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Anne-Laure Basdevant, Arvind Singh

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Anne-Laure Basdevant* and Arvind Singh†

Abstract

We consider a continuous time vertex reinforced jump process on a supercritical Galton-Watson tree. This process takes values in the set of vertices of the tree and jumps to a neighbouring vertex with rate proportional to the local time at that vertex plus a constant. In this paper, we complete results previously obtained by Davis and Volkov (2002,2004) and Collevecchio (2006,2009) by providing a characterization for the recurrence/transience of the walk.

Keywords. Reinforced processes; phase transition; random walks on trees; branching processes

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1 Introduction

The model of the continuous time vertex reinforced jump process (VRJP) introduced by Davis and Volkov [7] may be described in the following way: let G be a locally finite graph and pick $c > 0$. Call VRJP(c) a continuous time process $(X(t), t \geq 0)$ on the vertices of G, starting at time 0 at some vertex $v_0 \in G$ and such that, if X is at a vertex $v \in G$ at time t, then, conditionally on $(X(s), s \leq t)$, the process X jumps to a neighbour u of v with rate:

$$L_c(t,u) \stackrel{\text{def}}{=} c + \int_0^t \mathbf{1}_{\{X(s)=u\}} ds. \tag{1}$$

Equivalently, the walk stays at site v an exponential time of parameter $\sum_{u\sim v} L_c(t,u)$ and then jumps to a neighbour u with a probability proportional to $L_c(t,u)$.

The case $G = \mathbb{Z}$ was investigated by Davis and Volkov [7] who proved that, for any $c > 0$, the VRJP(c) is recurrent and the proportion of time spent at each site converges jointly to some non-degenerate distribution. In a subsequent article [8], the same authors studied the VRJP on more general graphs. They showed that when the G is a tree the walk can either be recurrent or transient. For a regular b-ary tree (more generally, a tree satisfying a so-called L-property), they proved the existence of two constants

$$0 < c_r(b) \leq c_t(b) \tag{2}$$

such that:

- For $c < c_r$, the VRJP(c) visits every vertex infinitely often a.s.
- For $c > c_t$, the VRJP(c) visits every vertex only a finite number of time a.s.

*Laboratoire Modal'X, Université Paris X, France. email: anne.laure.basdevant@normalesup.org
†Laboratoire de Mathématiques, Université Paris XI, France. email: arvind.singh@math.u-psud.fr

Although they did not prove that $c_r = c_t$, the computation of the bound c_t obtained in [8] already implies that the VRJP(1) is transient on a 4-ary tree. More recently, Collevecchio [5, 6] showed that the VRJP(1) on a 3-ary tree is also transient with positive speed (and a C.L.T. holds) and asked whether this result also holds for a VRJP(1) on a binary tree.

The main result of this paper states that, for almost every realization of an infinite super critical Galton-Watson tree with mean offspring distribution b, one has $c_t(b) = c_r(b)$ and recurrence occurs at the critical value. In fact, recalling Lyons-Pemantle's criterion for recurrence/transience of a random walk in random environment (RWRE) on a Galton-Watson tree (see Theorem 3 of [10]), Theorem 1.1 states that the phase transition of a VRJP(c) is exactly the same as that of a discrete time random walk in an i.i.d. random environment where the law of the environment is given by the random variable $m_c(\infty)$ defined below.

Concerning the discrete time model of the linearly edge reinforced random walk (LERRW), de Finetti's theorem implies, that any LERRW on an acyclic graph may be seen as a RWRE in a Dirichlet environment. However, the non-exchangeability of the increments of a VRJP forbids a direct interpretation of the process in terms of a time change of a RWRE and we do not have a convincing argument why the VRJP should have the same phase transition as a RWRE. For example, using Theorem 1.5 of [2], one can check that, on a regular tree, the random walk in the random environment defined by $m_c(\infty)$ always has a positive speed when it is transient. Does this result somehow imply that a transient VRJP always has positive speed?

Theorem 1.1. *For $c > 0$, let $m_c(\infty)$ denote a random variable on $(0, \infty)$ with density:*

$$\mathbf{P}\{m_c(\infty) \in dx\} \overset{def}{=} \frac{c \exp\left(-\frac{(c(x-1))^2}{2x}\right)}{\sqrt{2\pi x^3}} dx. \tag{3}$$

Define

$$\mu(c) \overset{def}{=} \inf_{a \in \mathbb{R}} \mathbf{E}\big[m_c(\infty)^a\big] = \frac{c}{\sqrt{2\pi}} \int_0^\infty x^{-1} \exp\left(-\frac{(c(x-1))^2}{2x}\right) dx. \tag{4}$$

Let $\mathbb{T}$ denote a Galton-Watson tree with mean $1 < b < \infty$. On the event that $\mathbb{T}$ is infinite, we have, for almost every realization of $\mathbb{T}$:

- *If $b\mu(c) \leq 1$, the VRJP(c) on $\mathbb{T}$ visits every vertex infinitely often a.s.*
- *If $b\mu(c) > 1$, the VRJP(c) on $\mathbb{T}$ visits every vertex only finitely many times a.s.*

For $c = 1$, we have $1/\mu(1) \simeq 1.095$. Therefore the VRJP(1) is transient on any regular b-ary tree with $b \geq 2$. Making a change of variable (see Appendix of [8]), the function μ may be rewritten in the form

$$\mu(c) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-y^2/2}}{\sqrt{1 + y^2/(4c^2)}} dy.$$

Thus, μ is continuous, strictly increasing on $[0, \infty)$ with $\lim_0 \mu = 0$ and $\lim_\infty \mu = 1$ (see Figure 1). Denoting by μ^{-1} its inverse, we get

Corollary 1.2. *For any supercritical Galton-Watson tree with mean $1 < b < \infty$, with the notation (2), we have, for almost every realization where the tree is infinite,*

$$c_t(b) = c_r(b) = \mu^{-1}(1/b).$$

In particular, the recurrence/transience phase transition for VRJP on the class of Galton-Watson tree is monotonic w.r.t. the reinforcement parameter c, i.e. *if the VRJP(c) is transient for some $c > 0$, then the VRJP($\tilde{c}$) is transient for any $\tilde{c} \geq c$.*

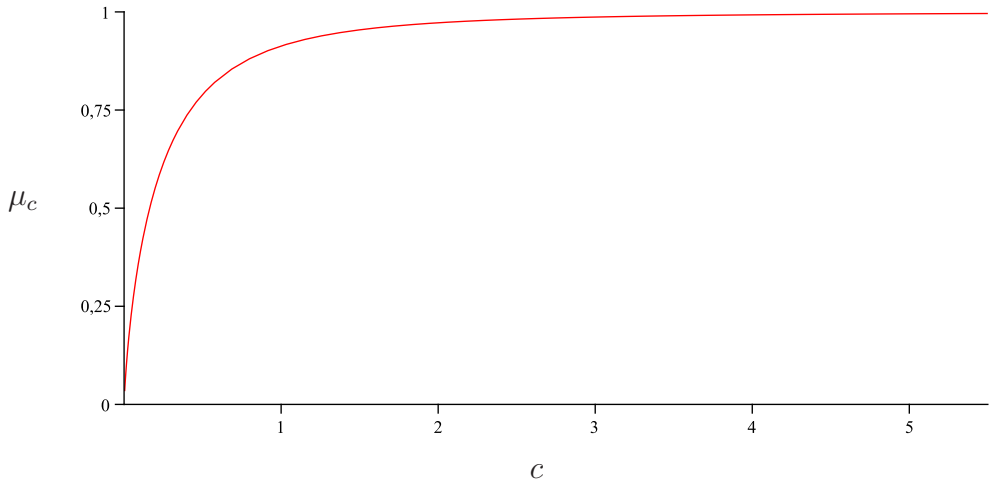

Figure 1: Graph of the function μ.

Let us note that, although this monotonicity result w.r.t. the parameter c seems quite natural, we do not know how to prove it without using the explicit computation of μ to assert that this function is monotonic. More generally, we do not know how to prove a similar result for an infinite graph which contains loops.

2 Preliminary results

In this section, we recall some important results concerning VRJP obtained by Davis and Volkov in [7, 8] which will play a key role in the proof of Theorem 1.1. We start with the so-called *restriction principle* for VRJP which follows from the lack of memory of the exponential law.

Proposition 2.1 (**Restriction principle**, Davis, Volkov [8])**.** *Let G be a connected graph and let G_1 be a connected subgraph with the property that for any path starting in any $v \in G\backslash G_1$ and ending in G_1, the first "port of entry" into G_1 is uniquely determined. Assume moreover that on each connected component of $G\backslash G_1$, the VRJP(c) is recurrent. Then the VRJP(c) on G starting at $v \in G_1$ restricted to G_1 has the same law as the VRJP(c) on the subgraph G_1 starting from the same point.*

We shall make intensive use of this result in the case where G is a rooted tree and G_1 is a subtree of G (for instance, the ball of radius N centered at the root).

2.1 VRJP on the graph $\{0, 1\}$

In view of the restriction principle stated above, many properties of the VRJP on an acyclic graph can be derived from the study of the VRJP on the simpler graph $G_0 \overset{\text{def}}{=} \{0, 1\}$. A detailed analysis of the VRJP on G_0 is undertaken in [7]. Consider a VRJP(c) on G_0, starting at 0. For $t \geq c$, define the stopping time

$$\xi(t) \overset{\text{def}}{=} \inf\{s > 0, L_c(s, 0) = t\}$$

and

$$A_c(t) \overset{\text{def}}{=} L_c(\xi(t), 1). \tag{5}$$

The quantity $A_c(t) - c$ corresponds to the time spent at site 1 before spending time $t - c$ at site 0. The variable $A_c(t)$ takes values in $[c, \infty)$ and has an atom at c. More precisely, denoting by $\mathcal{E}(c)$ an exponential random variable with parameter c, we have

$$\begin{aligned} \mathbf{P}\{A_c(t) = c\} &= \mathbf{P}\{\text{the VRJP}(c) \text{ does not jump before time } t - c\} \\ &= \mathbf{P}\{\mathcal{E}(c) > t - c\} \\ &= e^{-c(t-c)}. \end{aligned} \tag{6}$$

For $t > c$, the law of $A_c(t)$ conditioned on $\{A_c(t) > c\}$ is absolutely continuous w.r.t. the Lebesgue measure, with strictly positive density on (c, ∞). Considering only the time spent at site 1 before the first return to site 0, we get the lower bound:

$$\mathbf{P}\{A_c(t) \geq \alpha \mid A_c(t) > c\} \geq \mathbf{P}\{\mathcal{E}(t) > \alpha - c\} = e^{-(\alpha - c)t}. \tag{7}$$

For $t \geq c$, define

$$m_c(t) \stackrel{\text{def}}{=} \frac{A_c(t)}{t}.$$

It is proved in [7] that the process $(m_c(t), t \geq c)$ is a positive martingale which converges a.s. toward the random variable $m_c(\infty)$ defined in Theorem 1.1. The moments of $m_c(\infty)$ can be computed explicitly using (3). For $\theta \in \mathbb{R}$, we get

$$\mathbf{E}[m_c(\infty)^\theta] = \sqrt{\frac{2}{\pi}} c e^{c^2} K_{\theta - 1/2}(c^2) < \infty$$

where $K_\alpha(x)$ denotes the modified Bessel function of the second kind of order ν (*c.f.* [1] for details on this class of special functions). Using $K_\alpha = K_{-\alpha}$ and $K_\alpha \leq K_{\alpha'}$ for $0 \leq \alpha \leq \alpha'$, it follows that

$$\min_{\theta \in \mathbb{R}} \mathbf{E}\big[m_c(\infty)^\theta\big] = \mathbf{E}\Big[\sqrt{m_c(\infty)}\Big], \tag{8}$$

which entails the second equality of (4).

2.2 VRJP on trees

Let T be a deterministic locally bounded tree rooted at some vertex o. According to Theorem 3 of [8], any VRJP on T is either recurrent (every vertex is visited infinitely often a.s.) or transient (every vertex is visited only finitely many times a.s.). Moreover, we have the following characterization of recurrence and transience in terms of the local time of the walk at the root:

$$\text{The VRJP}(c) \text{ on } T \text{ is recurrent} \iff \lim_{t \to \infty} L_c(t, o) = \infty. \tag{9}$$

Define, for $t > c$,

$$\xi(t) \stackrel{\text{def}}{=} \inf\{s > 0, L_c(s, o) = t\},$$

and let $(v_0 = o, v_1, \ldots, v_n)$ be a nearest neighbour self-avoiding path starting from the root of T and ending at v_n. For $0 \leq k \leq n$, set

$$Z_k \stackrel{\text{def}}{=} L_c(\xi(t), v_k). \tag{10}$$

If T is a finite tree, then the VRJP(c) on T is recurrent. Applying the restriction principle to the subgraph $(v_0 = o, v_1, \ldots, v_n)$, it follows that the process $(Z_k)_{0 \leq k \leq n}$ is a Markov chain starting from $Z_0 = t$ with transition probabilities

$$\mathbf{P}\{Z_{k+1} \in E \mid Z_0, \ldots, Z_k = x\} = \mathbf{P}\{A_c(x) \in E\} \tag{11}$$

where A_c is the random variable defined in (5). Let us note that Z takes values in $[c, \infty)$ and that c is an absorbing point. Moreover, since $(A_c(t)/t)_{t \geq c}$ is a martingale starting from 1, the process Z is also a (positive) martingale. Therefore, Z_n converges a.s. as n tend to infinity and the limit is necessarily equal to c a.s.

3 Proof of Theorem 1.1

We first set some notation. Let $\mathcal{T}$ be the set of all locally finite rooted trees. Given a tree $T \in \mathcal{T}$, we denote its root by o. For $v \in T$, we use the notation $\overleftarrow{v}$ for the father of v and $\overrightarrow{v}^1, \overrightarrow{v}^2, \dots$ for the sons of v. We also denote by $|v|$ the height of the vertex v in the tree (*i.e.* its graph distance from the root). For $n \geq 0$, T_n will stand for the subtree of T of vertices of height smaller or equal to n.

In the following, ν will always denote a probability measure on the non-negative integers with finite mean $b > 1$ and $\mathbb{Q}_\nu$ will denote the probability measure on $\mathcal{T}$ under which the canonical r.v. $\mathbb{T}$ is a Galton-Watson tree with offspring distribution ν.

For $c > 0$, we consider on the same (possibly enlarged) probability space a process $X = (X(t), t \geq 0)$ and a collection of probability measures $(P_{T,c},\ T \in \mathcal{T})$ called quenched laws such that X under $P_{T,c}$ is a VRJP(c) on T with $X(0) = o$. The annealed probability is defined by

$$\mathbb{P}_{\nu,c} \overset{\text{def}}{=} P_{\mathbb{T},c} \otimes \mathbb{Q}_\nu.$$

We say that X under $\mathbb{P}_{\nu,c}$ is a VRJP(c) on a Galton-Watson tree with reproduction law ν. In the following, we shall omit the subscripts c, ν when it does not lead to confusion.

3.1 Restriction to trees without leaves

The Harris decomposition of a super-critical Galton-Watson tree states that conditionally on non extinction, $\mathbb{T}$ under $\mathbb{Q}_\nu$ can be generated in the following way:

- Generate a Galton-Watson tree $\mathbb{T}_g$ with no leaf called the backbone.
- Attach at each vertex v of $\mathbb{T}_g$ a random number N_v of i.i.d. subcritical trees $\mathbb{T}_l^1, \dots, \mathbb{T}_l^{N_v}$.

See for instance [3] for a precise description of the laws of N_v, $\mathbb{T}_g$ and $\mathbb{T}_l$. Let us simply note that the expected number of children per vertex of $\mathbb{T}_g$ is also equal to b. Consider now a VRJP(c) on $\mathbb{T}$ on the event that $\mathbb{T}$ is infinite. The restriction principle applied with $G = \mathbb{T}$ and $G_1 = \mathbb{T}_g$ implies that the VRJP(c) on $\mathbb{T}$ is transient if and only if the VRJP(c) on $\mathbb{T}_g$ is transient. Since the criterion for the transience/recurrence of the walk of Theorem 1.1 only depends on b, it suffices to prove the result for trees without leaves. In the sequel, we will always assume that this is the case, that is:

$$\nu(0) = 0.$$

3.2 Proof of recurrence when $b\mu(c) < 1$

In [8], Davis and Volkov prove that a VRJP(1) is recurrent when $b \leq 1.04$. In fact, their argument shows recurrence whenever $b\mu(c) < 1$ by simply fine-tuning some parameters. We provide below a sketch of the proof and we refer the reader to [8] for further details.

Consider a VRJP(1) on the non-negative integers $\{0, 1, \dots\}$ and denote by σ_n the first time the walk reaches level n. It is proved in the appendix of [8] that, for any $a > 1$,

$$\mathbf{P}\{L_1(\sigma_n, 0) < a^n\} \leq (\mathbf{E}[\sqrt{m_1(\infty)}]a^{1/2})^n. \tag{12}$$

Adapting the proof for any $c > 0$, it is immediate to check that, for any VRJP(c) on the non-negative integers,

$$\mathbf{P}\{L_c(\sigma_n, 0) < a^n\} \leq (\mathbf{E}[\sqrt{m_c(\infty)}]a^{1/2})^n = (\mu(c)a^{1/2})^n. \tag{13}$$

We now copy the argument of the proof of Theorem 5 of [8] using the bound (13) in place of (12). Let $T \in \mathcal{T}$ be an infinite tree. Let V_n denote the number of vertices of T of height n and set

$$G_n = L_c(\inf\{t > 0, |X_t| = n\}, o)$$

so that $G_n - c$ is the total time spent by X at the root before reaching a vertex of height n. Conditioning on the position of the VRJP(c) when it reaches level n and applying the restriction principle to the path connecting this vertex to the root, we find, using (13),

$$P_T\{G_n < a^n\} \leq (\mu(c)a^{1/2})^n V_n. \tag{14}$$

Assume now that the tree T satisfies

$$\liminf_{n\to\infty} V_n^{1/n} < \mu(c)^{-1}$$

then (14) yields, taking a sufficiently close to 1,

$$P_T\{G_{n_k} < a^{n_k}\} \leq (1-\varepsilon)^{n_k}$$

for some subsequence (n_k) and some $\varepsilon > 0$. Letting k go to infinity, we conclude that

$$\lim_{t\to\infty} L_c(t, o) = \infty. \quad P_T\text{-a.s.}$$

Thus, the VRJP(c) on T is recurrent according to (9). We conclude the proof for the VRJP(c) on the Galton-Watson tree $\mathbb{T}$ noticing that, when $b\mu(c) < 1$, we have for $\mathbb{Q}_\nu$-almost any tree $T \in \mathcal{T}$,

$$\lim_{n\to\infty} V_n^{1/n} = b < \mu(c)^{-1}.$$

3.3 The branching Markov Chain F

Recall that we assume $\nu(0) = 0$ so the tree $\mathbb{T}$ is infinite $\mathbb{Q}_\nu$-a.s. We introduce a branching Markov chain F indexed by the vertices of $\mathbb{T}$ and taking values in $[c, \infty)$,

$$F \overset{\text{def}}{=} (f(v), v \in \mathbb{T}) \in \bigcup_{T\in\mathcal{T}} [c, \infty]^T.$$

More precisely, the population at time n is indexed by $\{v \in \mathbb{T},\ |v| = n\}$ and the set of positions of the particles of F at time n is

$$F_n \overset{\text{def}}{=} (f(v),\ |v| = n).$$

Thus, the genealogy of this branching Markov chain is chosen to be exactly the Galton-Watson tree $\mathbb{T}$. In particular, under the annealed probability $\mathbb{P}$, each particle v splits, after a unit of time, into a random number B of particles $\overrightarrow{v}^1, \ldots, \overrightarrow{v}^B$ where B is distributed as ν. In order to characterize F, it remains to specify the law of the position $f(v)$ of the particles. We choose the dynamics of F, conditionally on its genealogy $\mathbb{T}$ in the following way:

(a) For any $n > 0$, conditionally on $(f(u),\ |u| < n)$, the random variables $(f(v), |v| = n)$ are independent.

(b) For any $v \neq o$, conditionally on $(f(u),\ |u| < |v|)$, the random variable $f(v)$ is distributed as $A_c(f(\overleftarrow{v}))$ where A_c is defined by (5).

We use the notation $\mathbb{P}_{x_0}$ for the annealed law where F starts with the initial particle o being located at $f(o) = x_0$. Note that, since the tree is infinite, the Markov chain F never becomes extinct. However, recalling that c is an absorbing point for A_c, it follows that if a particle v is located at $f(v) = c$, then all its descendants are also located at c. Thus, we will say that the process F *dies out* if there exists a time n such that all the particle at time n are at position c. Otherwise, we say that the process *survives*.

Proposition 3.1. *For any $x \leq y$, the process F under $\mathbb{P}_x$ is stochastically dominated by F under $\mathbb{P}_y$.*

Proof. Recalling (5), it is clear that $A_c(x) \leq A_c(y)$ for any $c \leq x \leq y$ and the result follows by induction. □

Proposition 3.2. *Let $x_0 > 0$ and $N > 0$ and let $(X^N(t), t \geq 0)$ denote a VRJP(c) on the finite subtree $\mathbb{T}_N = \{v \in \mathbb{T},\ |v| \leq N\}$, with $X^N(0) = o$. Set*

$$\xi^N(x_0) \stackrel{def}{=} \inf\{s > 0, L_c^N(s, o) = x_0\},$$

where L^N is defined as in (1) for X^N. Then, the random variables $(L_c^N(\xi^N(x_0), v),\ v \in \mathbb{T}_N)$ under $\mathbb{P}$ and $(f(v),\ v \in \mathbb{T}_N)$ under $\mathbb{P}_{x_0}$ have the same law.

Proof. Simply notice that since the $\mathbb{T}_N$ is finite, X^N is recurrent and ξ^N is finite a.s. and apply the restriction principle for VRJP. □

The VRJPs X on $\mathbb{T}$ and X^N on $\mathbb{T}_N$ coincide up to the first time they reach a site of height N, therefore,

$$\begin{aligned}\mathbb{P}\{X \text{ reaches level } N \text{ before spending time } x_0 - c \text{ at the origin}\}&\\ = \mathbb{P}\{X^N \text{ reaches level } N \text{ before spending time } x_0 - c \text{ at the origin}\}&\\ = \mathbb{P}_{x_0}\{\text{the process } F \text{ does not die out before time } N\}.&\end{aligned}$$

Letting N and then x_0 tend to infinity, and using (9), we get

$$\mathbb{P}\{X \text{ visits every vertex of } \mathbb{T} \text{ finitely many times}\} = \lim_{x_0 \to \infty}\uparrow\ \mathbb{P}_{x_0}\{F \text{ survives}\}. \tag{15}$$

The next proposition extends the $0-1$ law proved in [8] for deterministic trees to Galton-Watson trees.

Proposition 3.3 ($\mathbf{0-1}$ law for VRJP on Galton-Watson trees). *Let $\mathbb{T}$ be a Galton-Watson tree $\mathbb{T}$ without leaves and with mean $b > 1$. Then, for any $c > 0$, the VRJP(c) X on $\mathbb{T}$ is either recurrent or transient under the annealed law:*

$$\begin{aligned}\mathbb{P}\{X \textit{ visits every vertex of } \mathbb{T} \textit{ finitely many times}\}&\\ = 1 - \mathbb{P}\{X \textit{ visits every vertex of } \mathbb{T} \textit{ infinitely often}\} \in \{0, 1\}.&\end{aligned}$$

Proof. Since the $0-1$ law holds for any deterministic tree, we just need to show that the r.h.s. limit of (15) is either 0 or 1. Suppose that this limit is non-zero. We can find $x_0 > c$ and $\alpha > 0$ such that

$$\mathbb{P}_{x_0}\{F \text{ survives}\} \geq \alpha.$$

Given an interval I, let N_k^I denote the number of particles in F located inside I at time k *i.e.*

$$N_k^I \stackrel{\text{def}}{=} \#\{v \in \mathbb{T}, |v| = k \text{ and } f(v) \in I\}. \tag{16}$$

Since the particles in F evolve independently, conditionally on $(f(v), |v| \leq k)$, the process $(f(v), |v| \geq k)$ has the same law as the union of $\#\{v \in \mathbb{T},\ |v| = k\}$ independent branching Markov chains F starting from the positions $F_k = (f(v),\ |v| = k)$. Making use of the stochastic monotonicity of F w.r.t. to the position of the initial particle (Proposition 3.1), we deduce that, for any $\varepsilon > 0$, we can find m large enough such that, for any k and any x,

$$\begin{aligned}
\mathbb{P}_x\{F \text{ survives}\} &\geq \mathbb{P}_x\{N_k^{[x_0,\infty)} \geq m \text{ and } F \text{ survives}\} \\
&\geq \mathbb{P}_x\{N_k^{[x_0,\infty)} \geq m\}\,(1 - \mathbb{P}_{x_0}\{F \text{ dies out}\}^m) \\
&\geq \mathbb{P}_x\{N_k^{[x_0,\infty)} \geq m\}\,(1 - (1-\alpha)^m) \\
&\geq \mathbb{P}_x\{N_k^{[x_0,\infty)} \geq m\}(1 - \varepsilon).
\end{aligned} \tag{17}$$

On the one hand, we have, for any $y > c$,

$$\mathbb{P}_x\left\{f(v) > y \text{ for every } v \text{ of height } 1\right\} = \sum_{b=1}^{\infty} \nu(b)\mathbf{P}\{A_c(x)/x > y/x\}^b.$$

Since the sequence $A_c(x)/x$ converges as $x \to \infty$ towards a random variable which has no atom at 0 (*c.f.* Section 2.1), the previous equality implies

$$\lim_{x\to\infty} \mathbb{P}_x\left\{f(v) > y \text{ for every } v \text{ of height } 1\right\} = 1.$$

Using again the stochastic monotonicity of F w.r.t. its starting point, it follows by induction that, for any fixed k,

$$\lim_{x\to\infty} \mathbb{P}_x\{f(v) > x_0 \text{ for every } v \in \mathbb{T} \text{ s.t. } |v| = k\} = 1. \tag{18}$$

On the other hand, the tree $\mathbb{T}$ grows exponentially so that, for any m,

$$\lim_{k\to\infty} \mathbb{P}\{\#\{v \in \mathbb{T},\ |v| = k\} \geq m\} = 1. \tag{19}$$

Combining (18) and (19), we deduce that, for any m, we can find k and x large enough such that

$$\mathbb{P}_x\{N_k^{[x_0,\infty)} \geq m\} \geq 1 - \varepsilon. \tag{20}$$

which yields, using (17),

$$\mathbb{P}_x\{F \text{ survives}\} \geq (1-\varepsilon)^2.$$

□

3.4 Proof of Transience when $b\mu(c) > 1$

Let $(Z_n)_{n\geq 0}$ be a Markov chain on $[c, \infty)$ with transition probabilities given by (11) and denote by $\mathbf{P}_x$ the probability under which Z starts from $Z_0 = x$. Let $T \in \mathcal{T}$ and fix $v \in T$. It follows from the definition of the branching Markov chain F that

$$\mathbb{P}_x\{f(v) \in E \mid \mathbb{T} = T\} = \mathbf{P}_x\{Z_{|v|} \in E\}.$$

Let us for the time being admit that, for some $x_0 > c$, we have

$$\liminf_{n\to\infty} \mathbf{P}_{x_0}\{Z_n \geq x_0\}^{1/n} \geq \mu(c). \tag{21}$$

Recalling that $N_k^{[x_0,\infty)}$ denotes the number of particles of F located above level x_0 at time k, we find, when $\mu(c)b > 1$, that for k_0 large enough,

$$\begin{aligned}
\mathbb{E}_{x_0}[N_{k_0}^{[x_0,\infty)}] &= \mathbb{E}_{x_0}\Big[\sum_{|v|=k_0} \mathbf{1}_{\{f(v)\geq x_0\}}\Big] \\
&= \mathbb{E}[\#\{v\in\mathbb{T},\ |v|=k_0\}]\mathbf{P}_{x_0}\{Z_{k_0}\geq x_0\} \\
&= \Big(b\mathbf{P}_{x_0}\{Z_{k_0}\geq x_0\}^{1/k_0}\Big)^{k_0} \\
&\geq 2.
\end{aligned}$$

Just as in the proof of Proposition 3.3, making use of the branching property of F and keeping only the particles located above x_0 at times $k_0 n$, $n \geq 0$, it follows by induction that, under $\mathbb{P}_{x_0}$, the process $(N_{k_0 n}^{[x_0,\infty)})_{n\geq 0}$ stochastically dominates a classical Galton-Watson process with reproduction law $N_{k_0}^{[x_0,\infty)}$. Since $\mathbb{E}_{x_0}[N_{k_0}^{[x_0,\infty)}] \geq 2$, this Galton-Watson process has probability $\alpha > 0$ of non extinction, which implies

$$\mathbb{P}_{x_0}\{F \text{ survives}\} \geq \alpha.$$

We conclude using (15) and Proposition 3.3 that

$$\mathbb{P}\{X \text{ visits each vertex of } \mathbb{T} \text{ finitely many times}\} = 1.$$

It remains to prove (21) which is a consequence of

Lemma 3.4. *Let $(S(x), x\in\mathbb{R})$ be a collection of real-valued random variables. Assume that the following holds.*

(a) For any $x < y$, the random variable $x + S(x)$ is stochastically dominated by $y + S(y)$.

(b) $S(x)$ converges in law, as x tend to $+\infty$, towards a random variable $S(\infty)$ whose law is absolutely continuous w.r.t. the Lebesgue measure and $\mathbf{P}\{S(\infty) > 0\} > 0$.

(c) The Laplace transform $\phi(\lambda) \stackrel{def}{=} \mathbf{E}[e^{\lambda S(\infty)}]$ reaches its minimum at some point $\rho > 0$ which belongs to the non-empty interior of the its definition domain $\mathcal{D} \stackrel{def}{=} \{\lambda\in\mathbb{R}, \phi(\lambda) < \infty\}$.

Let $Y = (Y_n, n \geq 0)$ denote a real valued Markov chain with transition kernel $\mathbf{P}\{Y_{n+1}\in E \mid Y_n = y\} = \mathbf{P}\{S(y) + y \in E\}$ and let τ_x be the first time Y enters the interval $(-\infty, x)$. Denoting by $\mathbf{P}_x$ the law of Y starting from x, we have, for all x large enough,

$$\lim_{n\to\infty} \mathbf{P}_x\{\tau_x > n\}^{1/n} \geq \phi(\rho). \tag{22}$$

We apply the lemma to the Markov chain Y defined by

$$Y_n \stackrel{\text{def}}{=} \log Z_n.$$

According to (11), we have

$$\mathbf{P}\{Y_{n+1}\in E \mid Y_n = y\} = \mathbf{P}\{S(y) + y \in E\}$$

with $S(y) \stackrel{\text{def}}{=} \log m_c(\exp(y))$ and $S(\infty) \stackrel{\text{def}}{=} \log m_c(\infty)$ where m_c is the martingale of Section 2.1. On the one hand, Assumption (a) holds since $A_c(x) \leq A_c(y)$ for all $x \leq y$. On the other hand, the results of Davis and Volkov [7, 8] recalled in Section 2.1 imply that Assumptions (b),(c) also hold and

$$\inf_{\lambda \in \mathbb{R}} \mathbf{E}[e^{-\lambda S(\infty)}] = \mu(c).$$

Thus, we conclude that, for x_0 large enough,

$$\liminf_{n\to\infty} \mathbf{P}_{x_0}\{Z_n \geq x_0\}^{1/n} \geq \lim_{n\to\infty} \mathbf{P}_{\log x_0}\Big\{ \min_{1\leq i\leq n} Y_i \geq \log x_o \Big\}^{1/n} \geq \mu(c).$$

Proof of Lemma 3.4. Assumption (a) implies that for $x < y$, the Markov chain Y under $\mathbf{P}_x$ is stochastically dominated by Y under $\mathbf{P}_y$. Thus, using the Markov property, we get that, for any n, m,

$$\mathbf{P}_x\{\tau_x > n+m\} \geq \mathbf{P}_x\{\tau_x > n\}\mathbf{P}_x\{\tau_x > m\}.$$

The superadditivity of the sequence $\log \mathbf{P}_x\{\tau_x > n\}$ now implies that the limit in (22) exists. It remains to prove the lower bound for x large enough.

Set $g_x(t) \stackrel{\text{def}}{=} \mathbf{P}\{S(x) > t\}$ and $g(t) \stackrel{\text{def}}{=} \mathbf{P}\{S(\infty) > t\}$. In view of Assumption (b), as x goes to $+\infty$, g_x converges uniformly towards g. Define

$$\hat{g}_x(t) \stackrel{\text{def}}{=} \inf_{y\geq x} g_y(t).$$

For each x, the function $\hat{g}_x$ is càdlàg, non increasing, with $\lim_{t\to-\infty} \hat{g}_x(t) = 1$ and $\lim_{t\to+\infty} \hat{g}_x(t) = 0$. Thus, for each x, we can consider a random variable $\hat{S}(x)$ such that $\mathbf{P}\{\hat{S}(x) > t\} = \hat{g}_x(t)$. By construction the sequence of random variables $\hat{S}(x)$ is stochastically monotonic and converges in law toward the random variable $S(\infty)$. Let $\hat{Y}^x$ denote a random walk with step $\hat{S}(x)$ *i.e.* $\hat{Y}^x_{n+1} - \hat{Y}^x_n \stackrel{\text{law}}{=} \hat{S}(x)$.

By construction, the random variable $\hat{S}(x)$ is stochastically dominated by $S(y)$ for any $y \geq x$. Combining this fact and the stochastic monotonicity of the Markov chain Y w.r.t. its starting point, it follows by induction that the random walk $\hat{Y}^x$ started from x and killed when it enters the interval $(-\infty, x)$ is stochastically dominated by Y under $\mathbf{P}_x$. In particular, denoting by $\hat{\tau}^x_0$ the first time $\hat{Y}^x$ enters the interval $(-\infty, 0)$, it follows that $\hat{\tau}^x_0$ under $\mathbf{P}_0$ (*i.e.* the walk $\hat{Y}^x$ started from 0), is stochastically dominated by τ_x under $\mathbf{P}_x$. Hence,

$$\lim_{n\to\infty} \mathbf{P}_x\{\tau_x > n\}^{1/n} \geq \liminf_{n\to\infty} \mathbf{P}_0\{\hat{\tau}^x_0 > n\}^{1/n}. \tag{23}$$

Let $\hat{\phi}_x(\lambda) \stackrel{\text{def}}{=} \mathbf{E}[e^{\lambda \hat{S}(x)}]$ with definition domain $\hat{\mathcal{D}}_x \stackrel{\text{def}}{=} \{\lambda \in \mathbb{R}, \hat{\phi}_x(\lambda) < \infty\}$. Since, $\hat{S}(x)$ is stochastically dominated by $S(\infty)$, we have $\mathcal{D} \cap [0,\infty) \subset \hat{\mathcal{D}}_x \cap [0,\infty)$. According to Assumption (c), we can choose $a > 0$ such that $I_a \stackrel{\text{def}}{=} [\rho - a, \rho + a] \subset \mathcal{D} \cap [0,\infty)$. On I_a, as x goes to $+\infty$, the functions $\hat{\phi}_x$ converge uniformly toward ϕ. Making use of the strict convexity of a Laplace transform, it follows that, for all x large enough, the function $\hat{\phi}_x$ verifies Assumption (c) *i.e.* $\hat{\phi}_x$ reaches its minimum on $\hat{\mathcal{D}}_x$ at some point $\rho_x \in I_a$. Moveover, we have:

$$\lim_{x\to\infty} \hat{\phi}_x(\rho_x) = \phi(\rho). \tag{24}$$

Applying now Theorem 1 of [4] to the random walk $\hat{Y}^x$ with step distribution $\hat{S}(x)$ gives

$$\liminf_{n\to\infty} \mathbf{P}_0\{\hat{\tau}^x_0 > n\}^{1/n} = \hat{\phi}_x(\rho_x). \tag{25}$$

Combining (23) and (25), we get that

$$\lim_{n\to\infty} \mathbf{P}_x\{\tau_x > n\}^{1/n} \geq \hat{\phi}_x(\rho_x). \tag{26}$$

Assumption (b) also implies that, for some $\varepsilon, \eta > 0$ small enough, there exists x_0 such that, for all $x \geq x_0$, we have $\mathbf{P}\{S(x) > \varepsilon\} > \eta$, thus $\mathbf{P}\{Y_{n+1} > \varepsilon + Y_n \mid Y_n = x\} > \eta$. In particular, for $x > y > x_0$, the event $\mathcal{E}(x,y) = \{Y \text{ enters } [x,\infty) \text{ before entering } (-\infty, y)\}$ has a strictly positive probability under $\mathbf{P}_y$. Therefore, using again the Markov property and the stochastic monotonicity of Y w.r.t. its starting point, we get

$$\mathbf{P}_y\{\tau_y > n\} \geq \mathbf{P}_y\{\mathcal{E}(x,y)\}\mathbf{P}_x\{\tau_x > n\}$$

which yields

$$\lim_{n\to\infty} \mathbf{P}_y\{\tau_y > n\}^{1/n} \geq \lim_{n\to\infty} \mathbf{P}_x\{\tau_x > n\}^{1/n}. \tag{27}$$

Combining (24), (26) and (27), we conclude that, for $y \geq x_0$,

$$\lim_{n\to\infty} \mathbf{P}_y\{\tau_y > n\}^{1/n} \geq \lim_{x\to+\infty} \hat{\phi}_x(\rho_x) = \phi(\rho). \tag{28}$$

□

Remark 3.5. *Suppose that the VRJP(c) is recurrent on $\mathbb{T}$. Recall that $\xi(t)$ denotes the time where the local time of the walk at the origin reaches $t - c$. We can express $\xi(t)$ in terms of the branching Markov chain F and we get, using that $\mathbf{E}_t[Z_n] = t$ for all n,*

$$\mathbb{E}[\xi(t)] = \mathbb{E}_t\left[\sum_{v\in\mathbb{T}} (f(v) - c)\right] = \sum_{n=0}^{\infty} b^n \mathbf{E}_t[Z_n - c] = \sum_{n=0}^{\infty} b^n (t - c) = \infty \tag{29}$$

for any $t > c$. In particular, denoting by ζ_o the first time the walk returns to the root of the tree, it easily follows from (29), by conditioning on the time the walk makes its first jump and applying the restriction principle that any recurrent VRJP on $\mathbb{T}$ is "null" recurrent in the sense that $\mathbb{E}[\zeta_o] = \infty$.

3.5 The critical case $b\mu(c) = 1$.

The following proposition directly implies that the VRJP(c) on a Galton-Watson tree is recurrent in the critical case $b\mu(c) = 1$ since we already know that recurrence occurs when $b\mu(c) < 1$.

Proposition 3.6. *Assume that the VRJP(c) is transient on some Galton-Watson tree $\mathbb{T}$ without leaves and with mean $b > 1$. Then, there exists a Galton-Watson tree $\tilde{\mathbb{T}}$ (with leaves) with mean $1 < \tilde{b} < b$ such that the VRJP(c) on $\tilde{\mathbb{T}}$ is also transient on the event that $\tilde{\mathbb{T}}$ is infinite.*

The proof of Proposition 3.6 uses again the characterization of transience in terms of the positive probability of survival of the associated branching Markov chain F. Roughly speaking, we show that, conditionally on survival, the number of particles of F not located at c grows exponentially with time. This implies that the branching Markov chain on a small percolation of the original tree still survives with positive probability. Hence the VRJP on this percolated tree is also transient.

In the following, we assume as before that the Galton-Watson tree $\mathbb{T}$ with reproduction law ν has no leaves and has mean $b > 1$ so that it is infinite and grows exponentially. Recall the definition of the branching Markov chain $F = (f(v),\ v \in \mathbb{T})$ constructed in Section 3.3. We denote by $(\mathcal{F}_n)$ the natural filtration of F:

$$\mathcal{F}_n \stackrel{\text{def}}{=} \sigma(\mathbb{T}_n, (f(v),\ v \in \mathbb{T}_n)).$$

Lemma 3.7. *Recall the definition of N_n^I given in (16). Let $\mathcal{E}(x,k)$ be the event*

$$\mathcal{E}(x,k) \stackrel{def}{=} \{\textit{There exist infinitely many } n \textit{ such that } N_n^{[x,\infty)} \geq k\ \}.$$

For any starting point $x_0 > c$, we have,

$$\mathcal{E}(x_0,2) = \{F \textit{ survives}\} \quad \mathbb{P}_{x_0}\text{-a.s.}$$

Proof. The inclusion $\mathcal{E}(x_0,2) \subset \{F \text{ survives}\}$ is trivial. Let $\varepsilon > 0$ and set, for $k \leq n$,

$$B_{k,n} \stackrel{\text{def}}{=} \mathbb{E}_{x_0}\left[N_n^{(c,\infty)} \mathbf{1}_{\{N_k^{[c+\varepsilon,\infty)}=0, N_{k+1}^{[c+\varepsilon,\infty)}=0,\ldots,N_{n-1}^{[c+\varepsilon,\infty)}=0\}}\right].$$

Recall that each particle v of F evolves independently and gives birth to a random number B (with mean b) of children. Moreover, conditionally on $\mathcal{F}_n$, the positions $f(\vec{v}^1), \ldots, f(\vec{v}^B)$ of the children of a particle v at time n (*i.e.* $|v| = n$) are i.i.d. and distributed as $A_c(f(v))$. Thus, in view of (6), it follows that

$$\mathbb{E}[N_{n+1}^{(c,\infty)} \mid \mathcal{F}_n] \leq b(1-e^{-c\varepsilon}) N_n^{(c,\infty)} \qquad \text{on the event } \{N_n^{(c+\varepsilon,\infty)} = 0\}.$$

Choosing ε small enough such that $b(1-e^{-c\varepsilon}) < 1/2$, we get

$$\begin{aligned} B_{k,n+1} &= \mathbb{E}_{x_0}\left[\mathbb{E}[N_{n+1}^{(c,\infty)} \mid \mathcal{F}_n] \mathbf{1}_{\{N_k^{[c+\varepsilon,\infty)}=0, N_{k+1}^{[c+\varepsilon,\infty)}=0,\ldots,N_{n-1}^{[c+\varepsilon,\infty)}=0\}}\right] \\ &\leq \frac{1}{2}\mathbb{E}_{x_0}\left[N_n^{(c,\infty)} \mathbf{1}_{\{N_k^{[c+\varepsilon,\infty)}=0, N_{k+1}^{[c+\varepsilon,\infty)}=0,\ldots,N_{n-1}^{[c+\varepsilon,\infty)}=0\}}\right] \\ &\leq \frac{1}{2} B_{k,n}. \end{aligned}$$

which yields,

$$\mathbb{E}_{x_0}\left[\left(\sum_{n=k}^{\infty} N_n^{(c,\infty)}\right) \mathbf{1}_{\left\{N_i^{[c+\varepsilon,\infty)} = 0 \text{ for all } i \geq k\right\}}\right] \leq \sum_{n=k}^{\infty} B_{k,n} < \infty.$$

Therefore, the process F dies out $\mathbb{P}_{x_0}$-a.s. on the event $\left\{N_i^{[c+\varepsilon,\infty)} = 0 \text{ for all } i \geq k\right\}$. Taking the limit as k goes to infinity, we obtain

$$\mathcal{E}(c+\varepsilon,1) \supset \{F \text{ survives}\} \quad \mathbb{P}_{x_0}\text{-a.s.} \tag{30}$$

Let now $U_n \stackrel{\text{def}}{=} \mathbf{1}_{\{N_n^{[x_0,\infty)} \geq 2\}}$. Using the stochastic monotonicity of Proposition 3.1 and the facts that $\nu[2,\infty) > 0$ (since $b > 1$) and (7), we find that

$$\begin{aligned} \mathbb{E}[U_{n+1} \mid \mathcal{F}_n] &\geq \mathbb{E}[U_{n+1} \mathbf{1}_{\{N_n^{[c+\varepsilon,\infty)} \geq 1\}} \mid \mathcal{F}_n] \\ &\geq \mathbf{1}_{\{N_n^{[c+\varepsilon,\infty)} \geq 1\}} \mathbb{E}_{c+\varepsilon}[U_1] \\ &\geq \mathbf{1}_{\{N_n^{[c+\varepsilon,\infty)} \geq 1\}} \mathbb{P}_{c+\varepsilon}\left\{ \begin{array}{l} \text{the initial particle } o \text{ has at least two children} \\ \text{with } f(\vec{o}^1) \geq x_0 \text{ and } f(\vec{o}^2) \geq x_0 \end{array} \right\} \\ &= \mathbf{1}_{\{N_n^{[c+\varepsilon,\infty)} \geq 1\}} \nu[2,\infty) \mathbf{P}\{A_c(c+\varepsilon) > x_0\}^2 \\ &= C \mathbf{1}_{\{N_n^{[c+\varepsilon,\infty)} \geq 1\}} \end{aligned} \tag{31}$$

for some constant $C > 0$. Combining (30) and (31), we get

$$\sum_{n=1}^{\infty} \mathbb{E}[U_{n+1} \mid \mathcal{F}_n] = \infty \qquad \text{on the event } \{F \text{ survives}\}.$$

A direct application of the conditional Borel-Cantelli Lemma (*c.f.* [9]) yields

$$\sum_{n=1}^{\infty} U_n = \infty \qquad \text{on the event } \{F \text{ survives}\}.$$

which exactly means that $\mathcal{E}(x_0, 2) \supset \{F \text{ survives}\}$. □

Proof of Proposition 3.6. Assume that the VRJP(c) X on the Galton-Watson tree $\mathbb{T}$ with reproduction law ν is transient. According to Proposition 3.3 and (15), we have

$$\lim_{x\to\infty} \mathbb{P}_x\{F \text{ survives}\} = 1.$$

Define the (possibly infinite) $\mathcal{F}_n$-stopping time

$$\sigma_x \stackrel{\text{def}}{=} \inf\{k \geq 1, N_k^{[x,\infty)} \geq 2\}.$$

Using the result of the previous lemma, we get

$$\lim_{x\to\infty} \lim_{\gamma\to\infty} \mathbb{P}_x\{\sigma_x \leq \gamma\} = \lim_{x\to\infty} \mathbb{P}_x\{F \text{ survives}\} = 1. \tag{32}$$

Let now $\tilde{\mathbb{T}}$ be the tree obtained from $\mathbb{T}$ be removing each vertex (and its descendants) with probability $\eta > 0$. The tree $\tilde{\mathbb{T}}$ is again a Galton-Watson tree with mean $\tilde{b} = b(1-\eta) < b$. We denote by $\tilde{F}$ the restriction of F to $\tilde{\mathbb{T}}$,

$$\tilde{F} \stackrel{\text{def}}{=} (f(v),\ v \in \tilde{\mathbb{T}}).$$

The restriction principle states that $\tilde{F}$ is the branching Markov chain associated with the VRJP(c) on $\tilde{\mathbb{T}}$. Let $\tilde{M}$ be the number of particles in $\tilde{F}$ located above x at time σ_x,

$$\tilde{M} \stackrel{\text{def}}{=} \#\{v \in \tilde{\mathbb{T}}, |v| = \sigma_x, f(v) > x\}$$

(with the convention $\tilde{M} = 0$ when $\sigma_x = \infty$). We have,

$$\begin{aligned}
\mathbb{E}_x[\tilde{M}] &\geq \mathbb{E}_x[\tilde{M}\mathbf{1}_{\{\sigma_x \leq \gamma\}}\mathbf{1}_{\{\mathbb{T}_\gamma = \tilde{\mathbb{T}}_\gamma\}}] \\
&\geq 2\mathbb{P}_x\{\sigma_x \leq \gamma \text{ and } \mathbb{T}_\gamma = \tilde{\mathbb{T}}_\gamma\} \\
&\geq 2\left(\mathbb{P}_x\{\sigma_x \leq \gamma\} + \mathbb{Q}_\nu\{\mathbb{T}_\gamma = \tilde{\mathbb{T}}_\gamma\} - 1\right) \\
&\geq 2\left(\mathbb{P}_x\{\sigma_x \leq \gamma\} + \mathbb{Q}_\nu\{\#\mathbb{T}_\gamma \leq b^{2\gamma}\}(1-\eta)^{b^{2\gamma}} - 1\right).
\end{aligned} \tag{33}$$

Recalling that the distribution of offsprings ν has mean b, we get

$$\lim_{\gamma\to\infty} \mathbb{Q}_\nu\{\#\mathbb{T}_\gamma \leq b^{2\gamma}\} = 1. \tag{34}$$

Combining (32),(33) and (34), we can choose x, γ large enough and $\eta > 0$ small enough such that

$$\mathbb{E}_x[\tilde{M}] > 1.$$

Finally, using again the branching structure of $\tilde{F}$ and the stochastic monotonicity of the process w.r.t. the position of the initial particle, it follows by induction that the random variable $\#\{v \in \tilde{\mathbb{T}}, f(v) > x\}$ under $\mathbb{P}_x$ is stochastically larger than the total progeny of a Galton-Watson process with reproduction law $\tilde{M}$. Since $\mathbb{E}[\tilde{M}] > 1$, this process is supercritical hence

$$\mathbb{P}_x\{\tilde{F} \text{ survives}\} \geq \mathbb{P}_x\{\#\{v \in \tilde{\mathbb{T}}, f(v) > x\} = \infty\} > 0$$

which in turn implies that the VRJP(c) on the percolated tree $\tilde{\mathbb{T}}$ is transient on the event that $\tilde{\mathbb{T}}$ is infinite. □

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