



HAL
open science

Choosability of a weighted path and free-choosability of a cycle

Yves Aubry, Jean-Christophe Godin, Olivier Togni

► **To cite this version:**

Yves Aubry, Jean-Christophe Godin, Olivier Togni. Choosability of a weighted path and free-choosability of a cycle. 2010. hal-00484445v1

HAL Id: hal-00484445

<https://hal.science/hal-00484445v1>

Preprint submitted on 18 May 2010 (v1), last revised 16 May 2011 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

CHOOSABILITY OF A WEIGHTED PATH AND FREE-CHOOSABILITY OF A CYCLE

YVES AUBRY, JEAN-CHRISTOPHE GODIN AND OLIVIER TOGNI

ABSTRACT. A graph G with a list of colors $L(v)$ and weight $\omega(v)$ for each vertex v is (L, ω) -choosable if one can choose a subset of $\omega(v)$ colors from $L(v)$ for each vertex v , such that adjacent vertices receive disjoint color sets. In this paper, we give a necessary and sufficient condition for a weighted path to be (L, ω) -choosable for some lists L . Furthermore, we solve the problem of the free-choosability of a cycle.

1. INTRODUCTION

The concept of choosability of a graph, also called list colouring, has been introduced by Vizing in [11], and independently by Erdős, Rubin and Taylor in [3]. It contains of course the colorability as a particular case. Since its introduction, choosability has been extensively studied (see for example [1, 2, 9, 10, 5, 8] and very recently [7, 6]).

Following the work in [4], we introduce here the notion of a good list (it is an obvious necessary condition on the list for the graph to be choosable) and of a waterfall list (list adapted to find a choosability) and we show that any list of the first kind can be transformed into a list of the second one. We give as a first main result a necessary and sufficient condition for a weighted path with a given waterfall list L to be (L, ω) -choosable (Theorem 9).

In 1996, Voigt considered the following problem: let G be a graph and L a list assignment and assume that an arbitrary vertex $v \in V(G)$ is precoloured by a colour $f \in L(v)$. Is it always possible to complete this precolouring to a proper list colouring? This question leads to the concept of free-choosability introduced by Voigt in [12].

We investigate here the free-choosability of the first interesting case, namely the cycle. As an application of Theorem 9, we prove our second main result which gives a necessary and sufficient condition for a cycle to be free-choosable (Theorem 12). In order to get a concise statement, we introduce the free-choice ratio of a graph, in the same way that Alon, Tuza and Voigt in [1] introduced the choice ratio (which equals the so-called fractional chromatic number).

We recall in Section 2 the definitions of choosability and free-choosability.

Section 3 is devoted to the choosability of a weighted path and section 4 to the free-choosability of a cycle.

Date: May 18, 2010.

2010 Mathematics Subject Classification. 05C15, 05C38, 05C72.

Key words and phrases. Coloring, Choosability, Free-choosability, Cycles.

2. DEFINITIONS

Let $G = (V(G), E(G))$ be a graph where $V(G)$ is the set of vertices and $E(G)$ is the set of edges, and let a, b, n and e be integers.

Let L be a list of G i.e. a map: $L : V(G) \rightarrow \mathcal{P}(\mathbb{N})$ and let ω be a weight of G i.e. a map: $\omega : V(G) \rightarrow \mathbb{N}$.

Definition 1. An (L, ω) -choosability c of a graph G is a list of the weighted graph G such that for all $vv' \in E(G)$:

$$c(v) \subset L(v), \quad |c(v)| = \omega(v) \quad \text{and} \quad c(v) \cap c(v') = \emptyset.$$

We say that G is (L, ω) -choosable if there exists an (L, ω) -choosability c of G .

An (L, b) -choosability c of G is an (L, ω) -choosability of G such that for all $v \in V(G)$, we have $\omega(v) = b$.

A a -list L of G is a list of G such that for all $v \in V(G)$, we have $|L(v)| = a$.

Definition 2. G is said to be (a, b) -choosable if for any a -list L of G , there exists an (L, b) -choosability c of G .

Definition 3. G is said to be (a, b) -free-choosable if for any $v_0 \in V(G)$, and for any list L of G such that for any $v \in V(G) \setminus \{v_0\} : |L(v)| = a$ and $|L(v_0)| = b$, there exists an (L, b) -choosability c of G .

3. CHOOSABILITY OF A WEIGHTED PATH

The path P_{n+1} or length n is the graph with vertex set $V = \{v_0, v_1, \dots, v_n\}$ and edge set $E = \bigcup_{i=0}^{n-1} (v_i, v_{i+1})$. To simplify the notations $L(i)$ denotes $L(v_i)$ and $c(i)$ denotes $c(v_i)$.

Definition 4. For the path P_{n+1} of order n ,

- a waterfall list L is a list such that for all $i, j \in \{0, \dots, n\}$ with $|i - j| \geq 2$, $L(i) \cap L(j) = \emptyset$;
- two lists L and L' are similar if and only if P_{n+1} is (L, ω) -choosable whenever P_{n+1} is (L', ω) -choosable;
- the amplitude $A_{i,j}(L)$ (or $A_{i,j}$) of a list L is $A_{i,j}(L) = \bigcup_{k=i}^j L(k)$;
- a list L is good if $|L(i)| \geq \omega(i) + \omega(i+1)$ for any $i, 1 \leq i \leq n-1$.

Notice that another similar definition of a waterfall list is that any color is present only on one list or on two lists of consecutive vertices.

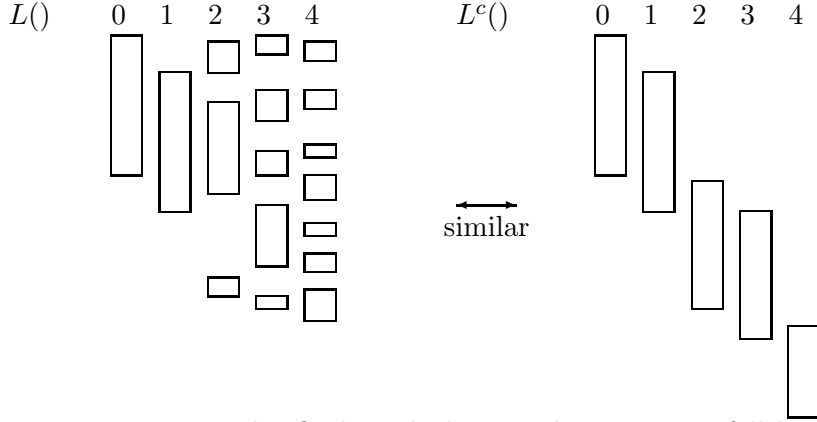


Fig. 1. Example of a list which is similar to a waterfall list.

Figure 1 shows a list L of P_5 (on the left), together with a similar waterfall list L^c (on the right).

Proposition 5. *For any good list L of P_{n+1} , there exists a similar waterfall list L^c with $|L^c(i)| = |L(i)|$ for all $i \in \{0, \dots, n\}$.*

Proof. We are going to transform a good list L of P_{n+1} into a waterfall list L^c and we will prove that L^c is similar with L .

First, remark that if a color $x \in L(i-1)$ but $x \notin L(i)$ for some $i, 1 \leq i \leq n-1$, then for any $j > i$, one can change the color x by a new color $y \notin A(L)$ in the list $L(j)$, without changing the choosability of the list. With this remark in hand, we can assume that L is such that any color x appears on the lists of consecutive vertices i_x, \dots, j_x .

Now, by permuting the colors if necessary, we can assume that if $x < y$ then $i_x < i_y$ or $i_x = i_y$ and $j_x \leq j_y$.

Repeat the following transformation:

1. Take the minimum color x for which $j_x \geq i_x + 2$ (i.e. the color x is present on at least three vertices $i_x, i_x + 1, i_x + 2, \dots, j_x$;

2. Replace color x by a new color y in lists $L(i_x + 2), \dots, L(j_x)$;

until the obtained list is a waterfall list (obviously, the number of iterations is always finite).

Now, we show that this transformation preserves the choosability of the list: Let L' be the list obtained from the list L by the above transformation.

If c is an (L, ω) -choosability of P_{n+1} then the choosability c' obtained from c by changing the color x by the color y in the color set $c(k)$ of each vertex $k \geq i_x + 2$ (containing x) is an (L', ω) -choosability since y is a new color.

Conversly, if c' is an (L', ω) -choosability of P_{n+1} , we consider two cases:

Case 1: $x \notin c'(i_x + 1)$ or $y \notin c'(i_x + 2)$. In this case, the choosability c obtained from c' by changing the color y by the color x in the color set $c'(k)$ of each vertex $k \geq i_x + 2$ (containing y) is an (L, ω) -choosability.

Case 2: $x \in c'(i_x + 1)$ and $y \in c'(i_x + 2)$. We have to consider two subcases:

- Subcase 1: $L'(i_x + 1) \not\subset (c'(i_x) \cup c'(i_x + 1) \cup c'(i_x + 2))$. There exists $z \in L'(i_x + 1) \setminus (c'(i_x) \cup c'(i_x + 1) \cup c'(i_x + 2))$ and the choosability c obtained from c' by changing the color x by the color z in $c'(i_x + 1)$ and replacing the color y by the color x in the color set $c'(k)$ of each vertex $k \geq i_x + 2$ (containing y) is an (L, ω) -choosability.
- Subcase 2: $L'(i_x + 1) \subset (c'(i_x) \cup c'(i_x + 1) \cup c'(i_x + 2))$. We have

$$|L'(i_x + 1)| = \left| \left((c'(i_x) \cup c'(i_x + 1) \cup c'(i_x + 2)) \cap L'(i_x + 1) \right) \right|.$$

As c' is an (L', ω) -choosability of P_{n+1} , we have

$$|L'(i_x + 1)| = |c'(i_x + 2) \cap L'(i_x + 1)| + |c'(i_x + 1) \cap L'(i_x + 1)| + \left| \left(c'(i_x) \setminus c'(i_x + 2) \right) \cap L'(i_x + 1) \right|,$$

$$|L'(i_x + 1)| - \omega(i_x + 1) - |c'(i_x + 2) \cap L'(i_x + 1)| = \left| \left(c'(i_x) \setminus c'(i_x + 2) \right) \cap L'(i_x + 1) \right|.$$

Since $y \in c'(i_x + 2)$ and $y \notin L'(i_x + 1)$, we obtain that

$$|c'(i_x + 2) \cap L'(i_x + 1)| \leq \omega(i_x + 2) - 1,$$

hence

$$\left(|L'(i_x + 1)| - \omega(i_x + 1) - \omega(i_x + 2) \right) + 1 \leq \left| \left(c'(i_x) \setminus c'(i_x + 2) \right) \cap L'(i_x + 1) \right|.$$

But, by hypothesis, L is a good list. Thus $|L(i_x + 1)| = |L'(i_x + 1)| \geq \omega(i_x + 1) + \omega(i_x + 2)$ and

$$1 \leq \left| \left(c'(i_x) \setminus c'(i_x + 2) \right) \cap L'(i_x + 1) \right|.$$

Consequently, there exists $z \in \left(c'(i_x) \setminus c'(i_x + 2) \right) \cap L'(i_x + 1)$. The choosability c is then constructed from c' by changing the color x by the color z in $c'(i_x + 1)$, the color z by the color x in $c'(i_x)$ and the color y by the color x in the set $c'(k)$ of each vertex $k \geq i_x + 2$.

□

Remark 6. Let L^c be a waterfall list of P_{n+1} and c be an (L^c, ω) -choosability of P_{n+1} . Then, it is clear that if $i, j \in \{0, \dots, n\}$ such that $i \neq j$ then

$$c(i) \cap c(j) = \emptyset \quad \text{and} \quad \left| \bigcup_{k=i}^j c(k) \right| = \sum_{k=i}^j \omega(k).$$

We now need to define the notion of waterfall choosability (see Figure 2 for an illustration). For a list L of P_{n+1} , set

$$I(i, j) := \begin{cases} \bigcap_{k=i}^j L(k) & \text{if } j \geq i \\ \emptyset & \text{otherwise} \end{cases}$$

and

$$I_p(i, j) := I(i, j) \setminus \left(L(i-1) \cup L(j+1) \right).$$

Definition 7. Let L^c be a waterfall list of P_{n+1} and c be an (L^c, ω) -choosability of P_{n+1} . We say that c is a waterfall (L^c, ω) -choosability of P_{n+1} if for all $i \in \{0, \dots, n\}$:

$$c(i) = K_{2i-1} \cup F_i \cup K_{2i}$$

with

$$K_{2i-1} \subset I_p(i-1, i) \setminus K_{2i-2} \text{ such that } |K_{2i-1}| = \min\{|I_p(i-1, i) \setminus K_{2i-2}|, \omega(i)\},$$

$$F_i \subset I_p(i, i) \text{ such that } |F_i| = \min\{|I_p(i, i)|, \omega(i) - |K_{2i-1}|\},$$

$$K_{2i} \subset I_p(i, i+1) \text{ such that } |K_{2i}| = \omega(i) - |K_{2i-1}| - |F_i|.$$

And with the convention $K_{-1} = K_{-2} = \emptyset$.

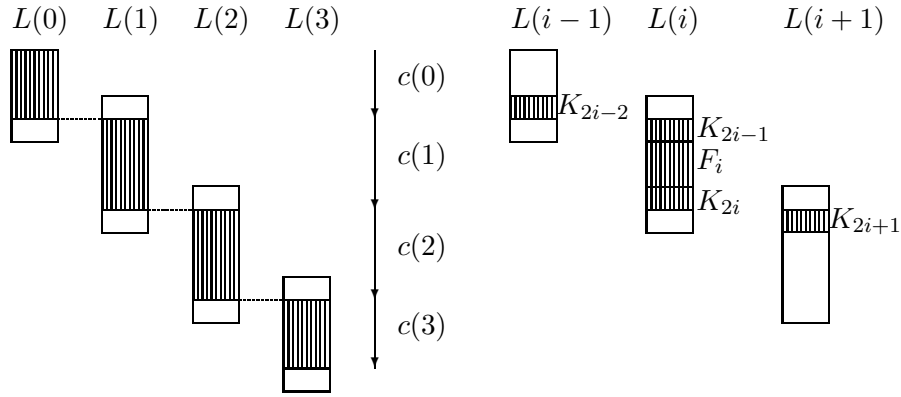


Fig. 2. Example of a waterfall choosability.

Lemma 8. Let L^c be a waterfall list of P_{n+1} . There exists an (L^c, ω) -choosability c of P_{n+1} if and only if there exists a waterfall (L^c, ω) -choosability c^c of P_{n+1} .

Proof. Under the hypothesis, if c^c is a waterfall (L^c, ω) -choosability of P_{n+1} then $c = c^c$ is an (L^c, ω) -choosability of P_{n+1} .

Conversely, if c is an (L^c, ω) -choosability of P_{n+1} then we construct by induction c_j for $j \in \{0, \dots, n\}$ such that: c_j is a waterfall (L^c, ω) -choosability of P_{j+1} , and c_j is an (L^c, ω) -choosability of P_{n+1} , and

$$c_j(i) = \begin{cases} c_{j-1}(i) & \text{if } i \in \{0, \dots, j-1\} \\ c(i) & \text{if } i \in \{j+1, \dots, n\}. \end{cases}$$

We begin by the construction of c_0 . Let f and δ be two integers such that $f = |c(0) \cap I_p(0, 0)|$ and $\delta = \min\{|I_p(0, 0)|, \omega(0)\}$. We have obviously $f \leq \delta$. Since

$$|c(0) \cap I_p(0, 1)| = |c(0)| - |c(0) \cap I_p(0, 0)| = \omega(0) - f \geq \delta - f$$

and

$$|I_p(0, 0) \setminus c(0)| = |I_p(0, 0)| - f \geq \delta - f,$$

we obtain that there exist two sets $k \subset c(0) \cap I_p(0, 1)$ and $r \subset I_p(0, 0) \setminus c(0)$ such that $|k| = |r| = \delta - f$. The invertible process between c_0 and c is the following one:

$$c_0(i) = \begin{cases} r \cup c(0) \setminus k & \text{if } i = 0 \\ c(i) & \text{otherwise} \end{cases}$$

By construction of r we have: $c_0(0) \cap c_0(1) = \emptyset$. Hence c_0 is a waterfall (L^c, ω) -choosability of P_1 and c_0 is an (L^c, ω) -choosability of P_{n+1} .

Let $j \geq 1$ and suppose that c_{j-1} is constructed. Recall that $K_{2j-2} = c_{j-1}(j-1) \cap I_p(j-1, j)$. In the same spirit as above, we consider two integers f and δ such that $f = |c_{j-1}(j) \cap I_p(j-1, j)|$ and $\delta = \min\{|I_p(j-1, j) \setminus K_{2j-2}|, \omega(j)\}$. We have of course $\delta \geq f$. Since

$$|c_{j-1}(j) \cap (I_p(j, j) \cup I_p(j, j+1))| = |c_{j-1}(j)| - f = \omega(j) - f \geq \delta - f,$$

and

$$|I_p(j-1, j) \setminus (K_{2j-2} \cup c_{j-1}(j))| = |I_p(j-1, j) \setminus K_{2j-2}| - f \geq \delta - f,$$

there exist two sets

$$k \subset c_{j-1}(j) \cap (I_p(j, j) \cup I_p(j, j+1))$$

and

$$r \subset I_p(j-1, j) \setminus (K_{2j-2} \cup c_{j-1}(j))$$

such that $|k| = |r| = \delta - f$. We construct an intermediary choosability c'_j using c_{j-1} with the following invertible process:

$$c'_j(i) = \begin{cases} r \cup c_{j-1}(j) \setminus k & \text{if } i = j \\ c_{j-1}(i) & \text{otherwise} \end{cases}$$

We obtain that $|c'_j(j) \cap I_p(j-1, j)| = \min\{|I_p(j-1, j) \setminus K_{2j-2}|, \omega(j)\}$.

Using the same idea, we continue. Recall that $K_{2j-1} = c'_j(j) \cap I_p(j-1, j)$. Let f' and δ' be two integers such that $f' = |c'_j(j) \cap I_p(j, j)|$ and $\delta' = \min\{|I_p(j, j)|, \omega(j) - |K_{2j-1}|\}$. We have $\delta' \geq f'$. Since

$$|c'_j(j) \cap I_p(j, j+1)| = (\omega(j) - |K_{2j-1}|) - f' \geq \delta' - f'$$

and

$$|I_p(j, j) \setminus c'_j(j)| = |I_p(j, j)| - f' \geq \delta' - f',$$

we obtain that there exist two sets $k' \subset c'_j(j) \cap I_p(j, j+1)$ and $r' \subset I_p(j, j) \setminus c'_j(j)$ such that $|k'| = |r'| = \delta' - f'$. Then, we construct c_j using c'_j with the following invertible process:

$$c_j(i) = \begin{cases} r' \cup c'_j(j) \setminus k' & \text{if } i = j \\ c'_j(i) & \text{otherwise} \end{cases}$$

Thus, c_j is a waterfall (L^c, ω) -choosability of P_{j+1} , and c_j is an (L^c, ω) -choosability of P_{n+1} . Finally, by induction, we construct $c_n = c^c$. \square

We have the following theorem which gives a necessary and sufficient condition for a weighted path to be (L^c, ω) -choosable.

Theorem 9. *Let L^c be a waterfall list of a weighted path P_{n+1} . Then P_{n+1} is (L^c, ω) -choosable if and only if:*

$$\forall i, j \in \{0, \dots, n\} : |A(i, j)(L^c)| \geq \sum_{k=i}^j \omega(k).$$

Proof. Let L^c be a waterfall list of a weighted path P_{n+1} . If c is a (L^c, ω) -choosability of P_{n+1} then

$$\forall i, j \in \{0, \dots, n\} : A(i, j) \supset \bigcup_{k=i}^j c(k).$$

Remark 6 shows that:

$$\forall i, j \in \{0, \dots, n\} : |A(i, j)| \geq \sum_{k=i}^j \omega(k).$$

Conversely, we use an induction on the length n of the path.

For $n = 0$, since $|L^c(0)| = |A(0, 0)| \geq \omega(0)$ we have that P_1 is (L^c, ω) -choosable.

Suppose now that the property is true for all k such that $0 \leq k \leq n$ and let L^c be a waterfall list of P_{n+2} such that:

$$\forall i, j \in \{0, \dots, n+1\} : |A(i, j)| \geq \sum_{k=i}^j \omega(k).$$

Consider the path P_{n+1} viewed as a subgraph of P_{n+2} . We have for all $i, j \in \{0, \dots, n\} : |A(i, j)| \geq \sum_{k=i}^j \omega(k)$. Hence, by induction, P_{n+1} is (L^c, ω) -choosable and Lemma 8 shows that there exists a waterfall (L^c, ω) -choosability c^c of P_{n+1} .

Recall that, by definition

$$\forall i \in \{0, \dots, n\} : c^c(i) = K_{2i-1} \cup F_i \cup K_{2i}.$$

We have two cases to study:

Case 1: there exists $t \in \{0, \dots, n\}$ such that $|K_{2t}| = 0$.

Consider the path P'_t viewed as the subgraph of P_{n+1} such that $V(P'_t) = \{v_{t+1}, \dots, v_{n+1}\}$. P'_t is a path of length $n - t \leq n$, hence, by induction, there exists an (L^c, ω) -choosability $c_{t'}$ of P'_t . Since

$$L^c(t+1) \setminus K_{2t} = L^c(t+1) \setminus c^c(t) = L^c(t+1),$$

there is no conflict between c^c and $c_{t'}$ since $c^c(v_t) \cap c_{t'}(v_{t+1}) = \emptyset$. Thus, we obtain an (L^c, ω) -choosability c' of P_{n+2} such that

$$c'(i) = \begin{cases} c^c(i) & \text{if } i \in \{0, \dots, t\} \\ c_{t'}(i) & \text{otherwise} \end{cases}$$

Case 2: for all $i \in \{0, \dots, n\} : |K_{2i}| > 0$.

By construction of c^c we have for all $i \in \{0, \dots, n\} : |K_{2i}| = \omega(i) - (|F_i| + |K_{2i-1}|)$, then

$$\forall i \in \{0, \dots, n\} : \omega(i) - |K_{2i-1}| > |F_i|.$$

Since for all $i \in \{0, \dots, n\}$ we have $|F_i| = \min\{|I_p(i, i)|, \omega(i) - |K_{2i-1}|\}$, hence we obtain

$$\forall i \in \{0, \dots, n\} : |F_i| = |I_p(i, i)| \quad (*)$$

Since for all $i \in \{0, \dots, n\} : |F_i| \geq 0$ et $\omega(i) - |K_{2i-1}| > |F_i|$ then $\omega(i) > |K_{2i-1}|$. By definition, we have for all $i \in \{0, \dots, n\} : |K_{2i-1}| = \min(|I_p(i-1, i)| - |K_{2i-2}|, \omega(i))$ hence we obtain that $|K_{2i-1}| = |I_p(i-1, i)| - |K_{2i-2}|$ and thus

$$\forall i \in \{0, \dots, n\} : |K_{2i-1}| + |K_{2i-2}| = |I_p(i-1, i)| \quad (**)$$

By construction of c^c we have

$$\forall i \in \{0, \dots, n\} : |K_{2i}| + |F_i| + |K_{2i-1}| = \omega(i),$$

and by convention $|K_{-1}| = 0$, hence we add all the previous equalities and we obtain:

$$|K_{2n}| + \sum_{i=0}^{n-1} (|K_{2i}| + |K_{2i+1}|) + \sum_{i=0}^n |F_i| = \sum_{i=0}^n \omega(i).$$

Now we use the equalities (*) and (**):

$$|K_{2n}| + \sum_{i=0}^{n-1} |I_p(i, i+1)| + \sum_{i=0}^n |I_p(i, i)| = \sum_{i=0}^n \omega(i).$$

By elementary computations, it can be shown that:

$$|A(0, n+1)| = \sum_{i=0}^{n+1} (|I_p(i, i)| + |I_p(i, i+1)|).$$

Hence (since $|I_p(n+1, n+2)| = 0$):

$$|K_{2n}| + |A(0, n+1)| - |I_p(n, n+1)| - |I_p(n+1, n+1)| = \sum_{i=0}^n \omega(i).$$

In other words, we have:

$$\omega(n+1) + \left(|A(0, n+1)| - \sum_{i=0}^{n+1} \omega(i) \right) = |I_p(n, n+1)| + |I_p(n+1, n+1)| - |K_{2n}|.$$

We use here the hypothesis $|A(0, n+1)| \geq \sum_{i=0}^{n+1} \omega(i)$, which implies

$$\omega(n+1) \leq |I_p(n, n+1)| + |I_p(n+1, n+1)| - |K_{2n}|.$$

Since $|I_p(n, n+1)| + |I_p(n+1, n+1)| = |L^c(n+1)|$ and $K_{2n} = c^c(n) \cap L^c(n+1) \subset L^c(n+1)$, then

$$\omega(n+1) \leq |L^c(n+1) \setminus c^c(n)|.$$

Finally, we choose $c(n+1) \subset L^c(n+1) \setminus c^c(n)$ such that $|c(n+1)| = \omega(n+1)$, and we construct an (L^c, ω) -choosability c' such that :

$$c'(i) = \begin{cases} c^c(i) & \text{if } i \in \{0, \dots, n\} \\ c(n+1) & \text{if } i = n+1 \end{cases}$$

□

The previous result has the following corollary when the list is a good waterfall list and $|L(n)| \geq \omega(n)$.

Corollary 10. *Let L^c be a good waterfall list of a weighted path P_{n+1} such that $|L^c(n)| \geq \omega(n)$. Then P_{n+1} is (L^c, ω) -choosable if and only if*

$$\forall j \in \{0, \dots, n\} : |A(0, j)(L^c)| \geq \sum_{k=0}^j \omega(k).$$

Proof. Under the hypothesis, if P_{n+1} is (L^c, ω) -choosable, then Theorem 9 proves in particular the result.

Conversely, since L^c is a waterfall list of P_{n+1} , we have:

$$\forall i, j \in \{1, \dots, n\} : |A(i, j)| = |\cup_{k=i}^j L^c(k)| \geq |\cup_{\substack{k=i \\ k-i \text{ even}}}^j L^c(k)| = \sum_{\substack{k=i \\ k-i \text{ even}}}^j |L^c(k)|.$$

Since L^c is a good list of P_{n+1} (for simplicity, we set $\omega(n+1) = 0$):

$$\forall i, j \in \{1, \dots, n\} : \sum_{\substack{k=i \\ k-i \text{ even}}}^j |L^c(k)| \geq \sum_{\substack{k=i \\ k-i \text{ even}}}^j (\omega(k) + \omega(k+1)) \geq \sum_{k=i}^j \omega(k),$$

then we obtain for all $i, j \in \{1, \dots, n\} : |A(i, j)| \geq \sum_{k=i}^j \omega(k)$. Since for all $j \in \{0, \dots, n\} : |A(0, j)| \geq \sum_{k=0}^j \omega(k)$, Theorem 9 concludes the proof. \square

Another interesting corollary holds for lists L such that $|L(0)| = |L(n)| = b$, and for all $i \in \{1, \dots, n-1\} : |L(i)| = a$. The function Even is defined for any real x by: Even(x) is the smallest even integer p such that $p \geq x$.

Corollary 11. *Let L be a list of P_{n+1} such that $|L(0)| = |L(n)| = b$, and $|L(i)| = a = 2b + e$ for all $i \in \{1, \dots, n-1\}$.*

If $n \geq \text{Even}\left(\frac{2b}{e}\right)$ then P_{n+1} is (L, b) -choosable.

Proof. The hypothesis implies that L is a good list of P_{n+1} , hence by Proposition 5, there exists a waterfall list L^c similar to L . So we get:

$$\forall i \in \{1, \dots, n-1\} : |L^c(i)| \geq 2b = \omega(i) + \omega(i+1)$$

and $|L^c(n)| \geq b = \omega(n)$. By Corollary 10 it remains to prove that:

$$\forall j \in \{0, \dots, n\} : |A(0, j)| \geq \sum_{k=0}^j \omega(k) = (j+1)b.$$

Case 1: $j = 0$. By hypothesis, we have $|A(0, 0)| = |L^c(0)| \geq b$.

Case 2: $j \in \{1, \dots, n-1\}$. Since L^c is a waterfall list of P_{n+1} we obtain that:
if j is even

$$|A(0, j)| \geq \sum_{\substack{k=0 \\ k \text{ even}}}^j |L^c(k)| = b + \sum_{\substack{k=2 \\ k \text{ even}}}^j 2b = b + \frac{j}{2} 2b = (j+1)b,$$

and if j is odd

$$|A(0, j)| \geq \sum_{\substack{k=0 \\ k \text{ odd}}}^j |L^c(k)| = \sum_{\substack{k=1 \\ k \text{ odd}}}^j 2b = \frac{j+1}{2} 2b = (j+1)b.$$

Hence for all $j \in \{0, \dots, n-1\}$: $|A(0, j)| \geq (j+1)b$.

Case 3: $j = n$. Since $n \geq \text{Even}\left(\frac{2b}{e}\right)$ by hypothesis, and

$$|A(0, n)| \geq \sum_{\substack{k=0 \\ k \text{ odd}}}^n |L^c(k)| = \begin{cases} a\frac{n}{2} & \text{if } n \text{ is even} \\ b + a\frac{n-1}{2} & \text{otherwise} \end{cases}$$

we deduce that $|A(0, n)| \geq (n+1)b$, which concludes the proof. \square

4. FREE-CHOOSABILITY OF A CYCLE

The cycle C_n of length n is the graph with vertex set $V = \{v_0, \dots, v_{n-1}\}$ and edge set $E = \bigcup_{i=0}^{n-1} (v_i, v_{i+1(\text{mod } n)})$.

Let $FC_h(x)$ be the set of graphs G which are (a, b) -free-choosable for all a, b such that $\frac{a}{b} \geq x$:

$$FC_h(x) = \{G, \text{ such that for all } \frac{a}{b} \geq x, G \text{ is } (a, b)\text{-free-choosable}\}.$$

Moreover, we can define the free-choice ratio $\text{fchr}(G)$ of a graph G by:

$$\text{fchr}(G) := \inf\left\{\frac{a}{b} \mid G \text{ is } (a, b)\text{-free-choosable}\right\}.$$

If $\lfloor x \rfloor$ denotes the greatest integer less or equal to the real x , we can state:

Theorem 12. *If C_n is a cycle of length n , then*

$$C_n \in FC_h\left(2 + \left\lfloor \frac{n}{2} \right\rfloor^{-1}\right).$$

Moreover, we have:

$$\text{fchr}(C_n) = 2 + \left\lfloor \frac{n}{2} \right\rfloor^{-1}.$$

Proof. Let a, b be two integers such that $a/b \geq 2 + \lfloor \frac{n}{2} \rfloor^{-1}$. Let C_n be a cycle of length n and L a a -list of C_n . Without loss generality, we can suppose that v_0 is the vertex chosen for the free-choosability and $L_0 \subset L(v_0)$ has b elements. It remains to construct an (L, b) -choosability c of C_n such that $c(v_0) = L_0$. Hence we have to construct an (L', b) -choosability c of P_{n+1} such that $L'(0) = L'(n) = L_0$ and for all $i \in \{1, \dots, n-1\}$, $L'(i) = L(v_i)$. We have $|L'(0)| = |L'(n)| = b$ and for all $i \in \{1, \dots, n-1\}$, $|L'(i)| = a$. Since $a/b \geq 2 + \lfloor \frac{n}{2} \rfloor^{-1}$ and $e = a - 2b$, we get $e/b \geq \lfloor \frac{n}{2} \rfloor^{-1}$ hence $n \geq \text{Even}(2b/e)$. Using Corollary 11, we get:

$$C_n \in FC_h\left(2 + \left\lfloor \frac{n}{2} \right\rfloor^{-1}\right).$$

Hence, we have that $\text{fchr}(C_n) \leq 2 + \lfloor \frac{n}{2} \rfloor^{-1}$. Moreover, let us prove that $M = 2 + \lfloor \frac{n}{2} \rfloor^{-1}$ is reached.

For n odd, Voigt has proved in [13] that the choice ratio $\text{chr}(C_n)$ of a cycle of odd length n is exactly M . Hence $\text{fchr}(C_n) \geq \text{chr}(C_n) = M$, and the result is proved.

For n even, let a, b be two integers such that $\frac{a}{b} < M$. We construct a counterexample for the free-choosability: let L be the list of C_n such that

$$L(i) = \begin{cases} \{1, \dots, a\} & \text{if } i \in \{0, 1\} \\ \{1 + \frac{i-1}{2}a, \dots, (\frac{i-1}{2} + 1)a\} & \text{if } i \neq n-1 \text{ is odd} \\ \{b+1 + \frac{i-2}{2}a, \dots, b + (\frac{i-2}{2} + 1)a\} & \text{if } i \text{ is even and } i \neq 0 \\ \{1, \dots, b, 1 + (\frac{n-4}{2} + 1)a, \dots, (\frac{n-4}{2} + 2)a - b\} & \text{if } i = n-1 \end{cases}$$

If we choose $c_0 = \{1, \dots, b\} \subset L(0)$, we can check that it does not exist an (L, b) -choosability of C_n such that $c(0) = c_0$, so we could not do better. \square

Remark 13. *In particular, the previous theorem implies that if $n \geq \text{Even}(\frac{2b}{e})$ then the cycle C_n of length n is $(2b + e, b)$ -free-choosable.*

Remark 14. *Erdős, Rubin and Taylor have stated in [3] the following question: If G is (a, b) -choosable, and $\frac{c}{d} > \frac{a}{b}$, does it imply that G is (c, d) -choosable? Gutner and Tarsi have shown in [6] that the answer is negative in general. If we consider the analogue question for free-choosability, then the previous theorem implies that it is true for the cycle.*

REFERENCES

- [1] N. Alon, Zs. Tuza, M. Voigt, *Choosability and fractional chromatic number*, Discrete Math. 165/166, (1997), 31-38.
- [2] O.V. Borodin, A.V. Kostochka, D.R. Woodall, *List edge and list colourings of multi-graph*, J. Combin. Theory Series B, 71 : 184-204, (1997).
- [3] P. Erdős, A.L Rubin and H. Taylor, *Choosability in graphs*, Proc. West-Coast Conf. on Combinatorics, Graph Theory and Computing, Congressus Numerantium XXVI, (1979), 125-157.
- [4] J.-C. Godin, *Coloration et choisissabilité des graphes et applications*, PhD thesis (in french), Université du Sud Toulon-Var, France (2009).
- [5] S. Gravier, *A Hajós-like theorem for list coloring*, Discrete Math. 152, (1996), 299-302.
- [6] S. Gutner and M. Tarsi, *Some results on $(a:b)$ -choosability*, Discrete Math. 309, (2009), 2260-2270.
- [7] F. Havet, *Choosability of the square of planar subcubic graphs with large girth*, Discrete Math. 309, (2009), 3553-3563.
- [8] F. Havet, *Channel assignment and multicolouring of the induced subgraphs of the triangular lattice*. Discrete Math. 233, (2001), 219-233.
- [9] C. Thomassen, *The chromatic number of a graph of girth 5 on a fixed surface*, J. Combin. Theory, (2003), 38-71.
- [10] Zs. Tuza and M. Voigt, *Every 2-choosable graph is $(2m, m)$ -choosable*, J. Graph Theory 22, (1996), 245-252.
- [11] V. G Vizing, *Coloring the vertices of a graph in prescribed colors (in Russian)*, Diskret. Analiz. No. 29, Metody Diskret. Anal. v Teorii Kodov i Shem 101 (1976), 3-10.
- [12] M. Voigt, *Choosability of planar graphs*, Discrete Math., 150, (1996), 457-460.
- [13] M. Voigt, *On list Colourings and Choosability of Graphs*, Abilitationsschrift, TU Ilmenau (1998).

INSTITUT DE MATHÉMATIQUES DE TOULON, UNIVERSITÉ DU SUD TOULON-VAR, FRANCE,
AND LABORATOIRE LE2I, UNIVERSITÉ DE BOURGOGNE, FRANCE

E-mail address: yves.aubry@univ-tln.fr, godinjeanchri@yahoo.fr and olivier.togni@u-bourgogne.fr