Some remarks on weighted logarithmic Sobolev inequality

Patrick Cattiaux, Arnaud Guillin, Liming Wu

SOME REMARKS ON WEIGHTED LOGARITHMIC SOBOLEV INEQUALITY

PATRICK CATTIAUX ♠ , **ARNAUD GUILLIN** ◇, AND **LI-MING WU** ◇

♠ Université de Toulouse

◇ Université Blaise Pascal

Abstract. We give here a simple proof of weighted logarithmic Sobolev inequality, for example for Cauchy type measures, with optimal weight, sharpening results of Bobkov-Ledoux [12]. Some consequences are also discussed.

Key words : Lyapunov functions, Talagrand transportation information inequality, logarithmic Sobolev inequality.

MSC 2000 : 26D10, 47D07, 60G10, 60J60.

1. Introduction

In a recent paper, Bobkov and Ledoux [12, Th. 3.4] proved that for a generalized Cauchy measure on $\mathbb{R}^n$, i.e.

$$d\nu_\beta(x) = \frac{1}{Z}(1+|x|^2)^{-\beta}dx$$

for $\beta > n/2$, the following weighted logarithmic Sobolev inequality holds, provided $\beta \geq (n+1)/2$: for any smooth bounded f

$$\mathrm{Ent}_{\nu_\beta}(f^2) = \nu_\beta\left(f^2 \log\left(\frac{f^2}{\nu_\beta(f^2)}\right)\right) \leq \frac{1}{\beta-1}\int |\nabla f(x)|^2(1+|x|^2)^2 d\nu_\beta(x).$$

Simple test functions however indicate that the weight $(1+|x|^2)^2$ is not optimal: one hopes $(1+|x|^2)\log(e+|x|^2)$ and that is what we will recover (with somewhat less precise constants).

It will be thus our purpose to prove inequalities of the type

$$\mathrm{Ent}_\mu(f^2) \leq c\int |\nabla f|^2 \omega d\mu$$

for some weight $\omega \geq 1$, and more generally weighted F-Sobolev inequalities with more general F's replacing the logarithm.

The (in a particular sense) case of weighted Poincaré inequalities is studied in [12] for Cauchy type measures and in [15] in more general situations. Consequences in terms of concentration of measure or isoperimetry are described in details in the latter reference.

Date: May 21, 2010.

It should also be interesting to look at weights that go to 0 at infinity (instead of weights bounded by 1 from below). Part of the results in [15] and in the present paper extend to this situation.

Our strategy will be the following:

(1) consider a dual form of the weighted logarithmic Sobolev inequality (or more generally F-Sobolev inequality): the Super weighted Poincaré inequality;
(2) use Lyapunov condition to prove these Super weighted Poincaré inequalities;
(3) show that these Super weighted Poincaré inequalities are equivalent to weighted F-Sobolev inequality (and in particular weighted logarithmic Sobolev inequality).

Let us then introduce the so called Super weighted Poincaré inequality for a probability measure μ, in a simple context, namely when the underlying Carré du champ is in fact the square length of the gradient. It is inspired from the pioneering works on Super Poincaré inequality by Wang [41]. Given a weight ω larger than 1, we say that μ satisfies a Super weighted Poincaré inequality if for all f smooth and bounded, there exists a non-increasing function β_ω such that for all $s > 0$

SwPI (1.1)
$$\int f^2 d\mu \leq s \int |\nabla f|^2 \omega d\mu + \beta_\omega(s)(\mu(|f|))^2 \,. \tag{1.1}$$

When $\omega = 1$, it is the usual Super Poincaré inequality which describes properties of the measure stronger than the usual Poincaré inequality. If we add some additional weight ω (tending to infinity as $|x| \to \infty$ for example) we will be able to give an inequality adapted to measures "above" and "below" Poincaré, being even able to play between the weight and β. Weighted Poincaré inequalities have been recently investigated by Bobkov-Ledoux [12] in particular for their interest in deviation inequalities, and by Cattiaux and al [15] showing their link with weak Poincaré inequalities and isoperimetric inequalities. They have been also intensively studied, in a converse form, in PDE theory to establish exponential convergence to equilibrium for fast diffusion equations (see [21, 8]). In parallel, Cattiaux and al [18] have studied Super Poincaré inequalities using Lyapunov conditions (see also [2, 3]). We will combine here these two approaches to study these Super weighted Poincaré inequalities.

2. Results and examples

2.1. **A Lyapunov condition for Super weighted Poincaré inequality.**

Lyapunov conditions appeared a long time ago in relation with the problem of convergence to equilibrium for Markov processes, see [36, 37, 38, 24] and references therein. They also have been used to study large and moderate deviations for empirical functionals of Markov processes (see Donsker-Varadhan [22, 23], Kontoyaniis-Meyn [34, 35], Wu [43], Guillin [31, 30],...) Their use to provide functional inequalities has been very recently deeply investigated with some success: Lyapunov-Poincaré inequalities [3], Poincaré inequalities [2], transportation inequalities [19], Super Poincaré inequalities [18], weighted and weak Poincaré inequalities [15] (also see the recent survey [17]). We will take advantage of the approach of these last two papers to build our main results, but let us first describe our framework.

Let E be some Polish state space, μ a probability measure and a μ-symmetric diffusion semigroup P_t with generator L on $L^2(E, \mu)$. The main assumption on L is that there exists some algebra $\mathcal{A}$ of bounded and uniformly continuous functions, containing constant functions,

which is in the domain of L in the graph norm of $\mathcal{L}$ on $L^2(\mu)$. It enables us to define a "carré du champ" Γ, *i.e.* for $f, g \in \mathcal{A}$, $L(fg) = fLg + gLf + 2\Gamma(f,g)$. We will also assume that Γ is a derivation (in each component), *i.e.* $\Gamma(fg,h) = f\Gamma(g,h) + g\Gamma(f,h)$, i.e. we are in the standard "diffusion" case in [1] and we refer to the introduction of [13] for more details. For simplicity we set $\Gamma(f) = \Gamma(f,f)$. Also, since L generates a diffusion, we have the following chain rule formula $\Gamma(\Psi(f), \Phi(g)) = \Psi'(f)\Phi'(g)\Gamma(f,g)$.
In particular if $E = \mathbb{R}^n$, $\mu(dx) = p(x)dx$ and $L = \Delta + \nabla \log p \cdot \nabla$, we may consider the algebra generated by C^∞ functions with compact support and the constant functions, as the interesting subalgebra $\mathcal{A}$, and then $\Gamma(f,g) = \nabla f \cdot \nabla g$.

Now we define the notion of ϕ-Lyapunov function. Let $W \geq 1$ be a smooth enough function on E and ϕ be a $\mathcal{C}^1$ positive increasing function defined on $\mathbb{R}^+$. We say that W is a ϕ-Lyapunov function if there is a family of increasing sets $(A_r)_{r\geq 0} \subset E$ such that $\bigcup_r A_r = E$ (we say that the family A_r is exhausting) and some $b \geq 0$ such that for some $r_0 > 0$

lyap (2.1)
$$LW \;\leq\; -\phi(W) \;+\; b\, \mathbb{1}_{A_{r_0}}\,. \tag{2.1}$$

This latter condition is sometimes called a "drift condition" but we prefer to call it Lyapunov condition. One has very different behavior depending on ϕ: if ϕ is linear then a Poincaré inequality is valid, whereas when ϕ is super-linear (or more generally in the form $\phi \times W$ where ϕ tends to infinity) we have stronger inequalities (Super Poincaré, ultracontractivity...), and finally if ϕ is sub-linear we are in the regime of weak Poincaré inequalities. We will cover setting in both weak and super Poincaré inequalities playing with the weight function.

We are now in position to state our main theorem:

mainTH **Theorem 2.1.** *Assume that L satisfies a Lyapunov condition (2.1), that μ satisfies some local Super Poincaré inequality, i.e. there exists β_{loc} decreasing in s (for all r) such that $\forall s > 0$*

SPloc (2.2)
$$\int_{A_r} f^2 d\mu \leq s \int \Gamma(f) d\mu + \beta_{loc}(r,s) \left(\int_{A_r} |f| d\mu \right)^2 . \tag{2.2}$$

We also introduce some $\psi : [1,\infty[\to [1,\infty[$ which is increasing and such that

$$0 < (\phi/\psi)'(W) \leq 1\,.$$

We finally assume that $G(r) := 1/(\inf_{A_r^c} \psi(W))$ goes to 0 as r goes to infinity.
Then μ satisfies a Super weighted Poincaré inequality, i.e. for all $s > 0$

result (2.3)
$$\int f^2 d\mu \leq 2s \int \frac{\Gamma(f)}{\left(\frac{\phi}{\psi}\right)'(W)}\, d\mu + \tilde{\beta}(s) \left(\int |f| d\mu \right)^2 \tag{2.3}$$

where

$$\tilde{\beta}(s) = c_{r_0}\, \beta_{loc}(G^{-1}(s), s/c_{r_0})$$

$G^{-1}(s) = \inf\{t > 0; G(t) > s\}$ *is the right inverse of* G *and*

$$c_{r_0} = 1 + b \frac{\sup_{A_{r_0}} (\psi/\phi)(W)}{\inf_{A_{r_0}^c} \psi(W)}.$$

Remark 2.2. In fact it is of course sufficient to verify some local Super weighted Poincaré inequality, but as the weight is usually bounded on the subset A_r considered, they are equivalent (up to the constants involved). And even, playing with r, as the weight is supposed to be greater than 1 they are implied by the local Super Poincaré inequalities as used here.

Remark 2.3. In the particular case where $\Gamma(f,g) = \nabla f \cdot \nabla g$ one can take more general Lyapunov condition, namely $\phi(W)$ may be replaced by $\phi \times W$ for some functional ϕ and the same for ψ appearing in the theorem. The modifications are straightforward but give hard to read result, and we let then the details for people needing such a framework.

Remark 2.4. In practice, A_r will often be level sets of the Lyapunov function W or balls of radius r. The local Super Poincaré inequality will then be obtained by perturbation of the Super weighted Poincaré inequality on balls for the underlying (Lebesgue) measure.

Proof. Let us begin with quite easy estimates: for $r \geq r_0$

$$\begin{aligned}
\int f^2 d\mu &= \int_{A_r} f^2 d\mu + \int_{A_r^c} f^2 d\mu \\
&= \int_{A_r} f^2 d\mu + \int_{A_r^c} \frac{\psi(W)\,\phi(W)}{\psi(W)\,\phi(W)} f^2 d\mu \\
&\leq \int_{A_r} f^2 d\mu + \frac{1}{\inf_{A_r^c}\psi(W)} \int f^2 \frac{\psi(W)}{\phi(W)}\,\phi(W)\,d\mu \\
&\leq \int_{A_r} f^2 d\mu + b\,\frac{\sup_{A_{r_0}}\left(\frac{\psi}{\phi}(W)\right)}{\inf_{A_r^c}\psi(W)} \int_{A_{r_0}} f^2 d\mu \\
&+ \frac{1}{\inf_{A_r^c}\psi(W)} \int \frac{-LW}{\frac{\phi}{\psi}(W)} f^2\,d\mu \\
&\leq \left(1 + b\,\frac{\sup_{A_{r_0}}\left(\frac{\psi}{\phi}(W)\right)}{\inf_{A_r^c}\psi(W)}\right) \int_{A_r} f^2 d\mu + \frac{1}{\inf_{A_r^c}\psi(W)} \int \frac{-LW}{\frac{\phi}{\psi}(W)} f^2\,d\mu\,.
\end{aligned}$$

Applying Lemma 2.5 below to the second term, the local Super Poincaré inequality and the fact that $(\phi/\psi)'(W) \leq 1$ to the first, we get

$$\begin{aligned}
\int f^2 d\mu \;\leq\; & \left(s\left(1 + b\frac{\sup_{A_{r_0}}(\psi/\phi)(W)}{\inf_{A_r^c}\psi(W)}\right) + \frac{1}{\inf_{A_r^c}\psi(W)}\right) \int \frac{\Gamma(f)}{\left(\frac{\phi}{\psi}\right)'(W)}\,d\mu \\
& + \beta_{loc}(r,s)\left(1 + b\frac{\sup_{A_{r_0}}(\psi/\phi)(W)}{\inf_{A_r^c}\psi(W)}\right)\left(\int |f|d\mu\right)^2 .
\end{aligned}$$

Recall now

$$c_{r_0} = 1 + b\frac{\sup_{A_{r_0}}(\psi/\phi)(W)}{\inf_{A_{r_0}^c}\psi(W)}$$

and $\tilde{s} = s c_{r_0}$ so that, since A_r^c is decreasing in r, the last inequality furnishes

$$\int f^2 d\mu \leq (\tilde{s} + G(r)) \int \frac{\Gamma(f)}{\left(\frac{\phi}{\psi}\right)'(W)}\,d\mu + \beta_{loc}(r, \tilde{s}/c_{r_0})c_{r_0}\left(\int |f|d\mu\right)^2 .$$

Choose now $r = G^{-1}(\tilde{s})$ to conclude. □

One crucial element of the proof above was the following lemma borrowed from [15] whose proof is reproduced here for completeness (showing also the necessity for L to be a diffusion)

:philyapounov **Lemma 2.5.** *Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a $\mathcal{C}^1$ increasing function. Then, for any $f \in \mathcal{A}$ and any positive $h \in D(\mathcal{E})$,*

$$\int \frac{-Lh}{\psi(h)}\, f^2\, d\mu \leq \int \frac{\Gamma(f)}{\psi'(h)} d\mu$$

Proof. Since L is μ-symmetric, using that Γ is a derivation and the chain rule formula, we have

$$\int \frac{-Lh}{\psi(h)}\, f^2\, d\mu \;=\; \int \Gamma\left(h, \frac{f^2}{\psi(h)}\right) d\mu = \int \left(\frac{2\, f\, \Gamma(f,h)}{\psi(h)} - \frac{f^2 \psi'(h)\Gamma(h)}{\psi^2(h)}\right) d\mu\, .$$

Since ψ is increasing and according to Cauchy-Schwarz inequality we get

$$\begin{aligned} \frac{f\,\Gamma(f,h)}{\psi(h)} &\leq \frac{f\sqrt{\Gamma(f)\Gamma(h)}}{\psi(h)} = \frac{\sqrt{\Gamma(f)}}{\sqrt{\psi'(h)}} \cdot \frac{f\sqrt{\psi'(h)\Gamma(h)}}{\psi(h)} \\ &\leq \frac{1}{2}\frac{\Gamma(f)}{\psi'(h)} + \frac{1}{2}\, \frac{f^2\psi'(h)\,\Gamma(h)}{\psi^2(h)}. \end{aligned}$$

The result follows. □

2.2. **Equivalence with weighted F-Sobolev inequality.** Let F be a continuous function, such that $\sup_{0<r<1} |rF(r)| < \infty$, $F(1) = 0$ and $\lim_{x\to+\infty} F(x) = +\infty$. We will say that the probability measure μ satisfies a defective weighted F-Sobolev inequality, with constants C_1 and C_2, and weight ω, if for all smooth and bounded f with $\mu(f^2) = 1$

$$\int f^2 F(f^2) d\mu \leq C_1 \int \Gamma(f)\, \omega d\mu + C_2.$$

Notice that, modifying if necessary the constant C_2 we may replace F by F_+. This inequality will be called tight, or simply a weighted F-Sobolev inequality if $C_2 = 0$.

When $\omega = 1$, it is known that if μ satisfies a defective F-Sobolev inequality and a Poincaré inequality, and with some (slight) additional assumptions on F, then μ satisfies a (tight) F-Sobolev inequality. The case $F = \log$ is known as Rothaus lemma, and the previous general result is obtained in [4] lemma 9 and Theorem 10.

The reader will easily check that the proofs in [4] extend to the weighted case, i.e. a weighted Poincaré inequality (with weight ω) and a weighted defective F-Sobolev inequality (with the same ω) imply a tight weighted F-Sobolev inequality, under the same assumptions than in [4] lemma 9. These assumptions are satisfied when $F(x) = \log_+(x)$ (see remark 15 in [4]). We thus have that a weighted log-Sobolev inequality implies a weighted $\log_+$-Sobolev inequality, and together with a weighted Poincaré inequality implies a tight weighted $\log_+$-Sobolev inequality, hence a tight weighted log-Sobolev inequality.

We shall use this line of reasoning in various situations below, without mentioning it explicitly.

Now let us make a simple remark: if in the Super weighted Poincaré inequality, we assume moreover that β_ω tends to a constant smaller than 1 as $s \to \infty$ (which is quite a very weak hypothesis), the Super weighted Poincaré inequality implies a weighted Poincaré inequality. Indeed applying (1.1) with $f = g - \mu(g)$ we get

$$(1-\beta_\omega(s))\,\mathrm{Var}_\mu(g) \le s\int \Gamma(g)\,\omega\,d\mu\,,$$

thanks to Cauchy-Schwarz inequality, and the result follows taking a large enough s for the left hand side to be positive.

The next proposition is adapted from the works of Wang [41] and Theorems 3.3.1 and 3.3.3 in [42]. We include its proof for the sake of completeness.

eqFsob **Proposition 2.6.** (1) *If μ satisfies a defective weighted F-Sobolev inequality with constants C_1, C_2, then there exist c_1, c_2 such that for all smooth bounded functions f and $\forall s > 0$*

$$\int f^2 d\mu \le s\int \Gamma(f)\omega d\mu + c_1 F^{-1}(c_2(1+1/s))\mu(|f|)^2$$

where $F^{-1}(s) = \inf\{r \ge 0;\ F(r) \ge s\}$.

(2) *If μ satisfies a Super weighted Poincaré inequality*

$$\int f^2 d\mu \le s\int \Gamma(f)\omega d\mu + \beta_\omega(s)\mu(|f|)^2$$

then μ satisfies a defective weighted F-Sobolev inequality with

$$F(r) = \frac{c_1(\epsilon)}{r}\int_0^r \xi(\epsilon t)dt - c_2(\epsilon)$$

for all $0<\epsilon<1$, where $c_1(\epsilon)$ and $c_2(\epsilon)$ are some constants, and

$$\xi(t) = \sup_{r>0}\left(\frac{1}{r} - \frac{\beta_\omega(r)}{rt}\right),$$

where $\beta_\omega^{-1}(t) = \inf\{r\ge 0; \beta_\omega(r) \le t\}$.

Proof. (1). As said before we may assume that $F \ge 0$, enlarging C_2 if necessary. Pick f with $\mu(|f|) = 1$. For all $r, t, a > 0$, it holds

$$rt \le rF(r^2/a) + t\sqrt{aF^{-1}(t)}\,.$$

We choose $a = \mu(f^2)$, $r = |f|$ and multiply the previous inequality by $|f|$, i.e.

$$t\,f^2 \le f^2\,F(f^2/\mu(f^2)) + |f|t\sqrt{\mu(f^2)\,F^{-1}(t)}\,.$$

Integrating this inequality with respect to μ yields

$$\mu(f^2F(f^2/\pi(f^2))) \ge t\mu(f^2) - t\sqrt{\mu(f^2)F^{-1}(t)}$$

and using the defective weighted F-Sobolev inequality :

$$(t-C_2)\mu(f^2) - t\sqrt{\mu(f^2)F^{-1}(t)} - C_1\int \Gamma(f)\,\omega d\mu \le 0\,.$$

Hence, for $t > C_2$,

$$\mu(f^2) \le \frac{2C_1}{t-C_2}\int \Gamma(f)\omega d\mu + \frac{t^2F^{-1}(t)}{(t-C_2)^2}\,,$$

and we write $r = 2C_1/(t - C_2)$ to conclude.
(2). The second part of the proof is inspired by capacity/measure criteria.
Pick f with $\mu(f^2) = 1$ and $\delta > 1$, consider $A_n = \{\delta^{n+1} > f^2 \geq \delta^n\}$ and

$$f_n = (|f| - \delta^{n/2})_+ \wedge (\delta^{(n+1)/2} - \delta^{n/2}).$$

Apply now the Super weighted Poincaré inequality to f_n,

$$\mu(f_n^2) \leq r\mu(\Gamma(f)\omega \mathbb{1}_{A_n}) + \beta(r)\mu(f_n)^2 \leq r\mu(\Gamma(f)\omega\mathbb{1}_{A_n}) + \beta_\omega(r)\mu(f^2 \geq \delta^n)\mu(f_n^2)$$

and since $\mu(f^2 \geq \delta^n) \leq 1/\delta^n$, we get

$$\begin{aligned}
\mu(\Gamma(f)\omega) &\geq \sum_{n\geq 0} \mu(\Gamma(f)\omega\mathbb{1}_{A_n}) \\
&\geq \sum_{n\geq 0} \xi(\delta^n)\mu(f_n^2) \\
&\geq \sum_{n\geq 0} \xi(\delta^n)\mu(f^2 \geq \delta^{n+1})(\delta^{(n+1)/2} - \delta^{n/2})^2 \\
&\geq \frac{(\sqrt{\delta}-1)^2}{1-\delta^{-1}} \sum_{n\geq 0} \int_{\delta^{n-1}}^{\delta^n} \xi(t)\mu(f^2 \geq \delta^2 t)dt \\
&\geq c_1 \int_0^\infty \xi(t))\mu(f^2 \geq \delta^2 t)dt - c_2 \\
&\geq c_3\pi(f^2F(f^2)) - c_2
\end{aligned}$$

which is what is needed. □

Using this result one sees that if a Super weighted (with weight ω) Poincaré inequality is valid with $\beta_\omega(s) = s^{-N}\, e^{c(1+1/s)}$ then a (ω) weighted logarithmic Sobolev inequality is valid. In the preceding subsection we have presented conditions to verify Super weighted Poincaré inequalities, we only have now to validate them through examples. It will be the purpose of the next subsection.

2.3. **Examples.** We consider here the $\mathbb{R}^n$ situation with $d\mu(x) = p(x)dx$ and $L = \Delta + \nabla \log p \cdot \nabla$, where p is smooth enough and positive, and $\cdot$ is the euclidean inner product. Recall the following elementary lemma whose proof can be found in [2]. This lemma will be helpful to deal with κ-concave measures.

lemfranck **Lemma 2.7.** *If V is convex and $\int e^{-V(x)}\, dx < +\infty$, then*

(1) *for all x, $x \cdot \nabla V(x) \geq V(x) - V(0)$,*
(2) *there exist $\delta > 0$ and $R > 0$ such that for $|x| \geq R$, $V(x) - V(0) \geq \delta\,|x|$.*

Another helpful result is the following result concerning the validity of a Super Poincaré inequality for Lebesgue measures on balls: for all $r > 0$ denote by $B(0, r)$ the euclidean ball in $\mathbb{R}^n$. Then there exists c_n such that for all smooth f and all $s > 0$,

SPIballs (2.4)
$$\int_{B(0,r)} f^2 dx \leq s \int_{B(0,r)} |\nabla f|^2 dx + c_n(1 + s^{-n/2}) \left(\int_{B(0,r)} |f| dx\right)^2.$$

Such an inequality will be particularly efficient when dealing with radial type measures, as perturbation argument to get the local Super Poincaré inequality will be easy to do.

Indeed we immediately obtain

muSPIballs

$$\begin{aligned} (2.5)\quad \int_{B(0,r)} f^2 d\mu \quad &\leq \quad s\int_{B(0,r)} |\nabla f|^2 d\mu \\ &+ \quad c_n \left(1+\left(\frac{s\,\inf_{B(0,r)} p}{\sup_{B(0,r)} p}\right)^{-n/2}\right)\left(\frac{\sup_{B(0,r)} p}{\inf^2_{B(0,r)} p}\right)\left(\int_{B(0,r)} |f| d\mu\right)^2 . \end{aligned}$$

For more general type of measures, it is not so difficult to get local inequalities for level sets of the potential, see [18, Prop. 3.6].

2.3.1. *Cauchy type measures.* Let $d\mu(x) = (V(x))^{-(n+\alpha)}\, dx$ for some positive convex function V and some $\alpha > 0$. Let us begin by establishing a Lyapunov condition:

lem:convex

Lemma 2.8. *Let $L = \Delta - (n+\alpha)(\nabla V/V)\nabla$ with V convex and $\alpha > 0$. Then, there exists $k \in (2, \alpha+2)$, $b, R > 0$ and function $W \geq 1$ such that*

$$LW \leq -\phi(W) + b\mathbb{1}_{B(0,R)}$$

with $\phi(u) = cu^{(k-2)/k}$ for some constant $c > 0$. Furthermore, one can choose $W(x) = |x|^k$ for x large.

Proof. Let $L = \Delta - (n+\alpha)(\nabla V/V)\nabla$ and choose $W \geq 1$ smooth, satisfying $W(x) = |x|^k$ for $|x|$ large enough and $k > 2$ that will be chosen later. For $|x|$ large enough we have

$$LW(x) = k\,(W(x))^{\frac{k-2}{k}}\left(n+k-2-\frac{(n+\alpha)\,x.\nabla V(x)}{V(x)}\right).$$

Using (1) in Lemma 2.7 (since $V^{-(n+\alpha)}$ is integrable, e^{-V} is also integrable) we have

$$n+k-2-\frac{(n+\alpha)\,x.\nabla V(x)}{V(x)} \leq k-2-\alpha+(n+\alpha)\frac{V(0)}{V(x)}.$$

Using (2) in Lemma 2.7 we see that we can choose $|x|$ large enough for $\frac{V(0)}{V(x)}$ to be less than ε, say $|x| > R_\varepsilon$. It remains to choose $k > 2$ and $\varepsilon > 0$ such that

$$k + n\varepsilon - 2 - \alpha(1-\varepsilon) \leq -\gamma$$

for some $\gamma > 0$. We have shown that, for $|x| > R_\varepsilon$,

$$LW \leq -k\gamma\phi(W),$$

with $\phi(u) = u^{\frac{k-2}{k}}$ (which is increasing since $k > 2$). A compactness argument achieves the proof. □

Consider now the case studied in [12] of the (generalized) Cauchy measure:

$$p(x) = Z_\beta^{-1}(1+|x|^2)^{-\beta}, \qquad \beta > n/2.$$

Lemma 2.8 gives us a Lyapunov conditions. Using (2.5) we get local Super Poincaré inequalities

$$\begin{aligned}\int_{B(0,R)} f^2 d\mu &\leq s \int_{B(0,R)} |\nabla f|^2 d\mu \\ &+ c_n \left(1 + s^{-n/2}\,(1+R^2)^{\beta\, n/2}\right)(1+R^2)^{2\beta}\, Z_\beta \left(\int_{B(0,R)} |f| d\mu\right)^2 .\end{aligned}$$

Choose now $\psi(v) = \log(v)$ for large v (and ψ smooth), Theorem 2.1 together with Proposition 2.6 thus furnishes (up to local modifications i.e for large $|x|$'s for example)

$$\phi(u) = u^{k-2/k}\,,\ \psi(u) = \log(u)\,,\ W(x) = |x|^k\,,\ (\psi(W))(x) = k \log|x|$$

hence

$$G(r) = \frac{1}{k\,\log r}\,,\ G^{-1}(s) = e^{1/ks}$$

so that

$$\left(\frac{\phi}{\psi}\right)'(u) \sim \frac{c}{u^{2/k}\,\log u}\,,\ \omega(x) \sim \left(\frac{1}{(\phi/\psi)'(W)}\right)(x) \sim c\,|x|^2\,\log|x|$$

and finally for small s

$$\beta_\omega(s) \sim s^{-\,n/2}\, e^{c/s}\,.$$

We have thus obtained

corcauchy **Corollary 2.9.** *Cauchy measures $\mu(dx) = Z_\beta^{-1}(1+|x|^2)^{-\beta}$ for $\beta > n/2$ verify the following weighted logarithmic Sobolev inequality: there exists $C = C(\beta, n)$ such that for all smooth bounded function f*

$$Ent_\mu(f^2) \leq C \int |\nabla f(x)|^2\,(1+|x|^2)\log(e+|x|^2) d\mu(x)\,.$$

We then obtain the correct order of magnitude of the weight in this inequality, compared to [12, Th.3.4]. However it has to be noted that we are loosing the pretty expression of the constant in front of the weighted energy. Note that in dimension 1, Barthe-Zhang [7] obtained the same weight.

2.3.2. *Exponential measure.* We will look at the exponential measure

$$\nu(dx) = Z_n^{-1}\, e^{-|x|} dx.$$

It is well known that the exponential measure satisfies a Poincaré inequality. It is also easy to see that considering $W(x) = e^{a|x|}$ for $|x| \geq R$, we get if $a < 1$ for R large enough

$$LW(x) = a\left(\frac{n-1}{|x|} + a - 1\right) W(x) \leq -\lambda W + b\mathbb{1}_{B(0,R)}$$

and thus the Lyapunov condition.

Using (2.5) with the choice $\psi(v) = \log(v)$ for large v (and ψ smooth), we get

Corollary 2.10. *The exponential measure ν satisfied the following weighted logarithmic Sobolev inequality: there exists $C = C(\beta, n)$ such that for all smooth bounded function f,*

$$Ent_\mu(f^2) \leq C \int |\nabla f(x)|^2\,(1+|x|)\, d\mu(x).$$

As a comparison, let us recall a result of Bobkov-Ledoux [11, Eq. (1.6)] which states that for the one sided exponential $\tilde{\nu}$ (in dimension one)

$$\mathrm{Ent}_{\tilde{\nu}}(f^2) \leq 4 \int x(f'(x))^2 d\tilde{\nu}.$$

We then recover in any dimension their result directly (they can only use tensorization to get n-dimensional version of this inequality) and may extend it to other potential.

Remark 2.11. Actually the proof above covers a very large class of measures satisfying a Poincaré inequality, namely measures $\mu(dx) = e^{-V}dx$ such that $V \to +\infty$ as $|x| \to +\infty$ and satisfying the following condition

$$\text{there exists } 0 < a < 1 \text{ such that } \liminf_{|x|\to+\infty} \left(a|\nabla V|^2 - \Delta V\right) = B > 0\,.$$

Indeed in this case we have $\phi(u) = \lambda u$ (for some $\lambda > 0$) and $W = e^{AV}$ for some well chosen positive constant A.
Choosing again $\psi(u) = \log u$ for large u's we obtain the weight $\omega(x) = |x|$ for large $|x|$'s. If we assume in addition that there exists some constant $c > 0$ such that for all R and all x such that $|x| = R$,

$$c \sup_{|y|=R} V(y) \leq V(x) \leq \frac{1}{c} \inf_{|y|\geq R} V(y)\,,$$

it is not difficult to see that $G^{-1}(s) \sim (\bar{V})^{-1}(1/s)$ where $\bar{V}(R) = \inf_{|y|\geq R} V(y)$ is increasing. Using (2.5) again we obtain that $\beta_\omega(s) \sim \exp(C/s)$ hence the same weighted logarithmic Sobolev inequality as in the previous corollary.
We do not know whether this is true for any measure satisfying the Poincaré inequality. Indeed we know that there exists some Lyapunov function W yielding a linear ϕ, but we do not know in full generality how to compare W and the potential V, so that we cannot give an explicit formula for β_ω. $\diamondsuit$

3. Properties and Applications

3.1. **Concentration of measure.** We will present here two different approaches to get concentration inequalities. The first one, due to Bobkov-Ledoux [12] uses controls on the growth of moments. As we obtain optimal weight by our approach, we will compare on some examples what are the implications of these better controls. The other one is based on the derivation of a suitable transportation cost information inequality following the approach of [9] based on Hamilton-Jacobi equation.

3.1.1. *Growth of moments and Deviation inequality.* We briefly recall here the main results concerning concentration inequality obtained by Bobkov-Ledoux [12, Th. 4.1,Cor. 4.2] and present their main result

Theorem 3.1 (Bobkov-Ledoux [12])**.** *Assume that the following weighted logarithmic Sobolev inequality is satisfied*

$$Ent_\mu(f^2) \leq 2 \int |\nabla f|^2 \omega d\mu.$$

Assume also that ω has a finite moment of order $p \geq 2$, then for any μ-centered 1-Lipshitz function f, one has

$$\|f\|_p \leq \sqrt{p-1}\|\omega\|_p.$$

It implies that if $\| \omega \|_p \leq C$,

$$\mu(|f| \geq t) \leq \begin{cases} 2e^{-t^2/2c^2e} & \text{if } 0 \leq t \leq C\sqrt{ep} \\ 2e^{-t/Ce} & \text{if } C\sqrt{ep} \leq t \leq Cep \\ 2\left(\frac{Cp}{t}\right)^p & \text{if } Cep \leq t \end{cases} \tag{3.1}$$

Remark now that the weight obtained by Bobkov-Ledoux for Cauchy measures ν_β is $\omega = (\beta-1)^{-1}(1+|x|^2)^2$ whereas ours is $\omega = C(1+|x|^2)\log(1+|x|^2)$ which thus allows integration for $L^p(\mu)$ for a larger p. In addition Corollary 2.9 is obtained for $\beta > n/2$ instead of $\beta \geq (n+1)/2$. Thus our result furnishes in principle a larger strip of Gaussian concentration. However the evaluation of C is quite bad here (due mainly to the local inequality). It thus raises the question of the optimal constant with our weight. In dimension 1, one may use the generalized Hardy inequality.

3.1.2. *Transportation inequality.* We give here another way to derive concentration inequality, based on transportation inequality, as derived from logarithmic Sobolev type inequality by Bobkov-Gentil-Ledoux [9] using Hamilton-Jacobi equation (see also [39] for a proof based on PDE and optimal transport, or [16] for a refined argument). Let us give quickly the argument adapted to our setting. First, let d_ω be the new Riemanian distance associated to ω, i.e. C_{xy} is the set of all absolutely continuous paths $\gamma : [0,1] \to \mathbb{R}^d$ such that $\gamma(0) = x$ and $\gamma(1) = y$ and

$$d_\omega(x,y) := \inf_{\gamma \in C_{x,y}} \int_0^1 \sqrt{\omega(\gamma(s))^{-1}\gamma'(s)^2} ds.$$

Thanks to results of Cutri-DaLio [20] or Dragoni [25] (in a more general setting, like possibly degenerate weight), the inf-convolution $Q_t^\omega f(x) := \inf\{f(y) + \frac{1}{t}d_\omega(x,y)\}$ is the viscosity solution of the weighted Hamilton-Jacobi equation

$$\begin{cases} \partial_t v + \frac{1}{2}\omega|\nabla v|^2 = 0 & \forall (x,t) \in \mathbb{R}^d \times]0,\infty[, \\ v = f & \forall (x,t) \in \mathbb{R}^d \times \{0\}. \end{cases} \tag{3.2}$$

Suppose now that μ satisfies a weighted logarithmic Sobolev inequality with weight $2\omega \geq 1$ (the factor 2 is only for a nice formulation of the result), we apply it to the function $f^2 = e^{tQ_t^\omega g}$ and denote $G(t) = \mu(f^2)$ so that we get, using that

$$tQ_tg = t\partial_t(tQ_tg) + \frac{1}{2}|\nabla(tQ_tg)|^2$$

the differential inequality

$$tG'(t) \leq G(t)\log(G(t)), \qquad G'(0) = \rho\mu(g).$$

It is now immediate to obtain that

$$\mu(e^{Q_1 g}) \leq e^{\mu(g)}$$

which is, by Bobkov-Goetze's result [10] an equivalent formulation for a T_2 inequality. Summarizing this argument, we get

Theorem 3.2. *Suppose that μ satisfies a weighted logarithmic Sobolev inequality with weight 2ω, i.e. for all nice f*

$$Ent_\mu(f^2) \leq 2 \int |\nabla f|^2 \, \omega d\mu,$$

then μ satisfies the following weighted Transportation-Information inequality (ωT_2): for all probability measure ν with $d\nu = f d\mu$

wTI

$$W_{2,\omega}^2(\nu, \mu) \leq Ent_\mu(f). \tag{3.3}$$

Here $W_{p,\omega}(\nu, \mu)$ is the L^p-Wasserstein distance between two probability measures ν, μ on E. Note that as usual, such a (ωT_2) inequality implies a (ωT_1) inequality: for all probability measure ν

$$W_{1,\omega}(\nu, \mu) := \sup_{\|f\|_{Lip(\omega)} \leq 1} \left(\int f d\nu - \int f d\mu \right) \leq \sqrt{\mathrm{Ent}_\mu \left(\frac{d\nu}{d\mu} \right)}$$

where $\|f\|_{Lip(\omega)} \leq 1$ means that $|f(x) - f(y)| \leq d_\omega(x, y)$.
The last inequality is equivalent to the fact that for all μ-centered function with $\|f\|_{Lip(\omega)} \leq 1$, $\forall r > 0$,

$$\mu(|f| \geq r) \leq 2e^{-r^2/2}.$$

3.2. **Entropic convergence.**

3.2.1. *The natural diffusion associated to the weighted energy.* As is well known, logarithmic Sobolev inequality are equivalent to the exponential decay in $\mathbb{L} \log \mathbb{L}$ of the diffusion semi group-associated to the Dirichlet form present in the inequality. We then get that a weighted logarithmic Sobolev inequality for the measure $d\mu = e^{-V(x)}dx$

$$\mathrm{Ent}_\mu(f^2) \leq \int |\nabla f|^2 \omega d\mu$$

implies that the semi-group (P_t^ω) with generator

$$L^\omega = \omega \Delta + (\nabla \omega - \omega \nabla V).\nabla$$

satisfies

$$\mathrm{Ent}_\mu(P_t^\omega f) \leq e^{-t/4} \mathrm{Ent}_\mu(f).$$

As this semigroup is reversible with respect to μ, it is certainly possible to use the results of [18], via also Lyapunov conditions, to get this convergence but it is far easier to get a Lyapunov condition on the generator L than on L^ω. Note that it may also be useful when one desires to sample from μ via a Langevin tempered diffusions type algorithm (see [24]): we provide here an easy way to find a diffusion coefficient leading to an exponential entropic convergence. It has to be noted that the approach is quite different than in Hwang&al [33] or Franke&al [26] where they add a divergence free drift to accelerate the diffusion. Moreover they are limited to cases where the initial measure μ satisfies a Poincaré inequality. One may also get deviation inequality for integral functional of this Markov process, once remarked that assuming weighted logarithmic Sobolev inequality implies a transportation cost (ωT_2) inequality, then we have using once again the weighted logarithmic Sobolev inequality: for all probability measure ν with $d\nu = f d\mu$

$$W_{2,\omega}^2(\nu, \mu) \leq 2 \int \frac{|\nabla f|^2}{f} \omega d\mu$$

which implies, by [32] that for all μ-centered function f with $\|f\|_{Lip(\omega)} \le 1$ and for $(X_t^\omega)_{t\ge 0}$ the Markov process with generator L^Ω: for all positive r

$$\mathbb{P}_\nu\left(\frac{1}{t}\int_0^t f(X_s^\omega)ds \ge r\right) \le e^{-r^2/4},$$

which may be useful in Monte-Carlo simulation.

3.2.2. *Link with weak logarithmic Sobolev inequality.* Two of the authors with I. Gentil introduced in [14] the weak logarithmic Sobolev inequalities, i.e. μ satisfies (WLSI) for some non increasing function β if for all bounded smooth function, $\forall s > 0$

WLSI (3.4)
$$\mathrm{Ent}_\mu(f^2) \le \beta(s)\int |\nabla f|^2 d\mu + sOsc(f)^2\,.$$

This is the weak counterpart of the classical Gross logarithmic Sobolev inequalities as weak Poincaré inequalities of [40] were for the usual Poincaré inequalities. These weak logarithmic Sobolev inequalities are particularly useful to assert the speed of convergence towards equilibrium (for the natural Markov process associated to μ) in entropy when dealing with particular initial measure (such as Dirac mass, not suitable to an L^2 analysis).
It was shown in [14] that weak logarithmic Sobolev inequalities are equivalent to some capacity/measure conditions. If in dimension one, these capacity/measure conditions can be translated into verifiable conditions, it is no more the case in larger dimensions and only a comparison, under some additional conditions, with Beckner inequalities (stronger than Poincaré) or weak Poincaré inequalities gave multidimensional examples. We will show here that weighted logarithmic Sobolev inequalities together with some concentration estimates, enable us to obtain weak logarithmic Sobolev inequalities, so that Lyapunov type conditions plus concentration give a new set of conditions for weak logarithmic Sobolev inequalities.

Theorem 3.3. *Assume that μ satisfies the following weighted logarithmic Sobolev inequality*

$$Ent_\mu(f^2) \le \int \omega|\nabla f|^2 d\mu$$

then μ satisfies a (WLSI) with function $\beta(s) = g^{-1}(s)$ where

$$g(r) = \mu(B_r^c)\left[2\mathfrak{c} + \log\left(1 + \frac{e^2}{\mu(B_r^c)}\right)\right] \tag{3.5}$$

with $B_r = \{x;\, \omega \le r\}$ and $\mathfrak{c} > 0$ explicit.

Proof. Let us first recall the result of Theorem 2.2 of [14] (taking advantage of Remark 2.3), that is a capacity measure condition for weak logarithmic Sobolev inequality.
To this end, let us recall the definition of the capacity of a given measurable set $A \subset \Omega$:

$$Cap_\mu(A,\Omega) := \inf\left\{\int |\nabla f|^2 d\mu;\ \mathbb{1}_A \le f \le \mathbb{1}_\Omega\right\}$$

where the infimum is taken over all Lipschitz functions. Finally if A is such that $\mu(A) < 1/2$ then

$$Cap_\mu(A) := \inf\{Cap_\mu(A,\Omega);\ A \subset \Omega, \mu(\Omega) \le 1/2\}.$$

A sufficient condition for (3.4) to hold is then: for every A with $\mu(A) < 1/2$,

capwlsi (3.6)
$$\forall s > 0, \qquad \frac{\mu(A)\log\left(1+\frac{e^2}{\mu(A)}\right)}{\beta(s)} \leq Cap_\mu(A).$$

We cannot use directly our weighted logarithmic Sobolev inequality with this notion of capacity so that we introduce the natural weighted capacity

$$\overline{Cap}_\mu(A,\Omega) := \inf\left\{\int |\nabla f|^2 \omega d\mu;\ 1_A \leq f \leq 1_\Omega\right\}$$

$$\begin{aligned}\overline{Cap}_\mu(A) &:= \inf\{\overline{Cap}_\mu(A,\Omega);\ A\subset\Omega, \mu(\Omega)\leq 1/2\}\\ &= \inf\left\{\int |\nabla f|^2\omega d\mu;\ f:\mathbb{R}^d\to[0,1], f1_A = 1, \mu(f=0)\geq 1/2\right\}\end{aligned}$$

Using Bobkov-Goetze's seminal work [10] or its refined version by Barthe-Roberto [6], the weighted logarithmic Sobolev inequality implies that for all A such that $\mu(A) < 1/2$ there exists $\mathfrak{c}$ such that

$$\mu(A)\log\left(1+\frac{e^2}{\mu(A)}\right) \leq \mathfrak{c}\,\overline{Cap}_\mu(A).$$

Consider now the set $B_r = \{x;\ \omega \leq r\}$, by a simple adaptation of the proof of Gozlan [29,], we get that if $A \subset B_r$

$$\overline{Cap}_\mu(A) \leq 2rCap_\mu(A) + 2\mu(B_r^c).$$

Remark now that the mapping $t \to t\log(1+e^2/t)$ is concave increasing for small values of t, so that for all A such that $\mu(A) \leq 1/2$

$$\begin{aligned}\mu(A)\log\left(1+\frac{e^2}{\mu(A)}\right) &\leq \mu(A\cap B_r)\log\left(1+\frac{e^2}{\mu(A\cap B_r)}\right) + \mu(A\cap B_r^c)\log\left(1+\frac{e^2}{\mu(A\cap B_r^c)}\right)\\ &\leq \mathfrak{c}\,\overline{Cap}_\mu(A\cap B_r) + \mu(B_r^c)\log\left(1+\frac{e^2}{\mu(B_r^c)}\right)\\ &\leq 2\mathfrak{c}r\,Cap_\mu(A) + \mu(B_r^c)\left[2\mathfrak{c} + \log\left(1+\frac{e^2}{\mu(B_r^c)}\right)\right].\end{aligned}$$

Setting $s = \mu(B_r^c)\left[2\mathfrak{c} + \log\left(1+\frac{e^2}{\mu(B_r^c)}\right)\right]$, we conclude the proof. □

If r is large enough, concentration result of the previous section will give upper bounds for the second term of the left hand side.

3.3. **Modified logarithmic Sobolev inequalities.** We will prove here that weighted logarithmic Sobolev inequalities imply modified logarithmic Sobolev inequalities (i.e. the energy is modified). These inequalities were initially introduced by Bobkov-Ledoux [11], where they show that a Poincaré inequality implies a logarithmic Sobolev inequality for a particular class of functions ($|\nabla f/f| \leq c < C_{c_{SG}}$ where c_{SG} is the spectral gap constant). These results were later extended to measures between exponential and Gaussian by Gentil and al [27, 28]. For recent results, giving nice conditions we will discuss later, see also [5].

Theorem 3.4. *Let H and H^y be a pair of dual convex Young functions, such that $H(|x|)/|x| \geq a > 0$ for large $|x|$ and $H^*(\epsilon|x|) \leq b(\epsilon)H^y(|x|)$ with $b(\epsilon) \to 0$ as $\epsilon \to 0$.*
Suppose now that the following weighted logarithmic Sobolev inequality holds

$$Ent_\mu(f^2) \leq \int |\nabla f|^2 \omega d\mu$$

for some weight $\omega \geq 1$, that a Poincaré inequality holds and that for some $\alpha > 0$

integcond
$$K := \int e^{\alpha H^y(\omega)} d\mu < \infty. \tag{3.7}$$

Then the following modified logarithmic Sobolev inequality holds

mlsi
$$Ent_\mu(f^2) \leq C \int \left(H\left(\epsilon^{-1}\left|\frac{\nabla f}{f}\right|^2\right) f^2 + |\nabla f|^2 \right) d\mu \tag{3.8}$$

for sufficiently small ϵ and some constant C (explicit in the proof).

Proof. Actually, it is sufficient to get a defective modified logarithmic Sobolev inequality, since a Poincaré inequality allows us to tighten a defective inequality thanks to [5, Th. 2.4]. We then have

$$\begin{aligned} \mathrm{Ent}_\mu(f^2) &\leq \int |\nabla f|^2 \omega d\mu \\ &= \int \epsilon^{-1}\left|\frac{\nabla f}{f}\right|^2 \epsilon\omega f^2 d\mu \\ &\leq \int H\left(\epsilon^{-1}\left|\frac{\nabla f}{f}\right|^2\right) f^2 d\mu + \int H^y(\epsilon\omega) f^2 d\mu. \end{aligned}$$

Choose now ϵ sufficiently small so that $b(\epsilon) \leq \alpha/2$ so that

$$\begin{aligned} \int H^*(\epsilon\omega) f^2 d\mu &\leq \frac{1}{2}\int \alpha H^y(\omega) f^2 d\mu \\ &\leq \frac{1}{2}\int (\alpha H^y(\omega) - \log K) f^2 d\mu + \frac{1}{2}\log K \int f^2 d\mu \\ &\leq \frac{1}{2}\mathrm{Ent}_\mu(f^2) + \frac{1}{2}\log K \int f^2 d\mu \end{aligned}$$

where we have used the variational formula for the entropy in the last line. Plugging the latter inequality in the preceding one, we obtain the defective modified logarithmic Sobolev inequality:

$$\mathrm{Ent}_\mu(f^2) \leq 2\int H\left(\epsilon^{-1}\left|\frac{\nabla f}{f}\right|^2\right) f^2 d\mu + \log(K)\int f^2 d\mu$$

which ends the proof. $\square$

One may then use the Lyapunov conditions used to derive a weighted logarithmic Sobolev inequality to get a generalization of Barthe-Kolesnikov [5, Th. 5.27,5.28].

Examples:
Consider the usual (for modified LSI) examples: $d\mu = Z_\alpha e^{-|x|^\beta}$ for $1 < \alpha \leq 2$ so that the

Poincaré inequality is valid. Using Lyapunov function $W(x) = e^{a|x|^\beta}$ for a less than one, one may easily derive the following Lyapunov condition:

$$LW \leq -c|x|^{2(\beta-1)}W + b1_{B(0,R)}$$

from which one deduces using $\psi(w) = \log(w)$ and Theorem 2.1 (and Prop. 2.6):

$$\mathrm{Ent}_\mu(f^2) \leq C \int |\nabla f|^2 \, (1 + |x|^{2-\beta}) d\mu.$$

Consider now the Young functions $H_\beta(x) = |x|^{\frac{\beta}{2(\beta-1)}}$ and $H^y_\beta(x) = c_\beta |x|^{\frac{\beta}{2-\beta}}$ so that $H^y_\beta(\epsilon\omega) = c_\beta \omega^{\frac{\beta}{2-\beta}} |x|^\beta$ which is easily seen to be integrable wrt μ for ϵ sufficiently small. We then get

$$\mathrm{Ent}_\mu(f^2) \leq C \int \left(\left| \frac{\nabla f}{f} \right|^{\frac{\beta}{\beta-1}} f^2 + |\nabla f|^2 \right) d\mu$$

for some constant C, which is a generalization in the multidimensional case of [27].

References

bakry [1] D. Bakry. L'hypercontractivité et son utilisation en théorie des semigroupes. In *Lectures on Probability theory. École d'été de Probabilités de St-Flour 1992*, volume 1581 of *Lecture Notes in Math.*, pages 1–114. Springer, Berlin, 1994.

BBCG [2] D. Bakry, F. Barthe, P. Cattiaux, and A. Guillin. A simple proof of the Poincaré inequality for a large class of probability measures. *Electronic Communications in Probability.*, 13:60–66, 2008.

BCG [3] D. Bakry, P. Cattiaux, and A. Guillin. Rate of convergence for ergodic continuous Markov processes : Lyapunov versus Poincaré. *J. Func. Anal.*, 254:727–759, 2008.

BCR1 [4] F. Barthe, P. Cattiaux, and C. Roberto. Interpolated inequalities between exponential and Gaussian, Orlicz hypercontractivity and isoperimetry. *Rev. Mat. Iber.*, 22(3):993–1066, 2006.

BK08 [5] F. Barthe and A. V. Kolesnikov. Mass transport and variants of the logarithmic Sobolev inequality. *J. Geom. Anal.*, 18(4):921–979, 2008.

bartr03sipm [6] F. Barthe and C. Roberto. Sobolev inequalities for probability measures on the real line. *Studia Math.*, 159(3), 2003.

BZ [7] F. Barthe and Z.. Zhang. Private communication. 2009.

blanchet [8] A. Blanchet, M. Bonforte, J. Dolbeault, G. Grillo, and J.L. Vázquez. Asymptotics of the fast diffusion equation via entropy estimates. *Arch. Ration. Mech. Anal.*, 191(2):347–385, 2009.

BGL [9] S. G. Bobkov, I. Gentil, and M. Ledoux. Hypercontractivity of Hamilton-Jacobi equations. *J. Math. Pu. Appli.*, 80(7):669–696, 2001.

bobkov-gotze [10] S. G. Bobkov and F. Götze. Exponential integrability and transportation cost related to logarithmic Sobolev inequalities. *J. Funct. Anal.*, 163(1):1–28, 1999.

BL97 [11] S.G. Bobkov and M. Ledoux. Poincaré inequalities and Talagrand concentration phenomenon for the exponential distribution. *Prob. Theor. Rel. Fields*, 107:383–400, 1997.

BLweight [12] S.G. Bobkov and M. Ledoux. Weighted Poincaré-type inequalities for Cauchy and other convex measures. *Ann. Probab.*, 37(2):403–427, 2009.

cat4 [13] P. Cattiaux. A pathwise approach of some classical inequalities. *Potential Analysis*, 20:361–394, 2004.

CGG [14] P. Cattiaux, I. Gentil, and A. Guillin. Weak logarithmic Sobolev inequalities and entropic convergence. *Probab. Theory Related Fields*, 139(3-4):563–603, 2007.

CGGR [15] P. Cattiaux, N. Gozlan, A. Guillin, and C. Roberto. Functional inequalities for heavy tailed distributions and applications to isoperimetry. available on Math. ArXiv. 0807.3112. [math PR], 2008.

CatGui1 [16] P. Cattiaux and A. Guillin. On quadratic transportation cost inequalities. *J. Math. Pures Appl.*, 88(4):341–361, 2006.

CGgre [17] P. Cattiaux and A. Guillin. Functional inequalities via Lyapunov conditions. to appear in Proceedings of the summer school on Optimal Transport (Grenoble 2009). available on Math. ArXiv. 1001.1822. [math PR], 2010.

CGWW [18] P. Cattiaux, A. Guillin, F-Y. Wang, and L. Wu. Lyapunov conditions for super Poincaré inequalities. *J. Funct. Anal.*, 256(6):1821–1841, 2009.

CGW [19] P. Cattiaux, A. Guillin, and L. Wu. A note on Talagrand transportation inequality and logarithmic Sobolev inequality. To appear in Prob. Theo. Rel. Fields., 2008.

CDL [20] A. Cutrì and F. Da Lio. Comparison and existence results for evolutive non-coercive first-order Hamilton-Jacobi equations. *ESAIM Control Optim. Calc. Var.*, 13(3):484–502, 2007.

DMC [21] J. Denzler and R. J. McCann. Fast diffusion to self-similarity: complete spectrum, long-time asymptotics and numerology. *Arch. Ration. Mech. Anal.*, 175(3):301–342, 2005.

DV3 [22] M. D. Donsker and S. R. S. Varadhan. Asymptotic evaluation of certain Markov process expectations for large time. III. *Comm. Pure Appl. Math.*, 29(4):389–461, 1976.

DV4 [23] M. D. Donsker and S. R. S. Varadhan. Asymptotic evaluation of certain Markov process expectations for large time. IV. *Comm. Pure Appl. Math.*, 36(2):183–212, 1983.

DFG [24] R. Douc, G. Fort, and A. Guillin. Subgeometric rates of convergence of f-ergodic strong Markov processes. *Stochastic Process. Appl.*, 119(3):897–923, 2009.

dragoni [25] F. Dragoni. Metric Hopf-Lax formula with semicontinuous data. *Discrete Contin. Dyn. Syst.*, 17(4):713–729, 2007.

FHPS [26] B. Franke, C.R. Hwang, H.M. Pai, and S.J. Sheu. The behavior of the spectral gap under growing drift. *Trans. Amer. Math. Soc*, 362(3):1325–1350, 2009.

GGM1 [27] I. Gentil, A. Guillin, and L. Miclo. Modified logarithmic Sobolev inequalities and transportation inequalities. *Probab. Theory Related Fields*, 133(3):409–436, 2005.

GGM2 [28] I. Gentil, A. Guillin, and L. Miclo. Modified logarithmic sobolev inequalities in null curvature. To appear in Revista Matematica Iberoamericana, 2006.

gozlan2 [29] N. Gozlan. Poincaré inequalities for non euclidean metrics and transportation inequalities. Preprint. Available on Math ArXiv 0707.2834[ps], 2007.

G2 [30] A. Guillin. Averaging principle of SDE with small diffusion: moderate deviations. *Ann. Probab.*, 31(1):413–443, 2003.

G1 [31] Arnaud Guillin. Moderate deviations of inhomogeneous functionals of Markov processes and application to averaging. *Stochastic Process. Appl.*, 92(2):287–313, 2001.

GLWY [32] A. Guillin, C. Léonard, L. Wu, and N. Yao. Transportation-information inequalities for Markov processes. *Probab. Theory Related Fields*, 144(3-4):669–695, 2009.

HHS [33] C.R. Hwang, S.Y. Hwang-Ma, and S.J Sheu. Accelerating diffusions. *Ann. Appl. Prob.*, 15:1433–1444, 2005/

KM1 [34] I. Kontoyiannis and S. P. Meyn. Spectral theory and limit theorems for geometrically ergodic Markov processes. *Ann. Appl. Probab.*, 13(1):304–362, 2003.

KM2 [35] I. Kontoyiannis and S. P. Meyn. Large deviations asymptotics and the spectral theory of multiplicatively regular Markov processes. *Electron. J. Probab.*, 10:no. 3, 61–123 (electronic), 2005.

MT [36] S. P. Meyn and R. L. Tweedie. *Markov chains and stochastic stability*. Communications and Control Engineering Series. Springer-Verlag London Ltd., London, 1993.

MT2 [37] S. P. Meyn and R. L. Tweedie. Stability of markovian processes II: continuous-time processes and sampled chains. *Adv. Appl. Proba.*, 25:487–517, 1993.

MT3 [38] S. P. Meyn and R. L. Tweedie. Stability of markovian processes III: Foster-Lyapunov criteria for continuous-time processes. *Adv. Appl. Proba.*, 25:518–548, 1993.

OV [39] F. Otto and C. Villani. Generalization of an inequality by Talagrand and links with the logarithmic Sobolev inequality. *J. Funct. Anal.*, 173(2):361–400, 2000.

RW [40] M. Röckner and F. Y. Wang. Weak Poincaré inequalities and L^2-convergence rates of Markov semigroups. *J. Funct. Anal.*, 185(2):564–603, 2001.

w00 [41] F. Y. Wang. Functional inequalities for empty essential spectrum. *J. Funct. Anal.*, 170(1):219–245, 2000.

Wbook [42] F. Y. Wang. *Functional inequalities, Markov processes and Spectral theory*. Science Press, Beijing, 2005.

wu1 [43] Liming Wu. Large and moderate deviations and exponential convergence for stochastic damping Hamiltonian systems. *Stochastic Process. Appl.*, 91(2):205–238, 2001.

Patrick CATTIAUX,, Institut de Mathématiques de Toulouse. CNRS UMR 5219., Université Paul Sabatier,, 118 route de Narbonne, F-31062 Toulouse cedex 09.

E-mail address: `cattiaux@math.univ-toulouse.fr`

Arnaud GUILLIN,, Laboratoire de Mathématiques, CNRS UMR 6620, Université Blaise Pascal, avenue des Landais 63177 Aubière.

E-mail address: `guillin@math.univ-bpclermont.fr`

Li-Ming WU,, Laboratoire de Mathématiques, CNRS UMR 6620, Université Blaise Pascal, avenue des Landais 63177 Aubière.

E-mail address: `wuliming@math.univ-bpclermont.fr`