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Exponential rarefaction of real curves with many components

Damien Gayet, Jean-Yves Welschinger

May 18, 2010

Abstract

Given a positive real Hermitian holomorphic line bundle L over a smooth real projective manifold X, the space of real holomorphic sections of the bundle L^d inherits for every $d \in \mathbb{N}^*$ a L^2 scalar product which induces a Gaussian measure. When X is a curve or a surface, we estimate the volume of the cone of real sections whose vanishing locus contains many real components. In particular, the volume of the cone of maximal real sections decreases exponentially as d grows to infinity.

Mathematics subject classification 2010: 14P25, 32U40, 60F10

Introduction

Let (X, c_X) be a smooth real projective manifold of dimension n and $(L, c_L) \xrightarrow{\pi} (X, c_X)$ be a real ample holomorphic line bundle. In particular, c_X and c_L are antiholomorphic involutions on X and L respectively, such that $c_X \circ \pi = \pi \circ c_L$. Let h be a real Hermitian metric on (L, c_L) with positive curvature ω. It induces a Kähler structure on (X, c_X). For every nonnegative integer d, this metric induces a Hermitian metric h^d on L^d and then a L^2-Hermitian product on the complex vector space $H^0(X, L^d)$ of holomorphic sections of L^d. This product is defined by $(\sigma, \tau) \in H^0(X, L^d) \times H^0(X, L^d) \mapsto \langle \sigma, \tau \rangle = \int_X h^d(\sigma, \tau) dx \in \mathbb{C}$, where $dx = \omega^n / \int_X \omega^n$ is the normalized volume induced by the Kähler form. Let $\mathbb{R}H^0(X, L^d)$ be the space of real sections $\{\sigma \in H^0(X, L^d) \mid c_L \circ \sigma = \sigma \circ c_X\}$ and $\Delta_k \subset H^0(X, L^d)$ (resp. $\mathbb{R}\Delta_k \subset \mathbb{R}H^0(X, L^d)$) be the discriminant locus (resp. its real part), that is the set of sections which do not vanish transversally. For every $\sigma \in H^0(X, L^d) \setminus \{0\}$, denote by $C_\sigma = \sigma^{-1}(0)$ the vanishing locus of σ and when σ is real, by $\mathbb{R}C_\sigma$ its real part. The divisor C_σ is smooth whenever $\sigma \in \mathbb{R}H^0(X, L^d) \setminus \mathbb{R}\Delta_d$. In this case, we denote by $b_0(\sigma) = b_0(\mathbb{R}C_\sigma)$ the number of connected components of $\mathbb{R}C_\sigma$.

0.1 Real projective surfaces

When X is two-dimensional, we know from Harnack-Klein inequality [9], [11] that $b_0(\mathbb{R}C_\sigma) \leq g(C_\sigma) + 1$, where equality holds for the so-called maximal curves. Here, the genus $g(C_\sigma)$ of these smooth curves C_σ gets computed by the adjunction formula and equals $g(C_\sigma) = \frac{1}{2}(d^2L^2 - dc_1(X).L + 2)$, where $c_1(X)$ denotes the first Chern class

of the surface X. For every $d \in \mathbb{N}^*$ and $a \in \mathbb{Q}_+^*$, denote by

$$\mathcal{M}_d^a = \{\sigma \in \mathbb{R}H^0(X, L^d) \setminus \mathbb{R}\Delta_d, \mid b_0(\mathbb{R}C_\sigma) \geq g(C_\sigma) + 1 - ad\};$$

it is an open cone in $\mathbb{R}H^0(X, L^d)$. The main purpose of this article is to prove the following

Theorem 1 *Let (X, c_X) be a smooth real projective surface and (L, c_L) be a real Hermitian holomorphic line bundle on X with positive curvature. Then for every sequence $d \in \mathbb{N}^* \mapsto a(d) \geq 1$ of rationals, there exist constants C, $D > 0$, such that*

$$\forall d \in \mathbb{N}^*, \ \mu(\mathcal{M}_d^{a(d)}) \leq Cd^6 e^{-D\frac{d}{a(d)}},$$

where $\mu(\mathcal{M}_d^{a(d)})$ denotes the Gaussian measure of $\mathcal{M}_d^{a(d)}$.

The Gaussian measure μ on the Euclidian space $(\mathbb{R}H^0(X, L^d), \langle\ ,\ \rangle)$ is defined by the formula

$$\forall A \subset \mathbb{R}H^0(X, L^d), \ \mu(A) = \frac{1}{\sqrt{\pi}^{N_d}} \int_A e^{-|x|^2} dx,$$

where dx denotes the Lebesgue measure associated to $\langle\ ,\ \rangle$. This Gaussian measure is a probability measure on $\mathbb{R}H^0(X, L^d)$ invariant under its isometry group.

Remark 1 *Likewise, the scalar product $\langle\ ,\ \rangle$ induces a Fubini-Study form on the linear system $P(\mathbb{R}H^0(X, L^d))$. The volume of the projection $P(\mathcal{M}_d^a)$ for the associated volume form just coincides with the measure $\mu(\mathcal{M}_k^a)$ computed in Theorem 1.*

In particular, when the sequence $a(d)$ is bounded, Theorem 1 implies that the measure of the set $\mathcal{M}_d^{a(d)}$ decreases exponentially with the degree d. This exponential rarefaction holds in particular for the set of maximal curves.

0.2 Real curves

When X is one-dimensional, we get the following result:

Theorem 2 *Let X be a closed smooth real curve and L be a real Hermitian holomorphic line bundle on X with positive curvature. Then for every positive sequence $(\epsilon(d))_{d\in\mathbb{N}}$ of rationals numbers, there exist constants C, $D > 0$ such that*

$$\forall d \in \mathbb{N}, \ \mu\{\sigma \in \mathbb{R}H^0(X, L^d) \setminus \mathbb{R}\Delta_d, \mid \#(\sigma^{-1}(0) \cap \mathbb{R}X) \geq \sqrt{d}\epsilon(d)\} \leq Cd^{3/2} e^{-D\epsilon^2(d)},$$

where μ denotes the Gaussian measure of the space $\mathbb{R}H^0(X, L^d)$.

0.3 Roots of polynomials

When (X, c_X) is the projective space of dimension $n \geq 1$, and (L, h, c_L) is the degree one holomorphic line bundle equipped with its standard Fubini-Study metric, the vector space $\mathbb{R}H^0(X, L^d)$ gets isomorphic to the space $\mathbb{R}_d[X_1, \cdots, X_n]$ of polynomials with n variables, real coefficients and degree at most d. The scalar product induced on $\mathbb{R}_d[X_1, \cdots, X_n]$ by this isomorphism is the one turning the basis $\left(\sqrt{\binom{d+n}{j}} x_1^{j_1} \cdots x_n^{j_n}\right)_{0\leq j_1+\cdots+j_n\leq d}$ into an orthonormal one, where $\binom{d+n}{j} = \frac{(d+n)!}{n!j_1!\cdots j_n!(d-j_1-\cdots-j_n)}$. Thus, the induced measure μ on $\mathbb{R}_d[X_1, \cdots, X_n]$ is the Gaussian measure associated to this basis. As a special case of our Theorems 1 and 2, we get the following

Corollary 1 *1. For every positive sequence $(\epsilon(d))_{d\in\mathbb{N}^*}$ of rational numbers, there exist positive constants C, D such that the measure of the space of polynomials $P \in \mathbb{R}_d[X]$ which have at least $\epsilon(d)\sqrt{d}$ real roots is bounded by $Cd^{3/2}\exp(-D\epsilon^2(d))$.*

2. For every sequence $d \in \mathbb{N}^* \mapsto a(d) \geq 1$ of rationals, there exist positive constants C, D such that the measure of the space of polynomials $P \in \mathbb{R}_d[X,Y]$ whose vanishing locus in $\mathbb{R}^2$ has at least $\frac{1}{2}d^2 - da(d)$ connected components is bounded from above by $Cd^6\exp(-D\frac{d}{a(d)})$.

0.4 Strategy of the proof of Theorems 1 and 2

Every curve $C_\sigma \subset X$, $\sigma \in \mathbb{R}H^0(X, L^d)\setminus\{0\}$, defines a current of integration which we renormalize by d, for its mass not to depend on $d \in \mathbb{N}^*$. In order to prove Theorems 1 and 2, we first obtain large deviation estimates for the random variable defined by this current. When d grows to infinity, the expectation of this variable converges outside of $\mathbb{R}X$ to the curvature form ω of L. These results thus go along the same lines as the one of Shiffman and Zelditch [14]. They make use in particular of the asymptotic isometry theorem of Tian [18] (see also [3] and [19]), as well as smoothness results [10] and behaviour close to the diagonal for the Bergman kernel [15],[1]. In order to deduce from these results informations on the random variable b_0, we use the theory of laminar currents introduced by Bedford, Lyubich and Smillie [2]. Indeed, we show as a corollary of a theorem of de Thélin [4] that every current in the closure of the ones arising from $\mathcal{M}_d^a$ (in particular every limit current of a sequence of real maximal curves) is weakly laminar outside of the real locus $\mathbb{R}X$, see Theorem 3. As a consequence, these currents remain in a compact set away from ω. At this point, our large deviation estimates provide the exponential decay.

0.5 Description of the paper

In the first paragraph, we bound the Markov moments needed for our large deviation estimates. In the second paragraph, we recall some elements of the theory of laminar currents in order to establish Theorem 3, that is laminarity outside of the real locus of currents in the closure of the union of the sets $\mathcal{M}_d^a$. We then get our estimates and their corollaries. We prove Theorems 1 and 2 in the third paragraph, dealing separately with the cases of bounded and unbounded sequences a. The last paragraph is devoted to some final discussion on the existence of real maximal curves on general real projective surfaces as well as on the expectation of the current of integration on real divisors.

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1 Markov-like functions on real linear systems of divisors

1.1 Case of projective spaces

Let $\mathcal{O}_{\mathbb{C}P^k}(1)$ be the degree one line bundle over $\mathbb{C}P^k$ and $||.||$ be its standard Hermitian metric with curvature the Fubini-Study Kähler form ω_{FS}. For every integer $m \geq 1$,

set

$$\begin{aligned} M^m_{\mathbb{C}P^k} : \mathbb{C}P^k &\to \mathbb{R} \\ z &\mapsto \int_{\mathbb{R}H^0(\mathbb{C}P^k, \mathcal{O}_{\mathbb{C}P^k}(1))} \left|\log ||\sigma(z)||^2\right|^m d\mu(\sigma), \end{aligned}$$

where $d\mu$ denotes the Gaussian measure of the Euclidian space $\mathbb{R}H^0(\mathbb{C}P^k, \mathcal{O}_{\mathbb{C}P^k}(1))$.

Proposition 1 *For every $m \geq 1$, the function $M^m_{\mathbb{C}P^k}$ satisfies*

$$\forall z \in \mathbb{C}P^k \setminus \mathbb{R}P^k,\ M^m_{\mathbb{C}P^k}(z) \leq \frac{4m!(k+1)}{1 - ||\tau(z)||},$$

where τ denotes the section of $\mathcal{O}_{\mathbb{C}P^k}(2)$ defined by $\tau(z_0, \cdots, z_k) = z_0^2 + \cdots + z_k^2$.

Remark 2 *The holomorphic section τ in Proposition 1 is invariant under the action of the group $PO_{k+1}(\mathbb{R})$ of real isometries of $\mathbb{C}P^k$. A slice of $\mathbb{C}P^k$ for this action is given by the interval $I = \{z_r = [1 : ir : 0 : \cdots : 0], 0 \leq r \leq 1\}$, where the end $r = 0$ (resp. $r = 1$) corresponds to the orbit $\mathbb{R}P^k$ (resp. $\tau^{-1}(0)$) of this action.*

Proof of Proposition 1. Both members of the inequality are invariants under the action of $PO_{k+1}(\mathbb{R})$, so that it is enough to prove it for z in the fundamental domain I. Let $\sigma_0, \cdots, \sigma_k$ be the orthonormal basis of $\mathbb{R}H^0(\mathbb{C}P^k, \mathcal{O}_{\mathbb{C}P^k}(1))$ given by $\sigma_i([z_0 : \cdots : z_k]) = \sqrt{k+1} z_i$. This basis induces the isometry

$$a = (a_0, \cdots, a_k) \in \mathbb{R}^{k+1} \mapsto \sigma_a = a_0\sigma_0 + \cdots + a_k\sigma_k \in \mathbb{R}H^0(\mathbb{C}P^k, \mathcal{O}_{\mathbb{C}P^k}(1)).$$

By definition, for every $a \in \mathbb{R}^{k+1}$ and $z \in \mathbb{C}P^k$,

$$||\sigma_a(z)||^2 = (k+1)\frac{|a_0z_0 + \cdots + a_kz_k|^2}{|z|^2}.$$

We deduce that for every $0 < r \leq 1$ and $m \geq 1$,

$$\begin{aligned} M^m_{\mathbb{C}P^k}(z_r) &= \int_{\mathbb{R}^{k+1}} \left|\log ||\sigma_a(z_r)||^2\right|^m d\mu(a) \\ &= \int_{\mathbb{R}^2} \left|\log\left((k+1)\frac{|a_0 + ira_1|^2}{1+r^2}\right)\right|^m \frac{e^{-|a|^2}}{\pi} da_0 da_1 \\ &= \int_0^\infty \int_0^{2\pi} \left|\log((k+1)\rho^2) + \log\frac{|\cos\theta + ir\sin\theta|^2}{1+r^2}\right|^m \frac{e^{-\rho^2}}{\pi}\rho d\rho d\theta \\ &\leq 2\int_0^\infty \left|\,|\log((k+1)\rho^2)| + \log(\frac{1+r^2}{r^2})\right|^m e^{-\rho^2}\rho d\rho, \end{aligned}$$

since for every $0 \leq r \leq 1$ and every $\theta \in [0, 2\pi]$, $r^2 \leq \cos^2\theta + r^2\sin^2\theta = |\cos\theta + ir\sin\theta|^2 \leq 1$. Hence,

$$M^m_{\mathbb{C}P^k}(z_r) \leq 2\int_0^1 (-\log(\frac{\rho}{\alpha})^2)^m e^{-\rho^2}\rho d\rho + 2\int_1^\infty (\log(\alpha\rho)^2)^m e^{-\rho^2}\rho d\rho,$$

where

$$\alpha^2 = (k+1)\frac{(1+r^2)}{r^2} = 2\frac{(k+1)}{1-||\tau(z_r)||}.$$

We now compute these two integrals.

$$\begin{aligned}
2\int_0^1 (-\log(\frac{\rho}{\alpha})^2)^m e^{-\rho^2}\rho d\rho &= 2\alpha^2 \int_0^{1/\alpha} (-\log\rho^2)^m e^{-\alpha^2\rho^2}\rho d\rho \\
&\leq 2\alpha^2 \int_0^1 (-\log\rho^2)^m \rho d\rho \\
&= \alpha^2 \int_0^\infty t^m e^{-t} dt \quad \text{with} \quad t = -\log\rho^2 \\
&= 2\frac{(k+1)m!}{1-||\tau(z_r)||}.
\end{aligned}$$

As for the second integral, we deduce from the estimate $\rho e^{-\rho^2} \leq e^{-1/\rho^2}/\rho^3$ valid for every $\rho \geq 1$, that

$$2\int_1^\infty (\log(\alpha\rho)^2)^m e^{-\rho^2}\rho d\rho \leq 2\int_1^\infty (\log(\alpha\rho)^2)^m \frac{e^{-1/\rho^2}}{\rho^3} d\rho.$$

With $t = 1/\rho$, the right hand side becomes $2\int_0^1 (-\log(\frac{t}{\alpha})^2)^m e^{-t^2} t dt \leq 2\frac{(k+1)m!}{1-||\tau(z)||}$.
□

1.2 Asymptotic results in the general case

Let X be a closed real Kähler manifold of dimension n and L be a real Hermitian line bundle over X with positive curvature ω. We denote by d_L the smallest integer such that L^d is very ample for every $d \geq d_L$ and by $\Phi_d : X \to P(H^0(X, L^d)^*)$ the associated embedding, where $x \in X$ is mapped to the set of linear forms that vanish on the hyperplane $\{\sigma \in H^0(X, L^d) \mid \sigma(x) = 0\}$. The L^2-Hermitian product of $H^0(X, L^d)$ induces a Fubini-Study metric on the complex projective space $P(H^0(X, L^d)^*)$ together with a Hermitian metric $||.||$ on the bundle $\mathcal{O}_{P(H^0(X,L^d)^*)}(1)$. We denote by h_{Φ_d} the pullback of $||.||$ on L^d under the canonical isomorphism $L^d \to \Phi_d^*\mathcal{O}_{P(H^0(X,L^d)^*)}(1)$. Recall that the latter is induced by the isomorphism

$$(x, \alpha) \in L^d \mapsto (\Phi_d(x), \alpha_x) \in \mathcal{O}_{P(H^0(X,L^d)^*)}(1),$$

where $\alpha_x : \sigma \in H^0(X, L^d) \mapsto \langle \sigma, \alpha\rangle_x \in \mathbb{C}$. In particular, the curvature form of h_{Φ_d} is $\Phi_d^*\omega_{FS}$, where ω_{FS} denotes the Fubini-Study form of $P(H^0(X, L^d)^*)$. The quotient h^d/h_{Φ_d} of these metrics of L^d is given by the function $x \in X \mapsto \sum_{i=1}^{N_d} h^d(\sigma_i(x), \sigma_i(x))$, where $(\sigma_1, \cdots, \sigma_{N_d})$ stands for any orthonormal basis of $H^0(X, L^d)$. Let $||.||_{\Phi_d}$ be the norm induced by h_{Φ_d}. For every $m \in \mathbb{N}^*$, we set

$$\begin{aligned}
M^m_{(X,L^d)} : X &\to \mathbb{R} \\
x &\mapsto \int_{\mathbb{R}H^0(X,L^d)} \left|\log ||\sigma(z)||^2_{\Phi_d}\right|^m d\mu(\sigma),
\end{aligned}$$

so that $M^m_{(X,L^d)} = M^m_{P(H^0(X,L^d)^*)} \circ \Phi_d$.

Proposition 2 *Let X be a closed real Kähler manifold of dimension n and L be a positive real Hermitian line bundle over X. For every sequence $(K_d)_{d\in\mathbb{N}^*}$ of compact subsets of $X \setminus \mathbb{R}X$ such that the sequence $(d\ dist(K_d, \mathbb{R}X)^2)_{d\in\mathbb{N}^*}$ grows to infinity, the sequence $|| \ ||\tau|| \circ \Phi_d||_{C^0(K_d)}$ converges to zero as d grows to infinity, where $\tau \in \mathbb{R}H^0(\mathcal{O}_{P(H^0(X,L^d)^*)}(2))$ is the section defined in Proposition 1. If K is a fixed compact, the same holds for any norm $C^q(K)$, $q \in \mathbb{N}$.*

The L^2-Hermitian product on $H^0(X, L^d)$ induces a scalar product on $\mathbb{R}H^0(X, L^d)$ and its dual $\mathbb{R}H^0(X, L^d)^*$. Let $\langle\ ,\ \rangle_{\mathbb{C}}$ be the extension of this scalar product to a complex bilinear product on $H^0(X, L^d)^*$. The section τ of $\mathcal{O}(2)_{P(H^0(X,L^d))^*}$ that appears in Propositions 1 and 2 is the one induced by $\sigma^* \in H^0(X, L^d)^* \mapsto \langle\sigma^*, \sigma^*\rangle_{\mathbb{C}} \in \mathbb{C}$.

Proof of Proposition 2. Let $D^* \in L^*$ be the open unit disc bundle for the metric h and $dxdv$ the product measure on D^*, where $dx = \omega^n/\int_X \omega^n$ is the measure on the base X of D^* and dv is the Lebesgue measure on the fibres. Let $L^2(D^*, \mathbb{C})$ be the space of complex functions of class L^2 on D^* for the measure $dxdv$ and $\mathcal{H}^2 \subset L^2(D^*, \mathbb{C})$ be the closed subspace of L^2 holomorphic functions on the interior of D^*. Every function $f \in \mathcal{H}^2$ has a unique expansion

$$f : (x, v) \in D^* \mapsto \sum_{d=0}^{\infty} a_d(x)v^d \in \mathbb{C},$$

where for every $d \geq 0$, $a_d \in H^0(X, L^d)$. The series $\sum_{d=0}^{\infty} a_d(x)v^d$ converges uniformly on every compact subset of D^* as well as in L^2-norm. Let B be the Bergman kernel of D^*, defined by the relation:

$$\forall (y, w) \in D^*, \forall f \in \mathcal{H}^2,\ f(y, w) = \int_{D^*} f(x, v)\overline{B((x, v), (y, w))}dxdv,$$

where this function $B : D^* \times D^* \to \mathbb{C}$ is holomorphic in the first variable and antiholomorphic in the second one. Kerzman [10] proved that the Bergman kernel can be extended smoothly up to the boundary outside of the diagonal of $\overline{D^*} \times \overline{D^*}$. Now, denote by $D^*_{X\setminus\mathbb{R}X} = \{(x, v) \in D^* \mid x \in X \setminus \mathbb{R}X\}$. The function

$$b : (x, v) \in D^*_{X\setminus\mathbb{R}X} \mapsto B((x, v), c_{L^*}(x, v)) \in \mathbb{C}$$

is holomorphic on $D^*_{X\setminus\mathbb{R}X}$ and can be extended smoothly on $\overline{D^*_{X\setminus\mathbb{R}X}}$. For every $d \geq 0$ let $(\sigma_{i,d})_{1\leq i\leq N_d}$ be an orthonormal basis of $\mathbb{R}H^0(X, L^d)$ and

$$e_{i,d} = \sqrt{\frac{d+1}{\pi}}\widehat{\sigma_{i,d}} \in \mathcal{H}^2,$$

where $\widehat{\sigma_{i,d}} : (x,v) \in D^* \mapsto \sigma_{i,d}(x)v^d \in \mathbb{C}$. The family $(e_{i,d})_{i,d}$ forms a Hilbertian basis of $\mathcal{H}^2$. Then,

$$\begin{aligned}
\forall (x,v) \in D^*_{X\setminus\mathbb{R}X},\ b(x,v) &= B((x,v), c_{L^*}(x,v)) \\
&= \sum_{d=0}^{\infty}\sum_{i=1}^{N_d} e_{i,d}(x,v)\overline{e_{i,d}(c_{L^*}(x,v))} \\
&= \sum_{d=0}^{\infty}\frac{d+1}{\pi}\sum_{i=1}^{N_d}\sigma_{i,d}(x)v^d\overline{\sigma_{i,d}(c(x))c_{L^*}(v)^d} \\
&= \sum_{d=0}^{\infty}\frac{d+1}{\pi}\sum_{i=1}^{N_d}\sigma_{i,d}^2(x)v^{2d}
\end{aligned}$$

since $\sigma_{i,d}$ is real, that is satisfies $\sigma_{i,d}\circ c = c_{L^d}\circ\sigma_{i,d}$. Note that $\tau\circ\Phi_d = \sum_{i=1}^{N_d}\sigma_{i,d}^2(x)$ and from Tian's asymptotic isometry theorem [18] (see also [3] and [19]) $||.||_{\Phi_d} \leq \dfrac{C}{d^n}||.||$. For a fixed compact K, the result thus just follows from Cauchy formula applied to b. In general, we may substitute L with L^d, K with K_d and deduce the result from Proposition 2.1 of [15] (see also [1]) since the sequence $(d\sup_{x\in K_d}\ \mathrm{dist}(x, c_X(x))^2)_{d\in\mathbb{N}^*}$ grows to infinity as d grows to infinity. $\square$

Proposition 2 and Proposition 1 imply the following

Corollary 2 *Let X be a real Kähler manifold and L be a real positive Hermitian line bundle over X. For every sequence $(K_d)_{d\in\mathbb{N}}$ of compact subsets of $X\setminus\mathbb{R}X$ such that the sequence $(d\ dist(K_d,\mathbb{R}X)^2)_{d\in\mathbb{N}^*}$ grows to infinity, there exists a positive constant c_K such that as soon as L^d is very ample,*

$$\forall m\in\mathbb{N}^*, \sup_{K_d} M^m_{(X,L^d)} \leq c_K m! N_d,$$

where $N_d = \dim H^0(X,L^d)$.

Remark 3 *Actually, in Corollary 2, $\lim_{d\to\infty} 1/N_d \sup_{K_d} M^m_{(X,L^d)} \leq 4m!$. Also, Proposition 2 and even the exponential decay of the quantity $\sup_K ||\tau||\circ\Phi_d$ are easy to establish in some cases, including the following ones.*

Projective spaces. When $X = \mathbb{C}P^n$, $\Phi_d : \mathbb{C}P^n \to \mathbb{C}P^{N_d-1}$ is equivariant with respect to the groups of real isometries $PO_{n+1}(\mathbb{R})$ and $PO_{N_d}(\mathbb{R})$. Since τ is invariant under these actions, $\tau\circ\Phi_d$ has to be a multiple of the section τ^d. Now $||\tau||_{\mathbb{R}P^n}\equiv 1$, so that $\tau\circ\Phi_d = \tau^d$ and $\sup_K||\tau\circ\Phi_d|| = \sup_K||\tau||^d$. This was observed by Macdonald [12] in the case $X = \mathbb{C}^n$.

Ellipsoid quadrics. Assume now that $X = \{[x_0 : \cdots : x_{n+1}] \in \mathbb{C}P^{n+1} \mid x_0^2 = x_1^2 + \cdots + x_{n+1}^2\}$ is the ellipsoid quadric and L the restriction of $\mathcal{O}_{\mathbb{C}P^{n+1}}(1)$ to X. Then, $\tau\circ\Phi_d$ is a multiple of the hyperplane section x_0^{2d}, since it is invariant under the group of isometries $O_{n+1}(\mathbb{R})$ acting on the coordinates $(x_1,\cdots,x_{n+1})$. At a real point $x = (x_0,\cdots,x_{n+1})$, $||\tau||\circ\Phi_d = 1$ and

$$||x_0^2|| = \frac{|x_0^2|}{|x_0|^2+\cdots+|x_{n+1}|^2} = \frac{1}{2}\left(\frac{x_0^2}{x_0^2+\cdots+x_{n+1}^2} + \frac{x_1^2+\cdots+x_{n+1}^2}{x_0^2+\cdots+x_{n+1}^2}\right) = \frac{1}{2},$$

hence $\tau \circ \Phi_d = 2^d x_0^{2d}$ and $\sup_K ||\sigma|| \circ \Phi_d = \sup_K (2||x_0^2||)^d$.

The hyperboloid surface. Finally, if $X = (\mathbb{C}P_1^1 \times \mathbb{C}P_2^1; \mathrm{Conj} \times \mathrm{Conj})$ is the hyperboloid quadric surface and $L = \mathcal{O}(a)_{\mathbb{C}P_1^1} \otimes \mathcal{O}(b)_{\mathbb{C}P_2^1}$ with $a, b > 0$, then $\tau \circ \Phi_d$ is $PO_2(\mathbb{R}) \times PO_2(\mathbb{R})$-invariant, hence a multiple of $(\tau_1^a \otimes \tau_2^b)^d$ where $\tau_i = \tau_{\mathbb{C}P_i^1} \in \mathcal{O}(2)$ for $i = 1, 2$. Computed at a real point, this multiple is one, so that $\tau \circ \Phi_d = (\tau_1^a \otimes \tau_2^b)^d$ and $\sup_K ||\sigma|| \circ \Phi_d = (\sup_K ||\tau_1^a \otimes \tau_2^b||)^d$.

2 Weakly laminar currents and large deviation estimates

2.1 Weakly laminar currents

Let (X, ω) be a smooth Kähler manifold and $\mathcal{T}_{L^2}^{(1,1)}$ be its space of closed positive currents of type $(1,1)$ and mass $L^2 = \int_X \omega \wedge \omega$. Recall that by definition such a current is a continuous linear form on the space of smooth two-forms that vanishes on forms of type $(2,0)$ and $(0,2)$ as well as on the exact forms. Moreover, the mass $\langle T, \omega \rangle$ equals L^2 and T is positive once evaluated on a positive $(1,1)$-form. In particular, T is of measure type, that is continuous for the sup norm on two-forms. The space $\mathcal{T}_{L^2}^{(1,1)}$, equipped with the weak topology, is a compact and convex space. For every $d \in \mathbb{N}^*$ and every $\sigma \in H^0(X, L^d) \setminus \{0\}$, denote by $Z_\sigma \in \mathcal{T}_{L^2}^{(1,1)}$ the current of integration

$$Z_\sigma : \phi \in \Omega^2(X) \mapsto \frac{1}{d} \int_{C_\sigma} \phi \in \mathbb{R},$$

where $C_\sigma = \sigma^{-1}(0)$. The following definition of weakly laminar currents was introduced in [2].

Definition 1 *A current $T \in \mathcal{T}_{L^2}^{(1,1)}$ is called weakly laminar in the open set $U \subset X$ iff there exist a family of embedded discs $(D_a)_{a \in \mathcal{A}}$ in U together with a measure da on $\mathcal{A}$, such that for every a and a' in $\mathcal{A}$, $D_a \cap D_{a'}$ is open in D_a and $D_{a'}$, and such that for every smooth two-form ϕ with support in U,*

$$\langle T, \phi \rangle = \int_{a \in \mathcal{A}} \left(\int_{D_a} \phi \right) da.$$

For every open subset U of X, denote by $Lam(U) \subset \mathcal{T}_{L^2}^{(1,1)}$ the subspace of closed positive currents of mass L^2 which are weakly laminar on U. For all $a \in \mathbb{Q}_+^*$, denote by $\mathcal{Z}^a$ the closure of the union $\cup_{d \in \mathbb{N}^*} \mathcal{Z}_d^a$ in $\mathcal{T}_{L^2}^{(1,1)}$, where $\mathcal{Z}_d^a$ denotes the image of the set

$$\mathcal{M}_d^a = \{\sigma \in \mathbb{R}H^0(X, L^d) \setminus \mathbb{R}\Delta_d \mid b_0(\mathbb{R}C_\sigma) \geq g(C_\sigma) + 1 - ad\}$$

under the map $\sigma \mapsto Z_\sigma \in \mathcal{T}_{L^2}^{(1,1)}$.

Theorem 3 *Let X be a closed real projective surface and L be a positive real Hermitian line bundle on X. Then, for every $a \in \mathbb{Q}_+^*$, the inclusion $\mathcal{Z}^a \subset Lam(X \setminus \mathbb{R}X)$ holds.*

In particular, every limit of a sequence of real maximal curves is weakly laminar outside of the real locus of the manifold.

Proof. This result is actually a direct consequence of Theorem 1 of [4]. Indeed, let $T \in \mathcal{Z}^a$ and $(Z_{\sigma_d})_{d\in\mathbb{N}^*}$ be a sequence of currents of integration which converges to T (we can indeed assume that $T \notin \bigcup_{d\in\mathbb{N}^*} \mathcal{Z}^a_d$). For every $d \in \mathbb{N}^*$, the genus of the complement $C_{\sigma_d} \setminus \mathbb{R}C_{\sigma_d}$ satisfies $g(C_{\sigma_d} \setminus \mathbb{R}C_{\sigma_d}) \leq ad$, while its area equals $d\int_X \omega^2$. Let B be a ball with compact closure in $X \setminus \mathbb{R}X$ and $A(C_{\sigma_d} \cap B)$ be the area of the restriction of C_{σ_d} to B . Without loss of generality, we can assume that $\frac{1}{d}A(C_{\sigma_d} \cap B)$ converges to $m_B \in [0, \int_X \omega^2]$. If $m_B = 0$, the restriction of T to B vanishes. Otherwise, $g(C_{\sigma_d} \cap B) = O(A(C_{\sigma_d} \cap B))$, where the area $A(C_{\sigma_d} \cap B)$ can be computed for the flat metric on the ball. From Theorem 1 of [4] we know that $1/m_B T_{|B}$ is weakly laminar. Hence the result. □

Lemma 1 *Let ω be a Kähler form on a complex surface X. Then ω is nowhere weakly laminar.*

Proof. Assume that there exists an open subset U of X and a measured family $(D_a)_{a\in\mathcal{A}}$ of embedded discs in U given by Definition 1 such that for every two-form ϕ with compact support in U,

$$\int_U \omega \wedge \phi = \int_{\mathcal{A}} \left(\int_{D_a} \phi \right) da.$$

For every two-form ψ defined and continuous on $\bigcup_{a\in\mathcal{A}} D_a$, we denote by T_ψ the current

$$\phi \in \Omega^2_c(U) \mapsto T_\psi(\phi) = \int_{a\in\mathcal{A}} \left(\int_{D_a} (\frac{\psi \wedge \phi}{\omega^2})\omega \right) da.$$

Then $T_\omega = \omega$, since for every $\phi \in \Omega^2_c(U)$,

$$T_\omega(\phi) = \int_{\mathcal{A}} \left(\int_{D_a} (\frac{\omega \wedge \phi}{\omega^2})\omega \right) da = \int_U (\frac{\omega \wedge \phi}{\omega^2})\omega^2 = \int_U \omega \wedge \phi.$$

Let ψ be the (1,1)-form defined along $\cup_{a\in\mathcal{A}} D_a$ in such a way that for every $a \in \mathcal{A}$ and $x \in D_a$, $T_x D_a$ lies in the kernel of ψ_x and $\psi_x \wedge \omega = \omega^2$. Then $T_\psi = \omega$, since

$$\forall \phi \in \Omega^2_c(U),\ T_\psi(\phi) = \int_{\mathcal{A}} \left(\int_{D_a} (\frac{\psi \wedge \phi}{\omega^2})\omega \right) da = \int_{\mathcal{A}} \left(\int_{D_a} \phi \right) da.$$

But ψ is integrable for both currents T_ψ and T_ω and $T_\psi(\psi) = 0$ while $T_\omega(\psi) = \omega^2$. This contradicts the equality $T_\psi = T_\omega$. □

2.2 Large deviation estimates

Proposition 3 *Let X be a real projective manifold of dimension n and L an ample real Hermitian line bundle over X. For every smooth $(2n-2)-$form ϕ with compact support in $X \setminus \mathbb{R}X$, every $d \geq d_L$ and every $\epsilon > 0$, the measure of the set*

$$\left\{ \sigma \in \mathbb{R}H^0(X, L^d) \setminus \mathbb{R}\Delta_d \mid \frac{1}{d} \left| \int_X \log ||\sigma(x)||^2_{\Phi_d} \partial\bar{\partial}\phi \right| \geq \epsilon \right\}$$

is bounded from above by the quantity

$$\frac{2c_{K_\phi}N_d}{Vol(K_\phi)}\exp\left(\frac{-\epsilon d}{2||\partial\bar\partial\phi||_{L^\infty}Vol\ (K_\phi)}\right),$$

where K_ϕ can denote both the support of ϕ or the support of $\partial\bar\partial\phi$, while c_{K_ϕ} and d_L are given in §1.2.

Recall that X is equipped with the volume form $dx=\omega^n/\int_X\omega^n$. The norm $||\partial\bar\partial\phi||_{L^\infty}$ and the volume $Vol(K_\phi)$ are computed with respect to this volume form. Recall also that N_d denotes the dimension of $\mathbb{R}H^0(X,L^d)$ and finally that from the Poincaré-Lelong formula, the current $1/(2i\pi d)\partial\bar\partial\log||\sigma(x)||^2_{\Phi_d}$ coincides with $\frac{1}{d}\Phi_d^*\omega_{FS}-Z_\sigma$.

Proof. We use Markov's trick. For every $\lambda>0$,

$$\left|\int_X\log||\sigma||^2_{\Phi_d}\partial\bar\partial\phi\right|\geq\epsilon d\Longleftrightarrow\exp\left(\lambda\left|\int_X\log||\sigma||^2_{\Phi_d}\partial\bar\partial\phi\right|\right)\geq e^{\lambda\epsilon d},$$

where

$$\exp\left(\lambda\left|\int_X\log||\sigma||^2_{\Phi_d}\partial\bar\partial\phi\right|\right)\ =\ \sum_{n=0}^{\infty}\frac{\lambda^m}{m!}\left|\int_X\log||\sigma||^2_{\Phi_d}\partial\bar\partial\phi\right|^m.$$

From Hölder's inequality we get

$$\begin{aligned}\left|\int_X\log||\sigma||^2_{\Phi_d}\partial\bar\partial\phi\right|^m &\leq \int_X\left|\log||\sigma||^2_{\Phi_d}\right|^m dx\left(\int_X\left|\partial\bar\partial\phi\right|^{\frac{m}{m-1}}dx\right)^{m-1}\\ &\leq \frac{1}{Vol(K_\phi)}\left(||\partial\bar\partial\phi||_{L^\infty}Vol(K_\phi)\right)^m\int_X\left|\log||\sigma||^2_{\Phi_d}\right|^m dx.\end{aligned}$$

As a consequence, for every $d\geq d_L$, the measure $\mu^d_\epsilon(\phi)$ of our set satisfies

$$\begin{aligned}e^{\lambda\epsilon d}\mu^d_\epsilon(\phi) &\leq \int_{\mathbb{R}H^0(X,L^d)}\exp\left(\lambda\left|\int_X\log||\sigma||^2_{\Phi_d}\partial\bar\partial\phi\ \right|d\mu(\sigma)\right)\\ &\leq \frac{1}{Vol(K_\phi)}\sum_{n=0}^{\infty}\frac{\lambda^m}{m!}||\partial\bar\partial\phi||^m_{L^\infty}Vol(K_\phi)^m\int_X M^m_{(X,L^d)}dx,\end{aligned}$$

where $M^m_{(X,L^d)}$ is defined in §1.2. Thanks to Corollary 2, the latter right hand side is bounded from above by $\frac{c_{K_\phi}N_d}{Vol(K_\phi)}\sum_{n=0}^{\infty}(\lambda||\partial\bar\partial\phi||_{L^\infty}Vol(K_\phi))^m$, that is

$$\frac{c_{K_\phi}N_d}{Vol(K_\phi)(1-\lambda||\partial\bar\partial\phi||_{L^\infty}Vol(K_\phi))}.$$

The result follows by choosing $\lambda=(2||\partial\bar\partial\phi||_{L^\infty}Vol(K_\phi))^{-1}$. □

Corollary 3 *Under the hypotheses of Proposition 3, let $\overset{\circ}{K}$ be a relatively compact open subset of $X\setminus\mathbb{R}X$. Then, there exist constants C_K, D_K, $\lambda_K>0$ and, for every $d\geq d_L$, a subset $\mathcal{A}^d_K\subset\mathbb{R}H^0(X,L^d)$ of measure bounded from above by $C_Ke^{-D_Kd}$ such that for every $\sigma\in\mathbb{R}H^0(X,L^d)\setminus\mathcal{A}^d_K$, the volume $A(C_\sigma\cap K)$ of $C_\sigma\cap K$ satisfies $A(C_\sigma\cap K)\geq\lambda_Kd$.*

Proof. Let $\overset{\circ}{K}_1$ be a relatively compact open subset of $\overset{\circ}{K}$ and $\chi : X \to [0,1]$ be a smooth cutoff function with support in K such that $\chi_{|K_1} \equiv 1$. Applied to $\phi = \chi\omega^{n-1}$ and $\epsilon = \pi \int_{K_1} \omega^n$, Proposition 3 provides us with constants C_K and D_K such that the set

$$\mathcal{A}_K^d = \{0\} \cup \left\{ \sigma \in \mathbb{R}H^0(X, L^d) \setminus \{0\} \mid \frac{1}{d} \left| \int_X \log ||\sigma||^2_{\Phi_d} \partial\bar{\partial}(\chi\omega^{n-1}) \right| \geq \pi \int_{K_1} \omega^n \right\}$$

is of measure bounded from above by $C_K e^{-D_K d}$, since N_d grows polynomially with d. From Poincaré-Lelong formula follows that for every $\sigma \in \mathbb{R}H^0(X, L^d) \setminus \mathcal{A}_K^d$,

$$\left| \frac{1}{d} \int_{C_\sigma} \chi\omega^{n-1} - \int_X \frac{1}{d} \Phi_d^* \omega_{FS} \wedge \chi\omega^{n-1} \right| \leq \frac{1}{2} \int_{K_1} \omega^n.$$

Now, from Tian's asymptotic isometry theorem [18] (see also [3] and [19]), $\frac{1}{d}\Phi_d^*\omega_{FS}$ converges to ω as d grows to ∞, so that for d large enough, $\int_X \frac{1}{d}\Phi_d^*\omega_{FS} \wedge \chi\omega^{n-1} \geq \int_{K_1} \omega^n$. As a consequence,

$$\frac{(n-1)!}{d} A(C_\sigma \cap K) \geq \frac{1}{d} \int_{C_\sigma} \chi\omega^{n-1} \geq \int_X \frac{1}{d} \Phi_d^* \omega_{FS} \wedge \chi\omega^{n-1} - \frac{1}{2} \int_{K_1} \omega^n.$$

The left hand side being bounded from below by a constant $(n-1)!\lambda_K$, the result follows. $\square$

For every $n \in \mathbb{N}^*$ and $\rho > 0$, denote by $B^{2n}(\rho) \subset \mathbb{C}^n$ the closed ball of radius ρ and volume $\pi^n \rho^{2n}/n!$. The standard Kähler form of $\mathbb{C}^n$ is denoted by ω_0.

Definition 2 *Let (X, ω) be a Kähler manifold of dimension n. By abuse, we define a ball of radius ρ of X to be the image of a holomorphic embedding $\psi_\rho : B^{2n}(\rho) \to X$ whose differential at the origin is isometric and which everywhere satisfies the inequalities $1/2\omega_0 \leq \psi_\rho^*\omega \leq 2\omega_0$.*

Corollary 4 *Under the hypotheses of Proposition 3, let $\overset{\circ}{K}$ be a relatively compact open subset of $X \setminus \mathbb{R}X$. Then, there exist constants D_K, λ_K^1, $\lambda_K^2 > 0$ such that for every ball B of radius $\rho > 0$ included in K and every $d \geq d_L$, there exists a set $\mathcal{A}_B^d \subset \mathbb{R}H^0(X, L^d)$ of measure*

$$\mu(\mathcal{A}_B^d) \leq \frac{2c_K N_d \int_X \omega^n}{\rho^{2n}} e^{-D_K d\rho^2}$$

such that for every $\sigma \in \mathbb{R}H^0(X, L^d) \setminus \mathcal{A}_B^d$, the volume of $C_\sigma \cap B$ satisfies $\lambda_K^1 d\rho^{2n} \leq A(C_\sigma \cap B) \leq \lambda_K^2 d\rho^{2n}$.

Recall that the constant c_K is given by Corollary 2, while d_L is defined in §1.2.

Proof. Let $\chi : \mathbb{C}^n \to [0,1]$ be a smooth cutoff function with support in the unit ball and such that $\chi^{-1}(1)$ contains the ball of radius $\sqrt{2/\pi}$. For any $\rho > 0$, let $\chi_\rho : x \in \mathbb{C}^n \mapsto \chi(x/\rho)$ be the associated cutoff function with support in the ball of radius ρ. Let $\psi : B \to B^{2n}(\rho) \subset \mathbb{C}^n$ be a biholomorphism given by Definition 2, and $B_1 = (\chi_\rho \circ \psi)^{-1}(1)$. Let $\phi = (\chi_\rho \circ \psi)\omega^{n-1}$, $K_\phi = supp(\phi)$, and for every $d \geq d_L$,

$$\mathcal{A}_B^d = \{0\} \cup \left\{ \sigma \in \mathbb{R}H^0(X, L^d) \setminus \{0\} \mid \frac{1}{d} \left| \int_X \log ||\sigma||^2_{\Phi_d} \partial\bar{\partial}\phi \right| \geq \pi \int_{B_1} \omega^n \right\}.$$

From Proposition 3 follows that

$$\mu(\mathcal{A}_B^d) \leq \frac{2c_{K_\phi}N_d}{Vol(K_\phi)} \exp\left(\frac{-\pi d \int_{B_1} \omega^n}{2||\partial\bar{\partial}\phi||_{L^\infty} Vol(K_\phi)}\right),$$

where

$$(\int_X \omega^n) Vol(K_\phi) = \int_{K_\phi} \omega^n \geq \frac{1}{2^n} \int_{\chi_\rho^{-1}(1)} \omega_0^n \geq \rho^{2n}.$$

The metric on B is bounded from above and below by the flat metric, see Definition 2. The quotient $\int_{B_1} \omega^n / Vol(K_\phi)$ is thus bounded from below by a positive constant, since $\int_{\chi_\rho^{-1}(1)} \omega_0^n / \int_{supp(\chi_\rho)} \omega_0^n$ does not depend on ρ. Likewise, $||\partial\bar{\partial}\phi||_{L^\infty}$ is bounded from above by a multiple of $\sup_{B^{2n}(\rho)} \left|\partial\bar{\partial}\chi_\rho \wedge \omega_0^{n-1}/\omega_0^n\right|$. The latter being of the order of $1/\rho^2$, we deduce the existence of a positive constant D_K such that

$$\mu(\mathcal{A}_B^d) \leq 2(\int_X \omega^n) \frac{c_K N_d}{\rho^{2n}} exp(-D_K \rho^2 d).$$

But for every $\sigma \in \mathbb{R}H^0(X, L^d) \setminus \mathcal{A}_B^d$, we have

$$\left|\frac{1}{d}\int_{C_\sigma} (\chi_\rho \circ \psi)\omega^{n-1} - \int_X \frac{1}{d}\Phi_d^* \omega_{FS} \wedge (\chi_\rho \circ \psi)\omega^{n-1}\right| \leq \frac{1}{2}\int_{B_1} \omega^n.$$

The term $\frac{1}{d}\int_X \Phi_d^* \omega_{FS} \wedge (\chi_\rho \circ \psi)\omega^{n-1}$ is greater than

$$\int_{B_1} \omega^n + \int_{B \setminus B_1} (\chi_\rho \circ \psi)\omega^{n-1} - ||(\frac{1}{d}\Phi_d^* \omega_{FS} - \omega) \wedge \omega^{n-1}||_{L^\infty} Vol(B).$$

From Tian's asymptotic isometry theorem [18], $\frac{1}{d}\Phi_d^* \omega_{FS}$ converges to ω as d grows to infinity. Together with Definition 2, this implies that for d large enough,

$$\frac{1}{d}\int_X \Phi_d^* \omega_{FS} \wedge (\chi_\rho \circ \psi)\omega^{n-1} \geq \int_{B_1} \omega^n$$

and

$$\frac{(n-1)!}{d\rho^{2n}} A(C_\sigma \cap B) \geq \frac{1}{d\rho^{2n}} \int_{C_\sigma} (\chi_\rho \circ \psi)\omega^{n-1} \geq \frac{1}{2\rho^{2n}} \int_{B_1} \omega^n.$$

The right hand side being bounded from below by a positive constant, we deduce the lower bound for $A(C_\sigma \cap B)$. Likewise, we deduce that $(n-1)!/(d\rho^{2n})A(C_\sigma \cap B) \leq 3/(2\rho^{2n}) \int_B \omega^n$. The right hand side being bounded from above by a positive constant, we deduce the upper bound for $A(C_\sigma \cap B)$ replacing B_1 by B in the proof. □

Lemma 2 *For every compact subset K of a n-dimensional Kähler manifold, there exist constants r_K and $n_K > 0$ such that for every $\rho > 0$ small enough, K can be covered by r_K/ρ^{2n} balls of radius ρ, in such a way that every point of K belongs to at most n_K balls.*

Proof. The lattice $\mathbb{Z}^{2n}$ acts by translations on $\mathbb{C}^n$. The orbit of the ball $B^{2n}(\sqrt{n})$ under this action covers $\mathbb{C}^n$ in such a way that every point belongs to a finite number

of balls. The images of this covering under homothetic transformations provides for every $\rho > 0$ a covering of $\mathbb{C}^n$ by balls of radius ρ such that every point belongs to a number of balls bounded independently of ρ. Let (X, ω) be a Kähler manifold. For every point $x \in K$, choose a holomorphic embedding $\phi_x : B' \to X$, where B' is a ball in $\mathbb{C}^n$ independent of x, $\phi_x(0) = x$ and ϕ_x is everywhere contracting. Let $B \subset B'$ be the ball of half radius. We extract a finite subcovering $\phi_1(B), \cdots, \phi_k(B)$ from the covering $(\phi_x(B))_{x \in K}$ of K. For every $j \in \{1, \cdots, k\}$ and every $p \in B$, there exists an affine expanding map $D_p^j : \mathbb{C}^n \to \mathbb{C}^n$ that fixes p and such that $\phi_j \circ D_p^j$ is an isometry at p. Then, there exists $\rho_0 > 0$ such that for every $0 < \rho \leq \rho_0$ and $1 \leq j \leq k$, the restriction to B of the covering of $\mathbb{C}^n$ by balls of radius ρ satisfies the following: for every ball $B_p(\rho)$ of this covering centered at $p \in B$, we have $D_p^j(B_p(\rho)) \subset B'$, and $\phi_j \circ D_p^j(B_p(\rho))$ is a ball of radius ρ of (X, ω). Since D_p^j is expanding, $D_p^j(B_p(\rho))$ contains $B_p(\rho)$, so that the union of these balls of radius ρ covers K. Moreover, the norm of D_p^j is uniformly bounded on B. Thus, there exists a constant $h > 1$ such that $D_p^j(B_p(\rho)) \subset B_p(h\rho)$ for every p and j. From this and the construction of our coverings of $\mathbb{C}^n$, we deduce the existence of a constant $n_K > 0$ independent of p such that for every point $x \in K$ and every covering of K by balls of radius ρ obtained in this way, x belongs to at most n_K balls of the covering. Finally, the existence of r_K follows from the construction of the covering of $\mathbb{C}^n$ we used. $\square$

3 Proof of the theorems

3.1 Proof of Theorem 2

Lemma 3 *Let X be a closed real one-dimensional Kähler manifold. There exist nonnegative constants η_0, E_1, E_2, E_3, E_4 and a family of smooth cutoff functions $\chi_\eta : X \to [0, 1]$ with support in $X \setminus \mathbb{R}X$, $0 < \eta \leq \eta_0$, such that for every $0 < \eta \leq \eta_0$,*

1. $E_1 \eta \leq Vol(supp(\partial\bar{\partial}\chi_\eta))$
2. $Vol(X \setminus \chi_\eta^{-1}(1)) < E_2 \eta$
3. $||\partial\bar{\partial}\chi_\eta||_{L^\infty} \leq E_3/\eta^2$
4. $dist(supp(\chi_\eta), \mathbb{R}X) \geq E_4 \eta$.

Proof. A neighborhood V of the real locus $\mathbb{R}X$ is the union of a finite number of annuli isomorphic to $A = \{z \in \mathbb{C} \mid 1 - \epsilon < |z| < 1 + \epsilon\}$. For every $\eta > 0$, choose χ_η such that $\chi_\eta(X \setminus V) = 1$ and the restriction of χ_η to A only depends on the modulus of $z \in A$. That is, for every $z \in A$, $\chi_\eta(z) = \rho_\eta(|z| - 1)$, where ρ_η is a function $]-\epsilon, \epsilon[\to [0, 1]$. Let $\rho : \mathbb{R} \to [0, 1]$ be an even function such that $\rho(x) = 1$ if $|x| \geq 1$ and $\rho(x) = 0$ if $|x| \leq 1/2$. For every $\eta > 0$, we set $\rho_\eta(x) = \rho(x/\eta)$. The family χ_η, $0 < \eta \leq \epsilon = \eta_0$ satisfies the required conditions. $\square$

Proof of Theorem 2. For every $d \in \mathbb{N}^*$, denote by

$$\mathcal{M}_d^{\epsilon(d)} = \{\sigma \in \mathbb{R}H^0(X, L^d) \setminus \mathbb{R}\Delta_d \mid \#(\sigma^{-1}(0) \cap \mathbb{R}X) \geq \sqrt{d}\epsilon(d)\}.$$

For every $\sigma \in \mathcal{M}_d^{\epsilon(d)}$, denote by $Z_\sigma : \phi \in C^0(X) \mapsto \frac{1}{d}\int_{C_\sigma} \phi \in \mathbb{R}$ the associated discrete measure, where $C_\sigma = \sigma^{-1}(0)$. Let $(\chi_\eta)_{0<\eta\leq\eta_0}$ be a family of real cutoff functions given by Lemma 3. By definition, for every $0 < \eta \leq \eta_0$ and every $\sigma \in \mathcal{M}_d^{\epsilon(d)}$,

$$\langle Z_\sigma, \chi_\eta \rangle = \frac{1}{d}\int_{C_\sigma} \chi_\eta \leq \int_X \omega - \frac{\epsilon(d)}{\sqrt{d}},$$

where ω denotes the curvature of L. Without loss of generality, we can assume that when d is large enough, $\epsilon(d)/\sqrt{d} \leq \eta_0$. We then set $\eta_d = \epsilon(d)/(2E_2\sqrt{d}\int_X \omega)$, where E_2 is given by Lemma 3. From Lemma 3, we deduce that for every $\sigma \in \mathcal{M}_d^{\epsilon(d)}$, $\langle \omega - Z_\sigma, \chi_{\eta_d} \rangle > \frac{\epsilon(d)}{2\sqrt{d}}$ and then from Poincaré-Lelong formula that

$$\frac{1}{d}\left|\int_X \log ||\sigma(x)||^2_{\Phi_d} \partial\bar{\partial}\chi_{\eta_d}\right| \geq \frac{\pi\epsilon(d)}{\sqrt{d}} - 2\pi||\frac{1}{d}\Phi^*\omega_{FS} - \omega||_{L^\infty}.$$

We know from Tian's asymptotic isometry theorem [18] that $d||\frac{1}{d}\Phi^*\omega_{FS} - \omega||_{L^\infty}$ is bounded, so that for d large enough, the right hand side is greater than $\epsilon(d)/\sqrt{d}$. For every d large enough, denote by K_d the support of $\partial\bar{\partial}\chi_{\eta_d}$. Without loss of generality, we can assume that $\epsilon(d)$ grows to infinity when d grows to infinity. By Lemma 3, so does $d\ \mathrm{dist}(K_d, \mathbb{R}X)^2$. Proposition 3 and Lemma 3 then provide the result. □

3.2 Proof of Theorem 1 when a is a bounded function

Let $a \in \mathbb{Q}_+^*$. We have to prove the existence of two positive constants C and D such that $\mu(\mathcal{M}_d^a) \leq Ce^{-Dd}$. From Theorem 3 we know that the compact $\mathcal{Z}^a = \overline{\bigcup_{d\in\mathbb{N}^*} \mathcal{Z}_d^a}$ introduced in §2.1 is included in $Lam(X \setminus \mathbb{R}X)$, whereas from Lemma 1, $\omega \notin Lam(X \setminus \mathbb{R}X)$. As a consequence, there exists a finite set $(\phi_j)_{j\in\mathcal{J}}$ of two-forms with compact support in $X \setminus \mathbb{R}X$ such that $\forall T \in \mathcal{Z}^a, \exists j \in \mathcal{J},\ |\langle \omega - T, \phi_j \rangle| > 1$. Moreover, Poincaré-Lelong formula writes

$$\forall d \geq d_L,\ \forall \sigma \in \mathbb{R}H^0(X, L^d) \setminus \{0\}, \frac{1}{2i\pi d}\partial\bar{\partial}\log ||\sigma||^2_{\Phi_d} = \frac{1}{d}\phi_d^*\omega_{FS} - Z_\sigma,$$

where ω_{FS} denotes the Fubini-Study form of $P(H^0(X, L^d))^*$ defined in §1.1, and Z_σ the current of integration defined in §2.1. From Tian's asymptotic isometry theorem [18] (see also [3] and [19]), $\frac{1}{d}\Phi_d^*\omega_{FS}$ converges to ω as d grows to ∞. Thus, there exists $d_1 \geq d_L$ such that

$$\forall d \geq d_1, \forall \sigma \in \mathcal{M}_d^a, \exists j \in \mathcal{J},\ \left| < \frac{1}{d}\partial\bar{\partial}\log ||\sigma||^2_{\Phi_d}, \phi_j > \right| > 2\pi.$$

From this relation and Proposition 3 we deduce

$$\begin{aligned} \mu(\mathcal{M}_d^a) &\leq \sum_{j\in\mathcal{J}} \mu\left\{\sigma \in \mathbb{R}H^0(X, L^d) \setminus \{0\} \mid \frac{1}{d}\left|\int_X \log ||\sigma||_{\Phi_d}\partial\bar{\partial}\phi_j\right| > 2\pi\right\} \\ &\leq \frac{2c_K N_d \#\mathcal{J}}{\inf Vol(supp(\phi_j))} \exp\left(-\frac{\pi d}{\max_{j\in\mathcal{J}} ||\partial\bar{\partial}\phi_j||_{L^\infty} Vol(K)}\right), \end{aligned}$$

where $K \subset X \setminus \mathbb{R}X$ is a compact containing all supports of the ϕ_j's, $j \in \mathcal{J}$, and c_K is given by Corollary 2. Hence the result. □

3.3 Proof of Theorem 1, general case

Lemma 4 *Under the hypotheses of Theorem 1, let $\overset{\circ}{K}$ be a relatively compact open subset of $X \setminus \mathbb{R}X$ equipped with a covering by balls given by Lemma 2. Let n_K be given by Lemma 2 and $\lambda_K > 0$, $\mathcal{A}_K^d \subset \mathbb{R}H^0(X, L^d)$ be given by Corollary 3. Then for every $d \geq d_L$ and $\sigma \in \mathcal{M}_d^{a(d)} \setminus \mathcal{A}_K^d$, there is a ball B of the covering such that the genus $g(C_\sigma \cap B)$ of $C_\sigma \cap B$ satisfies $g(C_\sigma \cap B) \leq \frac{n_K}{\lambda_K} a(d) A(C_\sigma \cap B)$, where $A(C_\sigma \cap B)$ denotes the area of $C_\sigma \cap B$.*

Recall that the integer d_L was defined in §1.2.

Proof. By definition, the genus $g(C_\sigma \cap B)$ is such that the Euler characteristic $\chi(C_\sigma \cap B)$ of this curve be given by the formula

$$\chi(C_\sigma \cap B) = 2b_0(C_\sigma \cap B) - 2g(C_\sigma \cap B) - r(C_\sigma \cap B)$$

where $b_0(C_\sigma \cap B)$ (resp. $r(C_\sigma \cap B)$) denotes the number of the connected components of $C_\sigma \cap B$ (resp. of $\partial(C_\sigma \cap B)$). In particular, for every $\sigma \in \mathcal{M}_d^{a(d)}$, $g(C_\sigma \cap B) \leq g(C_\sigma \setminus \mathbb{R}C_\sigma) \leq a(d)d$. Denote by $(B_i)_{i \in \mathcal{I}}$ the covering of K. Since $\sigma \notin \mathcal{A}_K^d$, Corollary 3 implies that $\sum_{i \in \mathcal{I}} A(C_\sigma \cap B_i) \geq A(C_\sigma \cap K) \geq \lambda_K d$. Now, from Lemma 2 we conclude that

$$\sum_{i \in \mathcal{I}} g(C_\sigma \cap B_i) \leq n_K g(C_\sigma \setminus \mathbb{R}C_\sigma) \leq n_K a(d) d.$$

Hence the result. □

Proof of Theorem 1. From §3.2, we can assume that the sequence $a(d)$ grows to infinity. For every $d \in \mathbb{N}^*$, we set $\rho_d = a(d)^{-1/2}$. Let $\overset{\circ}{K}$ be a relatively compact open subset of $X \setminus \mathbb{R}X$. For every d large enough, we cover K by balls of radius ρ_d as given by Lemma 2. This cover contains at most $r_K/\rho_d^4 = r_K a(d)^2$ balls. From Corollary 4, there is a subset $\mathcal{B}^d$ of $\mathbb{R}H^0(X, L^d)$ satisfying

$$\mu(\mathcal{B}^d) \leq 2 \int_X \omega^2 c_K r_K N_d a(d)^4 \exp\left(-D_K \frac{d}{a(d)}\right),$$

and such that for every $\sigma \in \mathbb{R}H^0(X, L^d) \setminus \mathcal{B}^d$ and every ball B of the cover,

$$\lambda_K^1 \frac{d}{a(d)^2} \leq A(C_\sigma \cap B) \leq \lambda_K^2 \frac{d}{a(d)^2}.$$

Let $\mathcal{A}_K^d \subset \mathbb{R}H^0(X, L^d)$ be the set given by Corollary 3. By Lemma 4, for every $\sigma \in \mathbb{R}H^0(X, L^d) \setminus \mathcal{A}_K^d$, there is a ball B_σ of our cover such that

$$g(C_\sigma \cap B_\sigma) \leq \frac{n_K}{\lambda_K} a(d) A(C_\sigma \cap B).$$

Without loss of generality, we can assume that $B_\sigma = B^4(\rho_d) \subset \mathbb{C}^2$ and that the area of C_σ is computed with respect to the standard metric ω_0 of $\mathbb{C}^2$. Denote by $\widetilde{C}_\sigma$ the image of $C_\sigma \cap B$ under the homothetic transformation with coefficient $1/\rho_d$, so that $\widetilde{C}_\sigma \subset B^4(1)$ and $g(\widetilde{C}_\sigma) \leq \frac{n_K}{\lambda_K} A(\widetilde{C}_\sigma)$.

Let $\mathcal{T}^{(1,1)}_{\pi^2}(B(1))$ be the space of positive closed currents of bidegree $(1,1)$ on the unit ball $B^4(1)$ with mass π^2, and $\widetilde{Z}_\sigma \in \mathcal{T}^{(1,1)}_{\pi^2}(B(1))$ the current of integration

$$\widetilde{Z}_\sigma : \phi \in \Omega^{(1,1)}_c(B^4(1)) \mapsto \widetilde{Z}_\sigma(\phi) = \frac{\pi^2}{A(\widetilde{C}_\sigma)} \int_{\widetilde{C}_\sigma} \phi.$$

We set

$$\mathcal{Z}^a = \overline{\bigcup_{d \geq d_L} \{\widetilde{Z}_\sigma, \sigma \in \mathbb{R}H^0(X, L^d) \setminus \mathcal{A}^d_K\}} \subset \mathcal{T}^{(1,1)}_{\pi^2}(B(1)).$$

By Theorem 1 of [4], $\mathcal{Z}^a$ is contained in the space of weakly laminar currents of the unit ball $B^4(1)$. In particular, from Lemma 1 we know that $\omega_0 \notin \mathcal{Z}^a$. Since $\overline{B^4(1)}$ is compact, $\mathcal{Z}^a$ is compact and there exists a finite number of two-forms $(\widetilde{\phi}_j)_{j \in \mathcal{J}}$ with compact support in $B^4(1)$ such that

$$\forall \lambda \in [\frac{\lambda^1_K}{\pi^2}, \frac{\lambda^2_K}{\pi^2}], \forall T \in \mathcal{Z}^a, \exists j \in \mathcal{J}, |\langle \lambda T - \omega_0, \widetilde{\phi}_j \rangle| > 1.$$

Applying this inequality to $T = \widetilde{Z}_\sigma$ and $\lambda = a(d)^2 A(C_\sigma \cap B)/\pi^2 d$, we get

$$\left| \frac{a(d)^2 A(C_\sigma \cap B)}{d A(\widetilde{C}_\sigma)} \int_{\widetilde{C}_\sigma} \widetilde{\phi}_j - \int_{B(1)} \omega_0 \wedge \widetilde{\phi}_j \right| > 1.$$

Denote by ϕ_j the pullback of $\widetilde{\phi}_j$ under the homothetic tranformation of coefficient $1/\rho_d$, so that the support of ϕ_j lies in $B^4(\rho_d)$. We get

$$\left| \frac{1}{d} \int_{C_\sigma} \phi_j - \int_{B^4(\rho_d)} \omega_0 \wedge \phi_j \right| > 1/a(d),$$

as long as $\sigma \notin \mathcal{B}^d$. Finally, since by Definition 2 $a(d) \int_{B_\sigma} (\omega - \omega_0) \wedge \phi_j$ converges to zero, we deduce for d large enough the relation

$$\forall \sigma \in \mathbb{R}H^0(X, L^d) \setminus (\mathcal{A}^d_K \cup \mathcal{B}^d),\ \exists j \in \mathcal{J},\ \left| \frac{1}{d} \int_{C_\sigma} \phi_j - \int_X \omega \wedge \phi_j \right| \geq 1/a(d).$$

Likewise, from Tian's asymptotic isometry theorem [18], $a(d) \int_X (\omega - \frac{1}{d}\Phi^*_d \omega_{FS}) \wedge \phi_j$ converges to zero as d grows to ∞. Applying Proposition 3 to every ball of our cover and every ϕ_j, $j \in \mathcal{J}$, with support in this ball, we finally obtain the existence of positive constants C, D, such that $\mu(\mathcal{M}^{a(d)}_d) \leq C N_d a(d)^4 \exp\left(-D \frac{d}{a(d)}\right)$. $\square$

4 Final remarks

4.1 Average current of integration

For every $k \geq 1$, denote by

$$\begin{aligned} E_{\mathbb{C}P^k} : \mathbb{C}P^k &\to \mathbb{R} \\ z &\mapsto \int_{\mathbb{R}H^0(\mathbb{C}P^k, \mathcal{O}_{\mathbb{C}P^k}(1))} \log ||\sigma(z)||^2 d\mu(\sigma) \end{aligned}$$

the expectation of the random variable $\sigma \mapsto \log ||\sigma||^2$.

Proposition 4 *For every $k \geq 1$ and $z \in \mathbb{C}P^k \setminus \mathbb{R}P^k$,*

$$E_{\mathbb{C}P^k}(z) = \log(\frac{k+1}{4}) + \int_0^\infty e^{-\rho} \log \rho d\rho + \log(1 + \sqrt{1 - ||\tau||^2(z)}),$$

where τ is the section introduced in Proposition 1.

This result is very close to Lemma 2.5 of [12].

Proof. As in the proof of Proposition 1 and using the notations of Remark 2, we get for every $0 < r \leq 1$:

$$\begin{aligned} E_{\mathbb{C}P^k}(z_r) &= \int_{\mathbb{R}^2} \log\left((k+1)\frac{|a_0 + ira_1|^2}{1+r^2}\right) \frac{e^{-|a|^2}}{\pi} da_0 da_1 \\ &= \log\left(\frac{k+1}{1+r^2}\right) + \int_0^\infty e^{-\rho} \log \rho d\rho + \frac{1}{2\pi}\int_0^{2\pi} \log|\cos\theta + ir\sin\theta|^2 d\theta \\ &= \log\left(\frac{k+1}{4(1+r^2)}\right) + \int_0^\infty e^{-\rho} \log \rho d\rho + \frac{1}{2\pi}\int_0^{2\pi} \log|e^{2i\theta}(1+r) + 1 - r|^2 d\theta. \end{aligned}$$

From Jensen formula, as soon as $r > 0$,

$$\frac{1}{2\pi}\int_0^{2\pi} \log|e^{2i\theta}(1+r) + 1 - r|^2 d\theta = \log|1-r|^2 + \log\left|\frac{1+r}{1-r}\right|^2 = \log(1+r)^2.$$

From this we deduce

$$\begin{aligned} E_{\mathbb{C}P^k}(z_r) &= \log(\frac{k+1}{4}) + \int_0^\infty e^{-\rho} \log \rho d\rho + \log\left(\frac{(1+r)^2}{1+r^2}\right) \\ &= \log(\frac{k+1}{4}) + \int_0^\infty e^{-\rho} \log \rho d\rho + \log\left(1 + \sqrt{1 - ||\tau||^2(z)}\right), \end{aligned}$$

since $||\tau||(z_r) = |\tau(z_r)|/|z_r|^2 = (1-r^2)/(1+r^2)$. The result follows from the invariance of $E_{\mathbb{C}P^k}$ and $||\tau||$ under the action of $PO_{k+1}(\mathbb{R})$, see Remark 2. □

Corollary 5 *For every $k \geq 1$ and every real line D in $\mathbb{C}P^k$, the restriction of the current $\frac{1}{2i\pi}\partial\bar{\partial}E_{\mathbb{C}P^k}$ to $D \setminus \mathbb{R}D$ coincides with the Fubini-Study form, while its restriction to the quadric $\{\tau = 0\}$ vanishes.*

Proof. Proposition 4 implies that the restriction of $E_{\mathbb{C}P^k}$ to the quadric $\{\tau = 0\}$ is constant, so that the current $\partial\bar{\partial}E_{\mathbb{C}P^k}$ vanishes on this quadric. In the same way, Proposition 4 implies that the restriction of $\frac{1}{2i\pi}\partial\bar{\partial}E_{\mathbb{C}P^k}$ to D does not depend on k. Thus, we may assume $k = 1$ and $D = \mathbb{C}P^1$. Now, every $\sigma \in \mathbb{R}H^0(\mathbb{C}P^1, \mathcal{O}_{\mathbb{C}P^1}(1))$ does not vanish on $\mathbb{C}P^1 \setminus \mathbb{R}P^1$, so that by definition

$$\begin{aligned} \forall z \in \mathbb{C}P^1 \setminus \mathbb{R}P^1, \frac{1}{2i\pi}\partial\bar{\partial}E_{\mathbb{C}P^k}(z) &= \int_{\mathbb{R}H^0(\mathbb{C}P^1, \mathcal{O}_{\mathbb{C}P^1}(1))} \frac{1}{2i\pi}\partial\bar{\partial}\log||\sigma(z)||^2 d\mu(\sigma) \\ &= \int_{\mathbb{R}H^0(\mathbb{C}P^1, \mathcal{O}_{\mathbb{C}P^1}(1))} \omega_{FS}(z) d\mu(\sigma) \\ &= \omega_{FS}(z). \end{aligned}$$

□

Now, let L be a real Hermitian line bundle with positive curvature on a smooth real Kähler manifold X of dimension $n \geq 1$. For every $d \in \mathbb{N}^*$ and every $(2n-2)$-form $\phi \in \Omega^{2n-2}(X)$, we denote by

$$Z^\phi : \sigma \in \mathbb{R}H^0(X, L^d) \setminus \{0\} \mapsto Z^\phi_\sigma = \frac{1}{d} \int_{C_\sigma} \phi \in \mathbb{R}$$

the associated random variable, where the space $\mathbb{R}H^0(X, L^d)$ is equiped with the L^2 Gaussian probability measure μ. We write

$$E_d(Z^\phi) = \int_{\mathbb{R}H^0(X,L^d)} Z^\phi_\sigma d\mu(\sigma)$$

for the expectation of this random variable, and $E_d(Z) : \phi \in \Omega^2(X) \mapsto E_d(Z^\phi) \in \mathbb{R}$ for the associated closed positive (1,1)-current.

Proposition 5 *Let L be a real Hermitian line bundle with positive curvature on a smooth closed real Kähler manifold X. Then, for every $d \geq d_L$,*

$$E_d(Z) = \frac{1}{d}\Phi_d^*\omega_{FS} - \frac{1}{2i\pi d}\Phi_d^*\partial\bar{\partial}E_{P(H^0(X,L^d)^*)}.$$

Moreover, the restriction of this current to the complement of the real locus converges to ω as d grows to infinity.

Recall that the embedding $\Phi_d : X \to P(H^0(X, L^d)^*)$, $d \geq d_L$, and the Fubini-Study form ω_{FS} of the projective space $P(H^0(X, L^d)^*)$ were introduced in §1.2.

Proof. Poincaré-Lelong formula provides for every $\sigma \in \mathbb{R}H^0(X, L^d) \setminus \{0\}$ the relation

$$\frac{1}{2i\pi d}\partial\bar{\partial}\log||\sigma||^2_{\Phi_d} = \frac{1}{d}\Phi_d^*\omega_{FS} - Z_\sigma.$$

The first part of Proposition 5 is obtained by integration of this relation on $\mathbb{R}H^0(X, L^d)$. Tian's asymptotic isometry theorem [18] implies that $\frac{1}{d}\Phi_d^*\omega_{FS}$ converges to the curvature ω of L. Proposition 2 combined with Proposition 4 imply that $\frac{1}{2i\pi d}\Phi_d^*\partial\bar{\partial}E_{P(H^0(X,L^d)^*)}$ converges to zero faster than every polynomial function in d, and even exponentially fast in the cases covered by Remark 3 (compare with [12]). Hence the result. □

Note that when the chosen probability space is the whole complex space $H^0(X, L^d)$, the expectation $\int_{H^0(X,L^d)} \log||\sigma(z)||^2 d\mu(\sigma)$ is a function of $z \in \mathbb{C}P^k$ invariant under the whole $PU_{k+1}(\mathbb{C})$, thus is constant. Hence, $E(Z^\phi_{\mathbb{C}}) = \frac{1}{d}\Phi_d^*\omega_{FS}$, see [14]. Moreover, Shiffman and Zelditch proved in [16] that the law of $Z^\phi_{\mathbb{C}}$ converges to a normal law as d grows to infinity, a result that was already obtained in dimension one in [17]. It would be here of interest to understand in more details the convergence of the law of Z^ϕ.

4.2 Existence of real maximal curves

An algebraic curve C of genus $g(C)$ is said to be maximal when the number of components of its real locus coincides with $g(C)+1$, the maximum allowed by Harnack-Klein

inequality [9], [11]. Our Theorem 1 proves, in particular, that if L is a real ample Hermitian line bundle over a real projective surface X, the measure of the set of real maximal curves linearly equivalent to L^d exponentially decreases as d increases. When $X = \mathbb{C}P^2$, Harnack [9] proved that such maximal curves exist in any degree. The study of these curves plays a central rôle in real algebraic geometry, at least since Hilbert included it in his 16th problem. Nevertheless, such curves do not always exist. For instance, if X is the product of two non maximal real curves, then for every ample real line bundle L over X and every $d \in \mathbb{N}^*$, the linear system $\mathbb{R}H^0(X, L^d)$ contains no maximal curve.

However, every real closed symplectic manifold (X, ω, c_X) of dimension four with rational form ω supports, when d is large enough, symplectic real surfaces Poincaré dual to $d\omega$ and whose real locus contains at least ϵd components, where ϵ depends on the manifold (X, ω, c_X), see [7]. Applying Harnack's method to these curves, we see that there always exist even symplectic real surfaces whose real locus contains at least $\epsilon' d^2$ connected components.

The following questions then arise. For every ample real line bundle L on a projective real surface X and every $d \in \mathbb{N}^*$, denote by

$$m(L^d) = \sup_{\sigma \in \mathbb{R}H^0(X, L^d) \setminus \mathbb{R}\Delta_d} b_0(\mathbb{R}C_\sigma)$$

the maximal number of connected components that a smooth real divisor linearly equivalent to L^d may contain. Then, denote by $\epsilon(X, L) = \limsup_{d \to \infty} \frac{1}{d^2} m(L^d)$, so that $0 \leq \epsilon(X, L) \leq \frac{1}{2} L.L$ by Harnack-Klein inequality and the adjunction formula. Is this quantity $\epsilon(X, L)$ bounded from below by a non negative constant independant of (X, L)? Does there exist a pair (X, L) with $\mathbb{R}X \neq \emptyset$ such that $\epsilon(X, L) < \frac{1}{2} L^2$? If not, what about the quantity $\limsup_{d \to \infty} \frac{1}{d}(m(L^d) - \frac{1}{2} d^2 L^2)$?

The same questions hold within the realm of four-dimensional real symplectic manifolds. Recall that the real symplectic surfaces built in [7] are obtained via Donaldson's method [5], so that their current of integration converges to ω as d grows to infinity. Theorem 3 provides an obstruction to get real maximal curves using this method (Donaldson's quantitative transversality gives another one, as observed in [7]). This phenomenon was in fact the starting point of our work.

This work raises several questions. It is known [6] that the expectation of the number of real roots of a real polynomial in one variable is $\sqrt{n}$. What is the expected value of $b_0(\mathbb{R}C_\sigma)$ in dimension two? How to improve Theorem 1 to get decays till this expectation, as in Theorem 2? What happens for values below this expectation? Note that for spherical harmonics on the two-dimensional sphere, such kinds of results have been obtained in [13]. More generally, what is the asymptotic law of the random variable b_0? What happens in higher dimensions?

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