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# On The Decidability Of MELL: Reachability In Petri Nets With Split/Join Transitions<sup>☆</sup>

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## Abstract

We define Petri nets with split and join transitions, a new model that extends Petri nets. We prove that reachability in this model without join transitions is equivalent to the decidability of MELL. We define a suitable notion of covering graph for the model, and prove its finiteness and effective constructibility.

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## Introduction

Introduction

### 1. Petri Nets with Split/Join Transitions

#### 1.1. Definition

**Definition 1.1.** Petri Nets with Split/Join Transitions.

A Petri net with split/join transitions (PNSJT) is a 4-tuple  $N = (P, T, W^-, W^+)$ , where

- $P$  is a finite set of places
- $T$  is a finite set of transitions
- $W^- : P \times T \rightarrow \mathbb{N} \cup \mathbb{N}^2$  is a pre-incidence function
- $W^+ : P \times T \rightarrow \mathbb{N} \cup \mathbb{N}^2$  is a post-incidence function,

such that:

1.  $\forall t \in T, \forall p, p' \in P : \text{arity}(W^-(p, t)) = \text{arity}(W^-(p', t))$ . This defines the pre-arity of  $t$ ,
2.  $\forall t \in T, \forall p, p' \in P : \text{arity}(W^+(p, t)) = \text{arity}(W^+(p', t))$ . This defines the post-arity of  $t$ ,
3. no transition  $t$  has both pre and post-arities equal to 2.

A transition  $t$  with pre-arity 2 is called a *join* transition, a transition  $t$  with post-arity 2 is called a *split* transition. A transition with pre and post -arity 1 is called a *regular* transition.

A PNSJT with only regular and split transitions is called a *Petri Nets with Split transitions* (PNST), and a PNSJT with only regular and join transitions is called a *Petri Nets with Join transitions* (PNJT).

A *single marking* of  $N$  is a mapping  $M : P \rightarrow \mathbb{N}$ . We say the single marking assigns to each place a number of *tokens*. A *marking* of  $N$  is a finite multiset  $\mathcal{M}$  of single markings.

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For any  $t \in T$ , we will write  $W^-(t)$  (respectively  $W_1^-(t)$ ,  $W_2^-(t)$ ) for the single marking  $p \rightarrow W^-(p, t)$  (resp.  $p \rightarrow W_1^-(p, t)$ ,  $p \rightarrow W_2^-(p, t)$ ), and  $W^+(t)$  (resp.  $W_1^+(t)$ ,  $W_2^+(t)$ ) for the single marking  $p \rightarrow W^+(p, t)$  (resp.  $p \rightarrow W_1^+(p, t)$ ,  $p \rightarrow W_2^+(p, t)$ ).

A *marked* PNSJT is a 5-tuple  $((P, T, W^-, W^+, \mathcal{M}))$ , where

- $N = (P, T, W^-, W^+)$  is a PNSJT,
- $\mathcal{M}$  is a marking of  $N$ .

### 1.2. Execution Semantics of a PNSJT

**Definition 1.2.** Enabling of a transition.

Let  $(N, \mathcal{M})$  be a marked PNSJT. Let  $t$  be a transition of  $N$ .  $t$  is *enabled* in  $(N, \mathcal{M})$  if and only if

- $t$  is a regular transition and there exists  $M \in \mathcal{M}$  such that, for all  $p \in P$ ,  $M(p) \geq W^-(p, t)$ . In this case we say that  $t$  is enabled in  $M$ , or
- $t$  is a split transition and there exists  $M \in \mathcal{M}$  such that, for all  $p \in P$ ,  $M(p) \geq W^-(p, t)$ . In this case we say that  $t$  is enabled in  $M$ , or
- $t$  is a join transition and there exist  $M_1, M_2 \in \mathcal{M}$  such that, for all  $p \in P$ ,  $M_1(p) \geq W^-(p, t)_1$  and  $M_2(p) \geq W^-(p, t)_2$ . In this case we say that  $t$  is enabled in  $M_1$  and  $M_2$ .

**Definition 1.3.** Firing of a transition.

Let  $(N, \mathcal{M})$  be a marked PNSJT, and  $t$  be a transition of  $N$  enabled in  $(N, \mathcal{M})$ . The *firing* of  $t$  in  $(N, \mathcal{M})$  is the relation

$$(N, \mathcal{M}) \rightarrow_t (N, \mathcal{M}'),$$

where

- if  $t$  is a regular transition enabled for one  $M \in \mathcal{M}$ ,  $M' = p \rightarrow M(p) - W^-(p, t) + W^+(p, t)$ , and  $\mathcal{M}' = \mathcal{M} \setminus \{M\} \uplus \{M'\}$ .
- if  $t$  is a split transition enabled for one  $M \in \mathcal{M}$ , let  $M_1$  and  $M_2$  be two single markings of  $N$  such that, for all  $p \in P$ ,  $M_1(p) - W^+(p, t)_1 \geq 0$ ,  $M_2(p) - W^+(p, t)_2 \geq 0$ , and  $M_1(p) + M_2(p) = M(p) - W^-(p, t) + W^+(p, t)_1 + W^+(p, t)_2$ . Then,  $\mathcal{M}' = \mathcal{M} \setminus \{M\} \uplus \{M_1\} \uplus \{M_2\}$ .
- if  $t$  is a join transition enabled for  $M_1$  and  $M_2$ , let  $M' = p \rightarrow M_1(p) - W_1^-(p, t) + M_2(p) - W_2^-(p, t) + W^+(p, t)$  and  $\mathcal{M}' = \mathcal{M} \setminus \{M_1\} \setminus \{M_2\} \uplus \{M'\}$ .

In words,

- firing a regular transition  $t$  in a single marking  $M$  consumes  $W^-(p, t)$  tokens from each of its input places  $p$ , and produces  $W^+(p, t)$  tokens in each of its output places  $p$ ,
- firing a split transition  $t$  in a single marking  $M$  consumes  $W^-(p, t)$  tokens from each of its input places  $p$ , splits the marking  $M$  in two new markings  $M_1$  and  $M_2$ , and produces  $W^+(p, t)_1$  tokens in each of its output places  $p$  in  $M_1$  and  $W^+(p, t)_2$  tokens in each of its output places  $p$  in  $M_2$ , and
- firing a join transition  $t$  in a couple  $(M_1, M_2)$  of single markings consumes  $W^-(p, t)_1$  tokens in each of its input places  $p$  in  $M_1$  and  $W^-(p, t)_2$  tokens in each of its input places  $p$  in  $M_2$ , sums the two markings into a new marking  $M'$ , and produces  $W^+(p, t)$  tokens in each of its output places  $p$ .

*Remark 1.4.* The execution semantics of a PNSJT with only regular transitions is exactly that of a classical petri net.

**Definition 1.5.** Promenade.

Let  $N$  be a PNSJT. A promenade on  $N$  is a labelled acyclic directed finite graph with in and out-degree at most two such that:

- each vertex is labelled with a single marking on  $N$ ,
- each edge is labelled with a transition of  $N$ ,
- for any vertex, the (possibly two) ingoing edges have the same label, as well as the (possibly two) outgoing edges,
- for any vertex  $v$  labelled with  $M$ , any outgoing edge  $e$  labelled with  $t$ ,  $t$  is enabled in  $M$ ,
- for any vertex  $v$  labelled with  $M$ , with indegree two, with parent nodes  $v_1$  labelled with  $M_1$  and  $v_2$  labelled with  $M_2$ , the ingoing edges are labelled with a join transition  $t$ , and we have:

$$(N, \{M_1\} \uplus \{M_2\}) \rightarrow_t (N, \{M\}),$$

- for any vertex  $v'$  labelled with  $M'$ , with indegree one, with parent node  $v$  labelled with  $M$  with outdegree one, the ingoing edge is labelled with a regular transition  $t$ , and we have:

$$(N, \{M\}) \rightarrow_t (N, \{M'\}),$$

- for any vertex  $v$  labelled with  $M$ , with outdegree two, with child nodes  $v_1$  labelled with  $M_1$  and  $v_2$  labelled with  $M_2$ , the outgoing edges are labelled with a split transition  $t$ , and we have:

$$(N, \{M\}) \rightarrow_t (N, \{M_1\} \uplus \{M_2\}).$$

**Definition 1.6.** Reachability Problem.

Let  $N$  be a PNSJT,  $\mathcal{M}_0, \mathcal{M}_1$  be two sets of single markings on  $N$ . The reachability problem for  $N, \mathcal{M}_0, \mathcal{M}_1$  is the following:

Does there exists a promenade  $P$  on  $N$  such that:

1. the set of labels of vertices of indegree 0 is  $\mathcal{M}_0$ , and
2. the set of labels of vertices of outdegree 0 is  $\mathcal{M}_1$ ?

The reachability problem for  $N, \mathcal{M}_0, \mathcal{M}_1$  can easily be reformulated as follows:

Does there exist two markings  $\mathcal{M}'_0, \mathcal{M}'_1$  such that:

1. the underlying set of  $\mathcal{M}'_0$  is  $\mathcal{M}_0$ ,
2. the underlying set of  $\mathcal{M}'_1$  is  $\mathcal{M}_1$ , and
3. there exists a finite sequence of transitions  $t_0, \dots, t_k$  of  $N$  such that

$$(N, \mathcal{M}'_0) \rightarrow_{t_0} \dots \rightarrow_{t_k} (N, \mathcal{M}'_1).$$

### 1.3. Correspondence between MELL and the PNST Reachability Problem

Roman capitals  $A, B$  stand for MELL formulas, which are given by the following grammar, where  $\otimes$  and  $\wp$  are duals,  $!$  and  $?$  are duals, and the neutral elements  $1$  and  $\perp$  are duals for the negation  $\perp$  accordingly to De Morgan laws:

$$\text{MELL:} \quad F ::= A \mid A^\perp \mid F \otimes F \mid F \wp F \mid !F \mid ?F \mid 1 \mid \perp$$

Greek capitals  $\Gamma, \Delta$  stand for sequents, which are multiset of formulas, so that exchange is implicit. The MELL (cut free) sequent calculus is given by the following rules:

$$\begin{array}{c}
\frac{}{\vdash A, A^\perp} \text{ (ax)} \quad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \wp B} \wp \quad \frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B} \otimes \\
\\
\frac{\vdash \Gamma}{\vdash \Gamma, ?A} ?W \quad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A} ?C \quad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A} ?D \quad \frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A} !P \\
\\
\frac{\vdash \Gamma}{\vdash \Gamma, \perp} \perp \quad \frac{}{\vdash 1} 1
\end{array}$$

The MELL decision problem is to decide, given a MELL sequent  $\Gamma$ , whether there exists a sequent calculus proof with conclusion  $\vdash \Gamma$ . We will simply write  $\vdash \Gamma$  for  $\Gamma$  satisfies the MELL problem.

**Lemma 1.7.** *For any of the  $\wp, ?W, ?C, ?D, \perp$  MELL sequent calculus rules  $(R)$  applied on a formula  $f$  with conclusion  $\Gamma_0$  and premise  $\Gamma_1$ , any MELL formula  $A$ ,  $(R)$  can be applied on  $f$  with conclusion  $\Gamma_0 \cup A$  and premise  $\Gamma_1 \cup A$ .*

*For any  $\otimes$  sequent calculus rules  $(R)$  applied on a formula  $f$  with conclusion  $\Gamma_0$  and premises  $\Gamma_1$  and  $\Gamma_2$ , any MELL formula  $A$ ,  $(R)$  can be applied on  $f$  with conclusion  $\Gamma_0 \cup A$  and premises  $\Gamma_1 \cup A$  and  $\Gamma_2$ , or on  $f$  with conclusion  $\Gamma_0 \cup A$  and premises  $\Gamma_1$  and  $\Gamma_2 \cup A$ .*

□

**Lemma 1.8.** *Let  $\Gamma$  be a MELL sequent with  $\vdash \Gamma$ , let  $F(\Gamma)$  be the set of sub-formulas of the formulas of  $\Gamma$ , and  $\mathcal{P}(F(\Gamma))$  be the set of subsets of  $F(\Gamma)$ .*

*There exists a MELL proof  $\Pi$  with conclusion  $\vdash \Gamma$  such that:*

1. *for any  $!P$  sequent calculus rule  $(R)$  applied with conclusion  $\zeta$  in  $\Pi$ ,  $\zeta$  equals its underlying set  $\Delta$ ,*
2. *for any  $!P$  sequent calculus rule  $(R_1)$  applied with conclusion  $\zeta$  in  $\Pi$ , any  $!P$  sequent calculus rule  $(R_2)$  applied with conclusion  $\zeta'$  in  $\Pi$  below  $(R_1)$ ,  $\zeta \neq \zeta'$ , and*
3. *in any path of  $\Pi$ , the number of  $!P$  rules is bounded by  $|\mathcal{P}(F(\Gamma))|$ .*

*Proof.* Assume  $\vdash \Gamma$ , and let  $\Pi$  be a MELL proof  $\Pi$  with conclusion  $\vdash \Gamma$ .

1. Let  $(R)$  be a  $!P$  sequent calculus rule applied with conclusion  $\zeta$  in  $\Pi$ . Then, it suffices to apply as many  $?W$  rules as necessary under  $(R)$ , and as many  $?C$  rules as necessary above  $(R)$ .
2. Let  $(R_1)$  and  $(R_2)$  be two  $!P$  rules applied in  $\Pi$  with the same conclusion  $\zeta$ ,  $(R_2)$  below  $(R_1)$ . Then, one can remove in  $\Pi$  the tree above  $(R_2)$  and replace it by the tree above  $(R_1)$ : the tree  $\Pi'$  obtained is still a MELL proof with conclusion  $\vdash \Gamma$ . Doing so for all such pairs of  $!P$  rules  $(R_1)$  and  $(R_2)$  yields the result.
3. follows directly from 1) and 2) above.

□

**Theorem 1.9.** *The PNST Reachability Problems and MELL reduce one to the other via many-one reductions.*

*Proof.* Let us first prove that PNST Reachability  $\preceq_1$  MELL.

Let  $N = (P, T, W^-, W^+)$  be a PNST,  $\mathcal{M}_0, \mathcal{M}_1$  be two sets of single markings on  $N$ . Since  $N$  is a PNST, i.e. has no join transition, we can without loss of generality assume that  $\mathcal{M}_0$  contains only one single marking  $M_0$ , and assume  $\mathcal{M}_1 = \{M_1, \dots, M_k\}$ . Let us now define the sequent  $\Gamma$  as follows:

1. let  $P = \{p_1 \cdots p_n\}$ . To each place  $p_i \in P$ , we associate a linear variable  $p_i$ ,
2. to a single marking  $M$  on  $N$ , we associate the formula  $F(M) = \left( \bigotimes_1^{M(p_1)} p_1 \right) \otimes \cdots \otimes \left( \bigotimes_1^{M(p_n)} p_n \right)$ ,  
where  $\bigotimes_1^k a$  denotes  $a \otimes \cdots \otimes a$   $k$  times if  $k \geq 1$ , and 1 if  $k = 0$ ,

3. to a regular transition  $r_i$  of  $N$ , we associate the following formula  $R_i = (F(W^-(r_i))) \otimes (F(W^+(r_i)))^\perp$ ,
4. to a split transition  $s_j$  of  $N$ , we associate the following formula  

$$S_j = (F(W^-(s_j))) \otimes \left( (F(W_1^+(s_j)))^\perp \otimes (F(W_2^+(s_j)))^\perp \right)$$
, and finally
5.  $\Gamma = \{?R_i\} \cup \{?S_j\} \cup F(M_0)^\perp \cup F(M_1) \cup \dots \cup F(M_k) \cup ?F(M_1) \cup \dots \cup ?F(M_k)$ .

Then,  $\vdash \Gamma$  if and only if  $N, \mathcal{M}_0, \mathcal{M}_1$  satisfy the PNST reachability problem.

Let us now prove that  $\text{MELL} \preceq_1 \text{PNST Reachability}$ .

Let  $\Gamma = F_1 \cup \dots \cup F_k$  be a MELL sequent. Let  $F(\Gamma)$  be the set of sub-formulas of the formulas of  $\Gamma$ , and  $\mathcal{P}(F(\Gamma))$  be the set of subsets of  $F(\Gamma)$ .

Let  $\xi$  be a sequent with underlying set  $\Xi \in \mathcal{P}(F(\Gamma))$ , such that a !P rule can be applied on a formula !A of  $\xi$ . Then, by Lemma 1.8,  $\vdash \xi$  if and only if  $\vdash \Xi$ . Let  $n(\Gamma) \leq |\mathcal{P}(F(\Gamma))|$  be the bound of Lemma 1.8, 3), and assume the possible !P rule applications in a proof of  $\vdash \Gamma$  are numbered from 1 to  $n(\Gamma)$ , and their conclusion sequents  $\Xi_1, \dots, \Xi_{n(\Gamma)}$ . Let also  $\mathcal{P} = \mathcal{P}(\{1 \dots n(\Gamma)\})$ . A set  $\pi \in \mathcal{P}$  will be denoted as an *index set* (of sequents conclusions of !P rule applications).

We define the PNST  $N(\Gamma) = (P, T, W^-, W^+)$  as follows:

1. to every  $\Delta \in \mathcal{P}(F(\Gamma))$ , every  $\pi \in \mathcal{P}$ , we associate a place  $p_{\Delta, \pi}$  in  $P$ . To every  $f \in \Delta$ , we associate a place  $p_{\Delta, \pi}^f$  in  $P$ . The sets of places  $\{p_{\Delta, \pi}, p_{\Delta, \pi}^{f_i}; f_i \in \Delta\}$  encodes the underlying set of formulas of any occurrence of a sequent  $\zeta$  that might occur in a proof with conclusion  $\vdash \Gamma$ , where  $\pi$  represents the index set of the conclusion sequents of all !P rules already present in the proof under the given occurrence of  $\zeta$ . The multiplicity of a formula in such a sequent will be encoded by the number of tokens that will be present in the corresponding place. For any sequent  $\zeta$  with underlying set  $\Delta$ , any  $\pi \in \mathcal{P}$ , denote by  $M(\zeta, \pi)$  the single marking encoding  $\zeta$ , where:
  - $M(\zeta, \pi)(p_{\Delta, \pi}) = 1$ ,
  - $M(\zeta, \pi)(p_{\Delta, \pi}^{f_i})$  is the multiplicity of  $f_i$  in  $\zeta$ , and
  - $M(\zeta, \pi)(p) = 0$  for any other place  $p$ .
2. Let  $\Delta \in \mathcal{P}(F(\Gamma))$ ,  $\zeta$  a MELL sequent with underlying set  $\Delta$ ,  $\pi \in \mathcal{P}$ ,  $f_i \in \zeta$  such that a sequent calculus rule  $(R_i)$  can be applied on  $f_i$  with conclusion  $\vdash \zeta$ .
  - Assume  $(R_i) = (ax)$ . To  $(R_i), \Delta, \pi$ , we associate a place  $q_{(R_i, \Delta, \pi)}$ , and
    - (a) a regular transition which takes one token in  $p_{\Delta, \pi}$ , one token in  $p_{\Delta, \pi}^{f_i}$ , one token in  $p_{\Delta, \pi}^{f_i^\perp}$ , places one token in  $q_{(R_i, \Delta, \pi)}$ , and leaves all other places untouched (lock), and
    - (b) a regular transition which takes one token in  $q_{(R_i, \Delta, \pi)}$ , and leaves all other places untouched (unlock).
  - Assume  $(R_i) = 1$ . To  $(R_i), \Delta, \pi$ , we associate a place  $q_{(R_i, \Delta, \pi)}$ , and
    - (a) a regular transition which takes one token in  $p_{\Delta, \pi}$ , one token in  $p_{\Delta, \pi}^1$ , places one token in  $q_{(R_i, \Delta, \pi)}$ , and leaves all other places untouched (lock), and
    - (b) a regular transition which takes one token in  $q_{(R_i, \Delta, \pi)}$ , and leaves all other places untouched (unlock).
  - Assume  $(R_i) = \wp$ ,  $f_i = A \wp B$ , and let  $\zeta_1$  be the premise of  $(R_i)$  with underlying set  $\Delta_1$ . To  $(R_i), \Delta, \Delta_1, \pi$ , we associate a place  $q_{(R_i, \Delta, \Delta_1, \pi)}$ , and
    - (a) a regular transition which takes one token from  $p_{\Delta, \pi}$ , one token from  $p_{\Delta, \pi}^{A \wp B}$ , places one token in  $q_{(R_i, \Delta, \Delta_1, \pi)}$ , one token in  $p_{\Delta_1, \pi}^A$ , one token in  $p_{\Delta_1, \pi}^B$ , and leaves all other places untouched (lock),
    - (b) for all  $f_j \in \Delta \cap \Delta_1$ , a regular transition which takes one token from  $q_{(R_i, \Delta, \Delta_1, \pi)}$ , one token from  $p_{\Delta, \pi}^{f_j}$ , places one token in  $q_{(R_i, \Delta, \Delta_1, \pi)}$ , one token in  $p_{\Delta_1, \pi}^{f_j}$ , and leaves all other places untouched. This transition is called a *transfer*, and
    - (c) a regular transition which takes one token from  $q_{(R_i, \Delta, \Delta_1, \pi)}$ , places one token in  $p_{\Delta_1, \pi}$ , and leaves all other places untouched (unlock).

- Assume  $(R_i) = ?W$ ,  $f_i = ?A$ , and let  $\zeta_1$  be the premise of  $(R_i)$  with underlying set  $\Delta_1$ . To  $(R_i), \Delta, \Delta_1, \pi$ , we associate a place  $q_{(R_i, \Delta, \Delta_1, \pi)}$ , and
  - (a) a regular transition which takes one token from  $p_{\Delta, \pi}$ , one token from  $p_{\Delta, \pi}^{?A}$ , places one token in  $q_{(R_i, \Delta, \Delta_1, \pi)}$ , and leaves all other places untouched (lock),
  - (b) for all  $f_j \in \Delta \cap \Delta_1$ , a regular transition which takes one token from  $q_{(R_i, \Delta, \Delta_1, \pi)}$ , one token from  $p_{\Delta, \pi}^{f_j}$ , places one token in  $q_{(R_i, \Delta, \Delta_1, \pi)}$ , one token in  $p_{\Delta_1, \pi}^{f_j}$ , and leaves all other places untouched (transfer), and
  - (c) a regular transition which takes one token from  $q_{(R_i, \Delta, \Delta_1, \pi)}$ , places one token in  $p_{\Delta_1, \pi}$ , and leaves all other places untouched (unlock).
- Assume  $(R_i) = ?C$ ,  $f_i = ?A$ , and let  $\zeta_1$  be the premise of  $(R_i)$  with underlying set  $\Delta_1$ . To  $(R_i), \Delta, \Delta_1, \pi$ , we associate a place  $q_{(R_i, \Delta, \Delta_1, \pi)}$ , and
  - (a) a regular transition which takes one token from  $p_{\Delta, \pi}$ , one token from  $p_{\Delta, \pi}^{?A}$ , places one token in  $q_{(R_i, \Delta, \Delta_1, \pi)}$ , two tokens in  $p_{\Delta_1, \pi}^{?A}$ , and leaves all other places untouched (lock),
  - (b) for all  $f_j \in \Delta \cap \Delta_1$ , a regular transition which takes one token from  $q_{(R_i, \Delta, \Delta_1, \pi)}$ , one token from  $p_{\Delta, \pi}^{f_j}$ , places one token in  $q_{(R_i, \Delta, \Delta_1, \pi)}$ , one token in  $p_{\Delta_1, \pi}^{f_j}$ , and leaves all other places untouched (transfer), and
  - (c) a regular transition which takes one token from  $q_{(R_i, \Delta, \Delta_1, \pi)}$ , places one token in  $p_{\Delta_1, \pi}$ , and leaves all other places untouched (unlock).
- Assume  $(R_i) = ?D$ ,  $f_i = ?A$ , and let  $\zeta_1$  be the premise of  $(R_i)$  with underlying set  $\Delta_1$ . To  $(R_i), \Delta, \Delta_1, \pi$ , we associate a place  $q_{(R_i, \Delta, \Delta_1, \pi)}$ , and
  - (a) a regular transition which takes one token from  $p_{\Delta, \pi}$ , one token from  $p_{\Delta, \pi}^{?A}$ , places one token in  $q_{(R_i, \Delta, \Delta_1, \pi)}$ , one token in  $p_{\Delta_1, \pi}^A$ , and leaves all other places untouched (lock),
  - (b) for all  $f_j \in \Delta \cap \Delta_1$ , a regular transition which takes one token from  $q_{(R_i, \Delta, \Delta_1, \pi)}$ , one token from  $p_{\Delta, \pi}^{f_j}$ , places one token in  $q_{(R_i, \Delta, \Delta_1, \pi)}$ , one token in  $p_{\Delta_1, \pi}^{f_j}$ , and leaves all other places untouched (transfer), and
  - (c) a regular transition which takes one token from  $q_{(R_i, \Delta, \Delta_1, \pi)}$ , places one token in  $p_{\Delta_1, \pi}$ , and leaves all other places untouched (unlock).
- Assume  $(R_i) = \perp$ ,  $f_i = \perp$ , and let  $\zeta_1$  be the premise of  $(R_i)$  with underlying set  $\Delta_1$ . To  $(R_i), \Delta, \Delta_1, \pi$ , we associate a place  $q_{(R_i, \Delta, \Delta_1, \pi)}$ , and
  - (a) a regular transition which takes one token from  $p_{\Delta, \pi}$ , one token from  $p_{\Delta, \pi}^\perp$ , places one token in  $q_{(R_i, \Delta, \Delta_1, \pi)}$ , and leaves all other places untouched (lock),
  - (b) for all  $f_j \in \Delta \cap \Delta_1$ , a regular transition which takes one token from  $q_{(R_i, \Delta, \Delta_1, \pi)}$ , one token from  $p_{\Delta, \pi}^{f_j}$ , places one token in  $q_{(R_i, \Delta, \Delta_1, \pi)}$ , one token in  $p_{\Delta_1, \pi}^{f_j}$ , and leaves all other places untouched (transfer), and
  - (c) a regular transition which takes one token from  $q_{(R_i, \Delta, \Delta_1, \pi)}$ , places one token in  $p_{\Delta_1, \pi}$ , and leaves all other places untouched (unlock).
- Assume  $(R_i) = \otimes$ ,  $f_i = A \otimes B$ , and let  $\zeta_1, \zeta_2$  be the two premises of  $(R_i)$  with underlying sets  $\Delta_1$  and  $\Delta_2$ . To  $(R_i), \Delta, \Delta_1, \Delta_2, \pi$ , we associate a place  $q_{(R_i, \Delta, \Delta_1, \pi)}$  and a place  $q_{(R_i, \Delta, \Delta_2, \pi)}$ , and
  - (a) a split transition which takes one token from  $p_{\Delta, \pi}$ , one token from  $p_{\Delta, \pi}^{A \otimes B}$ , places on its first projection one token in  $q_{(R_i, \Delta, \Delta_1, \pi)}$ , one token in  $p_{\Delta_1, \pi}^A$ , and on its second projection one token in  $q_{(R_i, \Delta, \Delta_2, \pi)}$ , one token in  $p_{\Delta_2, \pi}^B$ , and leaves all other places untouched (lock),
  - (b) for all  $f_j \in \Delta \cap \Delta_1$ , a regular transition which takes one token from  $q_{(R_i, \Delta, \Delta_1, \pi)}$ , one token from  $p_{\Delta, \pi}^{f_j}$ , places one token in  $q_{(R_i, \Delta, \Delta_1, \pi)}$ , one token in  $p_{\Delta_1, \pi}^{f_j}$ , and leaves all other places untouched (transfer),
  - (c) for all  $f_j \in \Delta \cap \Delta_2$ , a regular transition which takes one token from  $q_{(R_i, \Delta, \Delta_2, \pi)}$ , one token from  $p_{\Delta, \pi}^{f_j}$ , places one token in  $q_{(R_i, \Delta, \Delta_2, \pi)}$ , one token in  $p_{\Delta_2, \pi}^{f_j}$ , and leaves all other places untouched (transfer),

- (d) a regular transition which takes one token from  $q_{(R_i, \Delta, \Delta_1, \pi)}$ , places one token in  $p_{\Delta_1, \pi}$ , and leaves all other places untouched (unlock), and
  - (e) a regular transition which takes one token from  $q_{(R_i, \Delta, \Delta_2, \pi)}$ , places one token in  $p_{\Delta_2, \pi}$ , and leaves all other places untouched (unlock).
  - Assume  $(R_i) = !P$ ,  $f_i = !A$ , and let  $\zeta_1$  be the premise of  $(R_i)$  with underlying set  $\Delta_1$ . Then,  $\Delta = \Xi_j$  for some  $j$ . To  $(R_i)$ ,  $\Delta, \Delta_1, \pi$  with  $j \notin \pi$ , we associate a place  $q_{(R_i, \Delta, \Delta_1, \pi)}$ , and
    - (a) a regular transition which takes one token from  $p_{\Delta, \pi}$ , one token from  $p_{\Delta, \pi}^A$ , one token from  $p_{\Delta, \pi}^{f_i}$  for each of the  $f_i \in \Delta \cap \Delta_1$ , places one token in  $q_{(R_i, \Delta, \Delta_1, \pi)}$ , one token in  $p_{\Delta_1, \pi \uplus \{j\}}^A$ , one token in  $p_{\Delta, \pi \uplus \{j\}}^{f_i}$  for each of the  $f_i \in \Delta \cap \Delta_1$ , and leaves all other places untouched (lock),
    - (b) for all  $f_j \in \Delta \cap \Delta_1$ , a regular transition which takes one token from  $q_{(R_i, \Delta, \Delta_1, \pi)}$ , one token from  $p_{\Delta, \pi}^{f_j}$ , places one token in  $q_{(R_i, \Delta, \Delta_1, \pi)}$ , and leaves all other places untouched (transfer), and
    - (c) a regular transition which takes one token from  $q_{(R_i, \Delta, \Delta_1, \pi)}$ , places one token in  $p_{(\Delta_1, \pi \uplus \{j\})}$ , and leaves all other places untouched (unlock).
  - To  $(R_i)$ ,  $\Delta, \Delta_1, \pi$  with  $j \in \pi$ , we associate a place  $q_{(sink)}$ , and
    - (a) a regular transition which takes one token from  $p_{\Delta, \pi}$ , places one token in  $q_{(sink)}$ , and leaves all other places untouched (sink).
3. For each of the cases  $(R_i) = (ax), 1, \wp, ?W, ?C, ?D, \perp, \otimes, !P$  above, the places and transitions associated only depend on  $f_i, \Delta, \pi$  and on the underlying sets of the (possibly two) premises of the rule,  $\Delta_1$  and  $\Delta_2$ . The construction above is repeated for any possible combination of possible sets  $\Delta, \Delta_1, \Delta_2, \pi$  and formula  $f_i$ , where the number of such possible combinations is exponentially bounded.

Let  $\emptyset$  be the empty single marking. Let us show that  $\vdash \Gamma$  if and only if  $N(\Gamma), \{M(\Gamma, \emptyset)\}, \{\emptyset\}$  satisfies the PNST reachability problem.

Firstly, assume  $\vdash \Gamma$ . Then, there exists a MELLS sequent calculus proof  $\Pi$  with conclusion  $\vdash \Gamma$ . Assume  $\Pi$  satisfies the conditions of Lemma 1.8. Then,

1. for any rule application  $(R_i)$  in  $\Pi$ , with conclusion  $\Gamma_0$  and zero premise, if  $\pi$  denotes the index set of conclusion sequents of the  $!P$  rules occurring in  $\Pi$  below  $(R_i)$ ,  $N(\Gamma), \{M(\Gamma_0, \pi)\}, \{\emptyset\}$  satisfies the PNST reachability problem by the construction above.
2. For any rule application  $(R_i)$  other than  $!P$  in  $\Pi$ , with conclusion  $\Gamma_0$  and one premise  $\Gamma_1$ , if  $\pi$  denotes the index set of conclusion sequents of the  $!P$  rules occurring in  $\Pi$  below  $(R_i)$ ,  $N(\Gamma), \{M(\Gamma_0, \pi)\}, \{M(\Gamma_1, \pi)\}$  satisfies the PNST reachability problem by the construction above.
3. For any rule application  $(R_i)$  in  $\Pi$ , with conclusion  $\Gamma_0$  and two premises  $\Gamma_1, \Gamma_2$ , if  $\pi$  denotes the index set of conclusion sequents of the  $!P$  rules occurring in  $\Pi$  below  $(R_i)$ ,  $N(\Gamma), \{M(\Gamma_0, \pi)\}, \{M(\Gamma_1, \pi), M(\Gamma_2, \pi)\}$  satisfies the PNST reachability problem by the construction above.
4. For any rule application  $(R_i) = !P$  in  $\Pi$ , with conclusion  $\Gamma_0 = \Xi_j$  and one premise  $\Gamma_1$ , if  $\pi$  denotes the index set of conclusion sequents of the  $!P$  rules occurring in  $\Pi$  below  $(R_i)$ , by hypothesis  $j \notin \pi$ , and  $N(\Gamma), \{M(\Gamma_0, \pi)\}, \{M(\Gamma_1, \pi \uplus \{j\})\}$  satisfies the PNST reachability problem by the construction above.

The composition of these reachability results along the depth of  $\Pi$  yields a promenade with initial marking  $M(\Gamma, \emptyset)$  and all final markings  $\emptyset$ , thus,  $N(\Gamma), \{M(\Gamma, \emptyset)\}, \{\emptyset\}$  satisfies the PNST reachability problem.

Secondly, assume  $N(\Gamma), \{M(\Gamma, \emptyset)\}, \{\emptyset\}$  satisfies the PNST reachability problem. Then, there exists a promenade  $P$  on  $N(\Gamma)$ , where vertices of indegree 0 are labelled with  $M(\Gamma, \emptyset)$  and vertices of outdegree 0 are labelled with  $\emptyset$ . Since  $N(\Gamma)$  is a PNST, i.e. has no join transition, we can assume without loss of generality that  $P$  has only one node of indegree 0, i.e.  $P$  is a tree. It is easy to see that any path in the tree  $P$  can be decomposed in sequences  $(N(\Gamma), M_i) \rightarrow_{t_{lock}} \cdots \rightarrow_{t_{unlock}} (N(\Gamma), M_j)$ , where only transfer transitions occur between the lock and the unlock transitions. All the transitions in such a sequence involve the same rule  $(R_i)$  applied on the same formula  $f_i$ . We say that such a sequence  $s = (N(\Gamma), M_i) \rightarrow_{t_{lock}} \cdots \rightarrow_{t_{unlock}} (N(\Gamma), M_j)$  is *faithful* if the firing of the intermediate transfer (if applicable) transitions remove *all* tokens from  $M_i$ . Then, two cases arise:



1. In any path of  $P$ , any sequence  $s = (N(\Gamma), M_i) \rightarrow_{t_{lock}} \cdots \rightarrow_{t_{unlock}} (N(\Gamma), M_j)$  is faithful. In that case, the MELL sequents encoded by the markings  $M_i$  and the rules  $(R_i)$  induce naturally a proof  $\Pi$  of  $\Gamma$ , and  $\vdash \Gamma$ .
2. There exists a path  $c$  in  $P$  and an unfaithful sequence  $s_i = (N(\Gamma), M_i) \rightarrow_{t_{lock}} \cdots \rightarrow_{t_{unlock}} (N(\Gamma), M_{i+1})$  in this path. Assume without loss of generality that  $s_i$  is the first unfaithful sequence in  $c$ , and denote by  $(R_i)$ ,  $\Delta_i$ ,  $\pi$  the corresponding rule, conclusion sequent underlying set and index set. Since  $s_i$  is the first unfaithful sequence in  $c$ , there exists  $f \in \Delta$  such that  $M_i(p_{\Delta, \pi}^f) > 0$  and  $M_{i+1}(p_{\Delta, \pi}^f) > 0$ . Since all paths reach the empty single marking by hypothesis, there exists a path  $c' = s_1 \cdots s_k$  such that the sequence  $s_i$  is in  $c \cap c'$ , and the place  $p_{\Delta, \pi}^f$  is emptied by some transition in the sequence  $s_j = (N(\Gamma), M_j) \rightarrow_{t_{lock}} \cdots \rightarrow_{t_{unlock}} (N(\Gamma), M_{j+1})$ , with  $j > i$ , in  $c'$ . Then, the conclusion sequent underlying set and index set corresponding to  $s_j$  are  $\Delta_j = \Delta_i$  and  $\pi$ , while the rule  $(R_j)$  may differ from  $(R_i)$ . In other words, a token corresponding to a formula  $f$  is left behind during the simulation of the application of  $(R_i)$  on  $\zeta_i$  with underlying set  $\Delta_i$  and index set  $\pi$ , and inherited later on when simulating a rule  $(R_j)$  on  $\zeta_j$  with the same underlying set  $\Delta_j = \Delta_i$  and the same index set  $\pi$ . Consider the subpath  $s_i \cdots s_j$  of  $c'$ , with the corresponding sequence of rules  $(R_i) \cdots (R_j)$  and underlying sets  $\Delta_i \cdots \Delta_j$ . Note that for  $i \leq k < j$ , the index set corresponding to  $s_k$  is also  $\pi$ , by construction. Therefore, none of the rules  $(R_k)$  for  $i \leq k < j$  may be a !P rule, and, by Lemma 1.7, there exist sequences  $s'_1 \cdots s'_j$  with  $s'_k = (N(\Gamma), M'_k) \rightarrow_{t_{lock}} \cdots \rightarrow_{t_{unlock}} (N(\Gamma), M'_{k+1})$  such that:
  - (a) for any  $i \leq k < j$ , the rule corresponding to  $s'_k$  is  $(R_k)$ ,
  - (b) for any  $i \leq k < j$ , the conclusion sequent underlying set corresponding to  $s'_k$  is  $\Delta_k \cup \{f\}$ ,
  - (c) for any  $i \leq k < j$ ,  $s'_k$  is derived from  $s_k$  by changing the necessary indexes of places and transitions, and adding the necessary transfer transitions such that

- $M'_{k+1}(p_{\Delta_k \cup \{f\}, \pi}^f) = 0$  if  $\Delta_{k+1} \cup \{f\} \neq \Delta_k \cup \{f\}$ , and  $M'_{k+1}(p_{\Delta_{k+1} \cup \{f\}, \pi}^f) > 0$  (i.e. no token of  $p_{\Delta_k \cup \{f\}, \pi}^f$  is left behind).
- for any  $i \leq k \leq j$ , for any  $g \neq f$ ,  $M'_k(p_{\Delta_k \cup \{f\}, \pi}^g) = M_k(p_{\Delta_k, \pi}^g)$ .
- for any lock transition  $t$  being a split transition (i.e. corresponding to a  $\otimes$  rule), the single marking produced by the firing of  $t$  which is not in  $c'$  remains unchanged.

In words, the sequences  $s'_1 \cdots s'_j$  simulate the same rules as the sequences  $s_1 \cdots s_j$ , the same way, with the additional feature that they transfer inductively  $f$  from the conclusion sequent to the premise sequent.

The conditions above ensure that replacing the sequences  $s_i \cdots s_j$  by the sequences  $s'_i \cdots s'_j$  in  $P$  yield a graph  $P'$  that is also a promenade on  $N(\Gamma)$ , with (unique) vertex of indegree 0 labelled with  $M(\Gamma, \emptyset)$  and vertices of outdegree labelled with  $\emptyset$ . Moreover,  $P'$  has strictly less unfaithful sequences than  $P$ . Therefore, by induction on the number of unfaithful sequences in  $P$ , we can conclude that there exists a promenade  $P''$  on  $N(\Gamma)$ , with only faithful sequences and with (unique) vertex of indegree 0 labelled with  $M(\Gamma, \emptyset)$  and vertices of outdegree labelled with  $\emptyset$ . Thus,  $\vdash \Gamma$ . □

## 2. Vector Addition Systems with States and Split/Join Transitions

### 2.1. Definition

**Definition 2.1.** Vector Addition Systems with States and Split/Join Transitions.

A Vector Addition Systems with States and Split/Join Transitions (VASSSJT) is a 4-tuple  $S = (G, T, m, v)$ , where:

- $G = (Q, A)$  is a finite directed graph,
- $T \subseteq A \cup A^2$  is a set of *transitions*,
- $m \geq 1$  is a natural number,

- $v : T \rightarrow \mathbb{Z} \cup \{s\} \cup \{j\}$  is a function, such that
  1.  $v(t) \in \mathbb{Z}^m$  if and only if  $t \in A$ ,
  2.  $v(t) = s$  if and only if  $t = (a_1, a_2)$  and  $a_1$  and  $a_2$  share the same origin,
  3.  $v(t) = j$  if and only if  $t = (a_1, a_2)$  and  $a_1$  and  $a_2$  share the same destination,

The vertices of  $G$  are called its *states*. A transition  $t$  with  $v(t) \in \mathbb{Z}^m$  is called a *regular* transition, a transition  $t$  with  $v(t) = s$  is called a *split* transition, and a transition  $t$  with  $v(t) = j$  is called a *join* transition.

A VASSSJT with only regular and split transitions is called a *Vector Addition Systems with States and Split Transitions* (VASSST), and a VASSSJT with only regular and join transitions is called a *Vector Addition Systems with States and Join Transitions* (VASSJT).

**Definition 2.2.** Configuration of a VASSSJT.

Let  $S = (G, m, v)$  be a VASSSJT. A *single configuration* of  $G$  is a 2-tuple  $c = (q, x)$ , where  $q \in Q$  is a *state* and  $x \in \mathbb{N}^m$  is a *value*. A single configuration  $c$  is *positive* if and only if, for all  $0 \leq i \leq m$ ,  $c_i \geq 0$ . A *configuration* of  $S$  is a multiset  $\mathcal{C}$  of single configurations of  $S$ . A configuration is *positive* if and only if all its single configurations are positive.

## 2.2. Execution Semantics of a VASSSJT

**Definition 2.3.** Firing of a Transition.

Let  $S = (G, T, m, v)$  be a VASSSJT, and  $\mathcal{C}$  be a configuration of  $S$ . Let  $t$  be a transition of  $S$ . The *firing* of  $t$  in  $(S, \mathcal{C})$  is the relation:

$$(S, \mathcal{C}) \rightarrow_t (S, \mathcal{C}'),$$

where

- if  $v(t) \in \mathbb{Z}^m$ ,  $t = (q_0, q_1)$ , there exists  $c = (q_0, x_0) \in \mathcal{C}$ , and  $\mathcal{C}' = \mathcal{C} \setminus \{c\} \uplus \{(q_1, x_0 + v(t))\}$ .
- if  $v(t) = s$ ,  $t = ((q_0, q_1), (q_0, q'_1))$ , there exists  $c_0 = (q_0, x_0) \in \mathcal{C}$ , and  $\mathcal{C}' = \mathcal{C} \setminus \{c_0\} \uplus \{(q_1, x_1)\} \uplus \{(q'_1, x'_1)\}$ , with  $x_0 = x_1 + x'_1$ . Moreover, for any  $1 \leq i \leq m$ ,  $|(x_0)_i| = |(x_1)_i| + |(x'_1)_i|$ .
- if  $v(t) = j$ ,  $t = ((q_0, q_1), (q'_0, q_1))$ , there exist  $c_0 = (q_0, x_0) \in \mathcal{C}$  and  $c'_0 = (q'_0, x'_0) \in \mathcal{C}$ , and  $\mathcal{C}' = \mathcal{C} \setminus \{c_0\} \setminus \{c'_0\} \uplus \{(q_1, x_0 + x'_0)\}$ .

**Definition 2.4.** Promenade.

Let  $S = (G, T, m, v)$  be a VASSSJT. A *promenade* on  $S$  is a labelled acyclic directed finite graph with in and out-degree at most two such that:

- each vertex is labelled with a single configuration on  $S$ ,
- each edge is labelled with a transition of  $S$ ,
- for any vertex, the (possibly two) ingoing edges have the same label, as well as the (possibly two) outgoing edges,
- for any vertex  $v$  labelled with  $c$ , with indegree two, with parent nodes  $v_1$  labelled with  $c_1$  and  $v_2$  labelled with  $c_2$ , the ingoing edges are labelled with a join transition  $t$ , and we have:

$$(S, \{c_1\} \uplus \{c_2\}) \rightarrow_t (S, \{c\}),$$

- for any vertex  $v'$  labelled with  $c'$ , with indegree one, with parent node  $v$  labelled with  $c$  with outdegree one, the ingoing edge is labelled with a regular transition  $t$ , and we have:

$$(S, \{c\}) \rightarrow_t (S, \{c'\}),$$

- for any vertex  $v$  labelled with  $c$ , with outdegree two, with child nodes  $v_1$  labelled with  $c_1$  and  $v_2$  labelled with  $c_2$ , the outgoing edges are labelled with a split transition  $t$ , and we have:

$$(S, \{c\}) \rightarrow_t (S, \{c_1\} \uplus \{c_2\}).$$

A promenade  $P$  on  $S$  is *positive* if and only if all vertices are labelled with positive configurations.

**Definition 2.5.** (Positive) Reachability Problem.

Let  $S$  be a VASSSJT,  $\mathcal{C}_0, \mathcal{C}_1$  be two sets of single configurations on  $S$ . The reachability problem for  $S, \mathcal{C}_0, \mathcal{C}_1$  is the following:

Does there exist a promenade  $P$  on  $S$  such that:

1. the set of labels of vertices of indegree 0 is  $\mathcal{C}_0$ , and
2. the set of labels of vertices of outdegree 0 is  $\mathcal{C}_1$ ?

The positive reachability problem for  $S, \mathcal{C}_0, \mathcal{C}_1$  is the following:

Does there exist a positive promenade  $P$  on  $S$  satisfying the same conditions as above?

The reachability problem for  $S, \mathcal{C}_0, \mathcal{C}_1$  can easily be reformulated as follows:

Does there exist two configurations  $\mathcal{C}'_0, \mathcal{C}'_1$  such that:

1. the underlying set of  $\mathcal{C}'_0$  is  $\mathcal{C}_0$ ,
2. the underlying set of  $\mathcal{C}'_1$  is  $\mathcal{C}_1$ , and
3. there exists a finite sequence of transitions  $t_0, \dots, t_k$  of  $N$  such that

$$(S, \mathcal{C}'_0) \rightarrow_{t_0} \dots \rightarrow_{t_k} (S, \mathcal{C}'_1).$$

The positive reachability problem can also be reformulated as above, with the condition that all configurations along the chain  $(S, \mathcal{C}'_0) \rightarrow_{t_0} \dots \rightarrow_{t_k} (S, \mathcal{C}'_1)$  are positive.

### 2.3. VASSSJT and PNSJT simulate each other

- Proposition 2.6.**
1. The PNSJT reachability problem and the positive VASSSJT reachability problem reduce one to the other (via many-one reductions),
  2. The PNST reachability problem and the positive VASSST reachability problem reduce one to the other (via many-one reductions),
  3. The PNJT reachability problem and the positive VASSJT reachability problem reduce one to the other (via many-one reductions).

*Proof.* Let us first prove PNSJT reachability  $\preceq_1$  positive VASSSJT reachability.

Let  $N = (P, T, W^-, W^+)$  be a PNSJT,  $\mathcal{M}_0, \mathcal{M}_1$  be two sets of single markings on  $N$ . Let us define the VASSSJT  $S = (G, T', m, v)$ , with  $G = (Q, A)$ , as follows:

1.  $m = |P|$ , and we identify  $P$  with  $[1 \dots m]$ ,
2. to every regular transition  $r_i$  in  $T$ , we associate
  - a state  $q_i$  in  $Q$ ,
  - a regular transition  $(q, q_i)$  with  $v(q, q_i) = -W^-(r_i)$  (where we identify the function  $W^-(t_i) : [1 \dots m] \rightarrow \mathbb{N}$  with its vector in  $\mathbb{Z}^m$ ), and
  - a regular transition  $(q_i, q)$  with  $v(q_i, q) = W^+(r_i)$  in  $T'$ .
3. to every split transition  $s_j$  in  $T$ , we associate
  - three states  $o_j, t1_j$  and  $t2_j$  in  $Q$ ,
  - a regular transition  $(q, o_j)$  with  $v(q, o_j) = -W^-(s_j)$  in  $T'$ ,
  - a split transition  $((o_j, t1_j), (o_j, t2_j))$  in  $T'$ ,

- a regular transition  $(t1_j, q)$  with  $v(t1_j, q) = W_1^+(s_j)$ , and
  - a regular transition  $(t2_j, q)$  with  $v(t2_j, q) = W_2^+(s_j)$  in  $T'$ .
4. to every join transition  $j_k$  in  $T$ , we associate
- three states  $o1_k$  and  $o2_k$  and  $t_k$  in  $Q$ ,
  - a regular transition  $(q, o1_k)$  with  $v(q, o1_k) = -W_1^-(j_k)$ ,
  - a regular transition  $(q, o2_k)$  with  $v(q, o2_k) = -W_2^-(j_k)$  in  $T'$ ,
  - a join transition  $((o1_k, t_k), (o2_k, t_k))$  in  $T'$ , and
  - a regular transition  $(t_k, q)$  with  $v(t_k, q) = W^+(j_k)$  in  $T'$ .
5.  $Q$  is the disjoint union of  $\{q\}$ , and of the  $\{q_i\}$ ,  $\{o_j\}$ ,  $\{t1_j\}$ ,  $\{t2_j\}$ ,  $\{o1_k\}$ ,  $\{o2_k\}$  and  $\{t_k\}$ , defined above.

Let  $M$  be a single marking on  $N$ . To  $M$ , we associate the single configuration  $c(M) = (q, (M(1) \cdots M(m)))$  of  $S$ . Then,  $N$ ,  $\mathcal{M}_0$ ,  $\mathcal{M}_1$  satisfy the PNSJT reachability problem if and only if  $S$ ,  $c(\mathcal{M}_0)$ ,  $c(\mathcal{M}_1)$  satisfy the positive VASSSJT reachability problem.

Let us now prove positive VASSSJT reachability  $\preceq_1$  PNSJT reachability.

Let  $S = (G, T, m, v)$ , with  $G = (Q, A)$  be a VASSSJT,  $\mathcal{C}_0, \mathcal{C}_1$  be two sets of single configurations on  $S$ . Let us define the PNSJT  $N = (P, T', W^-, W^+)$  as follows:

1.  $P = Q \uplus \{1 \cdots m\}$ ,
2. to every regular transition  $r_i = (o_i, t_i)$  in  $T$ , we associate a regular transition  $r_i$  in  $T'$ , with
  - $W^-(q, r_i) = 0$  for  $q \in Q \setminus \{o_i\}$ ,
  - $W^-(o_i, r_i) = 1$ ,
  - $W^-(k, r_i) = -v(r_i)_k$  for  $k \in [1 \cdots m]$  and  $v(r_i)_k < 0$ ,  $W^-(k, r_i) = 0$  otherwise,
  - $W^+(q, r_i) = 0$  for  $q \in Q \setminus \{t_i\}$ ,
  - $W^+(t_i, r_i) = 1$ , and
  - $W^+(k, r_i) = v(r_i)_k$  for  $k \in [1 \cdots m]$  and  $v(r_i)_k \geq 0$ ,  $W^+(k, r_i) = 0$  otherwise.
3. to every split transition  $s_i = ((o_i, t1_i), (o_i, t2_i))$  in  $T$ , we associate a split transition  $s_i$  in  $T'$ , with
  - $W^-(q, r_i) = 0$  for  $q \in Q \setminus \{o_i\}$ ,
  - $W^-(o_i, r_i) = 1$ ,
  - $W^-(k, r_i) = 0$  for  $k \in [1 \cdots m]$ ,
  - $W_1^+(q, r_i) = 0$  for  $q \in Q \setminus \{t1_i\}$ ,
  - $W_1^+(t1_i, r_i) = 1$ ,
  - $W_1^+(k, r_i) = 0$  for  $k \in [1 \cdots m]$ ,
  - $W_2^+(q, r_i) = 0$  for  $q \in Q \setminus \{t2_i\}$ ,
  - $W_2^+(t2_i, r_i) = 1$ , and
  - $W_2^+(k, r_i) = 0$  for  $k \in [1 \cdots m]$ .
4. to every join transition  $j_i = ((o1_i, t_i), (o2_i, t_i))$  in  $T$ , we associate a join transition  $j_i$  in  $T'$ , with
  - $W_1^-(q, r_i) = 0$  for  $q \in Q \setminus \{o1_i\}$ ,
  - $W_1^-(o1_i, r_i) = 1$ ,
  - $W_1^-(k, r_i) = 0$  for  $k \in [1 \cdots m]$ ,
  - $W_2^-(q, r_i) = 0$  for  $q \in Q \setminus \{o2_i\}$ ,
  - $W_2^-(o2_i, r_i) = 1$ ,
  - $W_2^-(k, r_i) = 0$  for  $k \in [1 \cdots m]$ ,
  - $W^+(q, j_i) = 0$  for  $q \in Q \setminus \{t_i\}$ ,

- $W^+(t_i, j_i) = 1$ ,
- and  $W^+(k, j_i) = 0$  for  $k \in [1 \dots m]$ .

Let  $c = (q, x)$  be a single configuration of  $S$ . To  $c$ , we associate the single marking  $M(c)$  of  $S$ , where  $M(c)(q') = 0$  if  $q' \in Q \setminus \{q\}$ ,  $M(c)(q) = 1$ , and  $M(c)(k) = x_k$  for  $1 \leq k \leq m$ . Then,  $S, \mathcal{C}_0, \mathcal{C}_1$  satisfy the positive VASSSJT reachability problem if and only if  $N, M(\mathcal{C}_0), M(\mathcal{C}_1)$  satisfy the PNSJT reachability problem.

The same reductions apply for proving the other parts of the proposition. □

### 3. Karp and Miller Graph

#### 3.1. Definitions

**Definition 3.1.** Generalized configurations.

For  $m \geq 1$ , we consider the order relation  $\leq$  on  $(\mathbb{N} \cup \infty)^m$  defined as follows:

- for  $m = 1$ ,  $x \leq y$  if and only if  $x \leq y \in \mathbb{N}$ , or  $y = \infty$ , and
- for  $m \geq 1$ ,  $x \leq y$  if and only if, for all  $i = 1 \dots m$ ,  $x_i \leq y_i$ .

Given a VASSSJT  $S = (G, T, m, v)$ , with  $G = (Q, A)$ , a *generalized single configuration* of  $S$  is a 2-tuple  $g = (q, x)$ , where  $q \in Q$  is a *state* and  $x \in (\mathbb{N} \cup \infty)^m$  is a *generalized value*. A *generalized configuration* is a finite set  $\mathcal{G}$  of generalized single configurations of  $S$ , together with a multiplicity function  $M_{\mathcal{G}} : \mathcal{G} \rightarrow \mathbb{N} \cup \infty$ , i.e. a finite multiset whose elements may have finite or infinite multiplicity. For a generalized configuration  $\mathcal{G}$  and  $g \in \mathcal{C}$ , we write  $\mathcal{G} \setminus g$  for the generalized configuration  $\mathcal{G}'$  such that:

1. for all  $g' \neq g$ ,  $g' \in \mathcal{G}'$  if and only if  $g' \in \mathcal{G}$ . In that case  $M_{\mathcal{G}}(g') = M_{\mathcal{G}'}(g')$ , and
2. if  $M_{\mathcal{G}}(g) > 1$ ,  $g \in \mathcal{G}'$  with  $M_{\mathcal{G}'}(g) = M_{\mathcal{G}}(g) - 1$  (where, of course,  $\infty - 1 = \infty$ ),
3. if  $M_{\mathcal{G}}(g) = 1$ ,  $g \notin \mathcal{G}'$ .

For a generalized configuration  $\mathcal{G}$ ,  $k \in \mathbb{N} \uplus \infty$ , and a generalized single configuration  $g$ , we write  $\mathcal{G} \uplus^k c$  for the generalized configuration  $\mathcal{G}'$  such that:

1. for all  $g' \neq g$ ,  $g' \in \mathcal{G}'$  if and only if  $g' \in \mathcal{G}$ . In that case  $M_{\mathcal{G}}(g') = M_{\mathcal{G}'}(g')$ , and
2. if  $g \in \mathcal{G}$ ,  $g \in \mathcal{G}'$  with  $M_{\mathcal{G}'}(g) = M_{\mathcal{G}}(g) + k$ ,
3. if  $g \notin \mathcal{G}$ ,  $g \in \mathcal{G}'$  with  $M_{\mathcal{G}'}(g) = k$ .

And, for two generalized configurations  $\mathcal{G}, \mathcal{G}'$ , with  $\mathcal{G}' = \{g'_1, \dots, g'_k\}$ ,

$$\mathcal{G} \uplus \mathcal{G}' = \mathcal{G} \uplus^{M_{\mathcal{G}'}(g'_1)} g'_1 \dots \uplus^{M_{\mathcal{G}'}(g'_k)} g'_k, \text{ and}$$

$$\mathcal{G} \subseteq \mathcal{G}' \text{ iff } \forall g \in \mathcal{G}, g \in \mathcal{G}' \text{ with } M_{\mathcal{G}}(g) \leq M_{\mathcal{G}'}(g).$$

For a set  $\mathcal{D}$  of generalized single configurations, we write also  $\mathcal{D}$  for the generalized configuration with underlying set  $\mathcal{D}$ , where all elements have multiplicity 1.

We consider the order relation  $\preceq$  on generalized configurations defined as follows:  $\mathcal{G} \preceq \mathcal{G}'$  if and only if  $\mathcal{G}'$  can be derived in a finite number of steps from  $\mathcal{G}$  by:

- increasing some coordinate of an element  $g \in \mathcal{G}$  (while keeping other coordinates invariant), or
- performing some union  $\mathcal{G} \uplus^k g$  for some generalized single configuration  $g$ , with  $k \in \mathbb{N} \uplus \infty$ .

Moreover, we assume for the sake of simplicity that the reduction steps consisting in increasing some coordinate are not redundant (i.e. the same coordinate of the same single configuration is increased at most once in the derivation), that the reduction steps consisting in adding some single configuration are not redundant either (i.e. a single configuration is added at most once in the derivation), and that the former steps occur before the latter in the derivation. Note that this assumption does not change the definition, but allows to simplify the presentation of the liftings below.

**Definition 3.2.** Liftings.

Let  $S = (G, T, m, v)$ , with  $G = (Q, A)$  be a VASSSJT. Let  $\mathcal{T}$  be a connected, directed acyclic graph with vertices labelled with generalized configurations on  $S$ . Let  $d_n$  be a vertex in  $\mathcal{T}$  labelled with  $\mathcal{G}_n$ . Let  $\mathcal{G}_i$  be a generalized configuration on  $S$ . We write  $\mathcal{G}_i \ll \mathcal{G}_n$  if and only if  $\mathcal{G}_i \prec \mathcal{G}_n$  and there exists a path  $d_i, \dots, d_n$  in  $\mathcal{T}$ , where  $d_i \neq d_n$  is labelled with  $\mathcal{G}_i$ . Assume  $\mathcal{G}_i \ll_{d_n} \mathcal{G}_n$  and let  $\mathcal{G}_i = \mathcal{G}'_1 \rightarrow \dots \rightarrow \mathcal{G}'_m = \mathcal{G}_n$  be one corresponding finite non-empty derivation.

The *single lifting* of  $(d_n, d_i)$  in  $\mathcal{T}$ , denoted as  $\#(d_n, d_i)$  is the following operation. For all  $k = 1 \dots m - 1$ ,

- If, in the derivation step  $\mathcal{G}'_k \rightarrow \mathcal{G}'_{k+1}$ , the coordinate  $x_j$  of some  $g = (l, (p, x)) \in \mathcal{G}'_k$  has strictly increased by a value  $a$ , replace it by  $\infty$  in  $\mathcal{G}'_{k+1}, \dots, \mathcal{G}'_m$ . The coordinate  $x_j$  is denoted as an *augmentation coordinate* of the lifting, and  $a$  is its *augmentation gap*.
- If, in the derivation step  $\mathcal{G}'_k \rightarrow \mathcal{G}'_{k+1}$ , a generalized single configuration  $g$  has been added, with multiplicity  $a \in \mathbb{N}$ , give it multiplicity  $\infty$  in  $\mathcal{G}'_{k+1}, \dots, \mathcal{G}'_m$ . The configuration  $g$  is denoted as an *augmentation configuration* of the lifting, and  $a$  is its *augmentation gap*.
- Label  $d_n$  with the resulting generalized configuration  $\mathcal{G}_n$ .

The *augmentation gap* of the lifting is the minimal non-zero augmentation gap of the augmentation coordinates and of the augmentation configurations of the lifting.

Note that two single liftings commute:  $\forall i, j \leq n, \#(\#(d_n, d_i), d_j) = \#(\#(d_n, d_j), d_i)$ . Now, let  $J = \{j = j_1, \dots, j_k \text{ such that } \mathcal{G}_j \ll \mathcal{G}_n\}$ , let  $D^0 = d_n$  and for  $l = 1 \dots k$ ,  $D^l = \#(D^{l-1}, d_{j_l})$ . Then, the *lifting* of  $d_n$ , is  $\#(d_n) = D^k$ . Note that, when  $J = \emptyset$ ,  $k = 0$  and  $\#(d_n) = d_n$ .

**Definition 3.3.** Karp and Miller Graph.

Let  $S = (G, T, m, v)$ , with  $G = (Q, A)$  be a VASSSJT and  $\mathcal{D}$  be a set of single configurations. The *Karp and Miller graph*  $\mathcal{T} = (V, E)$  on  $(S, \mathcal{D})$  is a labelled directed graph constructed inductively as follows, where vertices are labelled with generalized configurations, and edges with transitions of  $S$  or (null).

1.  $\mathcal{T}$  has exactly one vertex  $\sigma_0$  of indegree 0, labelled with  $\emptyset$
2. For every  $c \in \mathcal{D}$ , let  $\sigma_c$  be a new vertex labelled with  $\{c\}$ , where  $c$  has multiplicity 1. Let  $(\sigma_0, \sigma_c)$  be a new edge, labelled with (null).
3. Let  $\sigma_i$  labelled with  $\mathcal{G}_i$  be a vertex of  $\mathcal{T}$ . If there exists a path  $\sigma_0, \dots, \sigma_i$  in  $\mathcal{T}$  and  $\sigma_j \neq \sigma_i$  labelled with the same  $\mathcal{G}_i$  in this path,  $\sigma_i$  has outdegree 0. Otherwise,
4. Let  $\sigma_i$  labelled with  $\mathcal{G}_i$  be a vertex of  $\mathcal{T}$ . Let  $(p_i, x_i) \in \mathcal{G}_i$ ,  $t = (p_i, p_{i+1}) \in T$  be a regular transition such that  $x_{i+1} = x_i + v(t) \geq 0$ . Then, let  $\sigma_{i+1}$  be a new vertex in  $V$  labelled with  $\mathcal{G}_{i+1} = \mathcal{G}_i \setminus (p_i, x_i) \uplus^1 (p_{i+1}, x_{i+1})$ , and  $(\sigma_i, \sigma_{i+1})$  be a new edge in  $E$  labelled with  $t$ . Mark one occurrence of  $(p_i, x_i) \in \mathcal{G}_i$  as the *origin* of  $(\sigma_i, \sigma_{i+1})$ , one occurrence of  $(p_{i+1}, x_{i+1})$  in  $\mathcal{G}_{i+1}$  as the *destination* of  $(\sigma_i, \sigma_{i+1})$ , and perform the lifting  $\#(\sigma_{i+1})$  in  $(V \uplus \sigma_{i+1}, E \uplus (\sigma_i, \sigma_{i+1}))$  (where we assume that the lifting preserves the marking).
5. Let  $\sigma_i$  labelled with  $\mathcal{G}_i$  be a vertex of  $\mathcal{T}$ . Let  $(p_i, x_i) \in \mathcal{G}_i$ ,  $t = ((p_i, p_{i+1}), (p_i, p'_{i+1})) \in T$  be a split transition. Let  $x_{i+1} + x'_{i+1} = x_i$ , such that, for  $j = 1 \dots m$ ,  $x_{ij} = \infty \Rightarrow x_{i+1j}, x'_{i+1j} = \infty$   $x'_{i+1j} \in \{0, \infty\}$ ?. Then, let  $\sigma_{i+1}$  be a new vertex in  $V$  labelled with  $\mathcal{G}_{i+1} = \mathcal{G}_i \setminus (p_i, x_i) \uplus^1 (p_{i+1}, x_{i+1}) \uplus^1 (p'_{i+1}, x'_{i+1})$ ,  $(\sigma_i, \sigma_{i+1})$  be a new edge in  $E$  labelled with  $t$ . Mark one occurrence of  $(p_i, x_i) \in \mathcal{G}_i$  as the *origin* of  $(\sigma_i, \sigma_{i+1})$ , one occurrence of  $(p_{i+1}, x_{i+1})$  in  $\mathcal{G}_{i+1}$  as a *destination* of  $(\sigma_i, \sigma_{i+1})$ , one occurrence of  $(p'_{i+1}, x'_{i+1})$  in  $\mathcal{G}_{i+1}$  as a *destination* of  $(\sigma_i, \sigma_{i+1})$ , and perform the lifting  $\#(\sigma_{i+1})$  in  $(V \uplus \sigma_{i+1}, E \uplus (\sigma_i, \sigma_{i+1}))$  (where we assume that the lifting preserves the marking).
6. Let  $\sigma_i$  labelled with  $\mathcal{G}_i$  be a vertex of  $\mathcal{T}$ . Let  $(p_i, x_i) \in \mathcal{G}_i$ ,  $(p'_i, x'_i) \in \mathcal{G}_i$ ,  $t = ((p_i, p_{i+1}), (p'_i, p_{i+1})) \in T$  be a join transition. Let  $x_{i+1} = x_i + x'_i$ . Then, let  $\sigma_{i+1}$  be a new vertex in  $V$  labelled with  $\mathcal{G}_{i+1} = \mathcal{G}_i \setminus (p_i, x_i) \setminus (p'_i, x'_i) \uplus^1 (p_{i+1}, x_{i+1})$ ,  $(\sigma_i, \sigma_{i+1})$  be a new edge in  $E$  labelled with  $t$ . Mark one occurrence

of  $(p_i, x_i) \in \mathcal{G}_i$  as an *origin* of  $(\sigma_i, \sigma_{i+1})$ , one occurrence of  $(p'_i, x'_i) \in \mathcal{G}_i$  as an *origin* of  $(\sigma_i, \sigma_{i+1})$ , one occurrence of  $(p_{i+1}, x_{i+1})$  in  $\mathcal{G}_{i+1}$  as the *destination* of  $(\sigma_i, \sigma_{i+1})$ , and perform the lifting  $\#(\sigma_{i+1})$  in  $(V \uplus \sigma_{i+1}, E \uplus (\sigma_i, \sigma_{i+1}))$  (where we assume that the lifting preserves the marking).

7. Let  $\sigma_i$  labelled with  $\mathcal{G}_i$ ,  $\sigma_j$  labelled with  $\mathcal{G}_j$  be two vertices of  $\mathcal{T}$ . Let  $(p_i, x_i) \in \mathcal{G}_i$ ,  $(p_j, x_j) \in \mathcal{G}_j$ ,  $t = ((p_i, p_{i+1}), (p_j, p_{i+1})) \in T$  be a join transition. Let  $x_{i+1} = x_i + x_j$ . Then, let  $\sigma_{i+1}$  be a new vertex in  $V$  labelled with  $\mathcal{G}_{i+1} = (\mathcal{G}_i \setminus (p_i, x_i)) \uplus (\mathcal{G}_j \setminus (p_j, x_j)) \uplus^1 (p_{i+1}, x_{i+1})$ ,  $(\sigma_i, \sigma_{i+1})$  and  $(\sigma_j, \sigma_{i+1})$  be two new edges in  $E$  labelled with  $t$ . Mark one occurrence of  $(p_i, x_i) \in \mathcal{G}_i$  as an *origin* of  $(\sigma_i, \sigma_{i+1})$ , one occurrence of  $(p_j, x_j) \in \mathcal{G}_j$  as an *origin* of  $(\sigma_j, \sigma_{i+1})$ , one occurrence of  $(p_{i+1}, x_{i+1})$  in  $\mathcal{G}_{i+1}$  as the *destination* of  $(\sigma_i, \sigma_{i+1})$  and of  $(\sigma_j, \sigma_{i+1})$ , and perform the lifting  $\#(\sigma_{i+1})$  in  $(V \uplus \sigma_{i+1}, E \uplus (\sigma_i, \sigma_{i+1}))$  (where we assume that the lifting preserves the marking).

The inductive construction above halts when no new vertex can be added with these rules. Note that, by construction,  $\mathcal{T}$  is connected and acyclic.

### 3.2. First results

**Proposition 3.4.** *The following statements are true:*

1. Let  $u_n \in (\mathbb{N} \uplus \infty)^m$ ,  $n \in \mathbb{N}$ , be an infinite sequence of  $m$ -tuples for some  $m \in \mathbb{N}$ . Then, there exists an infinite sub-sequence  $u'_n$ ,  $n \in \mathbb{N}$  of  $u_n$  that is increasing for the order relation  $\leq$ .
2. Let  $\mathcal{G}_n$ ,  $n \in \mathbb{N}$ , be an infinite sequence of generalized configurations. Then, there exists an infinite sub-sequence  $\mathcal{G}'_n$ ,  $n \in \mathbb{N}$  of  $\mathcal{G}_n$  that is increasing for the order relation  $\preceq$ .

*Proof.*

1. Let  $u_n \in (\mathbb{N} \uplus \infty)^m$ ,  $n \in \mathbb{N}$ , be an infinite sequence of  $m$ -tuples for some  $m \in \mathbb{N}$ . Then, two cases arise:
  - The sequence  $u_n$  is bounded by some  $u \in \mathbb{N}^m$ . In that case, there exists  $u' \leq u$  such that  $u'$  has infinitely many occurrences in  $u_n$ . The restriction of  $u_n$  to these occurrences is an infinite sequence that is (non-strictly) increasing.
  - The sequence  $u_n$  is unbounded. By induction on  $m$ :
    - (a) if  $m = 1$ , if  $\infty$  has infinitely many occurrences in  $u_n$ : the restriction of  $u_n$  to these occurrences is an infinite sequence that is (non-strictly) increasing. if  $\infty$  has only finitely many occurrences in  $u_n$ : there exists an infinite unbounded sub-sequence  $u'_n \in \mathbb{N}$ ,  $n \in \mathbb{N}$  with values in  $\mathbb{N}$ . Then,
      - i. if the sequence  $u'_n$  is bounded by some  $u \in \mathbb{N}^m$ , it admits as above an infinite sequence that is (non-strictly) increasing, and
      - ii. if the sequence  $u'_n$  is unbounded, it admits an infinite sub-sequence that is (strictly) increasing.
    - (b) Assume the result holds for  $m - 1$ . Consider by induction an infinite subsequence  $u'_n$ ,  $n \in \mathbb{N}$  of  $u_n$  such that its projection on the first  $m - 1$  coordinates is increasing, and apply the result for  $m = 1$  on the last coordinate of  $u'_n$ .
2. Let now  $\mathcal{G}_n$ ,  $n \in \mathbb{N}$ , be an infinite sequence of generalized configurations. Let  $\mathcal{G}'_n$ ,  $n \in \mathbb{N}$ , be an infinite sub-sequence of  $\mathcal{G}_n$ , such that the size of the underlying sets of  $\mathcal{G}'_n$ , is an infinite sequence of integer values that is increasing. Let us show that, for any  $\mathcal{G}'_i$  in the sequence  $\mathcal{G}'_n$ , there exists  $\mathcal{G}'_j$ ,  $i < j$  such that  $\mathcal{G}'_i \preceq \mathcal{G}'_j$ . For  $q \in Q$ , let  $v(q) \in \mathbb{N}^{|Q|}$  be the unary encoding of  $q$ . Let  $\mathcal{G}'_i = \{(q_1^i, x_1^i), \dots, (q_l^i, x_l^i)\}$  in the sequence, and, for  $j \geq i$ , let  $\mathcal{G}'_j = \{(q_1^j, x_1^j), \dots, (q_{l'}^j, x_{l'}^j)\}$  with, since the size of the sets increase,  $l' \geq l$ . Consider now only the  $l$  first elements of  $\mathcal{G}'_j$  in  $v(\mathcal{G}'_j) = (v(q_1^j), x_1^j, M(\mathcal{G}'_j)(q_1^j, x_1^j), \dots, v(q_l^j), x_l^j, M(\mathcal{G}'_j)(q_l^j, x_l^j)) \in (\mathbb{N} \uplus \infty)^{l \cdot (|Q| + m + 1)}$ . Consider now the sequence  $v(\mathcal{G}'_j)$ ,  $j \geq i$ . Then, by Proposition 3.4, 1), there exists an infinite sub-sequence  $v(\mathcal{G}''_j)$ ,  $j \geq i$  that is increasing for the order relation  $\leq$  on  $(\mathbb{N} \uplus \infty)^{l \cdot (|Q| + m + 1)}$ , and let  $\mathcal{G}''_n$ ,  $n \in \mathbb{N}$  be the corresponding sequence of generalized configurations. Then, by construction of  $v(\mathcal{G}''_j)$ ,  $\mathcal{G}'_i \preceq \mathcal{G}''$  for any  $\mathcal{G}'' = \mathcal{G}''_j$  in the sequence  $\mathcal{G}''_n$ . It follows that, for any  $\mathcal{G}'_i$  in the sequence  $\mathcal{G}'_n$ , there exists  $\mathcal{G}'_j$ ,  $i < j$  such that  $\mathcal{G}'_i \preceq \mathcal{G}'_j$ . Therefore, there exists an infinite sub-sequence of  $\mathcal{G}'_n$ ,  $n \in \mathbb{N}$  that is increasing for the order relation  $\preceq$ .

□

**Theorem 3.5.** *Let  $S = (G, T, m, v)$ , with  $G = (Q, A)$  be a VASSSJT and  $\mathcal{D}$  be a finite set of single configurations. The Karp and Miller graph  $\mathcal{T}$  on  $(S, \mathcal{D})$  is finite, and can be effectively constructed.*

*Proof.* Since  $\mathcal{T}$  is a connected, directed acyclic graph, it is finite if and only if its depth-first search tree is finite. By construction, every node in  $\mathcal{T}$  has a finite number of sons. Assume  $\mathcal{T}$  is infinite: then, by Koenig's Lemma, its depth-first search tree has an infinite branch. Denote by  $D = \mathcal{G}_n$ ,  $n \in \mathbb{N}$  the corresponding infinite derivation sequence of generalized configurations. By Proposition 3.4, 2), let  $\mathcal{G}'_n$ ,  $n \in \mathbb{N}$  be an infinite sub-sequence of  $\mathcal{G}_n$  that is increasing for the order relation  $\preceq$ .

Let  $\mathcal{G}'_i$  be one element in the sequence  $\mathcal{G}'_n$ , and define  $\mathcal{G}'_i = I(\mathcal{G}'_i) \uplus F(\mathcal{G}'_i)$ , where:

- $I(\mathcal{G}'_i) = \{g \in \mathcal{G}'_i : M_{\mathcal{G}'_i}(g) = \infty\}$ , with  $M_{I(\mathcal{G}'_i)} = M_{\mathcal{G}'_i}(g)$  for all  $g$ , and
- $F(\mathcal{G}'_i) = \{g \in \mathcal{G}'_i : M_{\mathcal{G}'_i}(g) \in \mathbb{N}\}$ , with  $M_{F(\mathcal{G}'_i)} = M_{\mathcal{G}'_i}(g)$  for all  $g$ .

Let  $Q = \{q_1, \dots, q_{|Q|}\}$ , let  $\mathcal{G}'_i = \{(q_1^i, x_1^i), \dots, (q_l^i, x_l^i)\}$ ,  $l \in \mathbb{N}$ , and define now:

- for  $j = 1, \dots, m$ , for  $q \in Q$ ,  $v_{j,q}(\mathcal{G}'_i) = (\sum_{(q,x) \in I(\mathcal{G}'_i): x_j \neq \infty} x_j, \sum_{(q,x) \in F(\mathcal{G}'_i): x_j \neq \infty} x_j \cdot M_{\mathcal{G}'_i}(q, x)) \in \mathbb{N}^2$ , and
- $v(\mathcal{G}'_i) = (v_{1,q_1}(\mathcal{G}'_i), v_{2,q_1}(\mathcal{G}'_i), \dots, v_{m,q_1}(\mathcal{G}'_i), v_{1,q_2}(\mathcal{G}'_i), \dots, v_{m,q_2}(\mathcal{G}'_i), \dots, v_{m,q_{|Q|}}(\mathcal{G}'_i)) \in \mathbb{N}^{2 \cdot m \cdot |Q|}$ .

Consider now the infinite sequence  $v(\mathcal{G}'_n)$ ,  $n \in \mathbb{N}$ . Then, by Proposition 3.4, 1), let  $v(\mathcal{G}''_n)$ ,  $n \in \mathbb{N}$  be an infinite sub-sequence of  $v(\mathcal{G}'_n)$  that is increasing for the order relation  $\leq$  on  $\mathbb{N}^{2 \cdot m \cdot |Q|}$ , and let  $\mathcal{G}''_n$ ,  $n \in \mathbb{N}$  be the corresponding infinite sequence of generalized configurations.

Two cases arise:

1.  $v(\mathcal{G}''_n)$  is bounded: there exists  $i \in \mathbb{N}$  such that, for all  $j > i$ ,  $v(\mathcal{G}''_i) = v(\mathcal{G}''_j)$ . Since the number of possible generalized configurations  $\mathcal{G}''$  with  $v(\mathcal{G}'') = v(\mathcal{G}''_i)$  is bounded, it follows that there exists an infinite sub-sequence of  $\mathcal{G}''_n$  of identical generalized configurations, which is contradictory with the construction of the Karp and Miller tree (case 3) of the construction in Definition 3.3).
2.  $v(\mathcal{G}''_n)$  is not bounded: there exists yet again an infinite subsequence  $v(\mathcal{G}^{(3)}_n)$ ,  $n \in \mathbb{N}$ , which is strictly increasing for the order relation  $\leq$  on  $\mathbb{N}^{2 \cdot m \cdot |Q|}$ . Consider the corresponding infinite sequence  $\mathcal{G}^{(3)}_n$ ,  $n \in \mathbb{N}$  of generalized configurations. For any  $i < j \in \mathbb{N}$ ,  $v(\mathcal{G}^{(3)}_i) < v(\mathcal{G}^{(3)}_j)$ , and  $\mathcal{G}^{(3)}_i \preceq \mathcal{G}^{(3)}_j$ . Then,  $F(\mathcal{G}^{(3)}_i) = F(\mathcal{G}^{(3)}_j)$ , otherwise, by construction, a single lifting needs to be performed on the finite part of  $\mathcal{G}^{(3)}_j$ , which would make  $v(\mathcal{G}^{(3)}_j)$  strictly decrease on its coordinates corresponding to  $F(\mathcal{G}^{(3)}_j)$ . Similarly, any derivation of  $\mathcal{G}^{(3)}_i \preceq \mathcal{G}^{(3)}_j$  consists in adding at least one single configuration with infinite multiplicity (otherwise, any derivation consisting only in increasing values would trigger a lifting that would make  $v$  strictly decrease). Consider now the infinite sequence of single configuration with infinite multiplicity added in each of the derivations  $\mathcal{G}^{(3)}_n \preceq \mathcal{G}^{(3)}_{n+1}$  for  $n \in \mathbb{N}$ . By Proposition 3.4, 1), it has an infinite sub-sequence that is increasing for the order  $\leq$  on  $(\mathbb{N} \uplus \infty)^m$ . If this infinite sub-sequence is bounded, it contains yet one more infinite sub-sequence of identical single configurations, which contradicts the construction of the graph (case 3) of the construction in Definition 3.3); if it is unbounded, it contains yet one more infinite sub-sequence that is strictly increasing, which contradicts the fact that the  $\mathcal{G}_i$  are lifted at each step in the derivation sequence (case 4), 5), 6) and 7) of the construction in Definition 3.3).

It follows by contradiction that the depth-first search tree of the Karp and Miller graph has no infinite branch, hence it is finite. The constructibility follows from its inductive definition. □

**Theorem 3.6.** *Let  $S = (G, T, m, v)$ , with  $G = (Q, A)$  be a VASSSJT and  $\mathcal{D}$  be a finite set of single configurations. Let  $\sigma$  be a node of the Karp and Miller graph  $\mathcal{T}$  on  $(S, \mathcal{D})$  labelled with  $\mathcal{G}^\sigma$ .*

*Let  $\mathcal{D}_\sigma \subseteq \mathcal{D}$  such that, for all  $c \in \mathcal{D}_\sigma$ , there exists a path  $\sigma_c, \dots, \sigma$  in  $\mathcal{T}$ .*

*Then, for any  $N \in \mathbb{N}$ , there exists a promenade  $P_\sigma^N$  of  $S$  such that:*



- the set of labels of the vertices of  $P_\sigma^N$  of in-arity 0 is  $\mathcal{D}_\sigma$ ,
- for all  $g = (q, x) \in \mathcal{G}^\sigma$  of multiplicity  $M_{\mathcal{G}^\sigma}(g) = k \in \mathbb{N}$ , there exists a set  $V(g) = \{v_1, \dots, v_k\}$  of vertices of  $P_\sigma^N$  of out-arity 0, where, for any  $v_i \in V(g)$ ,  $v_i$  is labelled with  $(q, t^i)$  where, for all  $j = 1, \dots, m$ :
  1.  $t_j^i \geq N$  if  $x_j = \infty$ , and
  2.  $t_j^i = x_j$  otherwise,
- for all  $g = (q, x) \in \mathcal{G}^\sigma$  of multiplicity  $M_{\mathcal{G}^\sigma}(g) = \infty$ , there exists a set  $V(g) = \{v_1, \dots, v_k\}$ ,  $k \geq N$ , of vertices of  $P_\sigma^N$  of out-arity 0, where, for any  $v_i \in V(g)$ ,  $v_i$  is labelled with  $(q, t^i)$  where, for all  $j = 1, \dots, m$ :
  1.  $t_j^i \geq N$  if  $x_j = \infty$ , and
  2.  $t_j^i = x_j$  otherwise,
- and the union of the sets  $V(g)$  above is the set of vertices of  $P_\sigma^N$  of out-arity 0.

*Proof.* For the sake of simplicity, we assume in the proof that all directed acyclic graphs are oriented from top to bottom.

Let  $\Sigma(\sigma) = \{\sigma_1, \dots, \sigma_t\}$  be the set of vertices in  $\mathcal{T} \setminus \sigma_0$  (recall Definition 3.3:  $\sigma_0$  is the only vertex of  $\mathcal{T}$  of in-degree 0) above  $\sigma$  (i.e, such that, for  $j = 1, \dots, t$ , there exists a path from  $\sigma_j$  to  $\sigma$  in  $\mathcal{T}$ ). Assume also that, for  $j = 1, \dots, t$ ,  $\sigma_j$  is labelled with  $\mathcal{G}^j$ . Let  $\mathcal{T}_\sigma$  be the restriction of  $\mathcal{T}$  to  $\Sigma(\sigma)$ , as given in Figure 2 for the VASSSJT  $S$  of Figure 1.

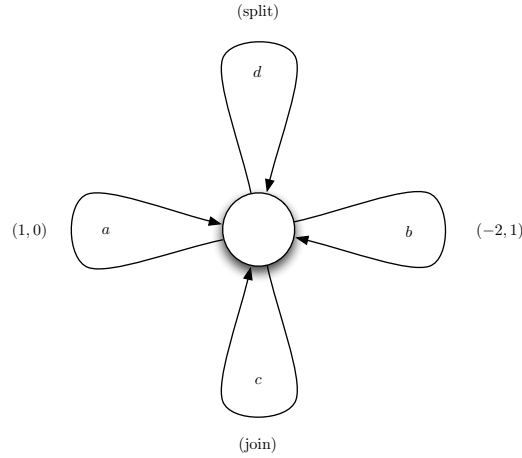


Figure 1: An example of VASSSJT  $S$ , with only one state, two regular transitions  $a$  with  $v(a) = (1, 0)$  and  $b$  with  $v(b) = (-2, 1)$ , a join transition  $c$  and a split transition  $d$ .

### Splitting generalized configurations into single ones:

Denote now by  $\mathcal{T}'_\sigma$  the directed acyclic graph defined inductively as follows, as in figure 3:

1. To  $\sigma$ , we associate a set  $v(\sigma)$  of nodes of out-degree 0, where, for any  $g \in \mathcal{G}^\sigma$ ,  $v(\sigma, g) \in v(\sigma)$  is labelled with  $g^{M_{\mathcal{G}^\sigma}(g)}$ .
2. Let  $v(\sigma_k) \in \mathcal{T}'_\sigma$ , where  $\sigma_k \in \mathcal{T}_\sigma$  has only one parent vertex  $\sigma_i \in \mathcal{T}_\sigma$ . Assume  $(\sigma_i, \sigma_k)$  is labelled with a transition  $t$ . Then, we let  $v(\sigma_i)$  be a new set of nodes in  $\mathcal{T}'_\sigma$ , where, for  $g_i \in \mathcal{G}^i$ ,

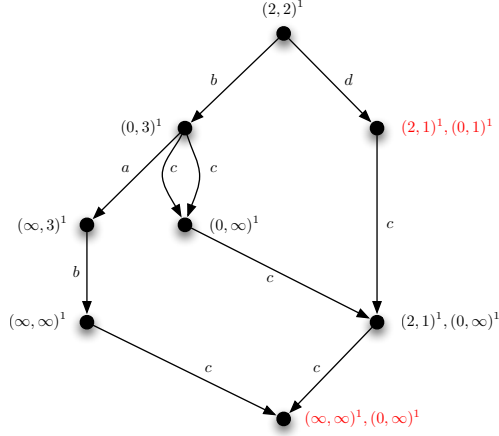


Figure 2: The graph  $\mathcal{T}_\sigma$  for  $(S, \{(2,2)\})$ , with one vertex  $\sigma$  labelled with  $(\infty, \infty)^1, (0, \infty)^1$ .

- If  $g_i$  is not marked in  $\mathcal{G}^i$  as the origin of  $(\sigma_i, \sigma_k)$ ,  $g_i$  yields  $g_k \in \mathcal{G}^k$  by the (possibly empty) lifting of  $\sigma_k$ . Then,  $v(\sigma_i, g_i)$  is a new node in  $v(\sigma_i)$  labelled with  $g_i^{M_{\mathcal{G}^i}(g_i)}$ , and  $(v(\sigma_i, g_i), v(\sigma_k, g_k))$  is a new edge labelled with  $(id)$ , with  $v(\sigma_k, g_k) \in v(\sigma_k)$ .
  - if  $g_i$  is marked in  $\mathcal{G}^i$  as the origin of  $(\sigma_i, \sigma_k)$ ,  $v(\sigma_i, g_i)$  is a new node in  $v(\sigma_i)$  labelled with  $g_i^{M_{\mathcal{G}^i}(g_i)}$ . Let  $g_k \in \mathcal{G}^k$  marked as the destination of  $(\sigma_i, \sigma_k)$ . Then,  $(v(\sigma_i, g_i), v(\sigma_k, g_k))$  is a new edge labelled with  $t$ , with  $v(\sigma_k, g_k) \in v(\sigma_k)$ .
3. Let  $v(\sigma_k) \in \mathcal{T}'_\sigma$ , where  $\sigma_k \in \mathcal{T}_\sigma$  has only two parent vertices  $\sigma_i \in \mathcal{T}_\sigma$  and  $\sigma_j \in \mathcal{T}_\sigma$ . Assume  $(\sigma_i, \sigma_k)$  and  $(\sigma_j, \sigma_k)$  are labelled with the same transition  $t$ . Then, we let  $v(\sigma_i)$  and  $v(\sigma_j)$  be two new sets of nodes in  $\mathcal{T}'_\sigma$ , where, for  $g_i \in \mathcal{G}^i$  (respectively  $g_j \in \mathcal{G}^j$ ),
- If  $g_i$  is not marked in  $\mathcal{G}^i$  as the origin of  $(\sigma_i, \sigma_k)$ ,  $g_i$  yields  $g_k \in \mathcal{G}^k$  by the (possibly empty) lifting of  $\sigma_k$ . Then,  $v(\sigma_i, g_i)$  is a new node in  $v(\sigma_i)$  labelled with  $g_i^{M_{\mathcal{G}^i}(g_i)}$ , and  $(v(\sigma_i, g_i), v(\sigma_k, g_k))$  is a new edge labelled with  $(id)$ , with  $v(\sigma_k, g_k) \in v(\sigma_k)$ .
  - if  $g_i$  is marked in  $\mathcal{G}^i$  as the origin of  $(\sigma_i, \sigma_k)$ ,  $v(\sigma_i, g_i)$  is a new node in  $v(\sigma_i)$  labelled with  $g_i^{M_{\mathcal{G}^i}(g_i)}$ . Let  $g_k \in \mathcal{G}^k$  marked as the destination of  $(\sigma_i, \sigma_k)$ . Then,  $(v(\sigma_i, g_i), v(\sigma_k, g_k))$  is a new edge labelled with  $t$ , with  $v(\sigma_k, g_k) \in v(\sigma_k)$ .
- (respectively,
- If  $g_j$  is not marked in  $\mathcal{G}^j$  as the origin of  $(\sigma_j, \sigma_k)$ ,  $g_j$  yields  $g_k \in \mathcal{G}^k$  by the (possibly empty) lifting of  $\sigma_k$ . Then,  $v(\sigma_j, g_j)$  is a new node in  $v(\sigma_j)$  labelled with  $g_j^{M_{\mathcal{G}^j}(g_j)}$ , and  $(v(\sigma_j, g_j), v(\sigma_k, g_k))$  is a new edge labelled with  $(id)$ , with  $v(\sigma_k, g_k) \in v(\sigma_k)$ .
  - if  $g_j$  is marked in  $\mathcal{G}^j$  as the origin of  $(\sigma_j, \sigma_k)$ ,  $v(\sigma_j, g_j)$  is a new node in  $v(\sigma_j)$  labelled with  $g_j^{M_{\mathcal{G}^j}(g_j)}$ . Let  $g_k \in \mathcal{G}^k$  marked as the destination of  $(\sigma_j, \sigma_k)$ . Then,  $(v(\sigma_j, g_j), v(\sigma_k, g_k))$  is a new edge labelled with  $t$ , with  $v(\sigma_k, g_k) \in v(\sigma_k)$ .

Note that, to a node  $\sigma_i \in \mathcal{T}_\sigma$ , we associate several sets  $v(\sigma_i) \in \mathcal{T}'_\sigma$ . The sets  $v(\sigma_i) \in \mathcal{T}'_\sigma$  are a partition of  $\mathcal{T}'_\sigma$ , denoted as  $V_\sigma$ . Note also that the graph  $\mathcal{T}''_\sigma$  obtained from  $\mathcal{T}'_\sigma$  by merging all nodes of all sets  $v(\sigma_i)$  is a tree, whose root is the node obtained by merging all nodes of  $v(\sigma)$ .

The promenade  $P_\sigma$ :

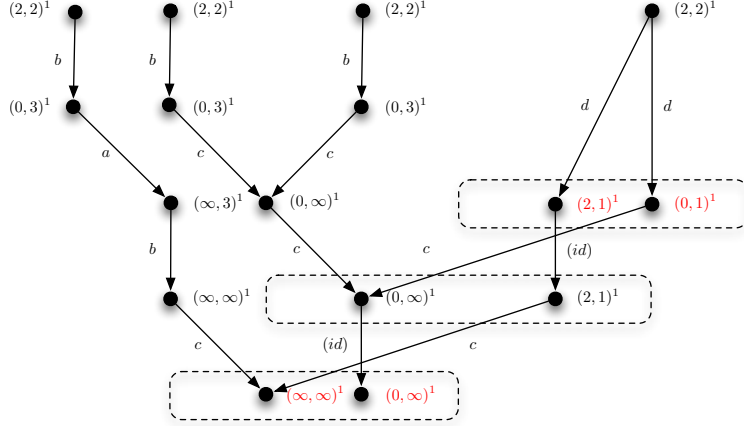


Figure 3: The graph  $T'_\sigma$  obtained from  $T_\sigma$ . The dotted boxes are the sets  $v(\sigma_i)$  of cardinality more than 2.

From now on, assume in first step that no generalized single configuration appears in  $T'_\sigma$  with multiplicity  $\infty$ . The case where  $\infty$  multiplicities occur will be treated in a second step.

Under this assumption, the graph  $T'_\sigma$  induces naturally a (non-necessarily positive) promenade  $P_\sigma$  of  $S$  as in Figure 4 (where, without loss of generality, we extend the notion of promenade with  $(id)$  transitions), together with a partition  $V'_\sigma$  of its nodes, by inductively firing from top to bottom the transitions labeling the edges of  $T'_\sigma$ , starting from the sets  $v(\sigma_i)$  with no in-going edge, corresponding to single configurations in  $\mathcal{D}$ . The only ambiguous case is when firing a split transition on a single configuration in  $P_\sigma$  corresponding to a single generalized configuration in  $T'_\sigma$  with  $\infty$  coordinates. In this case, by convention, the corresponding finite coordinates  $z_i$  of the single configuration in  $P_\sigma$  are split into  $z'_i$  and  $z''_i$  with  $|z'_i - z''_i| \leq 1$ . For a set  $v \in V_\sigma$ , denote by  $P(v)$  its image in  $P_\sigma$  by the construction above. Then, the image  $P(V_\sigma)$  of  $V_\sigma$  by the inductive construction above is a partition of the vertices of  $P_\sigma$ . Note that, as for  $T'_\sigma$ , the graph obtained from  $P_\sigma$  by merging all nodes that belong to the same set  $P(v) \in P(V_\sigma)$  into a single node is a tree.

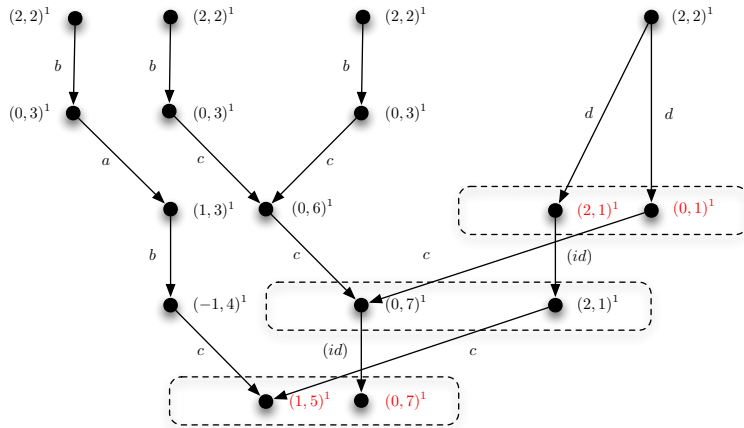


Figure 4: The promenade  $P_\sigma$  corresponding to the graph  $T'_\sigma$ .

Tree property:

Let  $P$  be a promenade of  $S$ , Let  $V$  be a partition of the vertices of  $P$ . We say that  $(P, V)$  satisfies the tree property if and only if the graph obtained from  $P$  by contracting all nodes that belong to a same set  $v \in V$  into a single node is a tree. Now, from  $P_\sigma$  and  $P(V_\sigma)$ , we will construct inductively a promenade that satisfies the theorem, together with a partition of its node, that satisfies the tree property.

**Repeating loops:**

Let  $P$  be a promenade of  $S$ ,  $V$  be a partition of the vertices of  $P$  such that  $(P, V)$  satisfies the tree property. Let  $v_i$  above  $v_j \in V$ . Denote by  $c_i$  the configuration of  $S$  labeling  $v_i$  (i.e,  $c_i$  is the multiset of the labels of the nodes in  $v_i$ ), and by  $c_j$  the configuration of  $S$  labeling  $v_j$ , and assume that  $c_j$  can be obtained from  $c_i$  by only modifying the values at some coordinates of some single configurations.

Then, the  $k$ -repetition of the chain  $v_i \cdots v_j$  in  $P$ , for  $k > 1$  is the promenade  $P_{i,j}^k$  obtained from  $P$  by:

1. isolating the subgraph  $P_{i,j}$  of  $P$  that is above  $v_j$ , and not above  $v_i$ . This is well defined since  $(P, V)$  satisfies the tree property. Let  $P_{i,j}^1, \dots, P_{i,j}^k$  be  $k$  copies of  $P_{i,j}$ . Denote by  $v_i^t$  the vertices of  $P_{i,j}^t$  corresponding to  $v_i \in P_{i,j}$ , and, similarly by  $v_j^t$  the vertices of  $P_{i,j}^t$  corresponding to  $v_j \in P_{i,j}$ , for  $t = 1, \dots, k$ .
2. identifying  $v_i^1$  with  $v_i$ ,  $v_j^t$  with  $v_j^{t+1}$  for  $t = 1, \dots, k-1$ , and  $v_j^k$  with  $v_j$ .
3. updating the labeling of the vertices from the top to the bottom, accordingly to the firing of the transitions.

Note that  $P_{i,j}^k$  admits a partition  $V_{i,j}^k$  of its vertices, which is the image of  $V$  by the construction above. Note also that  $(P_{i,j}^k, V_{i,j}^k)$  satisfies the tree property. For an illustration, see Figure 5.

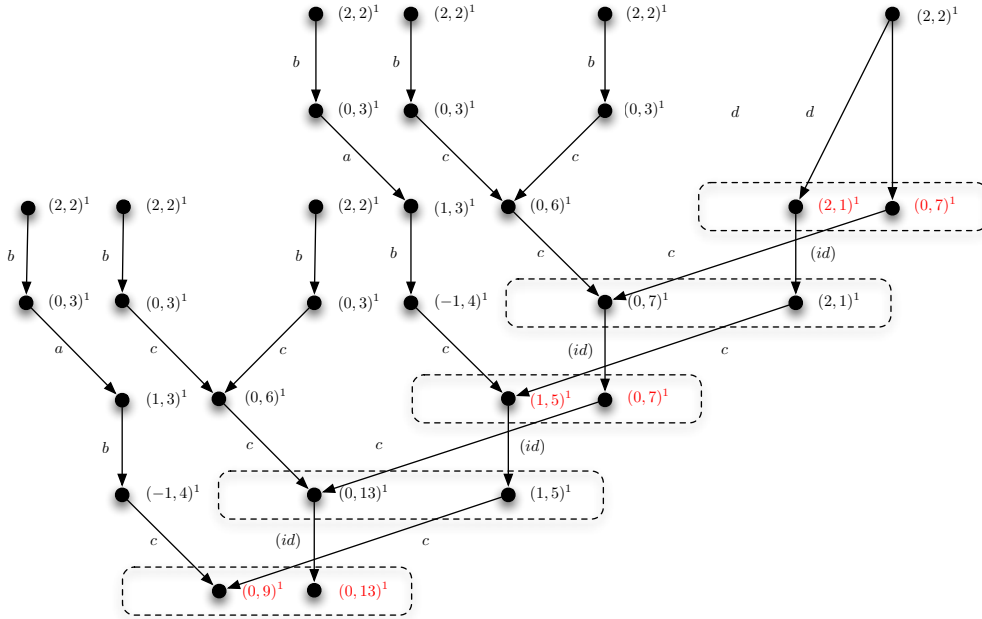


Figure 5: A 2-repetition on the promenade  $P_\sigma$ .

Now, Consider the promenade  $P_\sigma$ , and let  $P(v_i), P(v_j) \in P(V_\sigma)$ , such that

1.  $P(v_i)$  is above  $P(v_j)$  in  $P_\sigma$ , and

2.  $v_i \in \mathcal{T}'_\sigma$  is labelled with the generalized configuration  $g_i$ ,  $v_j \in \mathcal{T}'_\sigma$  is labelled with the generalized configuration  $g_j$ , with  $g_i \ll g_j$ .

Since no  $\infty$  multiplicity occurs in  $\mathcal{T}'_\sigma$ ,  $g_j$  is necessarily obtained from  $g_i$  only by augmenting values for some coordinates. By construction, these augmentation coordinates are labelled with  $\infty$  in  $g_j$ , and not in  $g_i$  (i.e. a lifting has been performed). Let  $a$  be the augmentation gap of this lifting. Assume  $P(v_i)$  is labelled with the configuration  $c_i$ , and  $P(v_j)$  by  $c_j$ . Then, remark the following:

- To any  $\infty$  value for some coordinate in  $g_i$  corresponds a finite value  $t_i$  for the same coordinate in  $c_i$ . This  $\infty$  value in  $g_i$  is inherited by construction on the same coordinate in  $g_j$ , and a finite value  $t_j$  corresponds to it in  $c_j$ . It may happen that  $t_j < t_i$ . It may even happen that  $t_j < 0$ . We denote  $t_i$  and  $t_j$  as finite coordinates of infinite heredity. See Figure 5 for an example.
- To any  $\infty$  augmentation coordinate in  $g_j$  corresponds a finite value  $t_j$  for the same coordinate in  $c_j$ . Since it is an augmentation coordinate of the lifting, this coordinate has a finite value  $t_i$  in  $g_i$  and in  $c_i$ , and  $t_i < t_j$ . See also Figure 5.

Now, in order to raise the augmentation coordinates of this lifting to an arbitrary level  $N \in \mathbb{N}$ , it suffices to perform the  $k$ -repetition of the chain  $P(v_i) \cdots P(v_j)$  in  $P_\sigma$ , for  $k = \lceil \frac{N}{a} \rceil$ . Of course, this  $k$ -repetition may decrease the values of the finite coordinates of infinite heredity, even to negative levels; it is therefore necessary to ensure that these finite values are high enough when performing this  $k$ -repetition, i.e. have been already leveled up by another  $k'$ -repetition above.

This yields the following inductive construction of a promenade of  $S$ , starting from  $P_\sigma$ :

**The algorithm building the promenade:**

Consider the promenade  $P$  returned by the following algorithm  $A$ :

```

input  $P_\sigma, N \in \mathbb{N}$ 
 $P \leftarrow P_\sigma$ 
 $N_{local} \leftarrow N$ 
For all  $v(\sigma_j) \in \mathcal{T}'_\sigma$ , from bottom to top, do
  For all  $v(\sigma_i) \in \mathcal{T}'_\sigma$  above  $v(\sigma_j)$ , from bottom to top, do
    Let  $g_i$  be the label of  $\sigma_i$ 
    Let  $g_j$  be the label of  $\sigma_j$ 
    If  $g_i \ll g_j$ , let  $a_{i,j}$  be the augmentation gap of the lifting
     $P \leftarrow$  the  $k$ -repetition of the chain  $P(v(\sigma_i)) \cdots P(v(\sigma_j))$  in  $P$ , with  $k = \lceil \frac{N_{local}}{a_{i,j}} \rceil$ 
     $N_{local} \leftarrow N_{local} + D_{i,j}$ , where  $D_{i,j}$  is the maximal decrease of finite coordinates
      of infinite heredity induced by the  $k$ -repetition of the step above
    end if
  end for
end for
return  $P$ .
```

Then, clearly,  $P$  satisfies Theorem 3.6.

### Introducing $\infty$ multiplicities

Augmentation multiplicities in  $\mathcal{T}'_\sigma$  can be leveled up to an arbitrary level in a promenade  $P$  exactly the same way as augmentation coordinates, i.e. by performing as many  $k$ -repetition as needed. However, the situation for  $\infty$  multiplicities in  $\mathcal{T}'_\sigma$ , and their respective counterparts in the promenade  $P_\sigma$  is more complicated than that of  $\infty$  coordinates: as noted above, it may happen that coordinates of  $\infty$  heredity in  $P_\sigma$  are decreased in a  $k$ -repetition, and reach negative values. The same applies for multiplicities of  $\infty$  heredity, yet it is for now unclear how to deal with "negative multiplicities", for what that means. Another complication comes from the fact that, to a single generalized configuration in  $\mathcal{T}'_\sigma$  of multiplicity greater than 1, several different single configurations (with different values for coordinates of  $\infty$  heredity) can correspond in the promenade.

One way to circumvent these difficulties is to merge all different configurations of the promenade corresponding to the same generalized configuration in  $\mathcal{T}'_\sigma$  into only one, by arbitrarily decreasing coordinates of  $\infty$  heredity to their minimal values among these different configurations. The one single configuration obtained can then be equipped with a "pseudo" multiplicity that we allow to take negative values. Such "pseudo" promenades with negative multiplicities will be used as intermediate steps in the inductive algorithm  $A$ , with the proviso that the resulting pseudo promenade has only positive multiplicities. Then, from this pseudo promenade, we will derive a promenade satisfying the theorem by undoing the merging of different configurations.

The definition of the "pseudo"-promenade  $Q_\sigma$  follows:

1. Let  $z \in v(\sigma_k) \in \mathcal{T}'_\sigma$  of indegree 0, labelled with the single configuration  $c_z$ . Then,  $v(\sigma_k) = \{z\}$ . To  $z$ , we associate  $Q(z) \in Q_\sigma$  labelled with  $c_z$ .
2. Let  $z \in v(\sigma_k) \in \mathcal{T}'_\sigma$  of indegree greater than 1. Then, to  $z$  in  $\mathcal{T}'_\sigma$ , we associate  $Q(z)$  in  $Q_\sigma$ , labelled with  $c_z$ , defined below.
  - If there exists  $x \in v(\sigma_i) \in \mathcal{T}'_\sigma$ , and an edge  $(x, z)$  in  $\mathcal{T}'_\sigma$ , labelled with  $t \neq (join)$ , there exists an edge  $(Q(x), Q(z))$  labelled with  $t$  in  $Q_\sigma$ . Assume  $Q(x)$  is labelled with  $c_x$  and let  $c_{x,z} = (q, t_1^{x,z}, \dots, t_m^{x,z})$  such that  $c_x \rightarrow_t c_{x,z}$ .
  - If there exist  $x \in v(\sigma_i) \in \mathcal{T}'_\sigma$ ,  $y \in v(\sigma_j) \in \mathcal{T}'_\sigma$  and edges  $(x, z)$  and  $(y, z)$  in  $\mathcal{T}'_\sigma$ , labelled with  $(join)$ , there exist edges  $(Q(x), Q(z))$  and  $(Q(y), Q(z))$  labelled with  $(join)$  in  $Q_\sigma$ . Assume  $Q(x)$  is labelled with  $c_x$ ,  $Q(y)$  is labelled with  $c_y$  and let  $c_{x,y,z} = (q, t_1^{x,y,z}, \dots, t_m^{x,y,z})$  such that  $\{c_x, c_y\} \rightarrow_t c_{x,y,z}$ .

Then,  $c_z = (q, t_1^z \dots, t_m^z)$ , where, for  $k = 1, \dots, m$ ,  $t_k^z$  is the minimal value of the  $t_k^{x,z}$  and of the  $t_k^{x,y,z}$  above.

**Repeating loops with augmentation configurations:**

Let  $Q$  be a pseudo-promenade of  $S$ ,  $V$  be a partition of the vertices of  $Q$  such that  $(Q, V)$  satisfies the tree property. Let  $v_i$  above  $v_j \in V$ . Denote by  $c_i$  the configuration of  $S$  labeling  $v_i$ , and by  $c_j$  the configuration of  $S$  labeling  $v_j$ , and let  $c'_j \subseteq c_j$  such that  $c_i \ll c'_j$  and  $c'_j$  can be obtained from  $c_j$  by only modifying the values at some coordinates of some single configurations, with  $v'_j \subseteq v_j$  the corresponding set of vertices.

Then, the  $k$ -repetition of the chain  $v_i \dots v_j$  in  $P$ , for  $k > 1$  is the promenade  $P_{i,j}^k$  obtained from  $P$  by:

1. isolating the subgraph  $P_{i,j}$  of  $P$  that is above  $v_j$ , and not above  $v_i$ . This is well defined since  $(P, V)$  satisfies the tree property. Let  $P_{i,j}^1, \dots, P_{i,j}^k$  be  $k$  copies of  $P_{i,j}$ . Denote by  $v_i^t$  the vertices of  $P_{i,j}^t$  corresponding to  $v_i \in P_{i,j}$ , by  $v_j^t$  the vertices of  $P_{i,j}^t$  corresponding to  $v_j \in P_{i,j}$ , and by  $v_j^t$  the vertices of  $P_{i,j}^t$  corresponding to  $v'_j \in P_{i,j}$ , for  $t = 1, \dots, k$ .
2. identifying  $v_i^1$  with  $v_i$ ,  $v_j^t$  with  $v_i^{t+1}$  for  $t = 1, \dots, k-1$ , and  $v_j^k$  with  $v'_j$ .
3. identifying also  $v_j \setminus v'_j$  with  $\cup_{t=1}^k v_j^t \setminus v_j^t$ .
4. updating the labeling of the vertices from the top to the bottom, accordingly to the firing of the transitions.

Now, the remark we have made on finite values of coordinates for the promenade  $P_\sigma$  above extend to multiplicities for the pseudo-promenade  $Q_\sigma$  as follows:

Let  $Q(v_i), Q(v_j) \in Q(V_\sigma)$ , such that

1.  $Q(v_i)$  is above  $Q(v_j)$  in  $Q_\sigma$ , and
2.  $v_i \in \mathcal{T}'_\sigma$  is labelled with the generalized configuration  $g_i$ ,  $v_j \in \mathcal{T}'_\sigma$  is labelled with the generalized configuration  $g_j$ , with  $g_i \ll g_j$ .

By construction,  $g_j$  is obtained from  $g_i$  only by augmenting values for some coordinates, and adding generalized single configurations. By construction, these augmentation coordinates are labelled with  $\infty$  in  $g_j$ , and not in  $g_i$  (i.e. a lifting has been performed), and the augmentation multiplicities are valued to  $\infty$  in

$g_j$  and not in  $g_i$  (if the corresponding generalized single configuration appears in  $g_i$ , which may not be the case). Let  $a$  be the augmentation gap of this lifting. Assume  $Q(v_i)$  is labelled with the configuration  $c_i$ , and  $Q(v_j)$  by  $c_j$ .

- To any  $\infty$  value for some coordinate in  $g_i$  corresponds a finite value  $t_i$  for the same coordinate in  $c_i$ . This  $\infty$  value in  $g_i$  is inherited by construction on the same coordinate in  $g_j$ , and a finite value  $t_j$  corresponds to it in  $c_j$ . It may happen that  $t_j < t_i$ . It may even happen that  $t_j < 0$ . We denote  $t_i$  and  $t_j$  as finite coordinates of infinite heredity.
- To any  $\infty$  value for some multiplicity in  $g_i$  corresponds a finite value  $k_i$  for the same multiplicity in  $c_i$ . This  $\infty$  multiplicity in  $g_i$  is inherited by construction in  $g_j$ , and a finite multiplicity  $k_j$  corresponds to it in  $c_j$ . It may happen that  $k_j < k_i$ . It may even happen that  $k_j < 0$ . We denote  $k_i$  and  $k_j$  as finite multiplicities of infinite heredity.
- To any  $\infty$  augmentation coordinate in  $g_j$  corresponds a finite value  $t_j$  for the same coordinate in  $c_j$ . Since it is an augmentation coordinate of the lifting, this coordinate has a finite value  $t_i$  in  $g_i$  and in  $c_i$ , and  $t_i < t_j$ .
- To any  $\infty$  augmentation multiplicity in  $g_j$  corresponds a finite value  $k_j$  in  $c_j$ . If, in the derivation of  $g_i \ll g_j$ , this augmentation multiplicity has been introduced by adding new occurrences of a generalized configuration derived from  $g_i$  only by augmenting coordinates, to this multiplicity corresponds a finite multiplicity  $k_i$  in  $c_i$ , and  $k_i < k_j$ . In the other case (the augmentation multiplicity has been introduced by adding a new single generalized configuration in the derivation of  $g_i \ll g_j$ , no finite multiplicity corresponds to it in  $c_i$ ).

Now, in order to raise the augmentation coordinates and augmentation multiplicities of this lifting to an arbitrary level  $N \in \mathbb{N}$ , it suffices to perform the  $k$ -repetition of the chain  $Q(v_i) \cdots Q(v_j)$  in  $Q_\sigma$ , for  $k = \lceil \frac{N}{a} \rceil$ . Of course, this  $k$ -repetition may decrease the values of the finite coordinates and multiplicities of infinite heredity, even to negative levels; it is therefore necessary to ensure that these finite values are high enough when performing this  $k$ -repetition, i.e have been already leveled up by another  $k'$ -repetition above. This is ensured by the same algorithm  $A$  as above, by taking as input  $Q_\sigma$  and  $N$ .

Now, it is clear that the output of Algorithm  $A$  on  $Q_\sigma$  and  $N$  is a pseudo-promenade  $Q_\sigma^N$  with only positive coordinates and positive multiplicities. From  $Q_\sigma^N$ , we can inductively build a positive promenade  $P_\sigma^N$ , by starting from the top-most configurations of  $Q_\sigma^N$ , and firing from top to bottom the transitions labeling the edges of  $Q_\sigma^N$ . Then, clearly, the promenade  $P_\sigma^N$  obtained satisfies Theorem 3.6. □