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► **To cite this version:**

Yves Aubry, Jean-Christophe Godin, Olivier Togni. Choosability of a weighted path and free-choosability of a cycle. 2010. hal-00484445v1

HAL Id: hal-00484445

<https://hal.science/hal-00484445v1>

Preprint submitted on 18 May 2010 (v1), last revised 16 May 2011 (v3)

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CHOOSABILITY OF A WEIGHTED PATH AND FREE-CHOOSABILITY OF A CYCLE

YVES AUBRY, JEAN-CHRISTOPHE GODIN AND OLIVIER TOGNI

ABSTRACT. A graph G with a list of colors $L(v)$ and weight $\omega(v)$ for each vertex v is (L, ω) -choosable if one can choose a subset of $\omega(v)$ colors from $L(v)$ for each vertex v , such that adjacent vertices receive disjoint color sets. In this paper, we give a necessary and sufficient condition for a weighted path to be (L, ω) -choosable for some lists L . Furthermore, we solve the problem of the free-choosability of a cycle.

1. INTRODUCTION

The concept of choosability of a graph, also called list colouring, has been introduced by Vizing in [11], and independently by Erdős, Rubin and Taylor in [3]. It contains of course the colorability as a particular case. Since its introduction, choosability has been extensively studied (see for example [1, 2, 9, 10, 5, 8] and very recently [7, 6]).

Following the work in [4], we introduce here the notion of a good list (it is an obvious necessary condition on the list for the graph to be choosable) and of a waterfall list (list adapted to find a choosability) and we show that any list of the first kind can be transformed into a list of the second one. We give as a first main result a necessary and sufficient condition for a weighted path with a given waterfall list L to be (L, ω) -choosable (Theorem 9).

In 1996, Voigt considered the following problem: let G be a graph and L a list assignment and assume that an arbitrary vertex $v \in V(G)$ is precoloured by a colour $f \in L(v)$. Is it always possible to complete this precolouring to a proper list colouring? This question leads to the concept of free-choosability introduced by Voigt in [12].

We investigate here the free-choosability of the first interesting case, namely the cycle. As an application of Theorem 9, we prove our second main result which gives a necessary and sufficient condition for a cycle to be free-choosable (Theorem 12). In order to get a concise statement, we introduce the free-choice ratio of a graph, in the same way that Alon, Tuza and Voigt in [1] introduced the choice ratio (which equals the so-called fractional chromatic number).

We recall in Section 2 the definitions of choosability and free-choosability.

Section 3 is devoted to the choosability of a weighted path and section 4 to the free-choosability of a cycle.

Date: May 18, 2010.

2010 Mathematics Subject Classification. 05C15, 05C38, 05C72.

Key words and phrases. Coloring, Choosability, Free-choosability, Cycles.

2. DEFINITIONS

Let $G = (V(G), E(G))$ be a graph where $V(G)$ is the set of vertices and $E(G)$ is the set of edges, and let a, b, n and e be integers.

Let L be a list of G i.e. a map: $L : V(G) \rightarrow \mathcal{P}(\mathbb{N})$ and let ω be a weight of G i.e. a map: $\omega : V(G) \rightarrow \mathbb{N}$.

Definition 1. An (L, ω) -choosability c of a graph G is a list of the weighted graph G such that for all $vv' \in E(G)$:

$$c(v) \subset L(v), \quad |c(v)| = \omega(v) \quad \text{and} \quad c(v) \cap c(v') = \emptyset.$$

We say that G is (L, ω) -choosable if there exists an (L, ω) -choosability c of G .

An (L, b) -choosability c of G is an (L, ω) -choosability of G such that for all $v \in V(G)$, we have $\omega(v) = b$.

A a -list L of G is a list of G such that for all $v \in V(G)$, we have $|L(v)| = a$.

Definition 2. G is said to be (a, b) -choosable if for any a -list L of G , there exists an (L, b) -choosability c of G .

Definition 3. G is said to be (a, b) -free-choosable if for any $v_0 \in V(G)$, and for any list L of G such that for any $v \in V(G) \setminus \{v_0\} : |L(v)| = a$ and $|L(v_0)| = b$, there exists an (L, b) -choosability c of G .

3. CHOOSABILITY OF A WEIGHTED PATH

The path P_{n+1} or length n is the graph with vertex set $V = \{v_0, v_1, \dots, v_n\}$ and edge set $E = \bigcup_{i=0}^{n-1} (v_i, v_{i+1})$. To simplify the notations $L(i)$ denotes $L(v_i)$ and $c(i)$ denotes $c(v_i)$.

Definition 4. For the path P_{n+1} of order n ,

- a waterfall list L is a list such that for all $i, j \in \{0, \dots, n\}$ with $|i - j| \geq 2$, $L(i) \cap L(j) = \emptyset$;
- two lists L and L' are similar if and only if P_{n+1} is (L, ω) -choosable whenever P_{n+1} is (L', ω) -choosable;
- the amplitude $A_{i,j}(L)$ (or $A_{i,j}$) of a list L is $A_{i,j}(L) = \bigcup_{k=i}^j L(k)$;
- a list L is good if $|L(i)| \geq \omega(i) + \omega(i+1)$ for any $i, 1 \leq i \leq n-1$.

Notice that another similar definition of a waterfall list is that any color is present only on one list or on two lists of consecutive vertices.

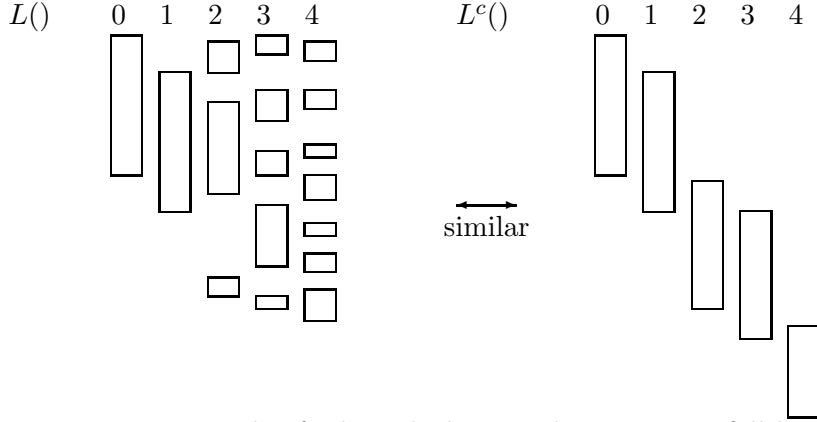


Fig. 1. Example of a list which is similar to a waterfall list.

Figure 1 shows a list L of P_5 (on the left), together with a similar waterfall list L^c (on the right).

Proposition 5. *For any good list L of P_{n+1} , there exists a similar waterfall list L^c with $|L^c(i)| = |L(i)|$ for all $i \in \{0, \dots, n\}$.*

Proof. We are going to transform a good list L of P_{n+1} into a waterfall list L^c and we will prove that L^c is similar with L .

First, remark that if a color $x \in L(i-1)$ but $x \notin L(i)$ for some $i, 1 \leq i \leq n-1$, then for any $j > i$, one can change the color x by a new color $y \notin A(L)$ in the list $L(j)$, without changing the choosability of the list. With this remark in hand, we can assume that L is such that any color x appears on the lists of consecutive vertices i_x, \dots, j_x .

Now, by permuting the colors if necessary, we can assume that if $x < y$ then $i_x < i_y$ or $i_x = i_y$ and $j_x \leq j_y$.

Repeat the following transformation:

1. Take the minimum color x for which $j_x \geq i_x + 2$ (i.e. the color x is present on at least three vertices $i_x, i_x + 1, i_x + 2, \dots, j_x$);
 2. Replace color x by a new color y in lists $L(i_x + 2), \dots, L(j_x)$;
- until the obtained list is a waterfall list (obviously, the number of iterations is always finite).

Now, we show that this transformation preserves the choosability of the list: Let L' be the list obtained from the list L by the above transformation.

If c is an (L, ω) -choosability of P_{n+1} then the choosability c' obtained from c by changing the color x by the color y in the color set $c(k)$ of each vertex $k \geq i_x + 2$ (containing x) is an (L', ω) -choosability since y is a new color.

Conversly, if c' is an (L', ω) -choosability of P_{n+1} , we consider two cases:

Case 1: $x \notin c'(i_x + 1)$ or $y \notin c'(i_x + 2)$. In this case, the choosability c obtained from c' by changing the color y by the color x in the color set $c'(k)$ of each vertex $k \geq i_x + 2$ (containing y) is an (L, ω) -choosability.

Case 2: $x \in c'(i_x + 1)$ and $y \in c'(i_x + 2)$. We have to consider two subcases:

- Subcase 1: $L'(i_x + 1) \not\subset (c'(i_x) \cup c'(i_x + 1) \cup c'(i_x + 2))$. There exists $z \in L'(i_x + 1) \setminus (c'(i_x) \cup c'(i_x + 1) \cup c'(i_x + 2))$ and the choosability c obtained from c' by changing the color x by the color z in $c'(i_x + 1)$ and replacing the color y by the color x in the color set $c'(k)$ of each vertex $k \geq i_x + 2$ (containing y) is an (L, ω) -choosability.
- Subcase 2: $L'(i_x + 1) \subset (c'(i_x) \cup c'(i_x + 1) \cup c'(i_x + 2))$. We have

$$|L'(i_x + 1)| = \left| \left((c'(i_x) \cup c'(i_x + 1) \cup c'(i_x + 2)) \cap L'(i_x + 1) \right) \right|.$$

As c' is an (L', ω) -choosability of P_{n+1} , we have

$$|L'(i_x + 1)| = |c'(i_x + 2) \cap L'(i_x + 1)| + |c'(i_x + 1) \cap L'(i_x + 1)| + \left| \left(c'(i_x) \setminus c'(i_x + 2) \right) \cap L'(i_x + 1) \right|,$$

$$|L'(i_x + 1)| - \omega(i_x + 1) - |c'(i_x + 2) \cap L'(i_x + 1)| = \left| \left(c'(i_x) \setminus c'(i_x + 2) \right) \cap L'(i_x + 1) \right|.$$

Since $y \in c'(i_x + 2)$ and $y \notin L'(i_x + 1)$, we obtain that

$$|c'(i_x + 2) \cap L'(i_x + 1)| \leq \omega(i_x + 2) - 1,$$

hence

$$\left(|L'(i_x + 1)| - \omega(i_x + 1) - \omega(i_x + 2) \right) + 1 \leq \left| \left(c'(i_x) \setminus c'(i_x + 2) \right) \cap L'(i_x + 1) \right|.$$

But, by hypothesis, L is a good list. Thus $|L(i_x + 1)| = |L'(i_x + 1)| \geq \omega(i_x + 1) + \omega(i_x + 2)$ and

$$1 \leq \left| \left(c'(i_x) \setminus c'(i_x + 2) \right) \cap L'(i_x + 1) \right|.$$

Consequently, there exists $z \in \left(c'(i_x) \setminus c'(i_x + 2) \right) \cap L'(i_x + 1)$. The choosability c is then constructed from c' by changing the color x by the color z in $c'(i_x + 1)$, the color z by the color x in $c'(i_x)$ and the color y by the color x in the set $c'(k)$ of each vertex $k \geq i_x + 2$.

□

Remark 6. Let L^c be a waterfall list of P_{n+1} and c be an (L^c, ω) -choosability of P_{n+1} . Then, it is clear that if $i, j \in \{0, \dots, n\}$ such that $i \neq j$ then

$$c(i) \cap c(j) = \emptyset \quad \text{and} \quad \left| \bigcup_{k=i}^j c(k) \right| = \sum_{k=i}^j \omega(k).$$

We now need to define the notion of waterfall choosability (see Figure 2 for an illustration). For a list L of P_{n+1} , set

$$I(i, j) := \begin{cases} \bigcap_{k=i}^j L(k) & \text{if } j \geq i \\ \emptyset & \text{otherwise} \end{cases}$$

and

$$I_p(i, j) := I(i, j) \setminus \left(L(i-1) \cup L(j+1) \right).$$

Definition 7. Let L^c be a waterfall list of P_{n+1} and c be an (L^c, ω) -choosability of P_{n+1} . We say that c is a waterfall (L^c, ω) -choosability of P_{n+1} if for all $i \in \{0, \dots, n\}$:

$$c(i) = K_{2i-1} \cup F_i \cup K_{2i}$$

with

$$K_{2i-1} \subset I_p(i-1, i) \setminus K_{2i-2} \text{ such that } |K_{2i-1}| = \min\{|I_p(i-1, i) \setminus K_{2i-2}|, \omega(i)\},$$

$$F_i \subset I_p(i, i) \text{ such that } |F_i| = \min\{|I_p(i, i)|, \omega(i) - |K_{2i-1}|\},$$

$$K_{2i} \subset I_p(i, i+1) \text{ such that } |K_{2i}| = \omega(i) - |K_{2i-1}| - |F_i|.$$

And with the convention $K_{-1} = K_{-2} = \emptyset$.

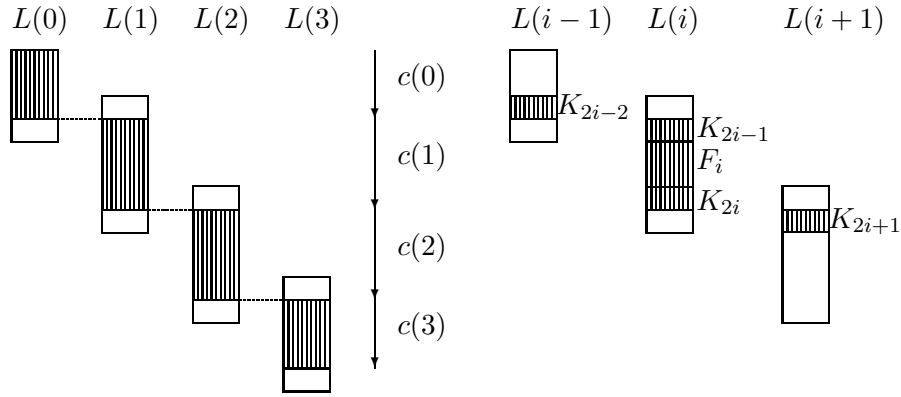


Fig. 2. Example of a waterfall choosability.

Lemma 8. Let L^c be a waterfall list of P_{n+1} . There exists an (L^c, ω) -choosability c of P_{n+1} if and only if there exists a waterfall (L^c, ω) -choosability c^c of P_{n+1} .

Proof. Under the hypothesis, if c^c is a waterfall (L^c, ω) -choosability of P_{n+1} then $c = c^c$ is an (L^c, ω) -choosability of P_{n+1} .

Conversely, if c is an (L^c, ω) -choosability of P_{n+1} then we construct by induction c_j for $j \in \{0, \dots, n\}$ such that: c_j is a waterfall (L^c, ω) -choosability of P_{j+1} , and c_j is an (L^c, ω) -choosability of P_{n+1} , and

$$c_j(i) = \begin{cases} c_{j-1}(i) & \text{if } i \in \{0, \dots, j-1\} \\ c(i) & \text{if } i \in \{j+1, \dots, n\}. \end{cases}$$

We begin by the construction of c_0 . Let f and δ be two integers such that $f = |c(0) \cap I_p(0, 0)|$ and $\delta = \min\{|I_p(0, 0)|, \omega(0)\}$. We have obviously $f \leq \delta$. Since

$$|c(0) \cap I_p(0, 1)| = |c(0)| - |c(0) \cap I_p(0, 0)| = \omega(0) - f \geq \delta - f$$

and

$$|I_p(0, 0) \setminus c(0)| = |I_p(0, 0)| - f \geq \delta - f,$$

we obtain that there exist two sets $k \subset c(0) \cap I_p(0, 1)$ and $r \subset I_p(0, 0) \setminus c(0)$ such that $|k| = |r| = \delta - f$. The invertible process between c_0 and c is the following one:

$$c_0(i) = \begin{cases} r \cup c(0) \setminus k & \text{if } i = 0 \\ c(i) & \text{otherwise} \end{cases}$$

By construction of r we have: $c_0(0) \cap c_0(1) = \emptyset$. Hence c_0 is a waterfall (L^c, ω) -choosability of P_1 and c_0 is an (L^c, ω) -choosability of P_{n+1} .

Let $j \geq 1$ and suppose that c_{j-1} is constructed. Recall that $K_{2j-2} = c_{j-1}(j-1) \cap I_p(j-1, j)$. In the same spirit as above, we consider two integers f and δ such that $f = |c_{j-1}(j) \cap I_p(j-1, j)|$ and $\delta = \min\{|I_p(j-1, j) \setminus K_{2j-2}|, \omega(j)\}$. We have of course $\delta \geq f$. Since

$$|c_{j-1}(j) \cap (I_p(j, j) \cup I_p(j, j+1))| = |c_{j-1}(j)| - f = \omega(j) - f \geq \delta - f,$$

and

$$|I_p(j-1, j) \setminus (K_{2j-2} \cup c_{j-1}(j))| = |I_p(j-1, j) \setminus K_{2j-2}| - f \geq \delta - f,$$

there exist two sets

$$k \subset c_{j-1}(j) \cap (I_p(j, j) \cup I_p(j, j+1))$$

and

$$r \subset I_p(j-1, j) \setminus (K_{2j-2} \cup c_{j-1}(j))$$

such that $|k| = |r| = \delta - f$. We construct an intermediary choosability c'_j using c_{j-1} with the following invertible process:

$$c'_j(i) = \begin{cases} r \cup c_{j-1}(j) \setminus k & \text{if } i = j \\ c_{j-1}(i) & \text{otherwise} \end{cases}$$

We obtain that $|c'_j(j) \cap I_p(j-1, j)| = \min\{|I_p(j-1, j) \setminus K_{2j-2}|, \omega(j)\}$.

Using the same idea, we continue. Recall that $K_{2j-1} = c'_j(j) \cap I_p(j-1, j)$. Let f' and δ' be two integers such that $f' = |c'_j(j) \cap I_p(j, j)|$ and $\delta' = \min\{|I_p(j, j)|, \omega(j) - |K_{2j-1}|\}$. We have $\delta' \geq f'$. Since

$$|c'_j(j) \cap I_p(j, j+1)| = (\omega(j) - |K_{2j-1}|) - f' \geq \delta' - f'$$

and

$$|I_p(j, j) \setminus c'_j(j)| = |I_p(j, j)| - f' \geq \delta' - f',$$

we obtain that there exist two sets $k' \subset c'_j(j) \cap I_p(j, j+1)$ and $r' \subset I_p(j, j) \setminus c'_j(j)$ such that $|k'| = |r'| = \delta' - f'$. Then, we construct c_j using c'_j with the following invertible process:

$$c_j(i) = \begin{cases} r' \cup c'_j(j) \setminus k' & \text{if } i = j \\ c'_j(i) & \text{otherwise} \end{cases}$$

Thus, c_j is a waterfall (L^c, ω) -choosability of P_{j+1} , and c_j is an (L^c, ω) -choosability of P_{n+1} . Finally, by induction, we construct $c_n = c^c$. \square

We have the following theorem which gives a necessary and sufficient condition for a weighted path to be (L^c, ω) -choosable.

Theorem 9. *Let L^c be a waterfall list of a weighted path P_{n+1} . Then P_{n+1} is (L^c, ω) -choosable if and only if:*

$$\forall i, j \in \{0, \dots, n\} : |A(i, j)(L^c)| \geq \sum_{k=i}^j \omega(k).$$

Proof. Let L^c be a waterfall list of a weighted path P_{n+1} . If c is a (L^c, ω) -choosability of P_{n+1} then

$$\forall i, j \in \{0, \dots, n\} : A(i, j) \supset \bigcup_{k=i}^j c(k).$$

Remark 6 shows that:

$$\forall i, j \in \{0, \dots, n\} : |A(i, j)| \geq \sum_{k=i}^j \omega(k).$$

Conversely, we use an induction on the length n of the path.

For $n = 0$, since $|L^c(0)| = |A(0, 0)| \geq \omega(0)$ we have that P_1 is (L^c, ω) -choosable.

Suppose now that the property is true for all k such that $0 \leq k \leq n$ and let L^c be a waterfall list of P_{n+2} such that:

$$\forall i, j \in \{0, \dots, n+1\} : |A(i, j)| \geq \sum_{k=i}^j \omega(k).$$

Consider the path P_{n+1} viewed as a subgraph of P_{n+2} . We have for all $i, j \in \{0, \dots, n\} : |A(i, j)| \geq \sum_{k=i}^j \omega(k)$. Hence, by induction, P_{n+1} is (L^c, ω) -choosable and Lemma 8 shows that there exists a waterfall (L^c, ω) -choosability c^c of P_{n+1} .

Recall that, by definition

$$\forall i \in \{0, \dots, n\} : c^c(i) = K_{2i-1} \cup F_i \cup K_{2i}.$$

We have two cases to study:

Case 1: there exists $t \in \{0, \dots, n\}$ such that $|K_{2t}| = 0$.

Consider the path P'_t viewed as the subgraph of P_{n+1} such that $V(P'_t) = \{v_{t+1}, \dots, v_{n+1}\}$. P'_t is a path of length $n - t \leq n$, hence, by induction, there exists an (L^c, ω) -choosability $c_{t'}$ of P'_t . Since

$$L^c(t+1) \setminus K_{2t} = L^c(t+1) \setminus c^c(t) = L^c(t+1),$$

there is no conflict between c^c and $c_{t'}$ since $c^c(v_t) \cap c_{t'}(v_{t+1}) = \emptyset$. Thus, we obtain an (L^c, ω) -choosability c' of P_{n+2} such that

$$c'(i) = \begin{cases} c^c(i) & \text{if } i \in \{0, \dots, t\} \\ c_{t'}(i) & \text{otherwise} \end{cases}$$

Case 2: for all $i \in \{0, \dots, n\} : |K_{2i}| > 0$.

By construction of c^c we have for all $i \in \{0, \dots, n\} : |K_{2i}| = \omega(i) - (|F_i| + |K_{2i-1}|)$, then

$$\forall i \in \{0, \dots, n\} : \omega(i) - |K_{2i-1}| > |F_i|.$$

Since for all $i \in \{0, \dots, n\}$ we have $|F_i| = \min\{|I_p(i, i)|, \omega(i) - |K_{2i-1}|\}$, hence we obtain

$$\forall i \in \{0, \dots, n\} : |F_i| = |I_p(i, i)| \quad (*)$$

Since for all $i \in \{0, \dots, n\} : |F_i| \geq 0$ et $\omega(i) - |K_{2i-1}| > |F_i|$ then $\omega(i) > |K_{2i-1}|$. By definition, we have for all $i \in \{0, \dots, n\} : |K_{2i-1}| = \min(|I_p(i-1, i)| - |K_{2i-2}|, \omega(i))$ hence we obtain that $|K_{2i-1}| = |I_p(i-1, i)| - |K_{2i-2}|$ and thus

$$\forall i \in \{0, \dots, n\} : |K_{2i-1}| + |K_{2i-2}| = |I_p(i-1, i)| \quad (**)$$

By construction of c^c we have

$$\forall i \in \{0, \dots, n\} : |K_{2i}| + |F_i| + |K_{2i-1}| = \omega(i),$$

and by convention $|K_{-1}| = 0$, hence we add all the previous equalities and we obtain:

$$|K_{2n}| + \sum_{i=0}^{n-1} (|K_{2i}| + |K_{2i+1}|) + \sum_{i=0}^n |F_i| = \sum_{i=0}^n \omega(i).$$

Now we use the equalities (*) and (**):

$$|K_{2n}| + \sum_{i=0}^{n-1} |I_p(i, i+1)| + \sum_{i=0}^n |I_p(i, i)| = \sum_{i=0}^n \omega(i).$$

By elementary computations, it can be shown that:

$$|A(0, n+1)| = \sum_{i=0}^{n+1} (|I_p(i, i)| + |I_p(i, i+1)|).$$

Hence (since $|I_p(n+1, n+2)| = 0$):

$$|K_{2n}| + |A(0, n+1)| - |I_p(n, n+1)| - |I_p(n+1, n+1)| = \sum_{i=0}^n \omega(i).$$

In other words, we have:

$$\omega(n+1) + \left(|A(0, n+1)| - \sum_{i=0}^{n+1} \omega(i) \right) = |I_p(n, n+1)| + |I_p(n+1, n+1)| - |K_{2n}|.$$

We use here the hypothesis $|A(0, n+1)| \geq \sum_{i=0}^{n+1} \omega(i)$, which implies

$$\omega(n+1) \leq |I_p(n, n+1)| + |I_p(n+1, n+1)| - |K_{2n}|.$$

Since $|I_p(n, n+1)| + |I_p(n+1, n+1)| = |L^c(n+1)|$ and $K_{2n} = c^c(n) \cap L^c(n+1) \subset L^c(n+1)$, then

$$\omega(n+1) \leq |L^c(n+1) \setminus c^c(n)|.$$

Finally, we choose $c(n+1) \subset L^c(n+1) \setminus c^c(n)$ such that $|c(n+1)| = \omega(n+1)$, and we construct an (L^c, ω) -choosability c' such that :

$$c'(i) = \begin{cases} c^c(i) & \text{if } i \in \{0, \dots, n\} \\ c(n+1) & \text{if } i = n+1 \end{cases}$$

□

The previous result has the following corollary when the list is a good waterfall list and $|L(n)| \geq \omega(n)$.

Corollary 10. *Let L^c be a good waterfall list of a weighted path P_{n+1} such that $|L^c(n)| \geq \omega(n)$. Then P_{n+1} is (L^c, ω) -choosable if and only if*

$$\forall j \in \{0, \dots, n\} : |A(0, j)(L^c)| \geq \sum_{k=0}^j \omega(k).$$

Proof. Under the hypothesis, if P_{n+1} is (L^c, ω) -choosable, then Theorem 9 proves in particular the result.

Conversely, since L^c is a waterfall list of P_{n+1} , we have:

$$\forall i, j \in \{1, \dots, n\} : |A(i, j)| = |\cup_{k=i}^j L^c(k)| \geq |\cup_{\substack{k=i \\ k-i \text{ even}}}^j L^c(k)| = \sum_{\substack{k=i \\ k-i \text{ even}}}^j |L^c(k)|.$$

Since L^c is a good list of P_{n+1} (for simplicity, we set $\omega(n+1) = 0$):

$$\forall i, j \in \{1, \dots, n\} : \sum_{\substack{k=i \\ k-i \text{ even}}}^j |L^c(k)| \geq \sum_{\substack{k=i \\ k-i \text{ even}}}^j (\omega(k) + \omega(k+1)) \geq \sum_{k=i}^j \omega(k),$$

then we obtain for all $i, j \in \{1, \dots, n\} : |A(i, j)| \geq \sum_{k=i}^j \omega(k)$. Since for all $j \in \{0, \dots, n\} : |A(0, j)| \geq \sum_{k=0}^j \omega(k)$, Theorem 9 concludes the proof. \square

Another interesting corollary holds for lists L such that $|L(0)| = |L(n)| = b$, and for all $i \in \{1, \dots, n-1\} : |L(i)| = a$. The function Even is defined for any real x by: Even(x) is the smallest even integer p such that $p \geq x$.

Corollary 11. *Let L be a list of P_{n+1} such that $|L(0)| = |L(n)| = b$, and $|L(i)| = a = 2b + e$ for all $i \in \{1, \dots, n-1\}$.*

If $n \geq \text{Even}\left(\frac{2b}{e}\right)$ then P_{n+1} is (L, b) -choosable.

Proof. The hypothesis implies that L is a good list of P_{n+1} , hence by Proposition 5, there exists a waterfall list L^c similar to L . So we get:

$$\forall i \in \{1, \dots, n-1\} : |L^c(i)| \geq 2b = \omega(i) + \omega(i+1)$$

and $|L^c(n)| \geq b = \omega(n)$. By Corollary 10 it remains to prove that:

$$\forall j \in \{0, \dots, n\} : |A(0, j)| \geq \sum_{k=0}^j \omega(k) = (j+1)b.$$

Case 1: $j = 0$. By hypothesis, we have $|A(0, 0)| = |L^c(0)| \geq b$.

Case 2: $j \in \{1, \dots, n-1\}$. Since L^c is a waterfall list of P_{n+1} we obtain that:
if j is even

$$|A(0, j)| \geq \sum_{\substack{k=0 \\ k \text{ even}}}^j |L^c(k)| = b + \sum_{\substack{k=2 \\ k \text{ even}}}^j 2b = b + \frac{j}{2} 2b = (j+1)b,$$

and if j is odd

$$|A(0, j)| \geq \sum_{\substack{k=0 \\ k \text{ odd}}}^j |L^c(k)| = \sum_{\substack{k=1 \\ k \text{ odd}}}^j 2b = \frac{j+1}{2} 2b = (j+1)b.$$

Hence for all $j \in \{0, \dots, n-1\}$: $|A(0, j)| \geq (j+1)b$.

Case 3: $j = n$. Since $n \geq \text{Even}\left(\frac{2b}{e}\right)$ by hypothesis, and

$$|A(0, n)| \geq \sum_{\substack{k=0 \\ k \text{ odd}}}^n |L^c(k)| = \begin{cases} a\frac{n}{2} & \text{if } n \text{ is even} \\ b + a\frac{n-1}{2} & \text{otherwise} \end{cases}$$

we deduce that $|A(0, n)| \geq (n+1)b$, which concludes the proof. \square

4. FREE-CHOOSABILITY OF A CYCLE

The cycle C_n of length n is the graph with vertex set $V = \{v_0, \dots, v_{n-1}\}$ and edge set $E = \bigcup_{i=0}^{n-1} (v_i, v_{i+1(\text{mod } n)})$.

Let $FC_h(x)$ be the set of graphs G which are (a, b) -free-choosable for all a, b such that $\frac{a}{b} \geq x$:

$$FC_h(x) = \{G, \text{ such that for all } \frac{a}{b} \geq x, G \text{ is } (a, b)\text{-free-choosable}\}.$$

Moreover, we can define the free-choice ratio $\text{fchr}(G)$ of a graph G by:

$$\text{fchr}(G) := \inf\left\{\frac{a}{b} \mid G \text{ is } (a, b)\text{-free-choosable}\right\}.$$

If $\lfloor x \rfloor$ denotes the greatest integer less or equal to the real x , we can state:

Theorem 12. *If C_n is a cycle of length n , then*

$$C_n \in FC_h\left(2 + \left\lfloor \frac{n}{2} \right\rfloor^{-1}\right).$$

Moreover, we have:

$$\text{fchr}(C_n) = 2 + \left\lfloor \frac{n}{2} \right\rfloor^{-1}.$$

Proof. Let a, b be two integers such that $a/b \geq 2 + \lfloor \frac{n}{2} \rfloor^{-1}$. Let C_n be a cycle of length n and L a a -list of C_n . Without loss generality, we can suppose that v_0 is the vertex chosen for the free-choosability and $L_0 \subset L(v_0)$ has b elements. It remains to construct an (L, b) -choosability c of C_n such that $c(v_0) = L_0$. Hence we have to construct an (L', b) -choosability c of P_{n+1} such that $L'(0) = L'(n) = L_0$ and for all $i \in \{1, \dots, n-1\}$, $L'(i) = L(v_i)$. We have $|L'(0)| = |L'(n)| = b$ and for all $i \in \{1, \dots, n-1\}$, $|L'(i)| = a$. Since $a/b \geq 2 + \lfloor \frac{n}{2} \rfloor^{-1}$ and $e = a - 2b$, we get $e/b \geq \lfloor \frac{n}{2} \rfloor^{-1}$ hence $n \geq \text{Even}(2b/e)$. Using Corollary 11, we get:

$$C_n \in FC_h\left(2 + \left\lfloor \frac{n}{2} \right\rfloor^{-1}\right).$$

Hence, we have that $\text{fchr}(C_n) \leq 2 + \lfloor \frac{n}{2} \rfloor^{-1}$. Moreover, let us prove that $M = 2 + \lfloor \frac{n}{2} \rfloor^{-1}$ is reached.

For n odd, Voigt has proved in [13] that the choice ratio $\text{chr}(C_n)$ of a cycle of odd length n is exactly M . Hence $\text{fchr}(C_n) \geq \text{chr}(C_n) = M$, and the result is proved.

For n even, let a, b be two integers such that $\frac{a}{b} < M$. We construct a counterexample for the free-choosability: let L be the list of C_n such that

$$L(i) = \begin{cases} \{1, \dots, a\} & \text{if } i \in \{0, 1\} \\ \{1 + \frac{i-1}{2}a, \dots, (\frac{i-1}{2} + 1)a\} & \text{if } i \neq n-1 \text{ is odd} \\ \{b+1 + \frac{i-2}{2}a, \dots, b + (\frac{i-2}{2} + 1)a\} & \text{if } i \text{ is even and } i \neq 0 \\ \{1, \dots, b, 1 + (\frac{n-4}{2} + 1)a, \dots, (\frac{n-4}{2} + 2)a - b\} & \text{if } i = n-1 \end{cases}$$

If we choose $c_0 = \{1, \dots, b\} \subset L(0)$, we can check that it does not exist an (L, b) -choosability of C_n such that $c(0) = c_0$, so we could not do better. \square

Remark 13. *In particular, the previous theorem implies that if $n \geq \text{Even}(\frac{2b}{e})$ then the cycle C_n of length n is $(2b + e, b)$ -free-choosable.*

Remark 14. *Erdős, Rubin and Taylor have stated in [3] the following question: If G is (a, b) -choosable, and $\frac{c}{d} > \frac{a}{b}$, does it imply that G is (c, d) -choosable? Gutner and Tarsi have shown in [6] that the answer is negative in general. If we consider the analogue question for free-choosability, then the previous theorem implies that it is true for the cycle.*

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