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# CHOOSABILITY OF A WEIGHTED PATH AND FREE-CHOOSABILITY OF A CYCLE 

YVES AUBRY, JEAN-CHRISTOPHE GODIN AND OLIVIER TOGNI


#### Abstract

A graph $G$ with a list of colors $L(v)$ and weight $\omega(v)$ for each vertex $v$ is $(L, \omega)$-choosable if one can choose a subset of $\omega(v)$ colors from $L(v)$ for each vertex $v$, such that adjacent vertices receive disjoint color sets. In this paper, we give a necessary and sufficient condition for a weighted path to be $(L, \omega)$-choosable for some lists $L$. Furthermore, we solve the problem of the free-choosability of a cycle.


## 1. Introduction

The concept of choosability of a graph, also called list colouring, has been introduced by Vizing in [11], and independently by Erdős, Rubin and Taylor in [3]. It contains of course the colorability as a particular case. Since its introduction, choosability has been extensively studied (see for example [1, 2, 2, $0,10,5,8]$ and very recently [7, [6]).

Following the work in [母], we introduce here the notion of a good list (it is an obvious necessary condition on the list for the graph to be choosable) and of a waterfall list (list adapted to find a choosability) and we show that any list of the first kind can be transformed into a list of the second one. We give as a first main result a necessary and sufficient condition for a weighted path with a given waterfall list $L$ to be ( $L, \omega$ )-choosable (Theorem (5).

In 1996, Voigt considered the following problem: let $G$ be a graph and $L$ a list assignment and assume that an arbitrary vertex $v \in V(G)$ is precoloured by a colour $f \in L(v)$. Is it always possible to complete this precolouring to a proper list colouring? This question leads to the concept of free-choosability introduced by Voigt in (12].

We investigate here the free-choosability of the first interesting case, namely the cycle. As an application of Theorem Q, we prove our second main result which gives a necessary and sufficient condition for a cycle to be free-choosable (Theorem 12). In order to get a concise statement, we introduce the free-choice ratio of a graph, in the same way that Alon, Tuza and Voigt in []] introduced the choice ratio (which equals the so-called fractional chromatic number).

We recall in Section 2 the definitions of choosability and free-choosability.
Section 3 is devoted to the choosability of a weighted path and section 4 to the free-choosability of a cycle.

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## 2. Definitions

Let $G=(V(G), E(G))$ be a graph where $V(G)$ is the set of vertices and $E(G)$ is the set of edges, and let $a, b, n$ and $e$ be integers.

Let $L$ be a list of $G$ i.e. a map: $L: V(G) \rightarrow \mathcal{P}(\mathbb{N})$ and let $\omega$ be a weight of $G$ i.e. a map: $\omega: V(G) \rightarrow \mathbb{N}$.

Definition 1. An $(L, \omega)$-choosability $c$ of a graph $G$ is a list of the weighted graph $G$ such that for all vv $\in E(G)$ :

$$
c(v) \subset L(v), \quad|c(v)|=\omega(v) \quad \text { and } \quad c(v) \cap c\left(v^{\prime}\right)=\emptyset .
$$

We say that $G$ is $(L, \omega)$-choosable if there exists an $(L, \omega)$-choosability $c$ of $G$.

An $(L, b)$-choosability $c$ of $G$ is an $(L, \omega)$-choosability of $G$ such that for all $v \in V(G)$, we have $\omega(v)=b$.

A $a$-list $L$ of $G$ is a list of $G$ such that for all $v \in V(G)$, we have $|L(v)|=a$.

Definition 2. $G$ is said to be ( $a, b$ )-choosable if for any a-list $L$ of $G$, there exists an $(L, b)$-choosability $c$ of $G$.

Definition 3. $G$ is said to be $(a, b)$-free-choosable if for any $v_{0} \in V(G)$, and for any list $L$ of $G$ such that for any $v \in V(G) \backslash\left\{v_{0}\right\}:|L(v)|=a$ and $\left|L\left(v_{0}\right)\right|=b$, there exists an $(L, b)$-choosability $c$ of $G$.

## 3. Choosability of a weighted path

The path $P_{n+1}$ or length $n$ is the graph with vertex set $V=\left\{v_{0}, v_{1}, \ldots, v_{n}\right\}$ and edge set $E=\bigcup_{i=0}^{n-1}\left(v_{i}, v_{i+1}\right)$. To simplify the notations $L(i)$ denotes $L\left(v_{i}\right)$ and $c(i)$ denotes $c\left(v_{i}\right)$.

Definition 4. For the path $P_{n+1}$ of order $n$,

- $a$ waterfall list $L$ is a list such that for all $i, j \in\{0, \ldots, n\}$ with $|i-j| \geq 2, L(i) \cap L(j)=\emptyset ;$
- two lists $L$ and $L^{\prime}$ are similar if and only if $P_{n+1}$ is $(L, \omega)$-choosable whenever $P_{n+1}$ is $\left(L^{\prime}, \omega\right)$-choosable;
- the amplitude $A_{i, j}(L)$ (or $A_{i, j}$ ) of a list $L$ is $A_{i, j}(L)=\cup_{k=i}^{j} L(k)$;
- a list $L$ is $\operatorname{good}$ if $|L(i)| \geq \omega(i)+\omega(i+1)$ for any $i, 1 \leq i \leq n-1$.

Notice that another similar definition of a waterfall list is that any color is present only on one list or on two lists of consecutive vertices.


Fig. 1. Example of a list which is similar to a waterfall list.

Figure 1 shows a list $L$ of $P_{5}$ (on the left), together with a similar waterfall list $L^{c}$ (on the right).

Proposition 5. For any good list $L$ of $P_{n+1}$, there exists a similar waterfall list $L^{c}$ with $\left|L^{c}(i)\right|=|L(i)|$ for all $i \in\{0, \ldots, n\}$.

Proof. We are going to transform a good list $L$ of $P_{n+1}$ into a waterfall list $L^{c}$ and we will prove that $L^{c}$ is similar with $L$.

First, remark that if a color $x \in L(i-1)$ but $x \notin L(i)$ for some $i, 1 \leq$ $i \leq n-1$, then for any $j>i$, one can change the color $x$ by a new color $y \notin A(L)$ in the list $L(j)$, without changing the choosability of the list. With this remark in hand, we can assume that $L$ is such that any color $x$ appears on the lists of consecutive vertices $i_{x}, \ldots, j_{x}$.

Now, by permuting the colors if necessary, we can assume that if $x<y$ then $i_{x}<i_{y}$ or $i_{x}=i_{y}$ and $j_{x} \leq j_{y}$.

Repeat the following transformation:

1. Take the minimum color $x$ for which $j_{x} \geq i_{x}+2$ (i.e. the color $x$ is present on at least three vertices $i_{x}, i_{x}+1, i_{x}+2, \ldots, j_{x}$;
2. Replace color $x$ by a new color $y$ in lists $L\left(i_{x}+2\right), \ldots, L\left(j_{x}\right)$;
until the obtained list is a waterfall list (obviously, the number of iterations is always finite).

Now, we show that this transformation preserves the choosability of the list: Let $L^{\prime}$ be the list obtained from the list $L$ by the above transformation.

If $c$ is an $(L, \omega)$-choosability of $P_{n+1}$ then the choosability $c^{\prime}$ obtained from $c$ by changing the color $x$ by the color $y$ in the color set $c(k)$ of each vertex $k \geq i_{x}+2$ (containing $x$ ) is an $\left(L^{\prime}, \omega\right)$-choosability since $y$ is a new color.

Conversly, if $c^{\prime}$ is an $\left(L^{\prime}, \omega\right)$-choosability of $P_{n+1}$, we consider two cases:
Case 1: $x \notin c^{\prime}\left(i_{x}+1\right)$ or $y \notin c^{\prime}\left(i_{x}+2\right)$. In this case, the choosability $c$ obtained from $c^{\prime}$ by changing the color $y$ by the color $x$ in the color set $c^{\prime}(k)$ of each vertex $k \geq i_{x}+2$ (containing $y$ ) is an ( $L, \omega$ )-choosability.

Case 2: $x \in c^{\prime}\left(i_{x}+1\right)$ and $y \in c^{\prime}\left(i_{x}+2\right)$. We have to consider two subcases:

- Subcase 1: $L^{\prime}\left(i_{x}+1\right) \not \subset\left(c^{\prime}\left(i_{x}\right) \cup c^{\prime}\left(i_{x}+1\right) \cup c^{\prime}\left(i_{x}+2\right)\right)$. There exists $z \in L^{\prime}\left(i_{x}+1\right) \backslash\left(c^{\prime}\left(i_{x}\right) \cup c^{\prime}\left(i_{x}+1\right) \cup c^{\prime}\left(i_{x}+2\right)\right)$ and the choosability $c$ obtained from $c^{\prime}$ by changing the color $x$ by the color $z$ in $c^{\prime}\left(i_{x}+1\right)$ and replacing the color $y$ by the color $x$ in the color set $c^{\prime}(k)$ of each vertex $k \geq i_{x}+2$ (containing $y$ ) is an ( $L, \omega$ )-choosability.
- Subcase 2: $L^{\prime}\left(i_{x}+1\right) \subset\left(c^{\prime}\left(i_{x}\right) \cup c^{\prime}\left(i_{x}+1\right) \cup c^{\prime}\left(i_{x}+2\right)\right)$. We have

$$
\left|L^{\prime}\left(i_{x}+1\right)\right|=\mid\left(\left(c^{\prime}\left(i_{x}\right) \cup c^{\prime}\left(i_{x}+1\right) \cup c^{\prime}\left(i_{x}+2\right)\right) \cap L^{\prime}\left(i_{x}+1\right) \mid .\right.
$$

As $c^{\prime}$ is an $\left(L^{\prime}, \omega\right)$-choosability of $P_{n+1}$, we have

$$
\begin{aligned}
& \left|L^{\prime}\left(i_{x}+1\right)\right|=\left|c^{\prime}\left(i_{x}+2\right) \cap L^{\prime}\left(i_{x}+1\right)\right|+\left|c^{\prime}\left(i_{x}+1\right) \cap L^{\prime}\left(i_{x}+1\right)\right|+\left|\left(c^{\prime}\left(i_{x}\right) \backslash c^{\prime}\left(i_{x}+2\right)\right) \cap L^{\prime}\left(i_{x}+1\right)\right| \\
& \left|L^{\prime}\left(i_{x}+1\right)\right|-\omega\left(i_{x}+1\right)-\left|c^{\prime}\left(i_{x}+2\right) \cap L^{\prime}\left(i_{x}+1\right)\right|=\left|\left(c^{\prime}\left(i_{x}\right) \backslash c^{\prime}\left(i_{x}+2\right)\right) \cap L^{\prime}\left(i_{x}+1\right)\right|
\end{aligned}
$$

Since $y \in c^{\prime}\left(i_{x}+2\right)$ and $y \notin L^{\prime}\left(i_{x}+1\right)$, we obtain that

$$
\left|c^{\prime}\left(i_{x}+2\right) \cap L^{\prime}\left(i_{x}+1\right)\right| \leq \omega\left(i_{x}+2\right)-1
$$

hence

$$
\left(\left|L^{\prime}\left(i_{x}+1\right)\right|-\omega\left(i_{x}+1\right)-\omega\left(i_{x}+2\right)\right)+1 \leq\left|\left(c^{\prime}\left(i_{x}\right) \backslash c^{\prime}\left(i_{x}+2\right)\right) \cap L^{\prime}\left(i_{x}+1\right)\right| .
$$

But, by hypothesis, $L$ is a good list. Thus $\left|L\left(i_{x}+1\right)\right|=\left|L^{\prime}\left(i_{x}+1\right)\right| \geq$ $\omega\left(i_{x}+1\right)+\omega\left(i_{x}+2\right)$ and

$$
1 \leq\left|\left(c^{\prime}\left(i_{x}\right) \backslash c^{\prime}\left(i_{x}+2\right)\right) \cap L^{\prime}\left(i_{x}+1\right)\right|
$$

Consequently, there exists $z \in\left(c^{\prime}\left(i_{x}\right) \backslash c^{\prime}\left(i_{x}+2\right)\right) \cap L^{\prime}\left(i_{x}+1\right)$. The choosability $c$ is then constructed from $c^{\prime}$ by changing the color $x$ by the color $z$ in $c^{\prime}\left(i_{x}+1\right)$, the color $z$ by the color $x$ in $c^{\prime}\left(i_{x}\right)$ and the color $y$ by the color $x$ in the set $c^{\prime}(k)$ of each vertex $k \geq i_{x}+2$.

Remark 6. Let $L^{c}$ be a waterfall list of $P_{n+1}$ and $c$ be an $\left(L^{c}, \omega\right)$-choosability of $P_{n+1}$. Then, it is clear that if $i, j \in\{0, \ldots, n\}$ such that $i \neq j$ then

$$
c(i) \cap c(j)=\emptyset \quad \text { and } \quad\left|\bigcup_{k=i}^{j} c(k)\right|=\sum_{k=i}^{j} \omega(k)
$$

We now need to define the notion of waterfall choosability (see Figure 2 for an illustration). For a list $L$ of $P_{n+1}$, set

$$
I(i, j):=\left\{\begin{array}{l}
\bigcap_{\emptyset=i}^{j} L(k) \text { if } j \geq i \\
\emptyset \quad \text { otherwise }
\end{array}\right.
$$

and

$$
I_{p}(i, j):=I(i, j) \backslash(L(i-1) \cup L(j+1))
$$

Definition 7. Let $L^{c}$ be a waterfall list of $P_{n+1}$ and c be an $\left(L^{c}, \omega\right)$-choosability of $P_{n+1}$. We say that $c$ is a waterfall $\left(L^{c}, \omega\right)$-choosability of $P_{n+1}$ if for all $i \in\{0, \ldots, n\}$ :

$$
c(i)=K_{2 i-1} \cup F_{i} \cup K_{2 i}
$$

with
$K_{2 i-1} \subset I_{p}(i-1, i) \backslash K_{2 i-2}$ such that $\left|K_{2 i-1}\right|=\min \left\{\left|I_{p}(i-1, i) \backslash K_{2 i-2}\right|, \omega(i)\right\}$, $F_{i} \subset I_{p}(i, i)$ such that $\left|F_{i}\right|=\min \left\{\left|I_{p}(i, i)\right|, \omega(i)-\left|K_{2 i-1}\right|\right\}$, $K_{2 i} \subset I_{p}(i, i+1)$ such that $\left|K_{2 i}\right|=\omega(i)-\left|K_{2 i-1}\right|-\left|F_{i}\right|$.
And with the convention $K_{-1}=K_{-2}=\emptyset$.


Fig. 2. Example of a waterfall choosability.

Lemma 8. Let $L^{c}$ be a waterfall list of $P_{n+1}$. There exists an $\left(L^{c}, \omega\right)$ choosability $c$ of $P_{n+1}$ if and only if there exists a waterfall $\left(L^{c}, \omega\right)$-choosability $c^{c}$ of $P_{n+1}$.

Proof. Under the hypothesis, if $c^{c}$ is a waterfall $\left(L^{c}, \omega\right)$-choosability of $P_{n+1}$ then $c=c^{c}$ is an $\left(L^{c}, \omega\right)$-choosability of $P_{n+1}$.

Conversely, if $c$ is an $\left(L^{c}, \omega\right)$-choosability of $P_{n+1}$ then we construct by induction $c_{j}$ for $j \in\{0, \ldots, n\}$ such that: $c_{j}$ is a waterfall $\left(L^{c}, \omega\right)$-choosability of $P_{j+1}$, and $c_{j}$ is an $\left(L^{c}, \omega\right)$-choosability of $P_{n+1}$, and

$$
c_{j}(i)= \begin{cases}c_{j-1}(i) & \text { if } i \in\{0, \ldots, j-1\} \\ c(i) & \text { if } i \in\{j+1, \ldots, n\} .\end{cases}
$$

We begin by the construction of $c_{0}$. Let $f$ and $\delta$ be two integers such that $f=\left|c(0) \cap I_{p}(0,0)\right|$ and $\delta=\min \left\{\left|I_{p}(0,0)\right|, \omega(0)\right\}$. We have obviously $f \leq \delta$. Since

$$
\left|c(0) \cap I_{p}(0,1)\right|=|c(0)|-\left|c(0) \cap I_{p}(0,0)\right|=\omega(0)-f \geq \delta-f
$$

and

$$
\left|I_{p}(0,0) \backslash c(0)\right|=\left|I_{p}(0,0)\right|-f \geq \delta-f
$$

we obtain that there exist two sets $k \subset c(0) \cap I_{p}(0,1)$ and $r \subset I_{p}(0,0) \backslash c(0)$ such that $|k|=|r|=\delta-f$. The inversible process between $c_{0}$ and $c$ is the following one:

$$
c_{0}(i)= \begin{cases}r \cup c(0) \backslash k & \text { if } i=0 \\ c(i) & \text { otherwise }\end{cases}
$$

By construction of $r$ we have: $c_{0}(0) \cap c_{0}(1)=\emptyset$. Hence $c_{0}$ is a waterfall $\left(L^{c}, \omega\right)$-choosability of $P_{1}$ and $c_{0}$ is an $\left(L^{c}, \omega\right)$-choosability of $P_{n+1}$.

Let $j \geq 1$ and suppose that $c_{j-1}$ is constructed. Recall that $K_{2 j-2}=$ $c_{j-1}(j-1) \cap I_{p}(j-1, j)$. In the same spirit as above, we consider two integers $f$ and $\delta$ such that $f=\left|c_{j-1}(j) \cap I_{p}(j-1, j)\right|$ and $\delta=\min \left\{\mid I_{p}(j-\right.$ $\left.1, j) \backslash K_{2 j-2} \mid, \omega(j)\right\}$. We have of course $\delta \geq f$. Since

$$
\left|c_{j-1}(j) \cap\left(I_{p}(j, j) \cup I_{p}(j, j+1)\right)\right|=\left|c_{j-1}(j)\right|-f=\omega(j)-f \geq \delta-f
$$

and

$$
\left|I_{p}(j-1, j) \backslash\left(K_{2 j-2} \cup c_{j-1}(j)\right)\right|=\left|I_{p}(j-1, j) \backslash K_{2 j-2}\right|-f \geq \delta-f
$$

there exist two sets

$$
k \subset c_{j-1}(j) \cap\left(I_{p}(j, j) \cup I_{p}(j, j+1)\right)
$$

and

$$
r \subset I_{p}(j-1, j) \backslash\left(K_{2 j-2} \cup c_{j-1}(j)\right)
$$

such that $|k|=|r|=\delta-f$. We construct an intermediary choosability $c_{j}^{\prime}$ using $c_{j-1}$ with the following inversible process:

$$
c_{j}^{\prime}(i)= \begin{cases}r \cup c_{j-1}(j) \backslash k & \text { if } i=j \\ c_{j-1}(i) & \text { otherwise }\end{cases}
$$

We obtain that $\left|c_{j}^{\prime}(j) \cap I_{p}(j-1, j)\right|=\min \left\{\left|I_{p}(j-1, j) \backslash K_{2 j-2}\right|, \omega(j)\right\}$.
Using the same idea, we continue. Recall that $K_{2 j-1}=c_{j}^{\prime}(j) \cap I_{p}(j-1, j)$. Let $f^{\prime}$ and $\delta^{\prime}$ be two integers such that $f^{\prime}=\left|c_{j}^{\prime}(j) \cap I_{p}(j, j)\right|$ and $\delta^{\prime}=$ $\min \left\{\left|I_{p}(j, j)\right|, \omega(j)-\left|K_{2 j-1}\right|\right\}$. We have $\delta^{\prime} \geq f^{\prime}$. Since

$$
\left|c_{j}^{\prime}(j) \cap I_{p}(j, j+1)\right|=\left(\omega(j)-\left|K_{2 j-1}\right|\right)-f^{\prime} \geq \delta^{\prime}-f^{\prime}
$$

and

$$
\left|I_{p}(j, j) \backslash c_{j}^{\prime}(j)\right|=\left|I_{p}(j, j)\right|-f^{\prime} \geq \delta^{\prime}-f^{\prime}
$$

we obtain that there exist two sets $k^{\prime} \subset c_{j}^{\prime}(j) \cap I_{p}(j, j+1)$ and $r^{\prime} \subset I_{p}(j, j) \backslash$ $c_{j}^{\prime}(j)$ such that $\left|k^{\prime}\right|=\left|r^{\prime}\right|=\delta^{\prime}-f^{\prime}$. Then, we construct $c_{j}$ using $c_{j}^{\prime}$ with the following inversible process:

$$
c_{j}(i)= \begin{cases}r^{\prime} \cup c_{j}^{\prime}(j) \backslash k^{\prime} & \text { if } i=j \\ c_{j}^{\prime}(i) & \text { otherwise }\end{cases}
$$

Thus, $c_{j}$ is a waterfall $\left(L^{c}, \omega\right)$-choosability of $P_{j+1}$, and $c_{j}$ is an $\left(L^{c}, \omega\right)$ choosability of $P_{n+1}$. Finally, by induction, we construct $c_{n}=c^{c}$.

We have the following theorem which gives a necessary and sufficient condition for a weighted path to be $\left(L^{c}, \omega\right)$-choosable.

Theorem 9. Let $L^{c}$ be a waterfall list of a weighted path $P_{n+1}$. Then $P_{n+1}$ is $\left(L^{c}, \omega\right)$-choosable if and only if:

$$
\forall i, j \in\{0, \ldots, n\}:\left|A(i, j)\left(L^{c}\right)\right| \geq \sum_{k=i}^{j} \omega(k)
$$

Proof. Let $L^{c}$ be a waterfall list of a weighted path $P_{n+1}$. If $c$ is a $\left(L^{c}, \omega\right)$ choosability of $P_{n+1}$ then

$$
\forall i, j \in\{0, \ldots, n\}: A(i, j) \supset \bigcup_{k=i}^{j} c(k)
$$

Remark 6 shows that:

$$
\forall i, j \in\{0, \ldots, n\}:|A(i, j)| \geq \sum_{k=i}^{j} \omega(k)
$$

Conversely, we use an induction on the length $n$ of the path.
For $n=0$, since $\left|L^{c}(0)\right|=|A(0,0)| \geq \omega(0)$ we have that $P_{1}$ is $\left(L^{c}, \omega\right)$ choosable.

Suppose now that the property is true for all $k$ such that $0 \leq k \leq n$ and let $L^{c}$ be a waterfall list of $P_{n+2}$ such that:

$$
\forall i, j \in\{0, \ldots, n+1\}:|A(i, j)| \geq \sum_{k=i}^{j} \omega(k)
$$

Consider the path $P_{n+1}$ viewed as a subgraph of $P_{n+2}$. We have for all $i, j \in\{0, \ldots, n\}:|A(i, j)| \geq \sum_{k=i}^{j} \omega(k)$. Hence, by induction, $P_{n+1}$ is $\left(L^{c}, \omega\right)$-choosable and Lemma 8 shows that there exists a waterfall $\left(L^{c}, \omega\right)$ choosability $c^{c}$ of $P_{n+1}$.

Recall that, by definition

$$
\forall i \in\{0, \ldots, n\}: c^{c}(i)=K_{2 i-1} \cup F_{i} \cup K_{2 i}
$$

We have two cases to study:

Case 1: there exists $t \in\{0, \ldots, n\}$ such that $\left|K_{2 t}\right|=0$.
Consider the path $P_{t}^{\prime}$ viewed as the subgraph of $P_{n+1}$ such that $V\left(P_{t}^{\prime}\right)=$ $\left\{v_{t+1}, \ldots, v_{n+1}\right\} . P_{t}^{\prime}$ is a path of length $n-t \leq n$, hence, by induction, there exists an $\left(L^{c}, \omega\right)$-choosability $c_{t^{\prime}}$ of $P_{t}^{\prime}$. Since

$$
L^{c}(t+1) \backslash K_{2 t}=L^{c}(t+1) \backslash c^{c}(t)=L^{c}(t+1)
$$

there is no conflict between $c^{c}$ and $c_{t^{\prime}}$ since $c^{c}\left(v_{t}\right) \cap c_{t^{\prime}}\left(v_{t+1}\right)=\emptyset$. Thus, we obtain an $\left(L^{c}, \omega\right)$-choosability $c^{\prime}$ of $P_{n+2}$ such that

$$
c^{\prime}(i)=\left\{\begin{array}{cc}
c^{c}(i) & \text { if } i \in\{0, \ldots, t\} \\
c_{t^{\prime}}(i) & \text { otherwise }
\end{array}\right.
$$

Case 2: for all $i \in\{0, \ldots, n\}:\left|K_{2 i}\right|>0$.
By construction of $c^{c}$ we have for all $i \in\{0, \ldots, n\}:\left|K_{2 i}\right|=\omega(i)-\left(\left|F_{i}\right|+\right.$ $\left|K_{2 i-1}\right|$ ), then

$$
\forall i \in\{0, \ldots, n\}: \omega(i)-\left|K_{2 i-1}\right|>\left|F_{i}\right|
$$

Since for all $i \in\{0, \ldots, n\}$ we have $\left|F_{i}\right|=\min \left\{\left|I_{p}(i, i)\right|, \omega(i)-\left|K_{2 i-1}\right|\right\}$, hence we obtain

$$
\begin{equation*}
\forall i \in\{0, \ldots, n\}:\left|F_{i}\right|=\left|I_{p}(i, i)\right| \tag{*}
\end{equation*}
$$

Since for all $i \in\{0, \ldots, n\}:\left|F_{i}\right| \geq 0$ et $\omega(i)-\left|K_{2 i-1}\right|>\left|F_{i}\right|$ then $\omega(i)>$ $\left|K_{2 i-1}\right|$. By definition, we have for all $i \in\{0, \ldots, n\}:\left|K_{2 i-1}\right|=\min \left(\mid I_{p}(i-\right.$ $1, i)\left|-\left|K_{2 i-2}\right|, \omega(i)\right)$ hence we obtain that $\left|K_{2 i-1}\right|=\left|I_{p}(i-1, i)\right|-\left|K_{2 i-2}\right|$ and thus

$$
\begin{equation*}
\forall i \in\{0, \ldots, n\}:\left|K_{2 i-1}\right|+\left|K_{2 i-2}\right|=\left|I_{p}(i-1, i)\right| \tag{**}
\end{equation*}
$$

By construction of $c^{c}$ we have

$$
\forall i \in\{0, \ldots, n\}:\left|K_{2 i}\right|+\left|F_{i}\right|+\left|K_{2 i-1}\right|=\omega(i),
$$

and by convention $\left|K_{-1}\right|=0$, hence we add all the previous equalities and we obtain:

$$
\left|K_{2 n}\right|+\sum_{i=0}^{n-1}\left(\left|K_{2 i}\right|+\left|K_{2 i+1}\right|\right)+\sum_{i=0}^{n}\left|F_{i}\right|=\sum_{i=0}^{n} \omega(i) .
$$

Now we use the equalities $(*)$ and $(* *)$ :

$$
\left|K_{2 n}\right|+\sum_{i=0}^{n-1}\left|I_{p}(i, i+1)\right|+\sum_{i=0}^{n}\left|I_{p}(i, i)\right|=\sum_{i=0}^{n} \omega(i) .
$$

By elementary computations, it can be shown that:

$$
|A(0, n+1)|=\sum_{i=0}^{n+1}\left(\left|I_{p}(i, i)\right|+\left|I_{p}(i, i+1)\right|\right) .
$$

Hence (since $\left.\left|I_{p}(n+1, n+2)\right|=0\right)$ :

$$
\left|K_{2 n}\right|+|A(0, n+1)|-\left|I_{p}(n, n+1)\right|-\left|I_{p}(n+1, n+1)\right|=\sum_{i=0}^{n} \omega(i) .
$$

In other words, we have:

$$
\omega(n+1)+\left(|A(0, n+1)|-\sum_{i=0}^{n+1} \omega(i)\right)=\left|I_{p}(n, n+1)\right|+\left|I_{p}(n+1, n+1)\right|-\left|K_{2 n}\right| .
$$

We use here the hypothesis $|A(0, n+1)| \geq \sum_{i=0}^{n+1} \omega(i)$, which implies

$$
\omega(n+1) \leq\left|I_{p}(n, n+1)\right|+\left|I_{p}(n+1, n+1)\right|-\left|K_{2 n}\right| .
$$

Since $\left|I_{p}(n, n+1)\right|+\left|I_{p}(n+1, n+1)\right|=\left|L^{c}(n+1)\right|$ and $K_{2 n}=c^{c}(n) \cap L^{c}(n+$ 1) $\subset L^{c}(n+1)$, then

$$
\omega(n+1) \leq\left|L^{c}(n+1) \backslash c^{c}(n)\right| .
$$

Finally, we choose $c(n+1) \subset L^{c}(n+1) \backslash c^{c}(n)$ such that $|c(n+1)|=\omega(n+1)$, and we construct an $\left(L^{c}, \omega\right)$-choosability $c^{\prime}$ such that:

$$
c^{\prime}(i)=\left\{\begin{array}{lrl}
c^{c}(i) & \text { if } i \in\{0, \ldots, n\} \\
c(n+1) & \text { if } i=n+1
\end{array}\right.
$$

The previous result has the following corollary when the list is a good waterfall list and $|L(n)| \geq \omega(n)$.
Corollary 10. Let $L^{c}$ be a good waterfall list of a weighted path $P_{n+1}$ such that $\left|L^{c}(n)\right| \geq \omega(n)$. Then $P_{n+1}$ is $\left(L^{c}, \omega\right)$-choosable if and only if

$$
\forall j \in\{0, \ldots, n\}:\left|A(0, j)\left(L^{c}\right)\right| \geq \sum_{k=0}^{j} \omega(k)
$$

Proof. Under the hypothesis, if $P_{n+1}$ is $\left(L^{c}, \omega\right)$-choosable, then Theorem 9 proves in particular the result.

Conversely, since $L^{c}$ is a waterfall list of $P_{n+1}$, we have:
$\forall i, j \in\{1, \ldots, n\}:|A(i, j)|=\left|\cup_{k=i}^{j} L^{c}(k)\right| \geq\left|\cup_{\substack{k=i \\ k-i \text { even }}}^{j} L^{c}(k)\right|=\sum_{\substack{k=i \\ k-i \text { even }}}^{j}\left|L^{c}(k)\right|$.
Since $L^{c}$ is a good list of $P_{n+1}$ (for simplicity, we set $\omega(n+1)=0$ ):

$$
\forall i, j \in\{1, \ldots, n\}: \sum_{\substack{k=i \\ k-i \text { even }}}^{j}\left|L^{c}(k)\right| \geq \sum_{\substack{k=i \\ k-i \text { even }}}^{j}(\omega(k)+\omega(k+1)) \geq \sum_{k=i}^{j} \omega(k)
$$

then we obtain for all $i, j \in\{1, \ldots, n\}:|A(i, j)| \geq \sum_{k=i}^{j} \omega(k)$. Since for all $j \in\{0, \ldots, n\}:|A(0, j)| \geq \sum_{k=0}^{j} \omega(k)$, Theorem 9 concludes the proof.

Another interesting corollary holds for lists $L$ such that $|L(0)|=|L(n)|=$ $b$, and for all $i \in\{1, \ldots, n-1\}:|L(i)|=a$. The function Even is defined for any real $x$ by: $\operatorname{Even}(x)$ is the smallest even integer $p$ such that $p \geq x$.
Corollary 11. Let $L$ be a list of $P_{n+1}$ such that $|L(0)|=|L(n)|=b$, and $|L(i)|=a=2 b+e$ for all $i \in\{1, \ldots, n-1\}$.

$$
\text { If } n \geq \operatorname{Even}\left(\frac{2 b}{e}\right) \text { then } P_{n+1} \text { is }(L, b)-\text { choosable. }
$$

Proof. The hypothesis implies that $L$ is a good list of $P_{n+1}$, hence by Proposition 5, there exists a waterfall list $L^{c}$ similar to $L$. So we get:

$$
\forall i \in\{1, \ldots, n-1\}:\left|L^{c}(i)\right| \geq 2 b=\omega(i)+\omega(i+1)
$$

and $\left|L^{c}(n)\right| \geq b=\omega(n)$. By Corollary 10 it remains to prove that:

$$
\forall j \in\{0, \ldots, n\}:|A(0, j)| \geq \sum_{k=0}^{j} \omega(k)=(j+1) b
$$

Case 1: $j=0$. By hypothesis, we have $|A(0,0)|=\left|L^{c}(0)\right| \geq b$.
Case 2: $j \in\{1, \ldots, n-1\}$. Since $L^{c}$ is a waterfall list of $P_{n+1}$ we obtain that:
if $j$ is even

$$
|A(0, j)| \geq \sum_{\substack{k=0 \\ k \text { even }}}^{j}\left|L^{c}(k)\right|=b+\sum_{\substack{k=2 \\ k \text { even }}}^{j} 2 b=b+\frac{j}{2} 2 b=(j+1) b
$$

and if $j$ is odd

$$
|A(0, j)| \geq \sum_{\substack{k=0 \\ k \text { odd }}}^{j}\left|L^{c}(k)\right|=\sum_{\substack{k=1 \\ k \text { odd }}}^{j} 2 b=\frac{j+1}{2} 2 b=(j+1) b
$$

Hence for all $j \in\{0, \ldots, n-1\}:|A(0, j)| \geq(j+1) b$.
Case 3: $j=n$. Since $n \geq \operatorname{Even}\left(\frac{2 b}{e}\right)$ by hypothesis, and

$$
|A(0, n)| \geq \sum_{\substack{k=0 \\
k \text { odd }}}^{n}\left|L^{c}(k)\right|=\left\{\begin{array}{lr}
a \frac{n}{2} & \text { if } n \text { is even } \\
b+a \frac{n-1}{2} & \text { otherwise }
\end{array}\right.
$$

we deduce that $|A(0, n)| \geq(n+1) b$, which concludes the proof.

## 4. Free-choosability of a cycle

The cycle $C_{n}$ of length $n$ is the graph with vertex set $V=\left\{v_{0}, \ldots, v_{n-1}\right\}$ and edge set $E=\bigcup_{i=0}^{n-1}\left(v_{i}, v_{i+1(\bmod n)}\right)$.

Let $F \mathcal{C}_{h}(x)$ be the set of graphs $G$ which are $(a, b)$-free-choosable for all $a, b$ such that $\frac{a}{b} \geq x$ :

$$
F \mathcal{C}_{h}(x)=\left\{G, \text { such that for all } \frac{a}{b} \geq x, G \text { is }(a, b) \text {-free-choosable }\right\}
$$

Moreover, we can define the free-choice ratio $\operatorname{fchr}(G)$ of a graph $G$ by:

$$
\operatorname{fchr}(G):=\inf \left\{\left.\frac{a}{b} \right\rvert\, G \text { is }(a, b) \text {-free-choosable }\right\}
$$

If $\lfloor x\rfloor$ denotes the greatest integer less or equal to the real $x$, we can state:
Theorem 12. If $C_{n}$ is a cycle of length $n$, then

$$
C_{n} \in F \mathcal{C}_{h}\left(2+\left\lfloor\frac{n}{2}\right\rfloor^{-1}\right)
$$

Moreover, we have:

$$
\operatorname{fchr}\left(C_{n}\right)=2+\left\lfloor\frac{n}{2}\right\rfloor^{-1}
$$

Proof. Let $a, b$ be two integers such that $a / b \geq 2+\left\lfloor\frac{n}{2}\right\rfloor^{-1}$. Let $C_{n}$ be a cycle of length $n$ and $L$ a $a$-list of $C_{n}$. Without loss generality, we can suppose that $v_{0}$ is the vertex choosen for the free-choosability and $L_{0} \subset L\left(v_{0}\right)$ has $b$ elements. It remains to construct an $(L, b)$-choosability $c$ of $C_{n}$ such that $c\left(v_{0}\right)=L_{0}$. Hence we have to construct an $\left(L^{\prime}, b\right)$-choosability $c$ of $P_{n+1}$ such that $L^{\prime}(0)=L^{\prime}(n)=L_{0}$ and for all $i \in\{1, \ldots, n-1\}, L^{\prime}(i)=L\left(v_{i}\right)$. We have $\left|L^{\prime}(0)\right|=\left|L^{\prime}(n)\right|=b$ and for all $i \in\{1, \ldots, n-1\},\left|L^{\prime}(i)\right|=a$. Since $a / b \geq 2+\left\lfloor\frac{n}{2}\right\rfloor^{-1}$ and $e=a-2 b$, we get $e / b \geq\left\lfloor\frac{n}{2}\right\rfloor^{-1}$ hence $n \geq \operatorname{Even}(2 b / e)$. Using Corollary 11, we get:

$$
C_{n} \in F \mathcal{C}_{h}\left(2+\left\lfloor\frac{n}{2}\right\rfloor^{-1}\right)
$$

Hence, we have that $\operatorname{fchr}\left(C_{n}\right) \leq 2+\left\lfloor\frac{n}{2}\right\rfloor^{-1}$. Moreover, let us prove that $M=2+\left\lfloor\frac{n}{2}\right\rfloor^{-1}$ is reached.

For $n$ odd, Voigt has proved in [13] that the choice ratio $\operatorname{chr}\left(C_{n}\right)$ of a cycle of odd length $n$ is exactly $M$. Hence $\operatorname{fchr}\left(C_{n}\right) \geq \operatorname{chr}\left(C_{n}\right)=M$, and the result is proved.
For $n$ even, let $a, b$ be two integers such that $\frac{a}{b}<M$. We construct a counterexample for the free-choosability: let $L$ be the list of $C_{n}$ such that

$$
L(i)=\left\{\begin{array}{lc}
\{1, \ldots, a\} & \text { if } i \in\{0,1\} \\
\left\{1+\frac{i-1}{2} a, \ldots,\left(\frac{i-1}{2}+1\right) a\right\} & \text { if } i \neq n-1 \text { is odd } \\
\left\{b+1+\frac{i-2}{2} a, \ldots, b+\left(\frac{i-2}{2}+1\right) a\right\} & \text { if } i \text { is even and } i \neq 0 \\
\left\{1, \ldots, b, 1+\left(\frac{n-4}{2}+1\right) a, \ldots,\left(\frac{n-4}{2}+2\right) a-b\right\} \quad \text { if } i=n-1
\end{array}\right.
$$

If we choose $c_{0}=\{1, \ldots, b\} \subset L(0)$, we can check that it does not exist an $(L, b)$-choosability of $C_{n}$ such that $c(0)=c_{0}$, so we could not do better.
Remark 13. In particular, the previous theorem implies that if $n \geq \operatorname{Even}\left(\frac{2 b}{e}\right)$ then the cycle $C_{n}$ of length $n$ is $(2 b+e, b)$-free-choosable.

Remark 14. Erdös, Rubin and Taylor have stated in (3] the following question: If $G$ is $(a, b)$-choosable, and $\frac{c}{d}>\frac{a}{b}$, does it imply that $G$ is $(c, d)$ choosable? Gutner and Tarsi have shown in [6] that the answer is negative in general. If we consider the analogue question for free-choosability, then the previous theorem implies that it is true for the cycle.

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