(\mathbb{R}P^2)/\Gamma_2(B_n(\mathbb{R}P^2))$  denote the Abelianisation of  $B_n(\mathbb{R}P^2)$ , and let  $\alpha: B_n(\mathbb{R}P^2) \rightarrow (B_n(\mathbb{R}P^2))^{\text{Ab}}$  be the canonical projection. Then we have the following short exact sequence:

$$1 \longrightarrow \Gamma_2(B_n(\mathbb{R}P^2)) \longrightarrow B_n(\mathbb{R}P^2) \xrightarrow{\alpha} (B_n(\mathbb{R}P^2))^{\text{Ab}} \longrightarrow 1. \quad (4)$$

We first prove the following result which deals with this short exact sequence.

**Proposition 6.** *Let  $n \in \mathbb{N}$ . Then  $(B_n(\mathbb{R}P^2))^{\text{Ab}} = B_n(\mathbb{R}P^2)/\Gamma_2(B_n(\mathbb{R}P^2)) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ , where the generators of the first (resp. second) copy of  $\mathbb{Z}_2$  is the image of the generators  $\sigma_i$  (resp.  $\rho_j$ ).*

*Proof.* This follows easily by Abelianising the presentation of  $B_n(\mathbb{R}P^2)$  given in Proposition 4. The generators  $\sigma_i$  (resp.  $\rho_j$ ) of  $B_n(\mathbb{R}P^2)$  are all identified by  $\alpha$  to a single generator  $\bar{\sigma} = \alpha(\sigma_i)$  (resp.  $\bar{\rho} = \alpha(\rho_j)$ ) of the first (resp. second)  $\mathbb{Z}_2$ -summand.  $\square$

We recall the following lemma from [GG5].

**Lemma 7** ([GG5]). *Let  $G$  be a group, and let  $\delta: H_2(G, \mathbb{Z}) \rightarrow H_2(G^{\text{Ab}}, \mathbb{Z})$  denote the homomorphism induced by Abelianisation. Then  $\Gamma_2(G) = \Gamma_3(G)$  if and only if  $\delta$  is surjective.*

We now come to the proof of Theorem 1.

*Proof of Theorem 1.* Since  $B_1(\mathbb{R}P^2) = \pi_1(\mathbb{R}P^2)$  and  $B_2(\mathbb{R}P^2) \cong \mathcal{Q}_{16}[\text{VB}]$ , parts (a) and (b) follow easily. Now suppose that  $n \geq 3$ . First observe that  $H_2(\mathbb{Z}_2 \oplus \mathbb{Z}_2) \cong \mathbb{Z}_2$ . By Lemma 7, if the homomorphism  $\delta$  is surjective then  $\Gamma_2(B_n(\mathbb{R}P^2)) = \Gamma_3(B_n(\mathbb{R}P^2))$ , and part (c) follows. Otherwise, if  $\delta$  is not surjective then it is trivial, and we obtain the following exact sequence:

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \Gamma_2(B_n(\mathbb{R}P^2))/\Gamma_3(B_n(\mathbb{R}P^2)) \longrightarrow H_1(B_n(\mathbb{R}P^2), \mathbb{Z}) \longrightarrow (B_n(\mathbb{R}P^2))^{\text{Ab}} \longrightarrow 1.$$

It follows that  $\mathbb{Z}_2 \longrightarrow \Gamma_2(B_n(\mathbb{R}P^2))/\Gamma_3(B_n(\mathbb{R}P^2))$  is an isomorphism. So we have the short exact sequence:

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow B_n(\mathbb{R}P^2)/\Gamma_3(B_n(\mathbb{R}P^2)) \longrightarrow \underbrace{H_1(B_n(\mathbb{R}P^2), \mathbb{Z})}_{\mathbb{Z}_2 \oplus \mathbb{Z}_2} \longrightarrow 1,$$

and hence the middle group, which we denote by  $H$ , is of order 8. Since the quotient  $\Gamma_2(B_n(\mathbb{R}P^2))/\Gamma_3(B_n(\mathbb{R}P^2))$  is non trivial, we conclude that  $H$  is non Abelian, and so is either  $\mathcal{Q}_8$  or the dihedral group  $\text{Dih}_8$ .

We claim that there is no surjective homomorphism  $B_n(\mathbb{R}P^2) \longrightarrow H$ . To see this, let  $\varphi: B_n(\mathbb{R}P^2) \longrightarrow H$  be a homomorphism. Since  $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$  for all  $1 \leq i \leq n-2$ , the  $\sigma_i$  are pairwise conjugate. Hence  $\varphi(\sigma_i)$  and  $\varphi(\sigma_j)$  are conjugate in  $H$  for all  $1 \leq i, j \leq n-1$ . But in both  $\mathcal{Q}_8$  and  $\text{Dih}_8$ , any two conjugate elements commute. Applying  $\varphi$  to the relation  $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$  and using induction yields  $\varphi(\sigma_i) = \varphi(\sigma_j)$  for all  $1 \leq i, j \leq n-1$ . If  $\varphi(\sigma_i) = 1$  then the relation  $\rho_{i+1} = \sigma_i^{-1} \rho_i \sigma_i^{-1}$  implies that  $\varphi(\rho_i) = \varphi(\rho_{i+1})$  for all  $1 \leq i \leq n-1$ , and thus  $\text{Im}(\varphi) = \langle \varphi(\rho_1) \rangle \neq H$ . So we may assume that  $\varphi(\sigma_i) \neq 1$ .

Suppose first that  $n \geq 4$ . Given  $1 \leq i \leq n$ , there exists  $1 \leq j \leq n-1$  such that  $\sigma_j$  commutes with  $\rho_i$ , and so  $\varphi(\sigma_j)$  commutes with  $\varphi(\rho_i)$  for all  $i$ . If  $n = 3$  then a similar analysis shows that  $\varphi(\rho_1)$  and  $\varphi(\rho_3)$  commute with the  $\varphi(\sigma_j)$ . Further,  $\rho_2 = \sigma_1^{-1} \rho_1 \sigma_1^{-1}$ , and hence  $\varphi(\rho_2)$  commutes with the  $\varphi(\sigma_j)$ . In both cases, we conclude that  $\text{Im}(\varphi)$  is contained in the centraliser of  $\varphi(\sigma_1)$  in  $H$ . A necessary condition for  $\varphi$  to be surjective is that  $\varphi(\sigma_1)$  be central in  $H$ , and so  $\varphi(\sigma_1)$  must be of order 2. Once more the relation  $\rho_{i+1} = \sigma_i^{-1} \rho_i \sigma_i^{-1}$  implies that  $\varphi(\rho_i) = \varphi(\rho_{i+1})$  for all  $1 \leq i \leq n-1$ , and hence  $\text{Im}(\varphi) = \langle \varphi(\rho_1), \varphi(\sigma_1) \rangle \neq H$ .

Thus no homomorphism  $B_n(\mathbb{R}P^2) \longrightarrow H$  is surjective, but this contradicts the surjectivity of the canonical projection  $B_n(\mathbb{R}P^2) \longrightarrow B_n(\mathbb{R}P^2)/\Gamma_3(B_n(\mathbb{R}P^2))$ . This completes the proof of part (c), and thus that of Theorem 1.  $\square$

We may obtain a much better description of  $\Gamma_2(B_3(\mathbb{R}P^2))$  as follows. This will be helpful in the analysis of the derived series in Section 3.

**Proposition 8.** *The group  $\Gamma_2(B_3(\mathbb{R}P^2))$  is isomorphic to  $(\mathbb{F}_3 \rtimes \mathcal{Q}_8) \rtimes \mathbb{Z}_3$ . The actions may be described as follows. Writing  $\mathcal{Q}_8 = \langle x, y \mid x^2 = y^2, yxy^{-1} = x^{-1} \rangle$ ,  $\mathbb{F}_3 = \mathbb{F}_3\langle z_1, z_2, z_3 \rangle$  and  $\mathbb{Z}_3 = \langle u \rangle$ , we have:*

$$\begin{array}{lll} xz_1x^{-1} = z_1^{-1} & xz_2x^{-1} = z_1^{-1}z_3^{-1}z_1 & xz_3x^{-1} = z_1^{-1}z_2^{-1}z_1 \\ yz_1y^{-1} = z_2z_3z_1 & yz_2y^{-1} = z_2^{-1} & yz_3y^{-1} = z_2z_3^{-1}z_2^{-1} \\ uz_1u^{-1} = x^2z_3z_1 & uz_2u^{-1} = x^2z_1^{-1} & uz_3u^{-1} = x^2z_2^{-1}z_1^{-1}z_3^{-1} \\ uxu^{-1} = xy & yuy^{-1} = x, & \end{array}$$

where  $u = (\rho_3\sigma_2\sigma_1)^4$ ,  $x = \rho_2\rho_1$ ,  $y = \rho_2B_{1,2}\rho_3^{-1}$ ,  $z_1 = \rho_3^2$ ,  $z_2 = B_{2,3}$  and  $z_3 = \rho_3B_{2,3}\rho_3^{-1}$ .

Remarks 9.

(a) The commutator subgroup of  $B_4(\mathbb{R}P^2)$  will be analysed in more detail in Section 3.

(b) Let  $n \geq 2$ . Recall from [GG3, Proposition 26] that there exist two elements  $a, b \in B_n(\mathbb{R}P^2)$  defined by:

$$\begin{cases} a = \rho_n \sigma_{n-1} \cdots \sigma_1 = \sigma_{n-1}^{-1} \cdots \sigma_1^{-1} \rho_1 \\ b = \rho_{n-1} \sigma_{n-2} \cdots \sigma_1 = \sigma_{n-2}^{-1} \cdots \sigma_1^{-1} \rho_1, \end{cases} \quad (5)$$

of order  $4n$  and  $4(n-1)$  respectively. These elements satisfy [GG3, Remark 27]:

$$b^{n-1} = \rho_{n-1} \cdots \rho_1 \text{ and } a^n = \rho_n \cdots \rho_1.$$

From [GG3, page 777], conjugation by  $a^{-1}$  permutes cyclically the following two collections of elements:

$$\begin{cases} \sigma_1, \dots, \sigma_{n-1}, a^{-1} \sigma_{n-1} a, \sigma_1^{-1}, \dots, \sigma_{n-1}^{-1}, a^{-1} \sigma_{n-1}^{-1} a, \text{ and} \\ \rho_1, \dots, \rho_n, \rho_1^{-1}, \dots, \rho_n^{-1}. \end{cases} \quad (6)$$

In particular,

$$\begin{cases} a^n \sigma_i a^{-n} = \sigma_i^{-1} & \text{for all } 1 \leq i \leq n-1 \\ a^n \rho_j a^{-n} = \rho_j^{-1} & \text{for all } 1 \leq j \leq n. \end{cases} \quad (7)$$

Further, for all  $1 \leq i \leq n$  [GG9],

$$\Delta_n \rho_i \Delta_n^{-1} = \rho_{n+1-i}^{-1}$$

in  $B_n(\mathbb{R}P^2)$ , which implies that

$$\Delta_n a \Delta_n^{-1} = \Delta_n \rho_n \sigma_{n-1} \cdots \sigma_1 \Delta_n^{-1} = \rho_1^{-1} \sigma_1 \cdots \sigma_{n-1} = a^{-1}. \quad (8)$$

These observations will be used frequently in what follows.

*Proof of Proposition 8.* Let  $n \geq 2$ , and let  $\alpha$  be the Abelianisation homomorphism of equation (4), where  $\alpha(\sigma_i) = \bar{\sigma}$  and  $\alpha(\rho_j) = \bar{\rho}$ . The permutation homomorphism  $\tau$  of equation (1) induces a homomorphism  $\bar{\tau}: (B_n(\mathbb{R}P^2))^{\text{Ab}} \rightarrow \langle \bar{\sigma} \rangle$ , and we obtain the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccc} & & 1 & & 1 & & 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & K & \longrightarrow & \Gamma_2(B_n(\mathbb{R}P^2)) & \xrightarrow{\tau'} & A_n = \Gamma_2(S_n) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & P_n(\mathbb{R}P^2) & \longrightarrow & B_n(\mathbb{R}P^2) & \xrightarrow{\tau} & S_n \longrightarrow 1 \\ & & \downarrow \alpha' & & \downarrow \alpha & & \downarrow h \\ 1 & \longrightarrow & \langle \bar{\rho} \rangle & \longrightarrow & (B_n(\mathbb{R}P^2))^{\text{Ab}} & \xrightarrow{\bar{\tau}} & \langle \bar{\sigma} \rangle \longrightarrow 1. \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 1 & & 1 & & 1 \end{array} \quad (9)$$

Here  $\tau'$  (resp.  $\alpha'$ ) is the restriction of  $\tau$  (resp.  $\alpha$ ) to  $\Gamma_2(B_n(\mathbb{R}P^2))$  (resp. to  $P_n(\mathbb{R}P^2)$ ),  $h$  is the homomorphism that to a transposition associates  $\bar{\sigma}$ , and  $K = \text{Ker}(\alpha') = \text{Ker}(\tau')$  is of index 2 in  $P_n(\mathbb{R}P^2)$  (recall from Proposition 6 that  $(B_n(\mathbb{R}P^2))^{\text{Ab}} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ , and  $\langle \bar{\rho} \rangle \cong \langle \bar{\sigma} \rangle \cong \mathbb{Z}_2$ ).

Now let  $n = 3$ . From [VB], we know that

$$P_3(\mathbb{R}P^2) \cong \mathbb{F}_2 \rtimes \mathcal{Q}_8. \quad (10)$$

Let us first determine generating sets of the two factors in terms of Van Buskirk's generators (this action was previously described in [GG4], but in terms of a different generating set). From the Fadell-Neuwirth short exact sequence (2), we have

$$1 \longrightarrow \pi_1(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \longrightarrow P_3(\mathbb{R}P^2) \xrightarrow{p_*} P_2(\mathbb{R}P^2) \cong \mathcal{Q}_8 \longrightarrow 1,$$

where  $\pi_1(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \cong \mathbb{F}_2$  is a free group of rank two with basis  $(\rho_3, B_{2,3})$ . The two elements  $a = \rho_3\sigma_2\sigma_1$  and  $b = \rho_2\sigma_1$  of equation (5) are of order 12 and 8 respectively, and satisfy:

$$b^2 = \rho_2\rho_1 \text{ and } a^3 = \rho_3\rho_2\rho_1. \quad (11)$$

From [GG9, Proposition 15], there is a copy of  $\mathcal{Q}_{16}$  in  $B_3(\mathbb{R}P^2)$  of the form  $\langle b, \Delta_3 a^{-1} \rangle$ , and by general arguments, one sees that it has two subgroups isomorphic to  $\mathcal{Q}_8$ , of the form  $\langle b^2, \Delta_3 a^{-1} \rangle$  and  $\langle b^2, b\Delta_3 a^{-1} \rangle$  respectively. We shall be interested in the latter copy since it is a subgroup of  $P_3(\mathbb{R}P^2)$ .

We have that  $a^4$  is of order 3,  $a^4 \in \text{Ker}(\alpha) = \Gamma_2(B_3(\mathbb{R}P^2))$ , and  $\tau(a^4) = \tau(a) = (1, 2, 3)$ . Since  $A_3 = \langle (1, 2, 3) \rangle$ , the correspondence  $(1, 2, 3) \mapsto a^4$  defines a section for  $\tau'$ , and hence

$$\Gamma_2(B_3(\mathbb{R}P^2)) \cong K \rtimes \mathbb{Z}_3 \quad (12)$$

from equation (9). Let us now study the structure of  $K$  in order to calculate the action.

By construction,  $K$  is the kernel of  $\alpha'$ , and so is an index 2 subgroup of  $P_3(\mathbb{R}P^2) \cong \mathbb{F}_2 \rtimes \mathcal{Q}_8$ . The homomorphism  $\alpha'$  is defined on the generators of  $P_3(\mathbb{R}P^2)$  (cf. Remark 5) by  $\rho_j \mapsto \bar{\rho}$  for  $j = 1, 2, 3$ , and for  $1 \leq i < j \leq 3$ ,  $B_{i,j}$  is sent to the trivial element of  $\langle \bar{\rho} \rangle$ . Since

$$b\Delta_3 a^{-1} = \sigma_1^{-1}\rho_1 \cdot \sigma_1\sigma_2\sigma_1 \cdot \sigma_1^{-1}\sigma_2^{-1}\rho_3^{-1} = \sigma_1^{-1}\rho_1\sigma_1\rho_3^{-1} = \rho_2\sigma_1^2\rho_3^{-1} = \rho_2B_{1,2}\rho_3^{-1}, \quad (13)$$

we see that our copy  $\langle b^2, b\Delta_3 a^{-1} \rangle$  of  $\mathcal{Q}_8$  lies in  $\text{Ker}(\alpha')$ . Thus we have a commutative diagram of short exact sequences:

$$\begin{array}{ccccccc} & & 1 & & 1 & & \\ & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \text{Ker}(p_*|_K) & \longrightarrow & K & \xrightarrow{p_*|_K} & P_2(\mathbb{R}P^2) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \parallel \\ 1 & \longrightarrow & \mathbb{F}_2(\rho_3, B_{2,3}) & \longrightarrow & P_3(\mathbb{R}P^2) & \xrightarrow{p_*} & P_2(\mathbb{R}P^2) \longrightarrow 1 \\ & & \downarrow \alpha'|_{\mathbb{F}_2(\rho_3, B_{2,3})} & & \downarrow \alpha' & & \\ 1 & \longrightarrow & \langle \bar{\rho} \rangle & \xlongequal{\quad} & \langle \bar{\rho} \rangle & & \\ & & \downarrow & & \downarrow & & \\ & & 1 & & 1 & & \end{array}$$

Note that we have used the following facts in order to construct this diagram:

(i)  $\alpha' \Big|_{\pi_1(\mathbb{R}P^2 \setminus \{x_1, x_2\})}$  is surjective onto  $\langle \bar{\rho} \rangle$  since  $\rho_3 \in \mathbb{F}_2(\rho_3, B_{2,3})$ .

(ii)  $p_*(b^2) = (\rho_2\rho_1)^2 = \Delta_2^2$  is equal to  $B_{1,2}$  in  $P_2(\mathbb{R}P^2)$ , and  $p_*(b\Delta_3a^{-1}) = \rho_2B_{1,2}$  by equation (13), and we conclude that  $p_* \Big|_K$  is surjective onto  $P_2(\mathbb{R}P^2)$ .

This second fact also implies that

$$K \cong K' \rtimes \mathcal{Q}_8, \quad (14)$$

where  $K' = \text{Ker}(p_* \Big|_K) = \text{Ker}(\alpha') \cap \mathbb{F}_2(\rho_3, B_{2,3}) = \text{Ker}(\alpha' \Big|_{\mathbb{F}_2(\rho_3, B_{2,3})})$  is of index two in  $\mathbb{F}_2(\rho_3, B_{2,3})$ . The application of the Reidemeister-Schreier rewriting process with Schreier transversal  $\{1, \rho_3\}$  to this restriction shows that  $K' = \mathbb{F}_3(\rho_3^2, B_{2,3}, \rho_3B_{2,3}\rho_3^{-1})$  is a free group of rank 3. Combining equations (12) and (14), we obtain  $K \cong (\mathbb{F}_3 \rtimes \mathcal{Q}_8) \rtimes \mathbb{Z}_3$ . The actions may be deduced from the action of  $\mathcal{Q}_8 = \langle b^2, b\Delta_3a^{-1} \rangle$  on  $\mathbb{F}_2(\rho_3, B_{2,3})$  which we now determine. Remark 9(b) and equation (11) imply that conjugation by  $b^2 = \rho_2\rho_1$  is given by:

$$\begin{cases} \rho_2\rho_1 \cdot \rho_3 \cdot \rho_1^{-1}\rho_2^{-1} = \rho_3^{-1}a^3\rho_3a^{-3}\rho_3 = \rho_3^{-1} \\ \rho_2\rho_1 \cdot B_{2,3} \cdot \rho_1^{-1}\rho_2^{-1} = \rho_3^{-1}a^3B_{2,3}a^{-3}\rho_3 = \rho_3^{-1}B_{2,3}^{-1}\rho_3. \end{cases} \quad (15)$$

From this, we deduce that under conjugation by  $b^2$ ,

$$\begin{cases} \rho_3 \longmapsto \rho_3^{-1} \\ B_{2,3}\rho_3 \longmapsto (B_{2,3}\rho_3)^{-1}. \end{cases} \quad (16)$$

As for conjugation by  $b\Delta_3a^{-1} = \rho_2B_{1,2}\rho_3^{-1}$ , we have

$$\begin{aligned} \rho_2B_{1,2}\rho_3^{-1} \cdot \rho_3 \cdot \rho_3B_{1,2}^{-1}\rho_2^{-1} &= \rho_2\rho_3\rho_2^{-1} = \rho_3\rho_2B_{2,3}^{-1}\rho_2^{-1} \quad \text{using Proposition 4} \\ &= \rho_3\rho_2\rho_1B_{2,3}^{-1}\rho_1^{-1}\rho_2^{-1}\rho_3^{-1}\rho_3 = a^3B_{2,3}^{-1}a^{-3}\rho_3 = B_{2,3}\rho_3 \end{aligned} \quad (17)$$

$$\begin{aligned} \rho_2B_{1,2}\rho_3^{-1} \cdot B_{2,3} \cdot \rho_3B_{1,2}^{-1}\rho_2^{-1} &= b\Delta_3a^{-1} \cdot B_{2,3} \cdot a\Delta_3^{-1}b^{-1} = b\Delta_3(a^{-1}B_{2,3}a)\Delta_3^{-1}b^{-1} \\ &= ba\sigma_1^2a^{-1}b^{-1} = \sigma_2a^2\sigma_1^2a^{-2}\sigma_2^{-1} \quad \text{from equation (5)} \\ &= B_{2,3}^{-1} \quad \text{from equation (6)}. \end{aligned} \quad (18)$$

Here we have used equation (8), as well as the standard property of  $\Delta_n$  (in  $B_n$ ) that  $\Delta_n\sigma_i\Delta_n^{-1} = \sigma_{n-i}$  for all  $1 \leq i \leq n-1$ . So under conjugation by  $b\Delta_3a^{-1}$ ,

$$\begin{cases} \rho_3 \longmapsto B_{2,3}\rho_3 \\ B_{2,3}\rho_3 \longmapsto \rho_3. \end{cases} \quad (19)$$

Relations (16) and (19) thus describe the action of  $\mathcal{Q}_8$  on  $\mathbb{F}_2(\rho_3, B_{2,3})$ , from which we may easily deduce its action on  $\mathbb{F}_3(\rho_3^2, B_{2,3}, \rho_3B_{2,3}\rho_3^{-1})$ :

$$\begin{aligned} \rho_2\rho_1 \cdot \rho_3^2 \cdot \rho_1^{-1}\rho_2^{-1} &= \rho_3^{-2} \\ \rho_2\rho_1 \cdot B_{2,3} \cdot \rho_1^{-1}\rho_2^{-1} &= \rho_3^{-2} \cdot \rho_3B_{2,3}^{-1}\rho_3^{-1} \cdot \rho_3^2 \\ \rho_2\rho_1 \cdot \rho_3B_{2,3}\rho_3^{-1} \cdot \rho_1^{-1}\rho_2^{-1} &= \rho_3^{-2} \cdot B_{2,3}^{-1} \cdot \rho_3^2 \\ \rho_2B_{1,2}\rho_3^{-1} \cdot \rho_3^2 \cdot \rho_3B_{1,2}^{-1}\rho_2^{-1} &= B_{2,3} \cdot \rho_3B_{2,3}\rho_3^{-1} \cdot \rho_3^2 \\ \rho_2B_{1,2}\rho_3^{-1} \cdot B_{2,3} \cdot \rho_3B_{1,2}^{-1}\rho_2^{-1} &= B_{2,3}^{-1} \\ \rho_2B_{1,2}\rho_3^{-1} \cdot \rho_3B_{2,3}\rho_3^{-1} \cdot \rho_3B_{1,2}^{-1}\rho_2^{-1} &= B_{2,3} \cdot \rho_3B_{2,3}^{-1}\rho_3^{-1} \cdot B_{2,3}^{-1} \end{aligned}$$

We now record the action of  $\mathbb{Z}_3$  on  $\mathbb{F}_3 \rtimes \mathcal{Q}_8$ . Since

$$\rho_3^{-2} = \sigma_2 \sigma_1^2 \sigma_2 = \sigma_2 \sigma_1^2 \sigma_2^{-1} \sigma_2^{-2} = B_{1,3} B_{2,3}$$

and

$$(\rho_2 \rho_1)^2 = (\rho_2 B_{1,2} \rho_3^{-1})^2 = b^4 = \Delta_3^2 = B_{1,2} B_{1,3} B_{2,3}, \quad (20)$$

we have

$$B_{1,2} = (\rho_2 \rho_1)^2 B_{2,3}^{-1} B_{1,3}^{-1} = (\rho_2 \rho_1)^2 \rho_3^2. \quad (21)$$

So

$$\begin{aligned} a^4 \cdot \rho_2 \rho_1 \cdot a^{-4} &= \rho_1^{-1} \rho_3 \quad \text{by equation (6)} \\ &= \rho_1^{-1} \rho_2^{-1} \cdot \rho_2 B_{1,2} \rho_3^{-1} \cdot \rho_3^2 B_{1,2}^{-1} \\ &= (\rho_2 \rho_1)^{-1} \cdot \rho_2 B_{1,2} \rho_3^{-1} \cdot (\rho_2 \rho_1)^{-2} \quad \text{by equation (21)} \\ &= \rho_2 \rho_1 \cdot \rho_2 B_{1,2} \rho_3^{-1} \quad \text{by equation (20),} \end{aligned}$$

and using the fact that  $(\rho_2 \rho_1)^2 = \Delta_3^2$  is of order 2 and is central in  $B_3(\mathbb{R}P^2)$ . Further,

$$\begin{aligned} a^4 \cdot \rho_2 B_{1,2} \rho_3^{-1} \cdot a^{-4} &= a^4 \cdot \rho_2 \rho_3^{-1} B_{1,2} \cdot a^{-4} = \rho_1^{-1} \rho_2 a^{-1} \sigma_2^2 a \quad \text{by equation (6)} \\ &= \rho_1^{-1} \rho_2 \rho_1^{-1} \sigma_1 \sigma_2 \sigma_2^2 \sigma_2^{-1} \sigma_1^{-1} \rho_1 \quad \text{by equation (5)} \\ &= \rho_1^{-1} \rho_2 \rho_1^{-1} \sigma_1 \sigma_2^2 \sigma_1 \sigma_1^{-2} \rho_1 = \rho_1^{-1} \cdot \rho_2 \rho_1 \sigma_1^{-2} \cdot \rho_1 = \rho_1^{-1} \rho_1 \rho_2 \rho_1 = \rho_2 \rho_1 \end{aligned}$$

using Proposition 4. This describes the action of  $\mathbb{Z}_3$  on the  $\mathcal{Q}_8$ -factor. As for the action of  $\mathbb{Z}_3$  on  $\mathbb{F}_2(B_{2,3}, \rho_3)$ , using relations (20) and (21), we have

$$\begin{aligned} a^4 B_{2,3} a^{-4} &= B_{1,2}^{-1} \quad \text{by equation (6)} \\ &= \Delta_3^2 \rho_3^{-2} \quad \text{by equations (20) and (21),} \end{aligned}$$

and

$$\begin{aligned} a^4 \rho_3 a^{-4} &= \rho_2^{-1} \quad \text{by equation (6)} \\ &= \rho_3^{-1} B_{1,2} \cdot B_{1,2}^{-1} \rho_3 \rho_2^{-1} = \rho_3^{-1} (\rho_2 \rho_1)^2 \rho_3^2 (\rho_2 B_{1,2} \rho_3^{-1})^{-1} \quad \text{by equation (21)} \\ &= \rho_3 (\rho_2 B_{1,2} \rho_3^{-1}) \quad \text{by equation (20)} \end{aligned}$$

Hence the action on  $\mathbb{F}_3 \left( \rho_3^2, B_{2,3}, \rho_3 B_{2,3} \rho_3^{-1} \right)$  is given by

$$\begin{aligned} a^4 \rho_3^2 a^{-4} &= \rho_3 (\rho_2 B_{1,2} \rho_3^{-1}) \rho_3 (\rho_2 B_{1,2} \rho_3^{-1}) \\ &= \rho_3 (\rho_2 B_{1,2} \rho_3^{-1}) \rho_3 (\rho_2 B_{1,2} \rho_3^{-1})^{-1} \Delta_3^2 \quad \text{by equation (20)} \\ &= \rho_3 B_{2,3} \rho_3 \Delta_3^2 \quad \text{by equation (17)} \\ &= \Delta_3^2 \cdot \rho_3 B_{2,3} \rho_3^{-1} \cdot \rho_3^2 \\ a^4 B_{2,3} a^{-4} &= \Delta_3^2 \cdot \rho_3^{-2} \\ a^4 \rho_3 B_{2,3} \rho_3^{-1} a^{-4} &= \rho_3 (\rho_2 B_{1,2} \rho_3^{-1}) \Delta_3^2 \rho_3^{-2} (\rho_2 B_{1,2} \rho_3^{-1})^{-1} \rho_3^{-1} \\ &= \Delta_3^2 \rho_3 (B_{2,3} \rho_3)^{-2} \rho_3^{-1} \quad \text{by equation (17)} \\ &= \Delta_3^2 B_{2,3}^{-1} \cdot \rho_3^{-2} \cdot \rho_3 B_{2,3}^{-1} \rho_3^{-1} \end{aligned}$$

Setting  $u = a^4$ ,  $x = \rho_2 \rho_1$ ,  $y = \rho_2 B_{1,2} \rho_3^{-1}$ ,  $z_1 = \rho_3^2$ ,  $z_2 = B_{2,3}$  and  $z_3 = \rho_3 B_{2,3} \rho_3^{-1}$  yields the desired actions, and completes the proof of Proposition 8.  $\square$

### 3 The derived series of $B_n(\mathbb{R}P^2)$

In this section, we study the derived series of  $B_n(\mathbb{R}P^2)$  and prove Theorems 2 and 3. We start by showing that for all  $n \neq 2, 3, 4$ ,  $(B_n(\mathbb{R}P^2))^{(1)}$  is perfect. We then study the cases  $n = 3, 4$  in more detail. If  $n = 3$ , we are able to determine completely the derived series of  $B_3(\mathbb{R}P^2)$ , and deduce that it is residually soluble. If  $n = 4$ , in Theorem 3 we obtain some partial results on the derived series of  $B_4(\mathbb{R}P^2)$  and its quotients.

*Proof of Theorem 2.* Cases (a) and (b) follow directly from the fact that  $B_1(\mathbb{R}P^2) \cong \mathbb{Z}_2$  and  $B_2(\mathbb{R}P^2) \cong \mathcal{Q}_{16}$ . Now consider case (d), i.e.  $n \geq 5$ . Let  $H \subseteq (B_n(\mathbb{R}P^2))^{(1)}$  be a normal subgroup of  $B_n(\mathbb{R}P^2)$  such that  $A = (B_n(\mathbb{R}P^2))^{(1)}/H$  is Abelian (notice that this condition is satisfied if  $H = (B_n(\mathbb{R}P^2))^{(2)}$ ), and let

$$\left\{ \begin{array}{l} \pi: B_n(\mathbb{R}P^2) \longrightarrow B_n(\mathbb{R}P^2)/H \\ \beta \longmapsto \tilde{\beta} \end{array} \right.$$

denote the canonical projection. Then the Abelianisation homomorphism  $\alpha$  of equation (4) factors through  $B_n(\mathbb{R}P^2)/H$ , in other words there exists a (surjective) homomorphism  $\hat{\alpha}: B_n(\mathbb{R}P^2)/H \longrightarrow (B_n(\mathbb{R}P^2))^{\text{Ab}}$  satisfying  $\alpha = \hat{\alpha} \circ \pi$ . So equation (4) induces the following short exact sequence:

$$1 \longrightarrow A \longrightarrow B_n(\mathbb{R}P^2)/H \xrightarrow{\hat{\alpha}} (B_n(\mathbb{R}P^2))^{\text{Ab}} \longrightarrow 1.$$

In particular,  $(B_n(\mathbb{R}P^2))^{\text{Ab}} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$  is a quotient of  $B_n(\mathbb{R}P^2)/H$ . We claim that the two are in fact isomorphic, which using the above short exact sequence will imply that  $(B_n(\mathbb{R}P^2))^{(1)} = H$ , and thus  $(B_n(\mathbb{R}P^2))^{(1)} = (B_n(\mathbb{R}P^2))^{(2)}$ . To prove the claim, first note that  $\widehat{\sigma}_1, \dots, \widehat{\sigma}_{n-1}, \widehat{\rho}_1, \dots, \widehat{\rho}_n$  generate  $B_n(\mathbb{R}P^2)/H$ . Since  $\alpha(\sigma_i) = \alpha(\sigma_1)$  for all  $1 \leq i \leq n-1$ , it follows that  $\hat{\alpha}(\widehat{\sigma}_i) = \hat{\alpha}(\widehat{\sigma}_1)$ . So there exist  $t_i \in A$ , with  $t_1 = 1$ , such that  $\widehat{\sigma}_i = t_i \widehat{\sigma}_1$ .

We now apply  $\pi$  to each of the relations of Proposition 4. First suppose that  $3 \leq i \leq n-1$ . Since  $\sigma_i$  commutes with  $\sigma_1$ , we have that  $\widehat{\sigma}_1 \cdot t_i \widehat{\sigma}_1 = t_i \widehat{\sigma}_1 \cdot \widehat{\sigma}_1$ , and hence  $t_i$  commutes with  $\widehat{\sigma}_1$ .

Now let  $4 \leq i \leq n-1$  (such an  $i$  exists since  $n \geq 5$ ). Since  $\sigma_i$  commutes with  $\sigma_2$ , we obtain  $t_i \widehat{\sigma}_1 \cdot t_2 \widehat{\sigma}_1 = t_2 \widehat{\sigma}_1 \cdot t_i \widehat{\sigma}_1$ . But  $A$  is Abelian, and so it follows from the previous paragraph that  $t_2$  commutes with  $\widehat{\sigma}_1$ . Applying this to the image under  $\pi$  of the relation  $\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2$ , we see that  $t_2 = t_2^2$ , and hence  $t_2 = 1$ .

Next, if  $i \geq 2$  then the relation  $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$  implies that  $t_i = t_{i+1}$ , and so  $t_2 = \dots = t_{n-1} = 1$ . Hence  $\widehat{\sigma}_1 = \widehat{\sigma}_2 = \dots = \widehat{\sigma}_{n-1}$ , and we denote this common element by  $\sigma$ .

Let  $1 \leq j \leq n$ . Then from Proposition 4 there exists  $1 \leq i \leq n-1$  such that  $\rho_j$  and  $\sigma_i$  commute. So in the quotient  $B_n(\mathbb{R}P^2)/H$ ,  $\widehat{\rho}_j$  commutes with  $\sigma$  for all  $1 \leq j \leq n$ . If  $1 \leq i \leq n-1$ , the relation  $\rho_{i+1} = \sigma_i^{-1} \rho_i \sigma_i^{-1}$  implies that  $\widehat{\rho}_{i+1} = \widehat{\rho}_i \sigma^{-2}$ . Hence  $\widehat{\rho}_{i+1} = \widehat{\rho}_1 \sigma^{-2i}$ . Projecting the relations  $\rho_{i+1}^{-1} \rho_i^{-1} \rho_{i+1} \rho_i = \sigma_i^2$  into  $B_n(\mathbb{R}P^2)/H$ , where  $1 \leq i \leq n-1$ , we obtain  $\sigma^2 = 1$ , and so  $\widehat{\rho}_i = \widehat{\rho}_1$  for all  $1 \leq i \leq n$ . Finally, by projecting the surface relation of  $B_n(\mathbb{R}P^2)$  into  $B_n(\mathbb{R}P^2)/H$ ,  $\widehat{\rho}_1^2 = \sigma^{2(n-1)} = 1$ . Therefore the group  $B_n(\mathbb{R}P^2)/H$  is a quotient of  $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ . But we know already that  $(B_n(\mathbb{R}P^2))^{\text{Ab}} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$  is a quotient of  $B_n(\mathbb{R}P^2)/H$ , and so this proves the claim. Taking  $H = (B_n(\mathbb{R}P^2))^{(2)}$ , it follows that the group  $(B_n(\mathbb{R}P^2))^{(1)}$  is perfect. A presentation of  $(B_n(\mathbb{R}P^2))^{(1)}$  will be given in Proposition 12. This proves part (d).

We now consider case (c), so  $n = 3$ . Parts (i) and (ii) are just restatements of the results obtained in Proposition 8. To prove part (iii), one may check easily using the presentation of Proposition 8 that the Abelianisation  $(B_3(\mathbb{R}P^2))^{(1)}/(B_3(\mathbb{R}P^2))^{(2)}$  of  $(B_3(\mathbb{R}P^2))^{(1)}$  is cyclic of order 3, generated by the Abelianisation of  $a^4$ . Since  $(B_3(\mathbb{R}P^2))^{(1)}$  is isomorphic to  $(\mathbb{F}_3 \rtimes \mathcal{Q}_8) \rtimes \mathbb{Z}_3$ , where the  $\mathbb{Z}_3$ -factor is generated by  $a^4$ , we obtain  $(B_3(\mathbb{R}P^2))^{(2)} \cong \mathbb{F}_3 \rtimes \mathcal{Q}_8$ , where the action is once more given by Proposition 8. To see that the quotient  $B_3(\mathbb{R}P^2)/(B_3(\mathbb{R}P^2))^{(2)}$  is isomorphic to  $\text{Dih}_{12}$ , note first that we have the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & (B_3(\mathbb{R}P^2))^{(1)} & \longrightarrow & B_3(\mathbb{R}P^2) & \longrightarrow & (B_3(\mathbb{R}P^2))^{\text{Ab}} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \parallel & & \\ 1 & \longrightarrow & (B_3(\mathbb{R}P^2))^{(1)}/(B_3(\mathbb{R}P^2))^{(2)} & \longrightarrow & B_3(\mathbb{R}P^2)/(B_3(\mathbb{R}P^2))^{(2)} & \longrightarrow & (B_3(\mathbb{R}P^2))^{\text{Ab}} & \longrightarrow & 1 \end{array}$$

Since  $(B_3(\mathbb{R}P^2))^{(1)}/(B_3(\mathbb{R}P^2))^{(2)} \cong \mathbb{Z}_3$  and  $(B_3(\mathbb{R}P^2))^{\text{Ab}} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ , it follows that the quotient  $B_3(\mathbb{R}P^2)/(B_3(\mathbb{R}P^2))^{(2)}$  is an extension of  $\mathbb{Z}_3$  by  $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ . We claim that the action is non trivial. To see this, we consider the conjugate of  $a^4$  (which is a coset representative of the generator of  $(B_3(\mathbb{R}P^2))^{(1)}/(B_3(\mathbb{R}P^2))^{(2)}$ ) by  $\sigma_1$  (which is a coset representative of  $\bar{\sigma} \in (B_3(\mathbb{R}P^2))^{\text{Ab}}$ ):

$$\begin{aligned} \sigma_1 a^4 \sigma_1^{-1} &= \sigma_1 (a^4 \sigma_1^{-1} a^{-4}) a^4 = \sigma_1 a^{-1} \sigma_2^{-1} a a^4 \quad \text{by equation (6)} \\ &= \sigma_1 \rho_1^{-1} \sigma_1 \sigma_2 \sigma_2^{-1} a^5 \quad \text{by equation (5)} \\ &= \rho_2^{-1} \rho_1^{-1} \rho_2^{-1} \rho_3^{-1} a^8 \quad \text{by Proposition 4 and equation (11)}. \end{aligned}$$

Now

$$\begin{aligned} \rho_2^{-1} \rho_1^{-1} \rho_2^{-1} \rho_3^{-1} &= \rho_3 (\rho_3^{-1} \rho_2^{-1} \rho_1^{-1} \rho_2^{-1}) \rho_3^{-1} = \rho_3 \left( \rho_3^{-2} B_{1,2} (\rho_2 B_{1,2} \rho_3^{-1})^{-1} (\rho_2 \rho_1)^{-1} \right) \rho_3^{-1} \\ &= \rho_3 \left( (\rho_2 \rho_1)^2 (\rho_2 B_{1,2} \rho_3^{-1})^{-1} (\rho_2 \rho_1)^{-1} \right) \rho_3^{-1} \quad \text{by equation (21)} \\ &= \rho_3 \left( (\rho_2 B_{1,2} \rho_3^{-1})^{-1} (\rho_2 \rho_1) \right) \rho_3^{-1} \quad \text{by equation (20)}. \end{aligned}$$

But  $(\rho_2 B_{1,2} \rho_3^{-1})^{-1} (\rho_2 \rho_1) \in (B_3(\mathbb{R}P^2))^{(2)}$  from Proposition 8, and since  $(B_3(\mathbb{R}P^2))^{(2)} \triangleleft B_3(\mathbb{R}P^2)$ , it follows that  $\rho_2^{-1} \rho_1^{-1} \rho_2^{-1} \rho_3^{-1} \in (B_3(\mathbb{R}P^2))^{(2)}$ . Thus  $\sigma_1 a^4 \sigma_1^{-1}$  is congruent modulo  $(B_3(\mathbb{R}P^2))^{(2)}$  to  $a^{-4}$ , and the action of  $\bar{\sigma}$  on  $(B_3(\mathbb{R}P^2))^{(1)}/(B_3(\mathbb{R}P^2))^{(2)}$  is multiplication by  $-1$ . In particular,  $B_3(\mathbb{R}P^2)/(B_3(\mathbb{R}P^2))^{(2)}$  is a non Abelian group of order 12. Of the three non-Abelian groups of order 12,  $B_3(\mathbb{R}P^2)/(B_3(\mathbb{R}P^2))^{(2)}$  cannot be isomorphic to  $A_4$  since the latter has no normal subgroup of order 3. It cannot be isomorphic to  $\text{Dic}_{12} = \mathbb{Z}_3 \rtimes \mathbb{Z}_4$  (with non-trivial action) either, since  $\text{Dic}_{12}$  has a unique subgroup of order 3 with quotient  $\mathbb{Z}_4$ . We conclude that  $B_3(\mathbb{R}P^2)/(B_3(\mathbb{R}P^2))^{(2)} \cong \text{Dih}_{12}$ . By the short exact sequence

$$1 \longrightarrow (B_3(\mathbb{R}P^2))^{(2)}/(B_3(\mathbb{R}P^2))^{(3)} \longrightarrow B_3(\mathbb{R}P^2)/(B_3(\mathbb{R}P^2))^{(3)} \longrightarrow B_3(\mathbb{R}P^2)/(B_3(\mathbb{R}P^2))^{(2)} \longrightarrow 1,$$

it follows that  $B_3(\mathbb{R}P^2)/(B_3(\mathbb{R}P^2))^{(3)}$  is an extension of  $\mathbb{Z}_2^4$  by  $\text{Dih}_{12}$ , so is of order 192. This proves part (iii).

Now let us prove (iv). The first part, that  $(B_3(\mathbb{R}P^2))^{(2)}/(B_3(\mathbb{R}P^2))^{(3)} \cong \mathbb{Z}_2^4$ , follows easily by Abelianising the presentation of  $(B_3(\mathbb{R}P^2))^{(2)} \cong \mathbb{F}_3 \rtimes \mathcal{Q}_8$  given in Proposition 8. Letting  $\varphi$  denote the Abelianisation homomorphism, one observes that  $\varphi(z_1) = (\bar{1}, \bar{0}, \bar{0}, \bar{0})$ ,  $\varphi(z_2) = \varphi(z_3) = (\bar{0}, \bar{1}, \bar{0}, \bar{0})$ ,  $\varphi(x) = (\bar{0}, \bar{0}, \bar{1}, \bar{0})$ , and  $\varphi(y) = (\bar{0}, \bar{0}, \bar{0}, \bar{1})$ . The restriction of  $\varphi$  to  $\mathbb{F}_3$  is surjective onto the subgroup  $H$  of  $(B_3(\mathbb{R}P^2))^{(2)}/(B_3(\mathbb{R}P^2))^{(3)} \cong \mathbb{Z}_2^4$  generated by the  $\varphi(z_i)$ , so  $H \cong \mathbb{Z}_2^2$ . The quotient  $Q$  of  $(B_3(\mathbb{R}P^2))^{(2)}/(B_3(\mathbb{R}P^2))^{(3)}$  by  $H$  is thus isomorphic to the subgroup of  $(B_3(\mathbb{R}P^2))^{(2)}/(B_3(\mathbb{R}P^2))^{(3)}$  generated by  $\varphi(x)$  and  $\varphi(y)$ , and so is also isomorphic to  $\mathbb{Z}_2^2$ . Since  $\mathcal{Q}_8 = \langle x, y \rangle$ ,  $\varphi$  induces a surjective homomorphism  $\bar{\varphi}: \mathcal{Q}_8 \rightarrow Q$  whose kernel is  $\langle x^2 \rangle$ . But as an element of  $(B_3(\mathbb{R}P^2))^{(2)}$ ,  $x^2 = \Delta_3^2 \in \text{Ker}(\varphi)$ , and denoting  $\text{Ker}(\varphi|_{\mathbb{F}_3})$  by  $L$ , we obtain the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccc}
& & 1 & & 1 & & 1 \\
& & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & L & \longrightarrow & (B_3(\mathbb{R}P^2))^{(3)} & \longrightarrow & \langle \Delta_3^2 \rangle \longrightarrow 1 \\
& & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & \mathbb{F}_3 & \longrightarrow & (B_3(\mathbb{R}P^2))^{(2)} & \longrightarrow & \mathcal{Q}_8 \longrightarrow 1 \\
& & \downarrow \varphi|_{\mathbb{F}_3} & & \downarrow \varphi & & \downarrow \\
1 & \longrightarrow & H & \longrightarrow & (B_3(\mathbb{R}P^2))^{(2)}/(B_3(\mathbb{R}P^2))^{(3)} & \longrightarrow & Q \longrightarrow 1 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 1 & & 1 & & 1
\end{array}$$

Since  $\Delta_3^2 = x^2 \in (B_3(\mathbb{R}P^2))^{(3)}$ , it follows that the upper short exact sequence splits, and the fact that  $\langle \Delta_3^2 \rangle$  is central implies that the splitting gives rise to a direct product. We conclude that  $(B_3(\mathbb{R}P^2))^{(3)} \cong L \oplus \mathbb{Z}_2$ . Now  $L$  is the kernel of the homomorphism  $\varphi|_{\mathbb{F}_3}: \mathbb{F}_3(z_1, z_2, z_3) \rightarrow H$  which under identification of  $H$  with  $\mathbb{Z}_2 \oplus \mathbb{Z}_2$  sends  $z_1$  to  $(\bar{1}, \bar{0})$ , say, and  $z_2$  and  $z_3$  to  $(\bar{0}, \bar{1})$ . An application of the Reidemeister-Schreier rewriting process shows that  $L$  is a free group  $\mathbb{F}_9$  of rank 9. Thus  $(B_3(\mathbb{R}P^2))^{(3)} \cong \mathbb{F}_9 \oplus \mathbb{Z}_2$  and  $(B_3(\mathbb{R}P^2))^{(3)}/(B_3(\mathbb{R}P^2))^{(4)} \cong \mathbb{Z}_9 \oplus \mathbb{Z}_2$ . It is then clear that  $(B_3(\mathbb{R}P^2))^{(i)} \cong (\mathbb{F}_9)^{(i-3)}$  for all  $i \geq 4$ . From the short exact sequence

$$1 \longrightarrow (B_3(\mathbb{R}P^2))^{(3)}/(B_3(\mathbb{R}P^2))^{(4)} \longrightarrow B_3(\mathbb{R}P^2)/(B_3(\mathbb{R}P^2))^{(4)} \longrightarrow B_3(\mathbb{R}P^2)/(B_3(\mathbb{R}P^2))^{(3)} \longrightarrow 1,$$

we see that  $B_3(\mathbb{R}P^2)/(B_3(\mathbb{R}P^2))^{(4)}$  is an extension of  $\mathbb{Z}_9 \oplus \mathbb{Z}_2$  by a group of order 192, so is infinite. This proves part (iv), and completes the proof of Theorem 2.  $\square$

We obtain easily the following corollary of Theorem 2:

**Corollary 10.** *Let  $n \in \mathbb{N}$ ,  $n \neq 4$ . Then  $B_n(\mathbb{R}P^2)$  is residually soluble if and only if  $n \leq 3$ .  $\square$*

We now turn our attention to the remaining case,  $n = 4$ .

*Proof of Theorem 3.* Part (a) follows from the first paragraph of the proof of Proposition 8. So let us prove part (b). For this, we shall study the following presentation of the group  $(B_4(\mathbb{R}P^2))^{(1)}$  which may be deduced from Proposition 12 (the notation  $\alpha, \beta$  etc. is that of Proposition 12):

**generators:**

$$\begin{array}{lll}
B_1 = \eta_2 = \rho_2 \sigma_1^{-1} \rho_1^{-1} \sigma_1^{-1} & C_1 = \eta_3 = \rho_3 \sigma_1^{-1} \rho_1^{-1} \sigma_1^{-1} & D_1 = \eta_4 = \rho_4 \sigma_1^{-1} \rho_1^{-1} \sigma_1^{-1} \\
B_2 = \kappa_2 = \sigma_1 \rho_2 \rho_1^{-1} \sigma_1^{-1} & C_2 = \kappa_3 = \sigma_1 \rho_3 \rho_1^{-1} \sigma_1^{-1} & D_2 = \kappa_4 = \sigma_1 \rho_4 \rho_1^{-1} \sigma_1^{-1} \\
B_3 = \theta_2 = \sigma_1 \rho_1 \rho_2 \sigma_1^{-1} & C_3 = \theta_3 = \sigma_1 \rho_1 \rho_3 \sigma_1^{-1} & D_3 = \theta_4 = \sigma_1 \rho_1 \rho_4 \sigma_1^{-1} \\
B_4 = \lambda_2 = \sigma_1 \rho_1 \sigma_1 \rho_2 & C_4 = \lambda_3 = \sigma_1 \rho_1 \sigma_1 \rho_3 & D_4 = \lambda_4 = \sigma_1 \rho_1 \sigma_1 \rho_4 \\
Y_1 = \alpha_2 = \sigma_2 \sigma_1^{-1} & Z_1 = \alpha_3 = \sigma_3 \sigma_1^{-1} & A_1 = \eta_1 = \rho_1 \sigma_1^{-1} \rho_1^{-1} \sigma_1^{-1} \\
Y_2 = \beta_2 = \sigma_1 \sigma_2 & Z_2 = \beta_3 = \sigma_1 \sigma_3 & A_3 = \theta_1 = \sigma_1 \rho_1 \rho_1 \sigma_1^{-1} \\
Y_3 = \gamma_2 = \sigma_1 \rho_1 \sigma_2 \sigma_1^{-1} \rho_1^{-1} \sigma_1^{-1} & Z_3 = \gamma_3 = \sigma_1 \rho_1 \sigma_3 \sigma_1^{-1} \rho_1^{-1} \sigma_1^{-1} & A_4 = \lambda_1 = \sigma_1 \rho_1 \sigma_1 \rho_1 \\
Y_4 = \tau_2 = \sigma_1 \rho_1 \sigma_1 \sigma_2 \rho_1^{-1} \sigma_1^{-1} & Z_4 = \tau_3 = \sigma_1 \rho_1 \sigma_1 \sigma_3 \rho_1^{-1} \sigma_1^{-1} & X_2 = \beta_1 = \sigma_1^2 \\
& & X_4 = \tau_1 = \sigma_1 \rho_1 \sigma_1^2 \rho_1^{-1} \sigma_1^{-1}.
\end{array}$$

For notational reasons, we set  $A_2 = X_1 = X_3 = 1$ .

**relators:**

$$\begin{array}{lll}
Z_2 X_2^{-1} Z_1^{-1} & (22) & X_4 Y_3 X_4 Y_4^{-2} & (29) & C_4 X_2^{-1} C_3^{-1} & (36) \\
X_2 Z_1 Z_2^{-1} & (23) & Y_1 Z_2 Y_1 Z_1^{-1} Y_2^{-1} Z_1^{-1} & (30) & X_4 C_3 C_4^{-1} & (37) \\
Z_4 X_4^{-1} Z_3^{-1} & (24) & Y_2 Z_1 Y_2 Z_2^{-1} Y_1^{-1} Z_2^{-1} & (31) & D_2 X_4^{-1} D_1^{-1} & (38) \\
X_4 Z_3 Z_4^{-1} & (25) & Y_3 Z_4 Y_3 Z_3^{-1} Y_4^{-1} Z_3^{-1} & (32) & X_2 D_1 D_2^{-1} & (39) \\
Y_2 Y_1^{-1} X_2^{-1} Y_1^{-1} & (26) & Y_4 Z_3 Y_4 Z_4^{-1} Y_3^{-1} Z_4^{-1} & (33) & D_4 X_2^{-1} D_3^{-1} & (40) \\
X_2 Y_1 X_2 Y_2^{-2} & (27) & C_2 X_4^{-1} C_1^{-1} & (34) & X_4 D_3 D_4^{-1} & (41) \\
Y_4 Y_3^{-1} X_4^{-1} Y_3^{-1} & (28) & X_2 C_1 C_2^{-1} & (35) & Y_1 Y_4^{-1} A_1^{-1} & (42) \\
\\
Y_2 A_1 Y_3^{-1} & (43) & Z_1 Z_4^{-1} A_1^{-1} & (50) & Z_4 B_3 Z_1^{-1} B_4^{-1} & (57) \\
Y_3 A_4 Y_2^{-1} A_3^{-1} & (44) & Z_2 A_1 Z_3^{-1} & (51) & B_1 X_4 X_2 & (58) \\
Y_4 A_3 Y_1^{-1} A_4^{-1} & (45) & Z_3 A_4 Z_2^{-1} A_3^{-1} & (52) & B_2 A_1^{-1} & (59) \\
Y_1 D_2 Y_4^{-1} D_1^{-1} & (46) & Z_4 A_3 Z_1^{-1} A_4^{-1} & (53) & X_4^{-1} A_4 X_2^{-1} B_3^{-1} & (60) \\
Y_2 D_1 Y_3^{-1} D_2^{-1} & (47) & Z_1 B_2 Z_4^{-1} B_1^{-1} & (54) & A_3 B_4^{-1} & (61) \\
Y_3 D_4 Y_2^{-1} D_3^{-1} & (48) & Z_2 B_1 Z_3^{-1} B_2^{-1} & (55) & Y_2^{-1} B_2 Y_4^{-1} C_1^{-1} & (62) \\
Y_4 D_3 Y_1^{-1} D_4^{-1} & (49) & Z_3 B_4 Z_2^{-1} B_3^{-1} & (56) & Y_1^{-1} B_1 Y_3^{-1} C_2^{-1} & (63) \\
\\
Y_4^{-1} B_4 Y_2^{-1} C_3^{-1} & (64) & B_2^{-1} A_3^{-1} B_3 X_4^{-1} & (72) & D_2^{-1} C_3^{-1} D_3 C_2 Z_4^{-1} Z_3^{-1} & (80) \\
Y_3^{-1} B_3 Y_1^{-1} C_4^{-1} & (65) & B_1^{-1} A_4^{-1} B_4 A_1 X_4^{-1} & (73) & D_1^{-1} C_4^{-1} D_4 C_1 Z_3^{-1} Z_4^{-1} & (81) \\
Z_1^{-1} C_1 Z_3^{-1} D_2^{-1} & (66) & C_4^{-1} B_1^{-1} C_1 B_4 Y_2^{-1} Y_1^{-1} & (74) & Y_2 Z_1 Z_2 Y_1 X_2 A_4^{-1} A_1^{-1} & (82) \\
Z_2^{-1} C_2 Z_4^{-1} D_1^{-1} & (67) & C_3^{-1} B_2^{-1} C_2 B_3 Y_1^{-1} Y_2^{-1} & (75) & X_2 Y_1 Z_2 Z_1 Y_2 X_2 A_3^{-1} & (83) \\
Z_3^{-1} C_3 Z_1^{-1} D_4^{-1} & (68) & C_2^{-1} B_3^{-1} C_3 B_2 Y_4^{-1} Y_3^{-1} & (76) & Y_4 Z_3 Z_4 Y_3 X_4 A_3^{-1} & (84) \\
Z_4^{-1} C_4 Z_2^{-1} D_3^{-1} & (69) & C_1^{-1} B_4^{-1} C_4 B_1 Y_3^{-1} Y_4^{-1} & (77) & X_4 Y_3 Z_4 Z_3 Y_4 A_1^{-1} A_4^{-1} & (85) \\
B_4^{-1} A_1^{-1} B_1 A_4 X_2^{-1} & (70) & D_4^{-1} C_1^{-1} D_1 C_4 Z_2^{-1} Z_1^{-1} & (78) & & \\
B_2 A_3 X_2^{-1} B_3^{-1} & (71) & D_3^{-1} C_2^{-1} D_2 C_3 Z_1^{-1} Z_2^{-1} & (79) & & 
\end{array}$$

We now Abelianise this presentation to deduce that  $(B_4(\mathbb{R}P^2))^{(1)}/(B_4(\mathbb{R}P^2))^{(2)} \cong \mathbb{Z}_3$ . We could do this directly, but it will be convenient for what follows to carry out a partial Abelianisation first. Let  $\Lambda$  denote the group obtained from the above presentation of  $(B_4(\mathbb{R}P^2))^{(1)}$  by adding the relations that the following generators commute pairwise:  $A_i, B_i, C_i, D_i, X_i, Z_i$  for  $i = 1, \dots, 4$  i.e. all of the generators of  $(B_4(\mathbb{R}P^2))^{(1)}$  commute pairwise, with the exception of the  $Y_i$ . From equations (22), (35), (36), (39) and (40), we have

$$X_2 = Z_2 Z_1^{-1} = C_2 C_1^{-1} = D_2 D_1^{-1} = C_3^{-1} C_4 = D_3^{-1} D_4 \quad (86)$$

and from equations (24), (34), (37), (38) and (41), we have

$$X_4 = Z_4 Z_3^{-1} = C_2 C_1^{-1} = D_2 D_1^{-1} = C_3^{-1} C_4 = D_3^{-1} D_4. \quad (87)$$

So

$$X_2 = X_4 \quad (88)$$

and

$$Z_2 Z_3 = Z_1 Z_4. \quad (89)$$

Now from equations (42), (43), (50), (51) and (59), we have

$$A_1 = B_2 = Y_1 Y_4^{-1} = Y_2^{-1} Y_3 = Z_1 Z_4^{-1} = Z_2^{-1} Z_3, \quad (90)$$

hence

$$Z_1 Z_2 = Z_3 Z_4. \quad (91)$$

Multiplying equations (89) and (91) yields

$$Z_1^2 = Z_3^2, \quad Z_2^2 = Z_4^2. \quad (92)$$

Further, from equations (52), (53), (58), (61), (86), (87) and (91), we have

$$B_4 = A_3 = Z_3 A_4 Z_2^{-1} = Z_1 Z_4^{-1} A_4 \quad (93)$$

$$B_1 = X_2^{-1} X_4^{-1} = Z_1 Z_2^{-1} Z_3 Z_4^{-1} = Z_1^2 Z_4^{-2}. \quad (94)$$

Substituting equations (90), (93) and (94) into equation (70) yields:

$$X_2 = B_4^{-1} A_1^{-1} B_1 A_4 = A_4^{-1} Z_4 Z_1^{-1} Z_4 Z_1^{-1} Z_1^2 Z_4^{-2} A_4 = 1.$$

Hence

$$X_2 = X_4 = 1, \quad Z_1 = Z_2, \quad Z_3 = Z_4, \quad Z_1^2 = Z_2^2 = Z_3^2 = Z_4^2 \quad (95)$$

$$C_1 = C_2, \quad D_1 = D_2, \quad C_3 = C_4, \quad D_3 = D_4, \quad B_1 = 1 \quad (96)$$

by equations (86), (87), (88), (92) and (94). From equations (26), (27), (28) and (29), we have  $Y_2 = Y_1^2$ ,  $Y_1 = Y_2^2$ ,  $Y_4 = Y_3^2$ ,  $Y_3 = Y_4^2$ , so

$$Y_2 = Y_1^{-1}, \quad Y_4 = Y_3^{-1}, \quad \text{and } Y_i^3 = 1 \text{ for all } i = 1, \dots, 4. \quad (97)$$

Using equations (30), (31) and (95), we see that

$$1 = Y_1 Z_2 Y_1 Z_1^{-1} Y_2^{-1} Z_1^{-1} = Y_1 Z_1 Y_1 Z_1^{-1} Y_1 Z_1^{-1} \quad (98)$$

and

$$1 = Y_2 Z_1 Y_2 Z_2^{-1} Y_1^{-1} Z_2^{-1} = Y_1^{-1} Z_1 Y_1^{-1} Z_1^{-1} Y_1^{-1} Z_1^{-1}. \quad (99)$$

Inverting equation (99) and conjugating by  $Z_1^{-1}$ , we obtain

$$1 = Y_1 Z_1 Y_1 Z_1^{-1} Y_1 Z_1. \quad (100)$$

Comparing equations (98) and (100) yields

$$Z_1^2 = 1. \quad (101)$$

From this and equations (90) and (95), it follows that

$$A_1^2 = 1. \quad (102)$$

By equations (60), (82), (90), (83), (93), (95), (97), (101) and (102), we see that

$$A_1 = A_4 = B_2 = B_3, \quad A_3 = B_4 = 1. \quad (103)$$

Now

$$C_1 = Y_2^{-1} B_2 Y_4^{-1} = Y_1 A_1 Y_3 = Y_1^{-1} Y_3^{-1} \quad (104)$$

by equations (42), (62), (97) and (104), and

$$C_3 = Y_4^{-1} B_4 Y_2^{-1} = Y_3 Y_1 \quad (105)$$

by equations (64), (97) and (103), hence

$$C_1 = C_2 = C_3^{-1} = C_4^{-1} \quad (106)$$

by equation (96). Similarly,

$$D_1 = Z_2^{-1} C_2 Z_4^{-1} = Z_1^{-1} Y_1^{-1} Y_3^{-1} Z_3^{-1} \quad (107)$$

by equations (67), (95), (96) and (104), and

$$D_3 = Z_4^{-1} C_4 Z_2^{-1} = Z_3^{-1} Y_3 Y_1 Z_1^{-1} \quad (108)$$

by equations (69), (95), (96) and (105), hence

$$D_1 = D_2 = D_3^{-1} = D_4^{-1} \quad (109)$$

by equations (95), (96) and (101). Using equations (75), (97), (102), (103), (105) and (106), we see that

$$1 = Y_1^{-1} Y_2^{-1} = B_3^{-1} C_2^{-1} B_2 C_3 = A_1 C_2^{-1} A_1 C_3 = C_3^2 = (Y_3 Y_1)^2.$$

Hence  $Y_3 Y_1 = Y_1^{-1} Y_3^{-1}$ , and so

$$C_1 = C_2 = C_3 = C_4, \quad D_1 = D_2 = D_3 = D_4 \quad (110)$$

by equations (106), (107), (108) and (109), as well as the fact that  $C_2, C_4, Z_2$  and  $Z_4$  commute pairwise. We deduce also from equations (109) and (110) that  $D_i^2 = 1$  for all  $i = 1, \dots, 4$ . Let  $C$  (resp.  $D$ ) denote the common value of the  $C_i$  (resp.  $D_i$ ), and let

$A = A_1 = A_4 = B_2 = B_3$ . Running through the relations (22)–(85) one by one, we see that our group  $\Lambda$  has generators  $Y_1, Y_3, Z_1, Z_3, A, C$  and  $D$  with the following defining relations:

$$\begin{cases} Y_1^3 = Y_3^3 = (Y_1 Z_1)^3 = (Y_3 Z_3)^3 = Z_1^2 = Z_3^2 = A^2 = C^2 = D^2 = 1 \\ A = CD = Y_1 Y_3 = Z_1 Z_3, C = Y_3 Y_1, Y_1 D Y_3 D = 1, (Y_3 Y_1)^2 = 1 \\ A, C, D, Z_1 \text{ and } Z_3 \text{ commute pairwise.} \end{cases} \quad (111)$$

Notice that we may write  $D = Y_3 Y_1 Z_3 Z_1$ , and so  $Y_1, Y_3, Z_1, Z_3$  generate  $\Lambda$ .

If we now Abelianise  $(B_4(\mathbb{R}P^2))^{(1)}$  completely by adding the relations that the  $Y_i$  commute pairwise with each of the generators of  $\Lambda$ , we see that  $A = C = D = Z_1 = Z_3 = 1$ ,  $Y_3 = Y_1^{-1}$ ,  $Y_1^3 = 1$ , and thus  $(B_4(\mathbb{R}P^2))^{(1)}/(B_4(\mathbb{R}P^2))^{(2)} \cong \mathbb{Z}_3$ . We underline the fact that under the complete Abelianisation of  $(B_4(\mathbb{R}P^2))^{(1)}$ , the generators  $A_i, B_i, C_i, D_i, X_i, Z_i$ ,  $i = 1, \dots, 4$ , of  $(B_4(\mathbb{R}P^2))^{(1)}$  are sent to the trivial element, and  $Y_1, Y_2^{-1}, Y_3^{-1}$  and  $Y_4$  are sent to the same generator of  $(B_4(\mathbb{R}P^2))^{(1)}/(B_4(\mathbb{R}P^2))^{(2)}$ . Taking  $b = \rho_3 \sigma_2 \sigma_1 = \sigma_2^{-1} \sigma_1^{-1} \rho_1 \in B_4(\mathbb{R}P^2)$  which we know to be of order 12, consider  $b^4$ . Since  $b^3 = \rho_3 \rho_2 \rho_1$  by Remark 9(b), we have

$$b^4 = \rho_3 \rho_2 \rho_1 \cdot \rho_3 \sigma_2 \sigma_1 = C_1 B_4 A_1 C_4 Y_1 X_2 \in (B_4(\mathbb{R}P^2))^{(1)}.$$

Under Abelianisation,  $b^4$  is thus sent to the  $(B_4(\mathbb{R}P^2))^{(2)}$ -coset of  $Y_1$  which is a generator of  $(B_4(\mathbb{R}P^2))^{(1)}/(B_4(\mathbb{R}P^2))^{(2)}$ . Since  $b^4$  is of order 3, it follows that the short exact sequence

$$1 \longrightarrow (B_4(\mathbb{R}P^2))^{(2)} \longrightarrow (B_4(\mathbb{R}P^2))^{(1)} \longrightarrow (B_4(\mathbb{R}P^2))^{(1)}/(B_4(\mathbb{R}P^2))^{(2)} \longrightarrow 1$$

splits, and hence

$$(B_4(\mathbb{R}P^2))^{(1)} \cong (B_4(\mathbb{R}P^2))^{(2)} \rtimes \mathbb{Z}_3,$$

where the action on  $(B_4(\mathbb{R}P^2))^{(2)}$  is given by conjugation by  $b^4$ . This proves part (b)(i). To prove part (b)(ii), consider the short exact sequence

$$1 \longrightarrow (B_4(\mathbb{R}P^2))^{(1)}/(B_4(\mathbb{R}P^2))^{(2)} \longrightarrow B_4(\mathbb{R}P^2)/(B_4(\mathbb{R}P^2))^{(2)} \longrightarrow B_4(\mathbb{R}P^2)/(B_4(\mathbb{R}P^2))^{(1)} \longrightarrow 1.$$

As in part (c)(iii) of the proof of Theorem 2, since  $(B_4(\mathbb{R}P^2))^{(1)}/(B_4(\mathbb{R}P^2))^{(2)} \cong \mathbb{Z}_3$  and  $B_4(\mathbb{R}P^2)/(B_4(\mathbb{R}P^2))^{(1)} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ , to prove that  $B_4(\mathbb{R}P^2)/(B_4(\mathbb{R}P^2))^{(2)} \cong \text{Dih}_{12}$ , it suffices to show that the action of  $B_4(\mathbb{R}P^2)/(B_4(\mathbb{R}P^2))^{(1)}$  on the kernel is non trivial. To achieve this, notice that the action by conjugation of  $\sigma_1$  (which is a representative of the generator  $\bar{\sigma}$  of  $B_4(\mathbb{R}P^2)/(B_4(\mathbb{R}P^2))^{(1)}$ ) on  $Y_1$  (which from above is a representative of a generator of  $(B_4(\mathbb{R}P^2))^{(1)}/(B_4(\mathbb{R}P^2))^{(2)}$ ) is given by  $\sigma_1 Y_1 \sigma_1^{-1} = \sigma_1 \sigma_2 \sigma_1^{-2} = Y_2 X_2^{-1}$ . Now modulo  $(B_4(\mathbb{R}P^2))^{(2)}$ ,  $Y_2 X_2^{-1}$  is congruent to  $Y_2$ , which in turn is congruent to  $Y_1^{-1}$ . The action of  $B_4(\mathbb{R}P^2)/(B_4(\mathbb{R}P^2))^{(1)}$  on  $(B_4(\mathbb{R}P^2))^{(1)}/(B_4(\mathbb{R}P^2))^{(2)}$  is thus non trivial, which proves that  $B_4(\mathbb{R}P^2)/(B_4(\mathbb{R}P^2))^{(2)} \cong \text{Dih}_{12}$ , and completes the proof of part (b)(ii).

To prove part (b)(iii), let  $n = 4$  in the commutative diagram (9) of short exact sequences. Recall that in the lower sequence,  $(B_4(\mathbb{R}P^2))^{\text{Ab}} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$  is generated by two elements  $\bar{\sigma}$  and  $\bar{\rho}$ ,  $\text{Ker}(\bar{\tau}) = \langle \bar{\rho} \rangle$ , and  $\bar{\tau}(\bar{\sigma})$ , which we also denote by  $\bar{\sigma}$ , is the generator of the quotient  $(B_4(\mathbb{R}P^2))^{\text{Ab}}/\langle \bar{\rho} \rangle$ . From the discussion following equation (9),

$K$  is of index 2 in  $P_4(\mathbb{R}P^2)$ . Furthermore, the homomorphism  $\alpha'$  sends the generator  $B_{i,j}$ ,  $1 \leq i < j \leq 4$  (resp.  $\rho_k$ ,  $1 \leq k \leq 4$ ) to the trivial element of  $\langle \bar{\rho} \rangle$  (resp. to  $\bar{\rho}$ ). This diagram may be continued vertically by taking commutator subgroups successively; in this way, we obtain the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccc}
1 & \longrightarrow & K'' & \longrightarrow & (B_4(\mathbb{R}P^2))^{(3)} & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & K' & \longrightarrow & (B_4(\mathbb{R}P^2))^{(2)} & \longrightarrow & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \\
& & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & K & \longrightarrow & (B_4(\mathbb{R}P^2))^{(1)} & \longrightarrow & A_4
\end{array} \tag{112}$$

The vertical arrows are inclusions, and  $K'$  (resp.  $K''$ ) is the kernel of the restriction of  $\tau$  to  $(B_4(\mathbb{R}P^2))^{(2)}$  (resp. to  $(B_4(\mathbb{R}P^2))^{(3)}$ ). So  $K'' = (B_4(\mathbb{R}P^2))^{(3)}$ , and since the index of  $(B_4(\mathbb{R}P^2))^{(2)}$  (resp.  $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ ) in  $(B_4(\mathbb{R}P^2))^{(1)}$  (resp.  $A_4$ ) is three, we deduce that  $K' = K$ , which proves part (b)(iii).

We now prove part (c). We start by studying the quotient  $(B_4(\mathbb{R}P^2))^{(1)}/(B_4(\mathbb{R}P^2))^{(3)}$ . As we saw above, the elements  $A_i, B_i, C_i, D_i, X_i, Z_i$ ,  $i = 1, \dots, 4$  of  $(B_4(\mathbb{R}P^2))^{(1)}$  are sent to the trivial element of  $(B_4(\mathbb{R}P^2))^{(1)}/(B_4(\mathbb{R}P^2))^{(2)}$ , and so belong to  $(B_4(\mathbb{R}P^2))^{(2)}$ . Hence considered as elements of  $(B_4(\mathbb{R}P^2))^{(1)}/(B_4(\mathbb{R}P^2))^{(3)}$  they commute pairwise (we shall not distinguish notationally between elements of  $(B_4(\mathbb{R}P^2))^{(1)}$  and their cosets in  $(B_4(\mathbb{R}P^2))^{(1)}/(B_4(\mathbb{R}P^2))^{(3)}$ ). These were precisely the relations that we added to those of  $(B_4(\mathbb{R}P^2))^{(1)}$  in order to obtain the presentation (111) of  $\Lambda$ , and thus the relations of  $\Lambda$  hold in  $(B_4(\mathbb{R}P^2))^{(1)}/(B_4(\mathbb{R}P^2))^{(3)}$ . In particular,  $(B_4(\mathbb{R}P^2))^{(1)}/(B_4(\mathbb{R}P^2))^{(3)}$  is a quotient of  $\Lambda$ .

Since  $Z_1, Z_3 \in (B_4(\mathbb{R}P^2))^{(2)}$ , we have that  $Y_1 Z_1 Y_1^{-1}, Y_1 Z_3 Y_1^{-1} \in (B_4(\mathbb{R}P^2))^{(2)}$ . Let  $G$  denote the group obtained from  $\Lambda$  by adding the following relations to the presentation (111) of  $\Lambda$ :

$$\left\{ \begin{array}{l} Z_1, Z_3, Y_1 Z_1 Y_1^{-1}, Y_1 Z_3 Y_1^{-1} \text{ commute pairwise} \\ \text{and commute with } Z_1, Z_3, A, C \text{ and } D. \end{array} \right. \tag{113}$$

Once more, considered as elements of  $B_4(\mathbb{R}P^2)$ ,  $Z_1, Z_3, Y_1 Z_1 Y_1^{-1}, Y_1 Z_3 Y_1^{-1}, A, C$  and  $D$  belong to  $(B_4(\mathbb{R}P^2))^{(2)}$ , and so the commutation relations of equation (113) of  $G$  also hold in  $(B_4(\mathbb{R}P^2))^{(1)}/(B_4(\mathbb{R}P^2))^{(3)}$ . This implies that  $(B_4(\mathbb{R}P^2))^{(1)}/(B_4(\mathbb{R}P^2))^{(3)}$  is also a quotient of  $G$ .

We now determine  $G$  and its relationship with  $(B_4(\mathbb{R}P^2))^{(1)}/(B_4(\mathbb{R}P^2))^{(3)}$ . Let  $L$  be the group with generators  $w_1, w_2, w_3, w_4, t$  and defining relations:

$$\left\{ \begin{array}{l} \text{for all } 1 \leq i, j \leq 4, w_i^2 = t^3 = 1, w_i w_j = w_j w_i, \\ t w_1 t^{-1} = w_2, t w_2 t^{-1} = w_1 w_2, t w_3 t^{-1} = w_4, t w_4 t^{-1} = w_3 w_4. \end{array} \right. \tag{114}$$

Clearly  $L$  is isomorphic to  $\mathbb{Z}_2^4 \rtimes \mathbb{Z}_3$ , where the action of conjugation by  $t$  on  $\langle w_1, \dots, w_4 \rangle$  permutes cyclically the elements  $w_1, w_2$  and  $w_1 w_2$  (resp.  $w_3, w_4$  and  $w_3 w_4$ ). We define a map  $\psi: L \rightarrow G$  on the generators of  $L$  as follows:

$$\psi(w_1) = Z_1, \psi(w_2) = Y_1 Z_1 Y_1^{-1}, \psi(w_3) = Z_3, \psi(w_4) = Y_1 Z_3 Y_1^{-1}, \psi(t) = Y_1.$$

Since  $Z_1^2 = Z_3^2 = 1$  and  $Y_1^3 = 1$  in  $G$ , we clearly have  $(\psi(w_i))^2 = (\psi(t))^3 = 1$  for  $i = 1, \dots, 4$ . The relations (113) of  $G$  imply that the  $\psi(w_i)$  commute pairwise. Further,  $\psi(t)\psi(w_1)(\psi(t))^{-1} = \psi(w_2)$  by definition, and

$$\begin{aligned}\psi(t)\psi(w_2)(\psi(t))^{-1} &= (\psi(t))^2\psi(w_1)(\psi(t))^{-2} = Y_1^2 Z_1 Y_1^{-2} = Y_1^{-1} Z_1 Y_1^{-2} = Z_1 Y_1 Z_1 Y_1^{-1} \\ &= \psi(w_1)\psi(w_2),\end{aligned}$$

using the relations  $Y_1^3 = 1$  and  $(Y_1 Z_1)^3 = 1$  of (111). Similar relations hold for  $w_3$  and  $w_4$ , and hence  $\psi$  extends to a homomorphism from  $L$  to  $G$ . Now  $\psi$  is surjective because the generating set  $\{Z_1, Z_3, Y_1, Y_3, A, C, D\}$  of  $G$  may be reduced to  $\{Z_1, Z_3, Y_1, Y_3\}$  using the relations (111). Thus  $G$  is a quotient of  $L$ , and hence  $(B_4(\mathbb{R}P^2))^{(1)}/(B_4(\mathbb{R}P^2))^{(3)}$  is also a quotient of  $L$ .

Let us now show that the groups  $L$  and  $(B_4(\mathbb{R}P^2))^{(1)}/(B_4(\mathbb{R}P^2))^{(3)}$  are isomorphic. Consider the map  $\varphi: (B_4(\mathbb{R}P^2))^{(1)} \rightarrow L$  defined on the generators of  $(B_4(\mathbb{R}P^2))^{(1)}$  as follows:

$$\left\{ \begin{array}{l} \varphi(X_2) = \varphi(X_4) = \varphi(A_3) = \varphi(B_1) = \varphi(B_4) = 1 \\ \varphi(A_1) = \varphi(A_4) = \varphi(B_2) = \varphi(B_3) = w_1 w_3 \\ \varphi(C_1) = \varphi(C_2) = \varphi(C_3) = \varphi(C_4) = w_1 w_2 w_3 w_4 \\ \varphi(D_1) = \varphi(D_2) = \varphi(D_3) = \varphi(D_4) = w_2 w_4 \\ \varphi(Z_1) = \varphi(Z_2) = w_1, \varphi(Z_3) = \varphi(Z_4) = w_3 \\ \varphi(Y_1) = t, \varphi(Y_2) = t^2, \varphi(Y_3) = (w_1 w_2 w_3 w_4) t^2, \varphi(Y_4) = (w_1 w_3) t. \end{array} \right. \quad (115)$$

A long but straightforward calculation shows that each of the relators (22)–(85) of  $(B_4(\mathbb{R}P^2))^{(1)}$  is sent to the trivial element of  $L$ , and hence  $\varphi$  extends to a surjective homomorphism of  $(B_4(\mathbb{R}P^2))^{(1)}$  onto  $L$ . Such a homomorphism sends  $((B_4(\mathbb{R}P^2))^{(1)})^{(2)} = (B_4(\mathbb{R}P^2))^{(3)}$  surjectively onto  $L^{(2)}$ . However  $L^{(2)}$  is trivial, so  $\varphi$  induces a surjective homomorphism  $\bar{\varphi}$  of  $(B_4(\mathbb{R}P^2))^{(1)}/(B_4(\mathbb{R}P^2))^{(3)}$  onto  $L$ , and hence  $L$  is a quotient of  $(B_4(\mathbb{R}P^2))^{(1)}/(B_4(\mathbb{R}P^2))^{(3)}$ . Since  $L$  is finite and  $(B_4(\mathbb{R}P^2))^{(1)}/(B_4(\mathbb{R}P^2))^{(3)}$  is a quotient of  $L$  by the previous paragraph, we conclude that

$$(B_4(\mathbb{R}P^2))^{(1)}/(B_4(\mathbb{R}P^2))^{(3)} \cong L \cong \mathbb{Z}_2^4 \rtimes \mathbb{Z}_3, \quad (116)$$

where the action is given by equation (114). Further,  $\psi: L \rightarrow G$  is surjective and  $(B_4(\mathbb{R}P^2))^{(1)}/(B_4(\mathbb{R}P^2))^{(3)}$  is a quotient of  $G$ , so  $G = (B_4(\mathbb{R}P^2))^{(1)}/(B_4(\mathbb{R}P^2))^{(3)}$ . An easy calculation shows that  $\psi^{-1} = \bar{\varphi}$ . From the short exact sequence

$$\begin{aligned} 1 \longrightarrow (B_4(\mathbb{R}P^2))^{(2)}/(B_4(\mathbb{R}P^2))^{(3)} \longrightarrow (B_4(\mathbb{R}P^2))^{(1)}/(B_4(\mathbb{R}P^2))^{(3)} \longrightarrow \\ (B_4(\mathbb{R}P^2))^{(1)}/(B_4(\mathbb{R}P^2))^{(2)} \longrightarrow 1, \end{aligned}$$

we see that  $(B_4(\mathbb{R}P^2))^{(2)}/(B_4(\mathbb{R}P^2))^{(3)} \cong \mathbb{Z}_2^4$ . It follows from the form of the isomorphism that the  $\mathbb{Z}_2$ -factors of  $(B_4(\mathbb{R}P^2))^{(2)}/(B_4(\mathbb{R}P^2))^{(3)}$  are generated by the elements  $Z_1, Z_3, Y_1 Z_1 Y_1^{-1}$  and  $Y_1 Z_3 Y_1^{-1}$ , and their images under  $\bar{\varphi}$  are  $w_1, w_3, w_2$  and  $w_4$  respectively. In particular,  $\bar{\varphi} \left( (B_4(\mathbb{R}P^2))^{(2)}/(B_4(\mathbb{R}P^2))^{(3)} \right) = \langle w_1, w_2, w_3, w_4 \rangle$ . This completes the proof of part (c).

We now prove part (d). Consider the commutative diagrams (9) and (112). Since  $K = K'$  from part (b)(iii) above, we have that  $K \subset (B_4(\mathbb{R}P^2))^{(2)} \cap P_4(\mathbb{R}P^2)$ . Conversely,

since the homomorphism  $(B_4(\mathbb{R}P^2))^{(2)} \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2$  of equation (112) is the restriction of the permutation homomorphism  $\tau: B_4(\mathbb{R}P^2) \longrightarrow S_4$  to  $(B_4(\mathbb{R}P^2))^{(2)}$ , it follows that any element of  $(B_4(\mathbb{R}P^2))^{(2)} \cap P_4(\mathbb{R}P^2)$  also belongs to  $K$ , and thus

$$K = (B_4(\mathbb{R}P^2))^{(2)} \cap P_4(\mathbb{R}P^2). \quad (117)$$

Further, from the upper exact sequence of equation (112),  $(B_4(\mathbb{R}P^2))^{(3)} \subset K$ , and since  $(B_4(\mathbb{R}P^2))^{(3)}$  is normal in  $B_4(\mathbb{R}P^2)$ , we obtain

$$1 \longrightarrow K/(B_4(\mathbb{R}P^2))^{(3)} \longrightarrow (B_4(\mathbb{R}P^2))^{(2)}/(B_4(\mathbb{R}P^2))^{(3)} \longrightarrow (B_4(\mathbb{R}P^2))^{(2)}/K \longrightarrow 1$$

by taking the quotient by  $(B_4(\mathbb{R}P^2))^{(3)}$  of the first two terms of the middle short exact sequence of equation (112). In particular,  $K/(B_4(\mathbb{R}P^2))^{(3)} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ , and we have a short exact sequence

$$1 \longrightarrow (B_4(\mathbb{R}P^2))^{(3)} \longrightarrow K \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 \longrightarrow 1. \quad (118)$$

Recall that the  $\mathbb{Z}_2$ -factors of  $(B_4(\mathbb{R}P^2))^{(2)}/(B_4(\mathbb{R}P^2))^{(3)}$  are generated by  $Z_1, Z_3, Y_1 Z_1 Y_1^{-1}$  and  $Y_1 Z_3 Y_1^{-1}$ . Using equation (117) and the expressions for  $Z_1, Z_3$  and  $Y_1$  in terms of the standard generators of  $B_4(\mathbb{R}P^2)$ , we conclude that

$$K/(B_4(\mathbb{R}P^2))^{(3)} = \left\{ 1, Z_1 Z_3, Y_1 Z_1 Z_3 Y_1^{-1}, Z_1 Z_3 Y_1 Z_1 Z_3 Y_1^{-1} \right\}. \quad (119)$$

We now apply the Reidemeister-Schreier rewriting process to the leftmost vertical short exact sequence of (9) to produce a set of generators of  $K$ . Taking  $\{1, \rho_1\}$  as a Schreier transversal of  $\langle \bar{\rho} \rangle$  in  $P_4(\mathbb{R}P^2)$  and  $\{B_{i,j}, \rho_k \mid 1 \leq i < j \leq 4, 1 \leq k \leq 4\}$  as a generating set of  $P_4(\mathbb{R}P^2)$ , we see that the following elements constitute a generating set of  $K$ :

$$\begin{cases} B_{1,2} = X_2, B_{1,3} = Y_1 X_2 Y_1^{-1}, B_{1,4} = Z_1 Y_2 X_2 Y_2^{-1} Z_1^{-1}, B_{2,3} = Y_1 Y_2 \\ B_{2,4} = Z_1 Y_2 Y_1 Z_1^{-1}, B_{3,4} = Z_1 Z_2, \rho_1 B_{1,2} \rho_1^{-1} = A_1 X_4 A_1^{-1} \\ \rho_1 B_{1,3} \rho_1^{-1} = A_1 Y_4 X_4 Y_4^{-1} A_1^{-1}, \rho_1 B_{1,4} \rho_1^{-1} = A_1 Z_4 Y_3 X_4 Y_3^{-1} Z_4^{-1} A_1^{-1} \\ \rho_1^2 = A_1 A_4, \rho_2^2 = B_1 B_4, \rho_3^2 = C_1 C_4, \rho_4^2 = D_1 D_4 \\ \rho_1 \rho_2 = A_1 B_4, \rho_1 \rho_3 = A_1 C_4, \rho_1 \rho_4 = A_1 D_4. \end{cases} \quad (120)$$

Note that we have also written each element in terms of the generators of the presentation of  $(B_4(\mathbb{R}P^2))^{(1)}$  given at the beginning of the proof, we have deleted  $\rho_1 B_{2,3} \rho_1^{-1}$  and  $\rho_1 B_{3,4} \rho_1^{-1}$  from the list of generators that appear initially in the process, and that for  $i = 2, 3, 4$ , we have replaced  $\rho_i \rho_1^{-1}$  by  $\rho_i^2 = \rho_i \rho_1^{-1} \cdot \rho_1 \rho_i$ . Since  $(B_4(\mathbb{R}P^2))^{(3)} \subset P_4(\mathbb{R}P^2)$ , we may consider the image of  $(B_4(\mathbb{R}P^2))^{(3)}$  in  $P_3(\mathbb{R}P^2)$  and  $P_2(\mathbb{R}P^2)$  under the projections

$$p_3: P_4(\mathbb{R}P^2) \longrightarrow P_3(\mathbb{R}P^2) \text{ and } p_2: P_3(\mathbb{R}P^2) \longrightarrow P_2(\mathbb{R}P^2)$$

obtained geometrically by forgetting the last string in each case. We claim that  $p_2 \circ p_3 \left( (B_4(\mathbb{R}P^2))^{(3)} \right) = \langle \rho_1 \rho_2 \rangle \cong \mathbb{Z}_4$ . To see this, we first use equation (118) and the Reidemeister-Schreier rewriting process to obtain a generating set for  $(B_4(\mathbb{R}P^2))^{(3)}$ . This is achieved as follows. From equation (120), we see that the elements of the set

$$\mathcal{T} = \left\{ 1, \rho_2 \rho_1^{-1}, \rho_2 \rho_1^{-1} \cdot \rho_3 \rho_1^{-1}, \rho_2 \rho_1^{-1} \cdot \rho_3 \rho_1^{-1} \cdot \rho_1 \rho_2^{-1} \right\}$$

belong to  $K$ . Equations (115) and (119) give rise to the following commutative diagram:

$$\begin{array}{ccccc}
K/(B_4(\mathbb{R}P^2))^{(3)} & \longrightarrow & (B_4(\mathbb{R}P^2))^{(2)}/(B_4(\mathbb{R}P^2))^{(3)} & \longrightarrow & (B_4(\mathbb{R}P^2))^{(1)}/(B_4(\mathbb{R}P^2))^{(3)} \\
\downarrow & & \downarrow & & \downarrow \\
\langle w_1w_3, w_2w_4 \rangle & \longrightarrow & \langle w_i \mid i = 1, \dots, 4 \rangle & \longrightarrow & L,
\end{array}$$

where the horizontal arrows are inclusions, and the vertical arrows are the isomorphisms induced by  $\bar{\varphi}: (B_4(\mathbb{R}P^2))^{(1)}/(B_4(\mathbb{R}P^2))^{(3)} \rightarrow L$ . Equation (120) yields  $\rho_2\rho_1^{-1} = B_1A_1^{-1}$  and  $\rho_3\rho_1^{-1} = C_1A_1^{-1}$ , and considering the  $K/(B_4(\mathbb{R}P^2))^{(3)}$ -cosets of these elements and applying equation (115), we obtain  $\bar{\varphi}(\rho_2\rho_1^{-1}) = w_1w_3$  and  $\bar{\varphi}(\rho_3\rho_1^{-1}) = w_2w_4$ . It follows that  $\mathcal{T}$  is a Schreier transversal for  $K/(B_4(\mathbb{R}P^2))^{(3)}$  in  $K$ , which enables us to write down a generating set  $\Sigma$  for  $(B_4(\mathbb{R}P^2))^{(3)}$ . However, to prove the claim, we do not need to study the whole list of generators. On the one hand, applying the description of the generators of  $K$  given by equation (120), the isomorphism  $\bar{\varphi}$  and equation (115), we see that the elements of

$$\mathcal{U} = \left\{ B_{1,2}, B_{1,3}, B_{1,4}, B_{2,3}, B_{2,4}, \rho_1 B_{1,2}\rho_1^{-1}, \rho_1 B_{1,3}\rho_1^{-1}, \rho_1 B_{1,4}\rho_1^{-1}, \rho_1^2, \rho_2^2, \rho_3^2, \rho_4^2 \right\}$$

belong to  $(B_4(\mathbb{R}P^2))^{(3)}$ , and appear as elements of  $\Sigma$ . Moreover, it is clear that these elements are mapped into  $\langle \Delta_2^2 \rangle$  under  $p_2 \circ p_3$  since  $\rho_1^2 = \rho_2^2 = \Delta_2^2$  in  $P_2(\mathbb{R}P^2)$ . The other elements of  $\Sigma$  obtained by applying the Reidemeister-Schreier process to an element  $u \in \mathcal{U}$  are just conjugates of  $u$  (the conjugating elements being the non-trivial elements of  $\mathcal{T}$ ), so also belong to  $(B_4(\mathbb{R}P^2))^{(3)}$ , and since  $\langle \Delta_2^2 \rangle$  is Abelian, these elements of  $\Sigma$  will lie in  $\langle \Delta_2^2 \rangle$ , which is contained in  $\langle \rho_1\rho_2 \rangle$ . Hence it suffices to consider the elements of  $\Sigma$  obtained by applying the Reidemeister-Schreier rewriting process to the three remaining elements  $\rho_1\rho_i$ ,  $i = 2, 3, 4$ , of equation (120). To do this, note that under identification of  $(B_4(\mathbb{R}P^2))^{(1)}/(B_4(\mathbb{R}P^2))^{(3)}$  with  $L$ , the elements of  $\mathcal{T}$  project respectively to  $1$ ,  $w_1w_3$ ,  $w_1w_2w_3w_4$  and  $w_2w_4$ , while  $\rho_1\rho_2$  projects to  $w_1w_3$ ,  $\rho_1\rho_3$  projects to  $w_2w_4$ , and  $\rho_1\rho_4$  projects to  $w_1w_2w_3w_4$ . The non-trivial elements of  $\Sigma$  arising as conjugates of  $\rho_1\rho_i$  are as follows:

- (a)  $i = 2$ :  $\rho_1\rho_2\rho_1\rho_2^{-1}, \rho_2^2, \rho_2\rho_1^{-1}\rho_3\rho_2^2\rho_3^{-1}\rho_1\rho_2^{-1}, \rho_2\rho_1^{-1}\rho_3\rho_2^{-1}\rho_1\rho_2\rho_1\rho_3^{-1}\rho_1\rho_2^{-1}$ .
- (b)  $i = 3$ :  $\rho_1\rho_3\rho_2\rho_3^{-1}\rho_1\rho_2^{-1}, \rho_2\rho_3\rho_1\rho_3^{-1}\rho_1\rho_2^{-1}, \rho_2\rho_1^{-1}\rho_3^2\rho_1\rho_2^{-1}, \rho_2\rho_1^{-1}\rho_3\rho_2^{-1}\rho_1\rho_3$ .
- (c)  $i = 4$ :  $\rho_1\rho_4\rho_1\rho_3^{-1}\rho_1\rho_2^{-1}, \rho_2\rho_4\rho_2\rho_3^{-1}\rho_1\rho_2^{-1}, \rho_2\rho_1^{-1}\rho_3\rho_4, \rho_2\rho_1^{-1}\rho_3\rho_2^{-1}\rho_1\rho_4\rho_1\rho_2^{-1}$ .

Under the projection  $p_2 \circ p_3$ , the elements for the cases  $i = 2, 3$  project to elements of  $\langle \Delta_2^2 \rangle$ , while those for the case  $i = 4$  project to  $\rho_1\rho_2$  or its inverse. We conclude that  $p_2 \circ p_3 \left( (B_4(\mathbb{R}P^2))^{(3)} \right) = \langle \rho_1\rho_2 \rangle \cong \mathbb{Z}_4$ , which proves the claim. Thus the restriction

$$p_2 \Big|_{p_3((B_4(\mathbb{R}P^2))^{(3)})} : p_3((B_4(\mathbb{R}P^2))^{(3)}) \longrightarrow \langle \rho_1\rho_2 \rangle$$

of  $p_2$  to  $p_3 \left( (B_4(\mathbb{R}P^2))^{(3)} \right)$  is surjective.

Now consider the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccc}
1 & \longrightarrow & \text{Ker} \left( p_2 \Big|_{p_3((B_4(\mathbb{R}P^2))^{(3)})} \right) & \longrightarrow & p_3 \left( (B_4(\mathbb{R}P^2))^{(3)} \right) & \xrightarrow{p_2 \Big|_{p_3((B_4(\mathbb{R}P^2))^{(3)})}} & \langle \rho_1 \rho_2 \rangle \longrightarrow 1 \\
& & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & \mathbb{F}_2(\rho_3, B_{2,3}) & \longrightarrow & P_3(\mathbb{R}P^2) & \xrightarrow{p_2} & P_2(\mathbb{R}P^2) \longrightarrow 1.
\end{array} \tag{121}$$

The lower short exact sequence is that of equation (2) with  $m = 2$  and  $n = 1$  (here  $p_* = p_2$ ), while the vertical arrows are inclusions. It follows that

$$\text{Ker} \left( p_2 \Big|_{p_3((B_4(\mathbb{R}P^2))^{(3)})} \right) = p_3 \left( (B_4(\mathbb{R}P^2))^{(3)} \right) \cap \mathbb{F}_2(\rho_3, B_{2,3}). \tag{122}$$

Since  $K \subset P_4(\mathbb{R}P^2)$ ,  $p_3$  restricts to  $K$ , and we have the following commutative diagram:

$$\begin{array}{ccc}
(B_4(\mathbb{R}P^2))^{(3)} & \xrightarrow{p_3 \Big|_{(B_4(\mathbb{R}P^2))^{(3)}}} & p_3 \left( (B_4(\mathbb{R}P^2))^{(3)} \right) \\
\downarrow & & \downarrow \\
K & \xrightarrow{p_3|_K} & P_3(\mathbb{R}P^2).
\end{array} \tag{123}$$

Again the vertical arrows are inclusions. Considered as elements of  $P_4(\mathbb{R}P^2)$ ,  $B_{i,j}$ ,  $1 \leq i < j \leq 3$  and  $\rho_k \rho_4$ ,  $1 \leq k \leq 3$ , belong to  $K$  by equation (9), and we deduce that the restriction of  $p_3$  to  $K$  is surjective. Since  $(B_4(\mathbb{R}P^2))^{(3)}$  is of index four in  $K$ , we conclude that  $p_3 \left( (B_4(\mathbb{R}P^2))^{(3)} \right)$  is of index at most four in  $P_3(\mathbb{R}P^2)$ .

Conversely, consider the Abelianisation of  $P_3(\mathbb{R}P^2)$ . From equation (10) and the action of  $\mathcal{Q}_8$  on  $\mathbb{F}_2(\rho_3, B_{2,3})$  described by equations (15), (17) and (18), we see that  $(B_3(\mathbb{R}P^2))^{\text{Ab}} \cong \mathbb{Z}_2^3$ , and that the Abelianisation homomorphism  $\pi: P_3(\mathbb{R}P^2) \rightarrow \mathbb{Z}_2^3$  sends each of  $\rho_i$ ,  $i = 1, 2, 3$ , to a distinct  $\mathbb{Z}_2$ -factor, and the  $B_{i,j}$ ,  $1 \leq i < j \leq 3$ , to the trivial element. Under  $p_3$ , the elements of  $\Sigma$  are sent to the trivial element of  $\mathbb{Z}_2^3$ , with the exception of those elements obtained via the Reidemeister-Schreier rewriting process using  $\rho_1 \rho_4$ , which are sent to  $(\bar{1}, \bar{1}, \bar{1})$ . It follows that

$$\pi \left( p_3 \left( (B_4(\mathbb{R}P^2))^{(3)} \right) \right) = \langle (\bar{1}, \bar{1}, \bar{1}) \rangle \cong \mathbb{Z}_2,$$

and so  $\pi \left( p_3((B_4(\mathbb{R}P^2))^{(3)}) \right)$  is of index four in  $\mathbb{Z}_2^3$ . We conclude from the following commutative diagram:

$$\begin{array}{ccc}
p_3 \left( (B_4(\mathbb{R}P^2))^{(3)} \right) & \longrightarrow & P_3(\mathbb{R}P^2) \\
\pi \downarrow & & \pi \downarrow \\
\pi \left( p_3 \left( (B_4(\mathbb{R}P^2))^{(3)} \right) \right) & \longrightarrow & \mathbb{Z}_2^3,
\end{array}$$

whose horizontal arrows are inclusions and whose vertical arrows are surjections, that  $p_3 \left( (B_4(\mathbb{R}P^2))^{(3)} \right)$  is of index at least four in  $P_3(\mathbb{R}P^2)$ . From the previous paragraph,

we conclude this index is exactly four, and since  $\langle \rho_1 \rho_2 \rangle$  is of index two in  $P_2(\mathbb{R}P^2)$ , it follows from equations (121) and (122) that  $p_3 \left( (B_4(\mathbb{R}P^2))^{(3)} \right) \cap \mathbb{F}_2(\rho_3, B_{2,3})$  is of index two in  $\mathbb{F}_2(\rho_3, B_{2,3})$ . Since  $B_{2,3} \in \Sigma$  (as an element of  $P_4(\mathbb{R}P^2)$ ), we have that  $B_{2,3} \in p_3 \left( (B_4(\mathbb{R}P^2))^{(3)} \right)$  (as an element of  $P_3(\mathbb{R}P^2)$ ). Thus under the canonical homomorphism

$$\mathbb{F}_2(\rho_3, B_{2,3}) \longrightarrow \mathbb{F}_2(\rho_3, B_{2,3}) / \left( p_3 \left( (B_4(\mathbb{R}P^2))^{(3)} \right) \cap \mathbb{F}_2(\rho_3, B_{2,3}) \right) \cong \mathbb{Z}_2,$$

$B_{2,3}$  is sent to  $\bar{0}$ , so  $\rho_3$  must be sent to  $\bar{1}$ , and hence the kernel of this homomorphism is given by:

$$p_3 \left( (B_4(\mathbb{R}P^2))^{(3)} \right) \cap \mathbb{F}_2(\rho_3, B_{2,3}) = \mathbb{F}_3(B_{2,3}, \rho_3^2, \rho_3 B_{2,3} \rho_3^{-1}). \quad (124)$$

From equation (120),  $\rho_4 \rho_3 \rho_2 \rho_1 = D_1 C_4 B_1 A_4$ , and using equation (115) and the isomorphism of equation (116), we see that  $\rho_4 \rho_3 \rho_2 \rho_1 \in (B_4(\mathbb{R}P^2))^{(3)}$ , and so  $\rho_3 \rho_2 \rho_1 \in p_3 \left( (B_4(\mathbb{R}P^2))^{(3)} \right)$  and  $\rho_2 \rho_1 \in p_2 \circ p_3 \left( (B_4(\mathbb{R}P^2))^{(3)} \right)$ . But we know that each of these three elements is of order four in its respective group [GG4, Proposition 26 and Remark 27], and hence it follows from the upper sequence of equation (121) and equation (124) that

$$p_3 \left( (B_4(\mathbb{R}P^2))^{(3)} \right) \cong \mathbb{F}_3 \rtimes \mathbb{Z}_4. \quad (125)$$

We shall determine the action shortly. Returning to equation (123), both of the horizontal restrictions  $p_3 \Big|_{(B_4(\mathbb{R}P^2))^{(3)}}$  and  $p_3 \Big|_K$  are surjective, and since  $(B_4(\mathbb{R}P^2))^{(3)}$  (resp.  $p_3((B_4(\mathbb{R}P^2))^{(3)})$ ) is of index four in  $K$  (resp.  $P_3(\mathbb{R}P^2)$ ), we obtain  $\text{Ker} \left( p_3 \Big|_{(B_4(\mathbb{R}P^2))^{(3)}} \right) = \text{Ker} (p_3 \Big|_K)$ . Thus from the upper homomorphism of equation (123) and equation (125), we have a short exact sequence

$$1 \longrightarrow \text{Ker} (p_3 \Big|_K) \longrightarrow (B_4(\mathbb{R}P^2))^{(3)} \xrightarrow{p_3 \Big|_{(B_4(\mathbb{R}P^2))^{(3)}}} \mathbb{F}_3 \rtimes \mathbb{Z}_4 \longrightarrow 1.$$

From equations (121), (122) and (124), a basis of the  $\mathbb{F}_3$ -factor of the quotient is given by  $\{B_{2,3}, \rho_3^2, \rho_3 B_{2,3} \rho_3^{-1}\}$ , and by (121) and the above discussion, we may take  $a^3 = \rho_3 \rho_2 \rho_1$  to be a generator of the  $\mathbb{Z}_4$ -factor. Using equation (7), we see that the action of  $\mathbb{Z}_4$  on  $\mathbb{F}_3$  is given by

$$\begin{cases} a^3 B_{2,3} a^{-3} = B_{2,3}^{-1} \\ a^3 \rho_3^2 a^{-3} = \rho_3^{-2} \\ a^3 \rho_3 B_{2,3} \rho_3^{-1} a^{-3} = \rho_3^{-2} \cdot \rho_3 B_{2,3}^{-1} \rho_3^{-1} \cdot \rho_3^2. \end{cases} \quad (126)$$

Consider the map  $s: \mathbb{F}_3 \rtimes \mathbb{Z}_4 \longrightarrow (B_4(\mathbb{R}P^2))^{(3)}$  defined on the generators of  $\mathbb{F}_3 \rtimes \mathbb{Z}_4$  by:

$$\begin{cases} x \longmapsto x & \text{for } x \in \{B_{2,3}, \rho_3^2, \rho_3 B_{2,3} \rho_3^{-1}\} \\ a^3 \longmapsto a^4. \end{cases}$$

Note that the elements on the right hand-side are considered to be elements in  $B_4(\mathbb{R}P^2)$ . Using the given generating set of  $K$ , we have

$$B_{2,3} = Y_1 Y_2, \rho_3^2 = C_1 C_4, \rho_3 B_{2,3} \rho_3^{-1} = C_1 Y_4 Y_3 C_1^{-1}, a^4 = \rho_4 \rho_3 \rho_2 \rho_1 = D_1 C_4 B_1 A_4.$$

By equations (115) and (116), we see that these elements belong to  $(B_4(\mathbb{R}P^2))^{(3)}$ , so the map  $s$  is well defined. Using equation (7) once more, we see that the action of  $a^4 = s(a^3)$  (which is of order 4) on  $s(x)$ ,  $x \in \{B_{2,3}, \rho_3^2, \rho_3 B_{2,3} \rho_3^{-1}\}$ , is also given by equation (126), up to replacing  $a^3$  by  $a^4$ . This shows that  $s$  extends to a homomorphism from  $\mathbb{F}_3 \rtimes \mathbb{Z}_4$  to  $(B_4(\mathbb{R}P^2))^{(3)}$ . It is then clear that  $s$  is a section for  $p_3 \big|_{(B_4(\mathbb{R}P^2))^{(3)}}$ , and hence

$$(B_4(\mathbb{R}P^2))^{(3)} \cong \text{Ker}(p_3 \big|_K) \rtimes (\mathbb{F}_3 \rtimes \mathbb{Z}_4).$$

From the following commutative diagram of short exact sequences,

$$\begin{array}{ccccccc} & & 1 & & 1 & & \\ & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \text{Ker}(p_3 \big|_K) \cong \mathbb{F}_5 & \longrightarrow & \text{Ker}(p_3) \cong \mathbb{F}_3 & \xrightarrow{\alpha' \big|_{\mathbb{F}_3}} & \langle \bar{\rho} \rangle \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \parallel \\ 1 & \longrightarrow & K & \longrightarrow & P_4(\mathbb{R}P^2) & \xrightarrow{\alpha'} & \langle \bar{\rho} \rangle \longrightarrow 1 \\ & & \downarrow p_3 \big|_K & & \downarrow p_3 & & \\ & & P_3(\mathbb{R}P^2) & \xlongequal{\quad\quad\quad} & P_3(\mathbb{R}P^2) & & \\ & & \downarrow & & \downarrow & & \\ & & 1 & & 1 & & \end{array}$$

we see that  $\text{Ker}(p_3 \big|_K)$  is also the kernel of the restriction of  $\alpha'$  to  $\text{Ker}(p_3)$  which is the free subgroup of  $P_4(\mathbb{R}P^2)$  of rank three with basis  $\{B_{1,4}, B_{2,4}, \rho_4\}$ . It thus follows that  $\text{Ker}(p_3 \big|_K)$  is a free group of rank five with basis  $\{B_{1,4}, B_{2,4}, \rho_4^2, \rho_4 B_{1,4} \rho_4^{-1}, \rho_4 B_{2,4} \rho_4^{-1}\}$ . We conclude that

$$\begin{aligned} (B_4(\mathbb{R}P^2))^{(3)} &\cong \mathbb{F}_5(B_{1,4}, B_{2,4}, \rho_4^2, \rho_4 B_{1,4} \rho_4^{-1}, \rho_4 B_{2,4} \rho_4^{-1}) \rtimes (\mathbb{F}_3(B_{2,3}, \rho_3^2, \rho_3 B_{2,3} \rho_3^{-1}) \rtimes \mathbb{Z}_4) \\ &\cong (\mathbb{F}_5(B_{1,4}, B_{2,4}, \rho_4^2, \rho_4 B_{1,4} \rho_4^{-1}, \rho_4 B_{2,4} \rho_4^{-1}) \rtimes \mathbb{F}_3(B_{2,3}, \rho_3^2, \rho_3 B_{2,3} \rho_3^{-1})) \rtimes \mathbb{Z}_4. \end{aligned} \quad (127)$$

As we already mentioned above,  $\rho_4 \rho_3 \rho_2 \rho_1$  belongs to  $(B_4(\mathbb{R}P^2))^{(3)}$ , and it projects to the generator of the  $\mathbb{Z}_4$ -factor of  $p_3((B_4(\mathbb{R}P^2))^{(3)})$ , so may be taken as a generator of the  $\mathbb{Z}_4$ -factor in equation (127). To determine completely  $(B_4(\mathbb{R}P^2))^{(3)}$ , it just remains to calculate the actions. By equation (7), the action of  $\mathbb{Z}_4$  on the given generators of  $\mathbb{F}_5 \rtimes \mathbb{F}_3$  is:

$$\left\{ \begin{array}{ll} B_{1,4} \longmapsto \rho_4^2 \cdot B_{1,4}^{-1} \cdot \rho_4^{-2} & B_{2,4} \longmapsto \rho_4^2 \cdot B_{1,4} \cdot B_{2,4}^{-1} \cdot B_{1,4}^{-1} \cdot \rho_4^{-2} \\ \rho_4^2 \longmapsto \rho_4^{-2} & \rho_4 B_{1,4} \rho_4^{-1} \longmapsto \rho_4 B_{1,4}^{-1} \rho_4^{-1} \\ \rho_4 B_{2,4} \rho_4^{-1} \longmapsto \rho_4 B_{1,4} \rho_4^{-1} \cdot \rho_4 B_{2,4}^{-1} \rho_4^{-1} \cdot \rho_4 B_{1,4}^{-1} \rho_4^{-1} & B_{2,3} \longmapsto B_{2,3}^{-1} \\ \rho_3^2 \longmapsto \rho_3^{-2} & \rho_3 B_{2,3} \rho_3^{-1} \longmapsto \rho_3^{-2} \cdot \rho_3 B_{2,3}^{-1} \rho_3^{-1} \cdot \rho_3^2, \end{array} \right. \quad (128)$$

and so the action of  $\mathbb{Z}_4$  by conjugation on the Abelianisation of  $\mathbb{F}_5 \rtimes \mathbb{F}_3$  is  $-\text{Id}$ . As for the action by conjugation of  $\mathbb{F}_3$  on  $\mathbb{F}_5$ , we have:

$$B_{2,3} : \begin{cases} B_{1,4} \mapsto B_{1,4} \\ B_{2,4} \mapsto \rho_4^2 \cdot B_{1,4} \cdot B_{2,4} \cdot B_{1,4}^{-1} \cdot \rho_4^{-2} \\ \rho_4^2 \mapsto \rho_4^2 \\ \rho_4 B_{1,4} \rho_4^{-1} \mapsto \rho_4 B_{1,4} \rho_4^{-1} \\ \rho_4 B_{2,4} \rho_4^{-1} \mapsto \rho_4^2 \cdot \rho_4 B_{1,4} \rho_4^{-1} \cdot \rho_4 B_{2,4} \rho_4^{-1} \cdot \rho_4 B_{1,4}^{-1} \rho_4^{-1} \cdot \rho_4^{-2} \end{cases} \quad (129)$$

$$\rho_3 B_{2,3} \rho_3^{-1} : \begin{cases} B_{1,4} \mapsto B_{2,4}^{-1} \cdot \rho_4^{-2} \cdot \rho_4 B_{2,4}^{-1} \rho_4^{-1} \cdot \rho_4 B_{1,4}^{-1} \rho_4^{-1} \cdot B_{2,4} \cdot \rho_4 B_{1,4} \rho_4^{-1} \cdot \\ \rho_4 B_{2,4} \rho_4^{-1} \cdot \rho_4^2 \cdot B_{1,4} \cdot \rho_4^{-2} \cdot \rho_4 B_{2,4}^{-1} \rho_4^{-1} \cdot \rho_4 B_{1,4}^{-1} \rho_4^{-1} \cdot B_{2,4}^{-1} \cdot \\ \rho_4 B_{1,4} \rho_4^{-1} \cdot \rho_4 B_{2,4} \rho_4^{-1} \cdot \rho_4^2 \cdot B_{2,4} \\ B_{2,4} \mapsto B_{2,4}^{-1} \cdot \rho_4^{-2} \cdot \rho_4 B_{2,4}^{-1} \rho_4^{-1} \cdot \rho_4 B_{1,4}^{-1} \rho_4^{-1} \cdot B_{2,4} \cdot \rho_4 B_{1,4} \rho_4^{-1} \cdot \\ \rho_4 B_{2,4} \rho_4^{-1} \cdot \rho_4^2 \cdot B_{2,4} \\ \rho_4^2 \mapsto B_{2,4}^{-1} \cdot B_{1,4}^{-1} \cdot \rho_4^{-2} \cdot \rho_4 B_{2,4}^{-1} \rho_4^{-1} \cdot \rho_4^2 \cdot B_{1,4} \cdot B_{2,4} \cdot \rho_4 B_{1,4}^{-1} \rho_4^{-1} \cdot \\ B_{2,4}^{-1} \cdot \rho_4 B_{1,4} \rho_4^{-1} \cdot \rho_4 B_{2,4} \rho_4^{-1} \cdot \rho_4^2 \cdot B_{2,4} \\ \rho_4 B_{1,4} \rho_4^{-1} \mapsto \rho_4 B_{1,4} \rho_4^{-1} \\ \rho_4 B_{2,4} \rho_4^{-1} \mapsto B_{2,4}^{-1} \cdot B_{1,4}^{-1} \cdot \rho_4^{-2} \cdot \rho_4 B_{2,4} \rho_4^{-1} \cdot \rho_4^2 \cdot B_{1,4} \cdot B_{2,4} \end{cases} \quad (130)$$

$$\rho_3^2 : \begin{cases} B_{1,4} \mapsto \rho_4 B_{1,4} \rho_4^{-1} \cdot \rho_4 B_{2,4} \rho_4^{-1} \cdot B_{2,4}^{-1} \cdot B_{1,4} \cdot B_{2,4} \cdot \rho_4 B_{2,4}^{-1} \rho_4^{-1} \cdot \\ \rho_4 B_{1,4}^{-1} \rho_4^{-1} \\ B_{2,4} \mapsto \rho_4 B_{1,4} \rho_4^{-1} \cdot \rho_4 B_{2,4} \rho_4^{-1} \cdot B_{2,4}^{-1} \cdot B_{1,4}^{-1} \cdot B_{2,4} \cdot B_{1,4} \cdot B_{2,4} \cdot \\ \rho_4 B_{2,4}^{-1} \rho_4^{-1} \cdot \rho_4 B_{1,4}^{-1} \rho_4^{-1} \\ \rho_4^2 \mapsto \rho_4 B_{1,4} \rho_4^{-1} \cdot \rho_4 B_{2,4} \rho_4^{-1} \cdot \rho_4^2 \cdot \rho_4 B_{2,4}^{-1} \rho_4^{-1} \cdot \rho_4 B_{1,4}^{-1} \rho_4^{-1} \\ \rho_4 B_{1,4} \rho_4^{-1} \mapsto \rho_4 B_{1,4} \rho_4^{-1} \\ \rho_4 B_{2,4} \rho_4^{-1} \mapsto \rho_4 B_{2,4} \rho_4^{-1}, \end{cases} \quad (131)$$

using the relation  $B_{1,4} B_{2,4} B_{3,4} = \rho_4^{-2}$  in  $P_4(\mathbb{R}P^2)$ . In all cases, the action of the given generators of  $\mathbb{F}_3$  on the Abelianisation of  $\mathbb{F}_5$  is trivial. We thus conclude that

$$\left( (B_4(\mathbb{R}P^2))^{(3)} \right)^{\text{Ab}} = (B_4(\mathbb{R}P^2))^{(3)} / (B_4(\mathbb{R}P^2))^{(4)} \cong \mathbb{Z}_2^8 \oplus \mathbb{Z}_4,$$

the  $\mathbb{Z}_2$ -factors arising from the fact that the action of  $\mathbb{Z}_4$  on  $\mathbb{F}_5 \rtimes \mathbb{F}_3$  is  $-\text{Id}$ . Consider the following short exact sequence:

$$1 \longrightarrow (B_4(\mathbb{R}P^2))^{(4)} \longrightarrow (B_4(\mathbb{R}P^2))^{(3)} \longrightarrow (B_4(\mathbb{R}P^2))^{(3)} / (B_4(\mathbb{R}P^2))^{(4)} \longrightarrow 1.$$

The  $\mathbb{Z}_4$ -factor of  $(B_4(\mathbb{R}P^2))^{(3)}$  is mapped bijectively onto the  $\mathbb{Z}_4$ -factor of the quotient  $(B_4(\mathbb{R}P^2))^{(3)} / (B_4(\mathbb{R}P^2))^{(4)}$ , so the kernel  $(B_4(\mathbb{R}P^2))^{(4)}$  of the projection  $(B_4(\mathbb{R}P^2))^{(3)} \longrightarrow (B_4(\mathbb{R}P^2))^{(3)} / (B_4(\mathbb{R}P^2))^{(4)}$  is the restriction of this projection to  $\mathbb{F}_5 \rtimes \mathbb{F}_3$ . From the form of the action of  $\mathbb{F}_3$  on  $\mathbb{F}_5$ , this restriction is the composition of the Abelianisation  $\mathbb{F}_5 \rtimes$

$\mathbb{F}_3 \longrightarrow \mathbb{Z}^5 \oplus \mathbb{Z}^3$ , followed by the homomorphism  $\mathbb{Z}^5 \oplus \mathbb{Z}^3 \longrightarrow \mathbb{Z}_2^5 \oplus \mathbb{Z}_2^3$  which takes the coordinates modulo 2. We see that

$$(B_4(\mathbb{R}P^2))^{(4)} \cong \mathbb{F}_{129} \rtimes \mathbb{F}_{17}, \quad (132)$$

where  $\mathbb{F}_{129}$  (resp.  $\mathbb{F}_{17}$ ) is the kernel of the restriction  $\mathbb{F}_5 \longrightarrow \mathbb{Z}_2^5$  (resp. of  $\mathbb{F}_3 \longrightarrow \mathbb{Z}_2^3$ ) of this composition to the first (resp. second) factor, and the action is that induced by that of  $\mathbb{F}_3$  on  $\mathbb{F}_5$ . It is then clear that for all  $i \geq 0$ ,  $(B_4(\mathbb{R}P^2))^{(4+i)} \cong (\mathbb{F}_{129} \rtimes \mathbb{F}_{17})^{(i)}$ . This completes the proof of part (d), and thus that of Theorem 3.  $\square$

*Remark 11.* In order to decide whether  $B_4(\mathbb{R}P^2)$  is residually soluble, it would be useful to know the form of the action in equation (132). If the product  $\mathbb{F}_{129} \rtimes \mathbb{F}_{17}$  were almost direct (*i.e.* the action of  $\mathbb{F}_{17}$  on the Abelianisation of  $\mathbb{F}_{129}$  were trivial) then  $B_4(\mathbb{R}P^2)$  would be residually soluble [FR1]. However, this is not the case. To see this, first consider the following basis  $(e_1, \dots, e_5)$  of  $\mathbb{F}_5$ :

$$\begin{aligned} e_1 &= B_{1,4}, e_2 = B_{2,4}, e_3 = \rho_4 B_{1,4} \rho_4^{-1} \cdot \rho_4 B_{2,4} \rho_4^{-1} \cdot \rho_4^2, \\ e_4 &= \rho_4 B_{1,4} \rho_4^{-1} \text{ and } e_5 = \rho_4 B_{2,4} \rho_4^{-1}. \end{aligned} \quad (133)$$

From equation (131), the action of  $\rho_3^2$  by conjugation on this basis is given by the following automorphism of  $\mathbb{F}_5$ :

$$\varphi_{\rho_3^2} : \begin{cases} e_1 \longmapsto e_4 e_5 e_2^{-1} e_1 e_2 e_5^{-1} e_4^{-1} \\ e_2 \longmapsto e_4 e_5 e_2^{-1} e_1^{-1} e_2 e_1 e_2 e_5^{-1} e_4^{-1} \\ e_3 \longmapsto e_4 e_5 e_3 e_5^{-1} e_4^{-1} \\ e_4 \longmapsto e_4 \\ e_5 \longmapsto e_5. \end{cases} \quad (134)$$

It follows from the form of the projection  $\mathbb{F}_3 \longrightarrow \mathbb{Z}_2^3$  that  $\rho_3^4$  belongs to the kernel  $\mathbb{F}_{17}$ . We will calculate the action of the corresponding automorphism  $\varphi_{\rho_3^4}$  on a certain element of  $\mathbb{F}_{129}$ . To do this, we first determine a basis of  $\mathbb{F}_{129}$  using the Reidemeister-Schreier rewriting process. A suitable transversal for the kernel of  $\mathbb{F}_5 \longrightarrow \mathbb{Z}_2^5$  relative to the basis of equation (133) is the word

$$\tau = e_1 e_2 e_1 e_3 e_1 e_2 e_1 e_4 e_1 e_2 e_1 e_3 e_1 e_2 e_1 e_5 e_1 e_2 e_1 e_3 e_1 e_2 e_1 e_4 e_1 e_2 e_1 e_3 e_1 e_2 e_1.$$

Let  $\tau_0$  denote the empty word, for  $i = 1, \dots, 31$ , let  $\tau_i$  be the subword of  $\tau$  consisting of the first  $i$  letters, and let  $\overline{w}$  denote the Schreier representative of the word  $w = w(e_1, \dots, e_5)$ . Deleting the trivial elements that appear in the set

$$\left\{ \tau_i e_j (\overline{\tau_i e_j})^{-1} \mid 0 \leq i \leq 31, 1 \leq j \leq 5 \right\},$$

gives rise to a basis of  $\mathbb{F}_{129}$ , and thus a basis for the Abelianisation  $\mathbb{Z}_{129}$  of  $\mathbb{F}_{129}$  (we shall not distinguish notationally between a basis element of  $\mathbb{F}_{129}$  and its projection in  $\mathbb{Z}_{129}$ ). Using equation (134), a long but straightforward calculation in  $\mathbb{Z}_{129}$  shows that

$$\begin{aligned} \varphi_{\rho_3^4}(\tau_5 e_3 \tau_2^{-1}) &= e_2^{-1} \tau_3^{-1} \cdot \tau_3 e_1 \tau_2^{-1} \cdot \tau_2 e_2 \tau_1^{-1} \cdot \tau_1 e_1 \cdot e_2 \tau_3^{-1} \cdot \tau_4 e_2^{-1} \tau_7^{-1} \cdot \tau_5 e_1 \tau_4^{-1} \cdot \tau_4 e_2 \tau_7^{-1} \cdot \\ &\quad \tau_7 e_1 \tau_6^{-1} \cdot \tau_6 e_2 \tau_5^{-1} \cdot \tau_5 e_3 \tau_2^{-1}. \end{aligned}$$

Each of the terms appearing on the right hand-side of this equality, as well as  $\tau_5 e_3 \tau_2^{-1}$ , belongs to the given basis of  $\mathbb{Z}_{129}$ , and so the induced action of  $\mathbb{F}_{17}$  on  $\mathbb{Z}_{129}$  is non trivial. This proves that the semi-direct product  $\mathbb{F}_{129} \rtimes \mathbb{F}_{17}$  is not almost direct. It thus remains an open question as to whether  $B_4(\mathbb{R}P^2)$  is residually soluble.

## 4 A presentation of $\Gamma_2(B_n(\mathbb{R}P^2))$ , $n \geq 3$

In this section, we derive a presentation of  $\Gamma_2(B_n(\mathbb{R}P^2))$  obtained using the Reidemeister-Schreier rewriting process.

**Proposition 12.** *Let  $n \geq 3$ . The following constitutes a presentation of the group  $\Gamma_2(B_n(\mathbb{R}P^2))$ :*

generators:

$$\begin{aligned} \alpha_i &= \sigma_i \sigma_1^{-1}, & \gamma_i &= \sigma_1 \rho_1 \sigma_i \sigma_1^{-1} \rho_1^{-1} \sigma_1^{-1} & \text{for } i = 2, \dots, n-1 \\ \beta_i &= \sigma_1 \sigma_i, & \tau_i &= \sigma_1 \rho_1 \sigma_1 \sigma_i \rho_1^{-1} \sigma_1^{-1} & \text{for } i = 1, \dots, n-1 \\ \eta_j &= \rho_j \sigma_1^{-1} \rho_1^{-1} \sigma_1^{-1}, & \theta_j &= \sigma_1 \rho_1 \rho_j \sigma_1^{-1}, & \lambda_j &= \sigma_1 \rho_1 \sigma_1 \rho_j & \text{for } j = 1, \dots, n \\ \kappa_j &= \sigma_1 \rho_j \rho_1^{-1} \sigma_1^{-1} & & & & \text{for } j = 2, \dots, n. \end{aligned}$$

To simplify the expression of the relators, we set  $\alpha_1 = \gamma_1 = \kappa_1 = 1$ .

relators:

(a) For all  $1 \leq i, j \leq n-1$ ,  $|i-j| \geq 2$ ,

$$\alpha_i \beta_j \beta_i^{-1} \alpha_j^{-1}, \quad \beta_i \alpha_j \alpha_i^{-1} \beta_j^{-1}, \quad \gamma_i \tau_j \tau_i^{-1} \gamma_j^{-1}, \quad \tau_i \gamma_j \gamma_i^{-1} \tau_j^{-1}.$$

(b) For all  $1 \leq i \leq n-2$ ,

$$\begin{aligned} \alpha_i \beta_{i+1} \alpha_i \alpha_{i+1}^{-1} \beta_i^{-1} \alpha_{i+1}^{-1}, & \quad \beta_i \alpha_{i+1} \beta_i \beta_{i+1}^{-1} \alpha_i^{-1} \beta_{i+1}^{-1}, \\ \gamma_i \tau_{i+1} \gamma_i \gamma_{i+1}^{-1} \tau_i^{-1} \gamma_{i+1}^{-1}, & \quad \tau_i \gamma_{i+1} \tau_i \tau_{i+1}^{-1} \gamma_i^{-1} \tau_{i+1}^{-1}. \end{aligned}$$

(c) For all  $1 \leq i \leq n-1$  and  $1 \leq j \leq n$  with  $j \neq i, i+1$ ,

$$\alpha_i \kappa_j \tau_i^{-1} \eta_j^{-1}, \quad \beta_i \eta_j \gamma_i^{-1} \kappa_j^{-1}, \quad \gamma_i \lambda_j \beta_i^{-1} \theta_j^{-1}, \quad \tau_i \theta_j \alpha_i^{-1} \lambda_j^{-1}.$$

(d) For all  $1 \leq i \leq n-1$ ,

$$\beta_i^{-1} \kappa_i \tau_i^{-1} \eta_{i+1}^{-1}, \quad \alpha_i^{-1} \eta_i \gamma_i^{-1} \kappa_{i+1}^{-1}, \quad \tau_i^{-1} \lambda_i \beta_i^{-1} \theta_{i+1}^{-1}, \quad \gamma_i^{-1} \theta_i \alpha_i^{-1} \lambda_{i+1}^{-1}.$$

(e) For all  $1 \leq i \leq n-1$ ,

$$\begin{aligned} \lambda_{i+1}^{-1} \eta_i^{-1} \eta_{i+1} \lambda_i \beta_i^{-1} \alpha_i^{-1}, & \quad \theta_{i+1}^{-1} \kappa_i^{-1} \kappa_{i+1} \theta_i \alpha_i^{-1} \beta_i^{-1}, \\ \kappa_{i+1}^{-1} \theta_i^{-1} \theta_{i+1} \kappa_i \tau_i^{-1} \gamma_i^{-1}, & \quad \eta_{i+1}^{-1} \lambda_i^{-1} \lambda_{i+1} \eta_i \gamma_i^{-1} \tau_i^{-1}. \end{aligned}$$

(f) (i) If  $n$  is even,

$$\begin{aligned} \beta_2 \alpha_3 \cdots \beta_{n-2} \alpha_{n-1} \beta_{n-1} \alpha_{n-2} \cdots \beta_3 \alpha_2 \beta_1 \lambda_1^{-1} \eta_1^{-1}, & \quad \beta_1 \alpha_2 \cdots \alpha_{n-2} \beta_{n-1} \alpha_{n-1} \beta_{n-2} \cdots \alpha_3 \beta_2 \theta_1^{-1} \\ \tau_2 \gamma_3 \cdots \tau_{n-2} \gamma_{n-1} \tau_{n-1} \gamma_{n-2} \cdots \tau_3 \gamma_2 \tau_1 \theta_1^{-1}, & \quad \tau_1 \gamma_2 \cdots \gamma_{n-2} \tau_{n-1} \gamma_{n-1} \tau_{n-2} \cdots \gamma_3 \tau_2 \eta_1^{-1} \lambda_1^{-1}. \end{aligned}$$

(ii) If  $n$  is odd,

$$\begin{aligned} \beta_2 \alpha_3 \cdots \alpha_{n-2} \beta_{n-1} \alpha_{n-1} \beta_{n-2} \cdots \beta_3 \alpha_2 \beta_1 \lambda_1^{-1} \eta_1^{-1}, & \quad \beta_1 \alpha_2 \cdots \beta_{n-2} \alpha_{n-1} \beta_{n-1} \alpha_{n-2} \cdots \alpha_3 \beta_2 \theta_1^{-1} \\ \tau_2 \gamma_3 \cdots \gamma_{n-2} \tau_{n-1} \gamma_{n-1} \tau_{n-2} \cdots \tau_3 \gamma_2 \tau_1 \theta_1^{-1}, & \quad \tau_1 \gamma_2 \cdots \tau_{n-2} \gamma_{n-1} \tau_{n-1} \gamma_{n-2} \cdots \gamma_3 \tau_2 \eta_1^{-1} \lambda_1^{-1}. \end{aligned}$$

*Remark 13.* The above presentation may be used to show that  $\Gamma_2(B_n(\mathbb{R}P^2))$  is perfect for  $n \geq 5$ .

*Proof.* Taking the standard presentation of  $B_n(\mathbb{R}P^2)$  given by Proposition 4, and the set  $\{1, \sigma_1, \sigma_1\rho_1, \sigma_1\rho_1\sigma_1\}$  as a Schreier transversal, we apply the Reidemeister-Schreier rewriting process to the short exact sequence (4). In this way, a generating set for  $\Gamma_2(B_n(\mathbb{R}P^2))$  is that given in the statement of the proposition. We record the following equalities for later use:

$$\left\{ \begin{array}{l} \sigma_1\rho_1\alpha_i\rho_1^{-1}\sigma_1^{-1} = \gamma_i \\ \sigma_1\rho_1\beta_i\rho_1^{-1}\sigma_1^{-1} = \tau_i \\ \sigma_1\rho_1\tau_i\rho_1^{-1}\sigma_1^{-1} = \sigma_1\rho_1\sigma_1\rho_1\sigma_1\sigma_i\rho_1^{-1}\sigma_1^{-1}\rho_1^{-1}\sigma_1^{-1} = \lambda_1\beta_i\lambda_1^{-1} \\ \sigma_1\rho_1\gamma_i\rho_1^{-1}\sigma_1^{-1} = \sigma_1\rho_1\sigma_1\rho_1\sigma_i\sigma_1^{-1}\rho_1^{-1}\sigma_1^{-1}\rho_1^{-1}\sigma_1^{-1} = \lambda_1\alpha_i\lambda_1^{-1} \\ \sigma_1\rho_1\lambda_i\rho_1^{-1}\sigma_1^{-1} = \sigma_1\rho_1\sigma_1\rho_1\sigma_1\rho_i\rho_1^{-1}\sigma_1^{-1} = \lambda_1\kappa_i \\ \sigma_1\rho_1\eta_i\rho_1^{-1}\sigma_1^{-1} = \sigma_1\rho_1\rho_i\sigma_1^{-1}\rho_1^{-1}\sigma_1^{-1}\rho_1^{-1}\sigma_1^{-1} = \theta_i\lambda_1^{-1} \\ \sigma_1\rho_1\kappa_i\rho_1^{-1}\sigma_1^{-1} = \sigma_1\rho_1\sigma_1\rho_i\rho_1^{-1}\sigma_1^{-1}\rho_1^{-1}\sigma_1^{-1} = \lambda_i\lambda_1^{-1} \\ \sigma_1\rho_1\theta_i\rho_1^{-1}\sigma_1^{-1} = \sigma_1\rho_1\rho_1\rho_i\sigma_1^{-1}\rho_1^{-1}\sigma_1^{-1} = \lambda_1\eta_i. \end{array} \right. \quad (135)$$

We now determine the relations of  $\Gamma_2(B_n(\mathbb{R}P^2))$  in terms of the given generating set. As we mentioned, we also set  $\alpha_1 = \kappa_1 = \gamma_1 = 1$ . For all  $1 \leq i, j \leq n-1$ ,  $|i-j| \geq 2$ , the relator  $\sigma_i\sigma_j\sigma_i^{-1}\sigma_j^{-1}$  gives rise to the following four relators, one for each element of the Schreier transversal:

$$\begin{aligned} 1 &= \sigma_i\sigma_j\sigma_i^{-1}\sigma_j^{-1} = \sigma_i\sigma_1^{-1} \cdot \sigma_1\sigma_j \cdot \sigma_i^{-1}\sigma_1^{-1} \cdot \sigma_1\sigma_j^{-1} = \alpha_i\beta_j\beta_i^{-1}\alpha_j^{-1} \\ 1 &= \sigma_1 \cdot \sigma_i\sigma_j\sigma_i^{-1}\sigma_j^{-1} \cdot \sigma_1^{-1} = \sigma_1\sigma_i \cdot \sigma_j\sigma_1^{-1} \cdot \sigma_1\sigma_i^{-1} \cdot \sigma_j^{-1}\sigma_1^{-1} = \beta_i\alpha_j\alpha_i^{-1}\beta_j^{-1} \\ 1 &= \sigma_1\rho_1 \cdot \sigma_i\sigma_j\sigma_i^{-1}\sigma_j^{-1} \cdot \rho_1^{-1}\sigma_1^{-1} = \sigma_1\rho_1\alpha_i\beta_j\beta_i^{-1}\alpha_j^{-1}\rho_1^{-1}\sigma_1^{-1} = \gamma_i\tau_j\tau_i^{-1}\gamma_j^{-1} \\ 1 &= \sigma_1\rho_1\sigma_1 \cdot \sigma_i\sigma_j\sigma_i^{-1}\sigma_j^{-1} \cdot \sigma_1^{-1}\rho_1^{-1}\sigma_1^{-1} = \sigma_1\rho_1\beta_i\alpha_j\alpha_i^{-1}\beta_j^{-1}\rho_1^{-1}\sigma_1^{-1} = \tau_i\gamma_j\gamma_i^{-1}\tau_j^{-1}. \end{aligned}$$

In the third and fourth equations, we have used equation (135). Similarly, from the relator  $\sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}^{-1}$ , for all  $1 \leq i \leq n-2$ , we obtain:

$$\begin{aligned} 1 &= \sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}^{-1} = \sigma_i\sigma_1^{-1} \cdot \sigma_1\sigma_{i+1} \cdot \sigma_i\sigma_1^{-1} \cdot \sigma_1\sigma_{i+1}^{-1} \cdot \sigma_i^{-1}\sigma_1^{-1} \cdot \sigma_1\sigma_{i+1}^{-1} \\ &= \alpha_i\beta_{i+1}\alpha_i\alpha_{i+1}^{-1}\beta_i^{-1}\alpha_{i+1}^{-1} \\ 1 &= \sigma_1 \cdot \sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}^{-1} \cdot \sigma_1^{-1} = \sigma_1\sigma_i \cdot \sigma_{i+1}\sigma_1^{-1} \cdot \sigma_1\sigma_i \cdot \sigma_{i+1}^{-1}\sigma_1^{-1} \cdot \sigma_1\sigma_i^{-1} \cdot \sigma_{i+1}^{-1}\sigma_1^{-1} \\ &= \beta_i\alpha_{i+1}\beta_i\beta_{i+1}^{-1}\alpha_i^{-1}\beta_{i+1}^{-1} \\ 1 &= \sigma_1\rho_1 \cdot \sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}^{-1} \cdot \rho_1^{-1}\sigma_1^{-1} = \sigma_1\rho_1\alpha_i\beta_{i+1}\alpha_i\alpha_{i+1}^{-1}\beta_i^{-1}\alpha_{i+1}^{-1}\rho_1^{-1}\sigma_1^{-1} \\ &= \gamma_i\tau_{i+1}\gamma_i\gamma_{i+1}^{-1}\tau_i^{-1}\gamma_{i+1}^{-1} \\ 1 &= \sigma_1\rho_1\sigma_1 \cdot \sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}^{-1} \cdot \sigma_1^{-1}\rho_1^{-1}\sigma_1^{-1} = \sigma_1\rho_1\beta_i\alpha_{i+1}\beta_i\beta_{i+1}^{-1}\alpha_i^{-1}\beta_{i+1}^{-1}\rho_1^{-1}\sigma_1^{-1} \\ &= \tau_i\gamma_{i+1}\tau_i\tau_{i+1}^{-1}\gamma_i^{-1}\tau_{i+1}^{-1}. \end{aligned}$$

For all  $1 \leq i \leq n-1$  and  $1 \leq j \leq n, j \neq i, i+1$ , the relator  $\sigma_i \rho_j \sigma_i^{-1} \rho_j^{-1}$  yields:

$$\begin{aligned}
1 &= \sigma_i \rho_j \sigma_i^{-1} \rho_j^{-1} = \sigma_i \sigma_1^{-1} \cdot \sigma_1 \rho_j \rho_1^{-1} \sigma_1^{-1} \cdot \sigma_1 \rho_1 \sigma_i^{-1} \sigma_1^{-1} \rho_1^{-1} \sigma_1^{-1} \cdot \sigma_1 \rho_1 \sigma_1 \rho_j^{-1} = \alpha_i \kappa_j \tau_i^{-1} \eta_j^{-1} \\
1 &= \sigma_1 \cdot \sigma_i \rho_j \sigma_i^{-1} \rho_j^{-1} \cdot \sigma_1^{-1} = \sigma_1 \sigma_i \cdot \rho_j \sigma_1^{-1} \rho_1^{-1} \sigma_1^{-1} \cdot \sigma_1 \rho_1 \sigma_1 \sigma_i^{-1} \rho_1^{-1} \sigma_1^{-1} \cdot \sigma_1 \rho_1 \rho_j^{-1} \sigma_1^{-1} \\
&= \beta_i \eta_j \gamma_i^{-1} \kappa_j^{-1} \\
1 &= \sigma_1 \rho_1 \cdot \sigma_i \rho_j \sigma_i^{-1} \rho_j^{-1} \cdot \rho_1^{-1} \sigma_1^{-1} = \sigma_1 \rho_1 \alpha_i \kappa_j \tau_i^{-1} \kappa_j^{-1} \rho_1^{-1} \sigma_1^{-1} = \gamma_i \lambda_j \beta_i^{-1} \theta_j^{-1} \\
1 &= \sigma_1 \rho_1 \sigma_1 \cdot \sigma_i \rho_j \sigma_i^{-1} \rho_j^{-1} \cdot \sigma_1^{-1} \rho_1^{-1} \sigma_1^{-1} = \sigma_1 \rho_1 \beta_i \eta_j \gamma_i^{-1} \kappa_j^{-1} \rho_1^{-1} \sigma_1^{-1} = \tau_i \theta_j \alpha_i^{-1} \lambda_j^{-1}.
\end{aligned}$$

For all  $1 \leq i \leq n-1$ , the relator  $\sigma_i^{-1} \rho_i \sigma_i^{-1} \rho_{i+1}^{-1}$  gives rise to:

$$\begin{aligned}
1 &= \sigma_i^{-1} \rho_i \sigma_i^{-1} \rho_{i+1}^{-1} = \sigma_i^{-1} \sigma_1^{-1} \cdot \sigma_1 \rho_i \rho_1^{-1} \sigma_1^{-1} \cdot \sigma_1 \rho_1 \sigma_i^{-1} \sigma_1^{-1} \rho_1^{-1} \sigma_1^{-1} \cdot \sigma_1 \rho_1 \sigma_1 \rho_{i+1}^{-1} \\
&= \beta_i^{-1} \kappa_i \tau_i^{-1} \eta_{i+1}^{-1} \\
1 &= \sigma_1 \cdot \sigma_i^{-1} \rho_i \sigma_i^{-1} \rho_{i+1}^{-1} \cdot \sigma_1^{-1} = \sigma_1 \sigma_i^{-1} \cdot \rho_i \sigma_1^{-1} \rho_1^{-1} \sigma_1^{-1} \cdot \sigma_1 \rho_1 \sigma_1 \sigma_i^{-1} \rho_1^{-1} \sigma_1^{-1} \cdot \sigma_1 \rho_1 \rho_{i+1}^{-1} \sigma_1^{-1} \\
&= \alpha_i^{-1} \eta_i \gamma_i^{-1} \kappa_{i+1}^{-1} \\
1 &= \sigma_1 \rho_1 \cdot \sigma_i^{-1} \rho_i \sigma_i^{-1} \rho_{i+1}^{-1} \cdot \rho_1^{-1} \sigma_1^{-1} = \sigma_1 \rho_1 \beta_i^{-1} \kappa_i \tau_i^{-1} \eta_{i+1}^{-1} \rho_1^{-1} \sigma_i^{-1} = \tau_i^{-1} \lambda_i \beta_i^{-1} \theta_{i+1}^{-1} \\
1 &= \sigma_1 \rho_1 \sigma_1 \cdot \sigma_i^{-1} \rho_i \sigma_i^{-1} \rho_{i+1}^{-1} \cdot \sigma_1^{-1} \rho_1^{-1} \sigma_1^{-1} = \sigma_1 \rho_1 \alpha_i^{-1} \eta_i \gamma_i^{-1} \kappa_{i+1}^{-1} \rho_1^{-1} \sigma_i^{-1} = \gamma_i^{-1} \theta_i \alpha_i^{-1} \lambda_{i+1}^{-1}.
\end{aligned}$$

From the relator  $\rho_{i+1}^{-1} \rho_i^{-1} \rho_{i+1} \rho_i \sigma_i^{-2}$ , for all  $1 \leq i \leq n-1$ , we obtain:

$$\begin{aligned}
1 &= \rho_{i+1}^{-1} \rho_i^{-1} \rho_{i+1} \rho_i \sigma_i^{-2} = \rho_{i+1}^{-1} \sigma_1^{-1} \rho_1^{-1} \sigma_1^{-1} \cdot \sigma_1 \rho_1 \sigma_1 \rho_i^{-1} \cdot \rho_{i+1} \rho_1^{-1} \rho_1^{-1} \sigma_1^{-1} \cdot \sigma_1 \rho_1 \sigma_1 \rho_i \cdot \\
&\quad \sigma_i^{-1} \sigma_1^{-1} \cdot \sigma_1 \sigma_i^{-1} = \lambda_{i+1}^{-1} \eta_i^{-1} \eta_{i+1} \lambda_i \beta_i^{-1} \alpha_i^{-1} \\
1 &= \sigma_1 \cdot \rho_{i+1}^{-1} \rho_i^{-1} \rho_{i+1} \rho_i \sigma_i^{-2} \cdot \sigma_1^{-1} = \sigma_1 \rho_{i+1}^{-1} \rho_1^{-1} \sigma_1^{-1} \cdot \sigma_1 \rho_1 \rho_i^{-1} \sigma_1^{-1} \cdot \sigma_1 \rho_{i+1} \rho_1^{-1} \sigma_1^{-1} \cdot \\
&\quad \sigma_1 \rho_1 \rho_i \sigma_1^{-1} \cdot \sigma_1 \sigma_i^{-1} \cdot \sigma_i^{-1} \sigma_1^{-1} = \theta_{i+1}^{-1} \kappa_i^{-1} \kappa_{i+1} \theta_i \alpha_i^{-1} \beta_i^{-1} \\
1 &= \sigma_1 \rho_1 \cdot \rho_{i+1}^{-1} \rho_i^{-1} \rho_{i+1} \rho_i \sigma_i^{-2} \cdot \rho_1^{-1} \sigma_1^{-1} = \sigma_1 \rho_1 \lambda_{i+1}^{-1} \eta_i^{-1} \eta_{i+1} \lambda_i \beta_i^{-1} \alpha_i^{-1} \rho_1^{-1} \sigma_1^{-1} \\
&= \kappa_{i+1}^{-1} \theta_i^{-1} \theta_{i+1} \kappa_i \tau_i^{-1} \gamma_i^{-1} \\
1 &= \sigma_1 \rho_1 \sigma_1 \cdot \rho_{i+1}^{-1} \rho_i^{-1} \rho_{i+1} \rho_i \sigma_i^{-2} \cdot \sigma_1^{-1} \rho_1^{-1} \sigma_1^{-1} = \sigma_1 \rho_1 \theta_{i+1}^{-1} \kappa_i^{-1} \kappa_{i+1} \theta_i \alpha_i^{-1} \beta_i^{-1} \rho_1^{-1} \sigma_1^{-1} \\
&= \eta_{i+1}^{-1} \lambda_i^{-1} \lambda_{i+1} \eta_i \gamma_i^{-1} \tau_i^{-1}.
\end{aligned}$$

Finally we come to the surface relator  $\sigma_1 \cdots \sigma_{n-1} \sigma_{n-1}^{-1} \cdots \sigma_1 \rho_1^{-2}$ . We deal with the cases  $n$  even and odd separately.

(a)  $n$  even. We have:

$$\begin{aligned}
1 &= \sigma_1 \cdots \sigma_{n-1} \sigma_{n-1} \cdots \sigma_1 \rho_1^{-2} = \sigma_1 \sigma_2 \cdot \sigma_3 \sigma_1^{-1} \cdots \sigma_1 \sigma_{n-2} \cdot \sigma_{n-1} \sigma_1^{-1} \cdot \sigma_1 \sigma_{n-1} \cdot \sigma_{n-2} \sigma_1^{-1} \cdots \\
&\quad \sigma_1 \sigma_3 \cdot \sigma_2 \sigma_1^{-1} \cdot \sigma_1^2 \cdot \rho_1^{-1} \sigma_1^{-1} \rho_1^{-1} \sigma_1^{-1} \cdot \sigma_1 \rho_1 \sigma_1 \rho_1^{-1} \\
&= \beta_2 \alpha_3 \cdots \beta_{n-2} \alpha_{n-1} \beta_{n-1} \alpha_{n-2} \cdots \beta_3 \alpha_2 \beta_1 \lambda_1^{-1} \eta_1^{-1} \\
1 &= \sigma_1 \cdot \sigma_1 \cdots \sigma_{n-1} \sigma_{n-1} \cdots \sigma_1 \rho_1^{-2} \cdot \sigma_1^{-1} = \sigma_1^2 \cdot \sigma_2 \sigma_1^{-1} \cdot \sigma_1 \sigma_3 \cdots \sigma_{n-2} \sigma_1^{-1} \cdot \sigma_1 \sigma_{n-1} \cdot \sigma_{n-1} \sigma_1^{-1} \cdot \\
&\quad \sigma_1 \sigma_{n-2} \cdots \sigma_3 \sigma_1^{-1} \cdot \sigma_1 \sigma_2 \cdot \sigma_1 \rho_1^{-2} \sigma_1^{-1} = \beta_1 \alpha_2 \cdots \alpha_{n-2} \beta_{n-1} \alpha_{n-1} \beta_{n-2} \cdots \alpha_3 \beta_2 \theta_1^{-1} \\
1 &= \sigma_1 \rho_1 \cdot \sigma_1 \cdots \sigma_{n-1} \sigma_{n-1} \cdots \sigma_1 \rho_1^{-2} \cdot \rho_1^{-1} \sigma_1^{-1} \\
&= \sigma_1 \rho_1 \cdot \beta_2 \alpha_3 \cdots \beta_{n-2} \alpha_{n-1} \beta_{n-1} \alpha_{n-2} \cdots \beta_3 \alpha_2 \beta_1 \lambda_1^{-1} \eta_1^{-1} \cdot \rho_1^{-1} \sigma_1^{-1} \\
&= \tau_2 \gamma_3 \cdots \tau_{n-2} \gamma_{n-1} \tau_{n-1} \gamma_{n-2} \cdots \tau_3 \gamma_2 \tau_1 \theta_1^{-1} \\
1 &= \sigma_1 \rho_1 \sigma_1 \cdot \sigma_1 \cdots \sigma_{n-1} \sigma_{n-1} \cdots \sigma_1 \rho_1^{-2} \cdot \sigma_1^{-1} \rho_1^{-1} \sigma_1^{-1} \\
&= \sigma_1 \rho_1 \cdot \beta_1 \alpha_2 \cdots \alpha_{n-2} \beta_{n-1} \alpha_{n-1} \beta_{n-2} \cdots \alpha_3 \beta_2 \theta_1^{-1} \cdot \rho_1^{-1} \sigma_1^{-1} \\
&= \tau_1 \gamma_2 \cdots \gamma_{n-2} \tau_{n-1} \gamma_{n-1} \tau_{n-2} \cdots \gamma_3 \tau_2 \eta_1^{-1} \lambda_1^{-1}.
\end{aligned}$$

(b)  $n$  odd. We have:

$$\begin{aligned}
1 &= \sigma_1 \cdots \sigma_{n-1} \sigma_{n-1} \cdots \sigma_1 \rho_1^{-2} = \sigma_1 \sigma_2 \cdot \sigma_3 \sigma_1^{-1} \cdots \sigma_{n-2} \sigma_1^{-1} \cdot \sigma_1 \sigma_{n-1} \cdot \sigma_{n-1} \sigma_1^{-1} \cdot \sigma_1 \sigma_{n-2} \cdots \\
&\quad \sigma_1 \sigma_3 \cdot \sigma_2 \sigma_1^{-1} \cdot \sigma_1^2 \cdot \rho_1^{-1} \sigma_1^{-1} \rho_1^{-1} \sigma_1^{-1} \cdot \sigma_1 \rho_1 \sigma_1 \rho_1^{-1} \\
&= \beta_2 \alpha_3 \cdots \alpha_{n-2} \beta_{n-1} \alpha_{n-1} \beta_{n-2} \cdots \beta_3 \alpha_2 \beta_1 \lambda_1^{-1} \eta_1^{-1} \\
1 &= \sigma_1 \cdot \sigma_1 \cdots \sigma_{n-1} \sigma_{n-1} \cdots \sigma_1 \rho_1^{-2} \cdot \sigma_1^{-1} = \sigma_1^2 \cdot \sigma_2 \sigma_1^{-1} \cdot \sigma_1 \sigma_3 \cdots \sigma_1 \sigma_{n-2} \cdot \sigma_{n-1} \sigma_1^{-1} \cdot \sigma_1 \sigma_{n-1} \cdot \\
&\quad \sigma_{n-2} \sigma_1^{-1} \cdots \sigma_3 \sigma_1^{-1} \cdot \sigma_1 \sigma_2 \cdot \sigma_1 \rho_1^{-2} \sigma_1^{-1} = \beta_1 \alpha_2 \cdots \beta_{n-2} \alpha_{n-1} \beta_{n-1} \alpha_{n-2} \cdots \alpha_3 \beta_2 \theta_1^{-1} \\
1 &= \sigma_1 \rho_1 \cdot \sigma_1 \cdots \sigma_{n-1} \sigma_{n-1} \cdots \sigma_1 \rho_1^{-2} \cdot \rho_1^{-1} \sigma_1^{-1} \\
&= \sigma_1 \rho_1 \cdot \beta_2 \alpha_3 \cdots \alpha_{n-2} \beta_{n-1} \alpha_{n-1} \beta_{n-2} \cdots \beta_3 \alpha_2 \beta_1 \lambda_1^{-1} \eta_1^{-1} \cdot \rho_1^{-1} \sigma_1^{-1} \\
&= \tau_2 \gamma_3 \cdots \gamma_{n-2} \tau_{n-1} \gamma_{n-1} \tau_{n-2} \cdots \tau_3 \gamma_2 \tau_1 \lambda_1^{-1} \lambda_1 \theta_1^{-1} \\
1 &= \sigma_1 \rho_1 \sigma_1 \cdot \sigma_1 \cdots \sigma_{n-1} \sigma_{n-1} \cdots \sigma_1 \rho_1^{-2} \cdot \sigma_1^{-1} \rho_1^{-1} \sigma_1^{-1} \\
&= \sigma_1 \rho_1 \cdot \beta_1 \alpha_2 \cdots \beta_{n-2} \alpha_{n-1} \beta_{n-1} \alpha_{n-2} \cdots \alpha_3 \beta_2 \theta_1^{-1} \cdot \rho_1^{-1} \sigma_1^{-1} \\
&= \tau_1 \gamma_2 \cdots \tau_{n-2} \gamma_{n-1} \tau_{n-1} \gamma_{n-2} \cdots \gamma_3 \tau_2 \eta_1^{-1} \lambda_1^{-1}.
\end{aligned}$$

This completes the proof of the proposition. □

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