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# CHOOSABILITY OF A WEIGHTED PATH AND FREE-CHOOSABILITY OF A CYCLE

YVES AUBRY, JEAN-CHRISTOPHE GODIN AND OLIVIER TOGNI

ABSTRACT. A graph  $G$  with a list of colors  $L(v)$  and weight  $\omega(v)$  for each vertex  $v$  is  $(L, \omega)$ -colorable if one can choose a subset of  $\omega(v)$  colors from  $L(v)$  for each vertex  $v$ , such that adjacent vertices receive disjoint color sets. In this paper, we give necessary and sufficient conditions for a weighted path to be  $(L, \omega)$ -colorable for some list assignments  $L$ . Furthermore, we solve the problem of the free-choosability of a cycle.

## 1. INTRODUCTION

The concept of choosability of a graph, also called list coloring, has been introduced by Vizing in [15], and independently by Erdős, Rubin and Taylor in [4]. It contains of course the colorability as a particular case. Since its introduction, choosability has been extensively studied (see for example [1, 3, 13, 14, 6] and more recently [7, 8]). Even for the original (unweighted) version, the problem proves to be difficult, and NP-complete for very restricted graph classes. Existing results for the weighted version only concern the case of constant weights (i.e.  $(a, b)$ -choosability), see [1, 4, 7, 14]. For the coloring problem of weighted graphs, quite a little bit more is known, see [12, 9, 10, 11].

This paper is among the first to consider list colorings of weighted graphs by studying conditions on the list assignment for a weighted path to be choosable. Starting from the idea that in a path, the lists of colors of non consecutive vertices do not interfere, and following the work in [5], we introduce here the notion of a waterfall list assignment of a weighted path. It is a list assignment such that any color is present only on one list or on two lists of consecutive vertices. We show that any list assignment (with some additional properties) can be transformed into a similar waterfall list assignment. We give as a first main result a necessary and sufficient condition for a weighted path with a given waterfall list  $L$  to be  $(L, \omega)$ -colorable (Theorem 12) and use it to derive  $(L, \omega)$ -colorability results for some general lists assignments.

In 1996, Voigt considered the following problem: let  $G$  be a graph and  $L$  a list assignment and assume that an arbitrary vertex  $v \in V(G)$  is precolored by a color  $f \in L(v)$ . Is it always possible to complete this precoloring to a proper list coloring? This question leads to the concept of free-choosability introduced by Voigt in [16].

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We investigate here the free-choosability of the first interesting case, namely the cycle. As an application of Theorem 12, we prove our second main result which gives a necessary and sufficient condition for a cycle to be  $(a, b)$ -free-choosable (Theorem 15). In order to get a concise statement, we introduce the free-choice ratio of a graph, in the same way that Alon, Tuza and Voigt in [1] introduced the choice ratio (which equals the so-called fractional chromatic number).

In addition to the results obtained in this paper, the study of waterfall lists may be of more general interest. For now on, the method is extended in [2] to be used in a reduction process, allowing to prove colorability results on triangle-free induced subgraphs of the triangular lattice.

We recall in Section 2 the known definitions of choosability and free-choosability of a graph and introduce the definitions of a waterfall list and of the similarity between two lists that are fundamental for this paper. In Section 3, we introduce new definitions and notations related to list assignments on paths which will be used in the proofs of preliminary results stated in this section and in the proofs of Section 4 which is devoted to the main results on choosability of a weighted path. Section 5 deal with the free-choosability of a cycle.

## 2. DEFINITIONS

Let  $G = (V(G), E(G))$  be a graph where  $V(G)$  is the set of vertices and  $E(G)$  is the set of edges, and let  $a, b, n$  and  $e$  be integers.

Let  $\omega$  be a weight function of  $G$  i.e. a map  $\omega : V(G) \rightarrow \mathbb{N}$  and let  $L$  be a list assignment of  $G$  i.e. a map  $L : V(G) \rightarrow \mathcal{P}(\mathbb{N})$ . By abuse of language and to simplify, we will just call  $L$  a list. If  $A$  is a finite set, we denote by  $|A|$  the cardinal of  $A$ .

A weighted graph  $(G, \omega)$  is a graph  $G$  together with a weight function  $\omega$  of  $G$ .

Let us recall the definitions of an  $(L, \omega)$ -colorable graph and an  $(a, b)$ -free-choosable graph which are essential in this paper.

**Definition 1.** *An  $(L, \omega)$ -coloring  $c$  of a graph  $G$  is a map that associate to each vertex  $v$  exactly  $\omega(v)$  colors from  $L(v)$  such that adjacent vertices receive disjoint color sets, i.e. for all  $v \in V(G)$ :*

$$c(v) \subset L(v), \quad |c(v)| = \omega(v),$$

and for all  $vv' \in E(G)$ :

$$c(v) \cap c(v') = \emptyset.$$

We say that  $G$  is  $(L, \omega)$ -colorable if there exists an  $(L, \omega)$ -coloring  $c$  of  $G$ .

Particular cases of  $(L, \omega)$ -colorability are of great interest. In order to introduce them, we define  $(L, b)$ -colorings and  $a$ -lists.

An  $(L, b)$ -coloring  $c$  of  $G$  is an  $(L, \omega)$ -coloring of  $G$  such that for all  $v \in V(G)$ , we have  $\omega(v) = b$ .

A  $a$ -list  $L$  of  $G$  is a list of  $G$  such that for all  $v \in V(G)$ , we have  $|L(v)| = a$ .

**Definition 2.**  *$G$  is said to be  $(a, b)$ -choosable if for any  $a$ -list  $L$  of  $G$ , there exists an  $(L, b)$ -coloring  $c$  of  $G$ .*

**Definition 3.**  $G$  is said to be  $(a, b)$ -free-choosable if for any  $v_0 \in V(G)$ , and for any list  $L$  of  $G$  such that for any  $v \in V(G) \setminus \{v_0\}$ , we have  $|L(v)| = a$  and  $|L(v_0)| = b$ , there exists an  $(L, b)$ -coloring  $c$  of  $G$ .

The path  $P_{n+1}$  of length  $n$  is the graph with vertex set  $V = \{v_0, v_1, \dots, v_n\}$  and edge set  $E = \bigcup_{i=0}^{n-1} \{v_i v_{i+1}\}$ . To simplify the notations,  $L(i)$  denotes  $L(v_i)$  and  $c(i)$  denotes  $c(v_i)$ .

By analogy with the flow of water in waterfalls, we define a waterfall list as follows:

**Definition 4.** A waterfall list  $L$  of a path  $P_{n+1}$  of length  $n$  is a list  $L$  such that for all  $i, j \in \{0, \dots, n\}$  with  $|i - j| \geq 2$ , we have  $L(i) \cap L(j) = \emptyset$ .

Notice that another similar definition of a waterfall list is that any color is present only on one list or on two lists of consecutive vertices. Figure 1 shows a list  $L$  of the path  $P_5$  (on the left), together with a similar waterfall list  $L^w$  (on the right).

We define now the similarity of two lists with respect to a weighted graph:

**Definition 5.** Let  $P_{n+1}$  be a weighted path of length  $n$ .

Two lists  $L$  and  $L'$  are said to be similar if this assertion is true:

$P_{n+1}$  is  $(L, w)$ -colorable  $\Leftrightarrow P_{n+1}$  is  $(L', w)$ -colorable.

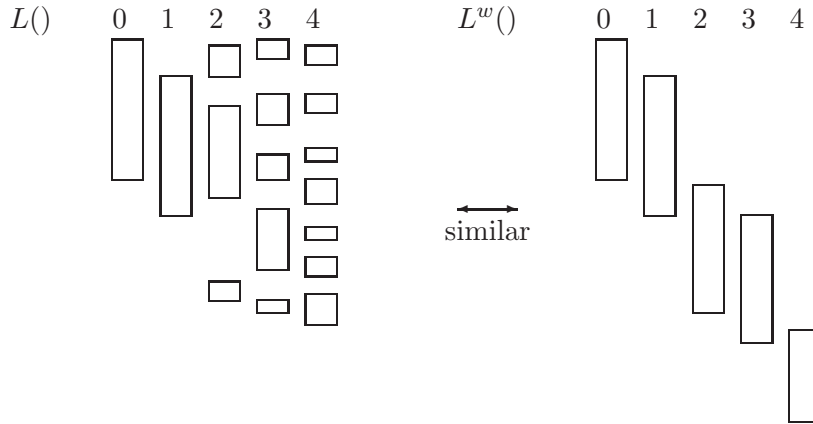


Fig. 1. Example of a list  $L$  which is similar to a waterfall list  $L^w$ .

### 3. PRELIMINARY RESULTS

The purpose of this section is to establish some lemmas and propositions in order to prove the main results in the next section. First, we need some other definitions and notations related to lists:

**Definition 6.** For a weighted path  $P_{n+1}$  of length  $n$ ,

- A list  $L$  is good if  $|L(i)| \geq \omega(i) + \omega(i + 1)$  for any  $i, 1 \leq i \leq n - 1$ .
- A waterfall list  $L$  is ordered if for any  $i \in \{0, \dots, n\}$ ,  $L(i) = [l_i, r_i]$  for some  $l_i, r_i \in \mathbb{N}$  and for any  $i \in \{0, \dots, n - 2\}$ ,  $l_{i+2} > r_i$ .

- A waterfall ordered list  $L$  is compact if  $L(0) = [1, r_0]$  and for any  $i \in \{1, \dots, n\}$ ,  $L(i) = [l_i, r_i]$ , with  $l_i \leq \sum_{k=0}^{i-1} \omega(i) + 1$ .
- The amplitude  $A(i, j)(L)$  (or  $A(i, j)$ ) of a list  $L$  is  $A(i, j)(L) = \cup_{k=i}^j L(k)$ .

As we will see in this section, it is more convenient to work with waterfall lists than with general lists. Hence, we first show that any good list can be transformed into a similar waterfall list.

**Proposition 7.** *For any good list  $L$  of  $P_{n+1}$ , there exists a similar waterfall list  $L^w$  with  $|L^w(i)| = |L(i)|$  for all  $i \in \{0, \dots, n\}$ .*

*Proof.* We are going to transform a good list  $L$  of  $P_{n+1}$  into a waterfall list  $L^w$  and we will prove that  $L^w$  is similar with  $L$ .

First, remark that if a color  $x \in L(i-1)$  but  $x \notin L(i)$  for some  $i$  with  $1 \leq i \leq n-1$ , then for any  $j > i$ , one can change the color  $x$  by a new color  $y \notin A(0, n)(L)$  in the list  $L(j)$ , without changing the choosability of the list. With this remark in hand, we can assume that  $L$  is such that any color  $x$  appears on the lists of consecutive vertices  $i_x, \dots, j_x$ .

Now, by permuting the colors if necessary, we can assume that if  $x < y$  then  $i_x < i_y$  or  $i_x = i_y$  and  $j_x \leq j_y$ .

Repeat the following transformation:

1. Take the minimum color  $x$  for which  $j_x \geq i_x + 2$  i.e. the color  $x$  is present on at least three vertices  $i_x, i_x + 1, i_x + 2, \dots, j_x$ ;
  2. Replace color  $x$  by a new color  $y$  in lists  $L(i_x + 2), \dots, L(j_x)$ ;
- until the obtained list is a waterfall list (obviously, the number of iterations is always finite).

Now, we show that this transformation preserves the choosability of the list. Let  $L'$  be the list obtained from the list  $L$  by the above transformation.

If  $c$  is an  $(L, \omega)$ -coloring of  $P_{n+1}$  then the coloring  $c'$  obtained from  $c$  by changing the color  $x$  by the color  $y$  in the color set  $c(k)$  of each vertex  $k \geq i_x + 2$  (containing  $x$ ) is an  $(L', \omega)$ -coloring since  $y$  is a new color.

Conversely, if  $c'$  is an  $(L', \omega)$ -coloring of  $P_{n+1}$ , we consider two cases:

*Case 1:*  $x \notin c'(i_x + 1)$  or  $y \notin c'(i_x + 2)$ . In this case, the coloring  $c$  obtained from  $c'$  by changing the color  $y$  by the color  $x$  in the color set  $c'(k)$  of each vertex  $k \geq i_x + 2$  (containing  $y$ ) is an  $(L, \omega)$ -coloring.

*Case 2:*  $x \in c'(i_x + 1)$  and  $y \in c'(i_x + 2)$ . We have to consider two subcases:

- Subcase 1:  $L'(i_x + 1) \not\subset (c'(i_x) \cup c'(i_x + 1) \cup c'(i_x + 2))$ . There exists  $z \in L'(i_x + 1) \setminus (c'(i_x) \cup c'(i_x + 1) \cup c'(i_x + 2))$  and the coloring  $c$  obtained from  $c'$  by changing the color  $x$  by the color  $z$  in  $c'(i_x + 1)$  and replacing the color  $y$  by the color  $x$  in the color set  $c'(k)$  of each vertex  $k \geq i_x + 2$  (containing  $y$ ) is an  $(L, \omega)$ -coloring.
- Subcase 2:  $L'(i_x + 1) \subset (c'(i_x) \cup c'(i_x + 1) \cup c'(i_x + 2))$ . We have

$$|L'(i_x + 1)| = \left| \left( (c'(i_x) \cup c'(i_x + 1) \cup c'(i_x + 2)) \cap L'(i_x + 1) \right) \right|.$$

As  $c'$  is an  $(L', \omega)$ -coloring of  $P_{n+1}$ , we have

$$|L'(i_x + 1)| = |c'(i_x + 2) \cap L'(i_x + 1)| + |c'(i_x + 1) \cap L'(i_x + 1)| + \left| \left( c'(i_x) \setminus c'(i_x + 2) \right) \cap L'(i_x + 1) \right|,$$

$$|L'(i_x+1)| - \omega(i_x+1) - |c'(i_x+2) \cap L'(i_x+1)| = \left| \left( c'(i_x) \setminus c'(i_x+2) \right) \cap L'(i_x+1) \right|.$$

Since  $y \in c'(i_x+2)$  and  $y \notin L'(i_x+1)$ , we obtain that

$$|c'(i_x+2) \cap L'(i_x+1)| \leq \omega(i_x+2) - 1,$$

hence

$$\left( |L'(i_x+1)| - \omega(i_x+1) - \omega(i_x+2) \right) + 1 \leq \left| \left( c'(i_x) \setminus c'(i_x+2) \right) \cap L'(i_x+1) \right|.$$

But, by hypothesis,  $L$  is a good list. Thus  $|L(i_x+1)| = |L'(i_x+1)| \geq \omega(i_x+1) + \omega(i_x+2)$  and

$$1 \leq \left| \left( c'(i_x) \setminus c'(i_x+2) \right) \cap L'(i_x+1) \right|.$$

Consequently, there exists  $z \in \left( c'(i_x) \setminus c'(i_x+2) \right) \cap L'(i_x+1)$ . The coloring  $c$  is then constructed from  $c'$  by changing the color  $x$  by the color  $z$  in  $c'(i_x+1)$ , the color  $z$  by the color  $x$  in  $c'(i_x)$  and the color  $y$  by the color  $x$  in the set  $c'(k)$  of each vertex  $k \geq i_x+2$ .

□

As mentioned in the next remark, an  $(L^w, \omega)$ -coloring of  $P_{n+1}$  has some nice properties.

**Remark 8.** *Let  $L^w$  be a waterfall list of  $P_{n+1}$  and  $c$  be an  $(L^w, \omega)$ -coloring of  $P_{n+1}$ . Then, it is clear that if  $i, j \in \{0, \dots, n\}$  such that  $i \neq j$  then*

$$c(i) \cap c(j) = \emptyset \quad \text{and} \quad \left| \bigcup_{k=i}^j c(k) \right| = \sum_{k=i}^j \omega(k).$$

We now present two results about similarities between waterfall, ordered and compact lists.

**Lemma 9.** *For any waterfall list  $L^w$  of  $P_{n+1}$ , there exists a similar ordered waterfall list  $L^o$  with  $|A(i, j)(L^o)| = |A(i, j)(L^w)|$  for all  $i, j \in \{0, \dots, n\}$ .*

*Proof.* As  $L^w$  is a waterfall list, then each color is present on at most two lists of consecutive vertices. Hence one can permute two colors  $x$  and  $y$  without changing the choosability of the weighted path. To obtain a similar ordered waterfall list from  $L = L^w$ , we will apply the following three operations as many times as necessary:

- decrease by one each of the colors that are greater than  $x$  in the lists  $L(i)$  if there exists  $x \in [1, \max(A(0, n))]$  such that for all  $i \in \{0, \dots, n\}$ ,  $x \notin L(i)$ ;
- permute colors  $x$  and  $y$  if  $x \in L(i) \setminus L(i+1)$ ,  $y < x$ , and  $y \in L(i) \cap L(i+1)$ ;
- permute colors  $x$  and  $y$  if  $x \in L(i)$  and  $y \in L(j)$  such that  $i < j$  and  $x > y$ .

Moreover, we have  $|A(i, j)(L^o)| = |A(i, j)(L^w)|$  for all  $i, j \in \{0, \dots, n\}$  since none of the above operations modifies the amplitude of the list.

□

**Lemma 10.** *For any ordered waterfall list  $L^o$  of  $P_{n+1}$ , there exists a similar compact waterfall list  $L^c$  with  $|L^c(i)| = |L^o(i)|$  for all  $i \in \{0, \dots, n\}$ .*

*Proof.* We are going to construct a compact waterfall list  $L^c$  from  $L^o$  in this way (see Fig. 2): set

$$L^c(0) = L^o(0)$$

and for all  $i \in \{1, \dots, n\}$ ,

$$L^c(i) = [\ell_i - d_i, r_i - d_i],$$

with  $d_i = \max(0, \ell_i - \sum_{k=0}^{i-1} \omega(k) - 1)$ .

It is easily seen that if  $c$  is an  $(L^o, \omega)$ -coloring of  $P_{n+1}$ , then the coloring  $c'$  given by  $c'(i) = \{x - d_i | x \in c(i)\}$  for each  $i \in \{0, \dots, n\}$  is an  $(L^c, \omega)$ -coloring of  $P_{n+1}$  and conversely, if  $c$  is an  $(L^c, \omega)$ -coloring of  $P_{n+1}$ , then the coloring  $c'$  given by  $c'(i) = \{x + d_i | x \in c(i)\}$  is an  $(L^o, \omega)$ -coloring of  $P_{n+1}$ .  $\square$

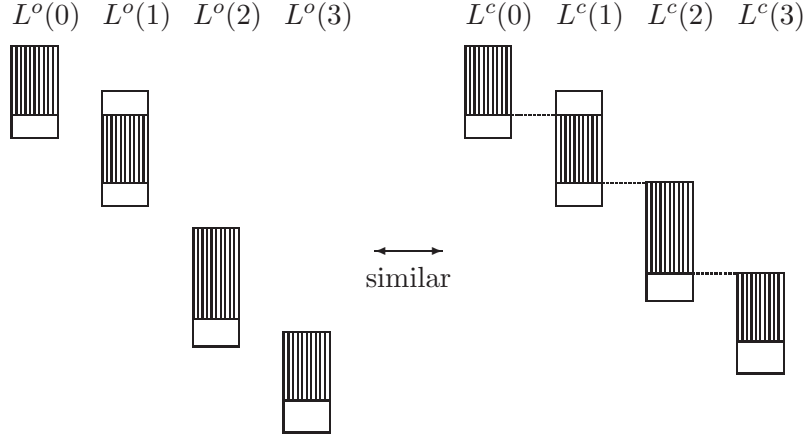


Fig. 2. Example of an ordered waterfall list  $L^o$  which is similar to a compact waterfall list  $L^c$ .

As we will show in the next Proposition, a weighted path with a compact waterfall list  $L^c$  is  $(L^c, \omega)$ -colorable if and only if the greedy coloring algorithm (that assigns for  $i = 0, 1, \dots, n$  the smallest  $\omega(i)$  colors of the list  $L^c(i)$  to vertex  $i$  with respect to the colors of vertex  $i - 1$ ) succeeds in finding a coloring.

**Proposition 11.** *Let  $L^c$  be a compact waterfall list of a weighted path  $P_{n+1}$ . Then  $P_{n+1}$  is  $(L^c, \omega)$ -colorable if and only if:*

$$\forall j \in \{0, \dots, n\}, |A(0, j)(L^c)| \geq \sum_{k=i}^j \omega(k).$$

*Proof.* Firstly, if  $c$  is an  $(L^c, \omega)$ -coloring of  $P_{n+1}$  then

$$\forall j \in \{0, \dots, n\}, A(0, j)(L^c) \supset \bigcup_{k=0}^j c(k).$$

Remark 8 shows that:

$$\forall j \in \{0, \dots, n\}, |A(0, j)(L^c)| \geq \sum_{k=0}^j \omega(k).$$

Secondly, it is easily seen that if  $\forall j \in \{0, \dots, n\}$ ,  $|A(0, j)(L^c)| \geq \sum_{k=0}^j \omega(k)$  then  $P_{n+1}$  is  $(L^c, \omega)$ -colorable since the set of colors used by the greedy coloring algorithm on this compacted list is an interval, i.e. the intervals of colors of two consecutive vertices are consecutive (see Fig. 2).  $\square$

#### 4. CHOOSABILITY OF A WEIGHTED PATH

We are ready to state the following theorem which gives a necessary and sufficient condition for a weighted path to be  $(L^w, \omega)$ -colorable where  $L^w$  is a waterfall list. This is not a big restriction to consider waterfall lists because good lists introduced in Section 3 are those well adapted to list coloring problems, and, thanks to Proposition 7, it remains to consider waterfall lists.

Then, the general problem of list coloring for a weighted path reduces to a very easy condition to verify:

**Theorem 12.** *Let  $L^w$  be a waterfall list of a weighted path  $P_{n+1}$ . Then  $P_{n+1}$  is  $(L^w, \omega)$ -colorable if and only if:*

$$\forall i, j \in \{0, \dots, n\}, \left| \bigcup_{k=i}^j L^w(k) \right| \geq \sum_{k=i}^j \omega(k).$$

*Proof.* Let  $\omega(i, j) = \sum_{k=i}^j \omega(k)$  and recall that  $A(i, j)(L) = \bigcup_{k=i}^j L(k)$ .

By Lemma 9, there exists an ordered waterfall list  $L^o$  which is similar with  $L^w$  and by Lemma 10, there exists a compact waterfall list  $L^c$  which is similar with  $L^o$  (and thus with  $L^w$ ).

Now, with Proposition 11, we have that  $P_{n+1}$  is  $(L^c, \omega)$ -colorable if and only if:

$$\forall j \in \{0, \dots, n\}, |A(0, j)(L^c)| \geq \omega(0, j).$$

By Lemma 9, we can assume that for all  $i, j \in \{0, \dots, n\}$ , we have  $|A(i, j)(L^w)| = |A(i, j)(L^o)|$ . Hence it remains to show that

$$\forall i, j \in \{0, \dots, n\}, |A(i, j)(L^o)| \geq \omega(i, j) \Leftrightarrow |A(0, j)(L^c)| \geq \omega(0, j).$$

First, we show that the left-hand inequality implies the right one. Let  $L^o(i) = [l_i^o, r_i^o]$  and  $L^c(i) = [l_i^c, r_i^c]$  and assume that  $\forall i, j \in \{0, \dots, n\}$ , we have  $|A(i, j)(L^o)| \geq \omega(i, j)$ . Let  $j \in \{0, \dots, n\}$  and let  $i$  be the largest integer not greater than  $j$  such that  $l_i^o > \omega(0, i-1)$  (by convention,  $\omega(0, -1) = 0$ ). Then, by construction,  $l_i^c = \omega(0, i-1) + 1$  and by the choice of  $i$ , we have  $|A(i, j)(L^c)| = |A(i, j)(L^o)| \geq \omega(i, j)$ . Consequently,

$$|A(0, j)(L^c)| = \omega(0, i-1) + |A(i, j)(L^c)| \geq \omega(0, i-1) + \omega(i, j) = \omega(0, j).$$

Conversely, suppose that for all  $j \in \{0, \dots, n\}$ ,  $|A(0, j)(L^c)| \geq \omega(0, j)$ . As  $|A(i, j)(L^o)| \geq |A(i, j)(L^c)|$ , then if  $i = 0$  we have  $|A(0, j)(L^o)| \geq \omega(0, j)$  and if  $i > 0$ ,  $|A(i, j)(L^o)| \geq |A(i, j)(L^c)| \geq |A(0, j)(L^c)| - \omega(0, i-1) \geq \omega(0, j) - \omega(0, i-1) = \omega(i, j)$ .  $\square$

The previous result has the following corollary when the list is a good waterfall list and  $|L(n)| \geq \omega(n)$ .



**Corollary 13.** *Let  $L^w$  be a waterfall list of a weighted path  $P_{n+1}$  such that  $|L^w(i)| \geq \omega(i) + \omega(i+1)$  for any  $i, 1 \leq i \leq n-1$  and  $|L^w(n)| \geq \omega(n)$ . Then  $P_{n+1}$  is  $(L^w, \omega)$ -colorable if and only if*

$$\forall j \in \{0, \dots, n\}, \left| \bigcup_{k=0}^j L^w(k) \right| \geq \sum_{k=0}^j \omega(k).$$

*Proof.* Under the hypothesis, if  $P_{n+1}$  is  $(L^w, \omega)$ -colorable, then Theorem 12 proves in particular the result.

Conversely, since  $L^w$  is a waterfall list of  $P_{n+1}$ , we have:

$$\forall i, j \in \{1, \dots, n\}, |A(i, j)| = \left| \bigcup_{k=i}^j L^w(k) \right| \geq \left| \bigcup_{\substack{k=i \\ k-i \text{ even}}}^j L^w(k) \right| = \sum_{\substack{k=i \\ k-i \text{ even}}}^j |L^w(k)|.$$

Since  $L^w$  is a good list of  $P_{n+1}$  (for simplicity, we set  $\omega(n+1) = 0$ ):

$$\forall i, j \in \{1, \dots, n\}, \sum_{\substack{k=i \\ k-i \text{ even}}}^j |L^w(k)| \geq \sum_{\substack{k=i \\ k-i \text{ even}}}^j (\omega(k) + \omega(k+1)) \geq \sum_{k=i}^j \omega(k),$$

then we obtain for all  $i, j \in \{1, \dots, n\}$ ,  $|A(i, j)| \geq \sum_{k=i}^j \omega(k)$ . Since for all  $j \in \{0, \dots, n\}$ ,  $|A(0, j)| \geq \sum_{k=0}^j \omega(k)$ , Theorem 12 concludes the proof.  $\square$

Another interesting corollary holds for lists  $L$  such that  $|L(0)| = |L(n)| = b$ , and for all  $i \in \{1, \dots, n-1\}$ ,  $|L(i)| = a$ . The function  $\text{Even}$  is defined for any real  $x$  by:  $\text{Even}(x)$  is the smallest even integer  $p$  such that  $p \geq x$ .

**Corollary 14.** *Let  $L$  be a list of  $P_{n+1}$  such that  $|L(0)| = |L(n)| = b$ , and  $|L(i)| = a = 2b + e$  for all  $i \in \{1, \dots, n-1\}$  (with  $e \neq 0$ ).*

*If  $n \geq \text{Even}\left(\frac{2b}{e}\right)$  then  $P_{n+1}$  is  $(L, b)$ -colorable.*

*Proof.* The hypothesis implies that  $L$  is a good list of  $P_{n+1}$ , hence by Proposition 7, there exists a waterfall list  $L^w$  similar to  $L$ . So we get:

$$\forall i \in \{1, \dots, n-1\}, |L^w(i)| \geq 2b = \omega(i) + \omega(i+1)$$

and  $|L^w(n)| \geq b = \omega(n)$ . By Corollary 13 it remains to prove that:

$$\forall j \in \{0, \dots, n\}, |A(0, j)| \geq \sum_{k=0}^j \omega(k) = (j+1)b.$$

*Case 1:*  $j = 0$ . By hypothesis, we have  $|A(0, 0)| = |L^w(0)| \geq b$ .

*Case 2:*  $j \in \{1, \dots, n-1\}$ . Since  $L^w$  is a waterfall list of  $P_{n+1}$  we obtain that:

if  $j$  is even

$$|A(0, j)| \geq \sum_{\substack{k=0 \\ k \text{ even}}}^j |L^w(k)| = b + \sum_{\substack{k=2 \\ k \text{ even}}}^j 2b = b + \frac{j}{2} 2b = (j+1)b,$$

and if  $j$  is odd

$$|A(0, j)| \geq \sum_{\substack{k=0 \\ k \text{ odd}}}^j |L^w(k)| = \sum_{\substack{k=1 \\ k \text{ odd}}}^j 2b = \frac{j+1}{2} 2b = (j+1)b.$$

Hence for all  $j \in \{0, \dots, n-1\}$ ,  $|A(0, j)| \geq (j+1)b$ .

Case 3:  $j = n$ . Since  $n \geq \text{Even}\left(\frac{2b}{e}\right)$  by hypothesis, and

$$|A(0, n)| \geq \sum_{\substack{k=0 \\ k \text{ odd}}}^n |L^w(k)| = \begin{cases} a\frac{n}{2} & \text{if } n \text{ is even} \\ b + a\frac{n-1}{2} & \text{otherwise} \end{cases}$$

we deduce that  $|A(0, n)| \geq (n+1)b$ , which concludes the proof.  $\square$

For example, let  $P_{n+1}$  be the path of length  $n$  with a list  $L$  such that  $|L(0)| = |L(n)| = 4$ , and  $|L(i)| = 9$  for all  $i \in \{1, \dots, n-1\}$ . Then the previous Corollary tells us that we can find an  $(L, 4)$ -coloring of  $P_{n+1}$  whenever  $n \geq 8$ . In other words, if  $n \geq 8$ , we can choose 4 colors on each vertex such that adjacent vertices receive disjoint colors. If  $|L(i)| = 11$  for all  $i \in \{1, \dots, n-1\}$ , then  $P_{n+1}$  is  $(L, 4)$ -colorable whenever  $n \geq 4$ .

## 5. FREE-CHOOSABILITY OF A CYCLE

The cycle  $C_n$  of length  $n$  is the graph with vertex set  $V = \{v_0, \dots, v_{n-1}\}$  and edge set  $E = \bigcup_{i=0}^{n-1} \{v_i v_{i+1(\text{mod } n)}\}$ .

Let  $FC_h(x)$  be the set of graphs  $G$  which are  $(a, b)$ -free-choosable for all  $a, b$  such that  $\frac{a}{b} \geq x$ :

$$FC_h(x) = \{G \mid \forall \frac{a}{b} \geq x, G \text{ is } (a, b)\text{-free-choosable}\}.$$

Moreover, we can define the free-choice ratio  $\text{fchr}(G)$  of a graph  $G$  by:

$$\text{fchr}(G) := \inf\left\{\frac{a}{b} \mid G \text{ is } (a, b)\text{-free-choosable}\right\}.$$

If  $\lfloor x \rfloor$  denotes the greatest integer less or equal to the real  $x$ , we can state:

**Theorem 15.** *If  $C_n$  is a cycle of length  $n$ , then*

$$C_n \in FC_h\left(2 + \left\lfloor \frac{n}{2} \right\rfloor^{-1}\right).$$

Moreover, we have:

$$\text{fchr}(C_n) = 2 + \left\lfloor \frac{n}{2} \right\rfloor^{-1}.$$

*Proof.* Let  $a, b$  be two integers such that  $a/b \geq 2 + \lfloor \frac{n}{2} \rfloor^{-1}$ . Let  $C_n$  be a cycle of length  $n$  and  $L$  a  $a$ -list of  $C_n$ . Without loss generality, we can suppose that  $v_0$  is the vertex chosen for the free-choosability and  $L_0 \subset L(v_0)$  has  $b$  elements. It remains to construct an  $(L, b)$ -coloring  $c$  of  $C_n$  such that  $c(v_0) = L_0$ . Hence we have to construct an  $(L', b)$ -coloring  $c$  of  $P_{n+1}$  such that  $L'(0) = L'(n) = L_0$  and for all  $i \in \{1, \dots, n-1\}$ ,  $L'(i) = L(v_i)$ . We have  $|L'(0)| = |L'(n)| = b$  and for all  $i \in \{1, \dots, n-1\}$ ,  $|L'(i)| = a$ . Since

$a/b \geq 2 + \lfloor \frac{n}{2} \rfloor^{-1}$  and  $e = a - 2b$ , we get  $e/b \geq \lfloor \frac{n}{2} \rfloor^{-1}$  hence  $n \geq \text{Even}(2b/e)$ . Using Corollary 14, we get:

$$C_n \in FC_h(2 + \lfloor \frac{n}{2} \rfloor^{-1}).$$

Hence, we have that  $\text{fchr}(C_n) \leq 2 + \lfloor \frac{n}{2} \rfloor^{-1}$ . Moreover, let us prove that  $M = 2 + \lfloor \frac{n}{2} \rfloor^{-1}$  is reached.

For  $n$  odd, Voigt has proved in [17] that the choice ratio  $\text{chr}(C_n)$  of a cycle of odd length  $n$  is exactly  $M$ . Hence  $\text{fchr}(C_n) \geq \text{chr}(C_n) = M$ , and the result is proved.

For  $n$  even, let  $a, b$  be two integers such that  $\frac{a}{b} < M$ . We construct a counterexample for the free-choosability: let  $L$  be the list of  $C_n$  such that

$$L(i) = \begin{cases} \{1, \dots, a\} & \text{if } i \in \{0, 1\} \\ \{1 + \frac{i-1}{2}a, \dots, (\frac{i-1}{2} + 1)a\} & \text{if } i \neq n-1 \text{ is odd} \\ \{b + 1 + \frac{i-2}{2}a, \dots, b + (\frac{i-2}{2} + 1)a\} & \text{if } i \text{ is even and } i \neq 0 \\ \{1, \dots, b, 1 + (\frac{n-4}{2} + 1)a, \dots, (\frac{n-4}{2} + 2)a - b\} & \text{if } i = n-1 \end{cases}$$

If we choose  $c_0 = \{1, \dots, b\} \subset L(0)$ , we can check that it does not exist an  $(L, b)$ -coloring of  $C_n$  such that  $c(0) = c_0$ , so we could not do better.  $\square$

**Remark 16.** *In particular, the previous theorem implies that if  $n \geq \text{Even}(\frac{2b}{e})$  then the cycle  $C_n$  of length  $n$  is  $(2b + e, b)$ -free-choosable.*

**Remark 17.** *Erdős, Rubin and Taylor have stated in [4] the following question: If  $G$  is  $(a, b)$ -colorable, and  $\frac{c}{d} > \frac{a}{b}$ , does it imply that  $G$  is  $(c, d)$ -colorable? Gutner and Tarsi have shown in [7] that the answer is negative in general. If we consider the analogue question for free-choosability, then the previous theorem implies that it is true for the cycle.*

## REFERENCES

- [1] N. Alon, Zs. Tuza, M. Voigt, *Choosability and fractional chromatic number*, Discrete Math. 165/166, (1997), 31-38.
- [2] Y. Aubry, J.-C. Godin and O. Togni, *Extended core and choosability of a graph*, arXiv:1006.2958v1 [cs.DM], 2010.
- [3] O.V. Borodin, A.V. Kostochka, D.R. Woodall, *List edge and list colourings of multi-graph*, J. Combin. Theory Series B, 71 : 184-204, (1997).
- [4] P. Erdős, A.L Rubin and H. Taylor, *Choosability in graphs*, Proc. West-Coast Conf. on Combinatorics, Graph Theory and Computing, Congressus Numerantium XXVI, (1979), 125-157.
- [5] J.-C. Godin, *Coloration et choisissabilité des graphes et applications*, PhD thesis (in french), Université du Sud Toulon-Var, France (2009).
- [6] S. Gravier, *A Hajós-like theorem for list coloring*, Discrete Math. 152, (1996), 299-302.
- [7] S. Gutner and M. Tarsi, *Some results on  $(a:b)$ -choosability*, Discrete Math. 309, (2009), 2260-2270.
- [8] F. Havet, *Choosability of the square of planar subcubic graphs with large girth*. Discrete Math. 309, (2009), 3553-3563.
- [9] F. Havet, *Channel assignment and multicolouring of the induced subgraphs of the triangular lattice*. Discrete Math. 233, (2001), 219-233.
- [10] F. Havet, J. Zerovnik. *Finding a five bicoloring of a triangle-free subgraph of the triangular lattice*. Discrete mathematics 244, (2002), 103-108.

- [11] M. Kchikech and O. Togni. *Approximation algorithms for multicoloring powers of square and triangular meshes*, Discrete Math. and Theoretical Computer Science, Vol. 8 (1):159-172, 2006.
- [12] C. McDiarmid and B. Reed. *Channel assignement and weighted coloring*. Networks, 36, (2000), 114-117.
- [13] C. Thomassen, *The chromatic number of a graph of girth 5 on a fixed surface*, J. Combin. Theory, (2003), 38-71.
- [14] Zs. Tuza and M. Voigt, *Every 2-choosable graph is  $(2m,m)$ -choosable*, J. Graph Theory 22, (1996), 245-252.
- [15] V. G Vizing, *Coloring the vertices of a graph in prescribed colors (in Russian)*, Diskret. Analiz. No. 29, Metody Diskret. Anal. v Teorii Kodov i Shem 101 (1976), 3-10.
- [16] M. Voigt, *Choosability of planar graphs*, Discrete Math., 150, (1996), 457-460.
- [17] M. Voigt, *On list Colourings and Choosability of Graphs*, Abilitationsschrift, TU Ilmenau (1998).

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