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# REGULARITY AND CROSSINGS OF SHOT NOISE PROCESSES

HERMINE BIERMÉ AND AGNÈS DESOLNEUX

**ABSTRACT.** In this paper, we consider shot noise processes and their expected number of level crossings. When the kernel response function is sufficiently smooth, the crossings mean number function is obtained through an integral formula. Moreover, as the intensity increases, or equivalently as the number of shots becomes larger, a normal convergence to the classical Rice's formula for Gaussian processes is obtained. The Gaussian kernel function is studied in detail and two different regimes are exhibited.

## 1. INTRODUCTION

In this paper, we will consider a *shot noise process* which is a real-valued random process given by

$$(1) \quad X(t) = \sum_i \beta_i g(t - \tau_i), \quad t \in \mathbb{R}$$

where  $g$  is a given (deterministic) measurable function (it will be called the *kernel function* of the shot noise), the  $\{\tau_i\}$  are the points of a Poisson point process on the line of intensity  $\lambda\nu(ds)$ , where  $\lambda > 0$  and  $\nu$  is a positive  $\sigma$ -finite measure on  $\mathbb{R}$ , and the  $\{\beta_i\}$  are independent copies of a random variable  $\beta$  (called the *impulse*), independent of  $\{\tau_i\}$ .

Shot noise processes are related to many problems in physics as they result from the superposition of “shot effects” which occur at random. Fundamental results were obtained by Rice in [24]. Daley in [10] gave sufficient conditions on the kernel function to ensure the convergence of the formal series in a preliminary work. General results, including sample paths properties, were given by Rosiński [25] in a more general setting. In most of the literature the measure  $\nu$  is the Lebesgue measure on  $\mathbb{R}$  such that the shot noise process is a stationary one. In order to derive more precise sample paths properties and especially crossings rates, mainly two properties have been extensively exhibited and used. The first one is the Markov property, which is valid, choosing a non continuous positive causal kernel function that is 0 for negative time. This is the case in particular of the exponential kernel  $g(t) = e^{-t}\mathbf{1}_{t \geq 0}$  for which explicit distributions and crossings rates can be obtained [22]. A simple formula for the expected numbers of level crossings is valid for more general kernels of this type but resulting shot noise processes are non differentiable [4, 17]. The infinitely divisible property is the second main tool. Actually, this allows to establish convergence to a Gaussian process as the intensity increases [23, 16]. Sample paths properties of Gaussian processes have been extensively studied and fine results are known concerning the level crossings of smooth Gaussian processes (see [2, 9] for instance).

The goal of the paper is to study the crossings of a shot noise process in the general case when the kernel function  $g$  is smooth. In this setting we lose Markov's property but the shot noise process inherits smoothness properties. Integral formulas for the number of level crossings of such processes was generalized to the non Gaussian case by [19] but assumptions rely on properties of densities of distributions, which may not be valid for shot noise processes. We derive integral formulas for the crossings mean number function and pay a special interest in the continuity of this function with respect to the level. Exploiting further on normal convergence, we exhibit a Gaussian regime for the

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mean crossings function when the intensity goes to infinity. A particular example, which is studied in detail, concerns the Gaussian shot noise process where  $\beta = 1$  almost surely and  $g$  is a Gaussian kernel of width  $\sigma$ :

$$g(t) = g_\sigma(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-t^2/2\sigma^2}.$$

Such a model has many applications because it is solution of the heat equation (we consider  $\sigma$  as a variable), and it thus models a diffusion from random sources (the points of the Poisson point process).

The paper is organized as follows. In Section 2, we first give general properties (moments, covariance, regularity) of a shot noise process defined by (1). In Section 3, we study the question of the existence and the continuity of a probability density for  $X(t)$ . Such a question is important to obtain a Rice's formula for crossings. In Section 4, we give an explicit formula for crossings of a shot noise process in terms of its characteristic function (which can be controlled thanks to estimates on oscillatory integrals). One of the difficulties is to obtain results for the crossings of a given level  $\alpha$  and not only for almost every  $\alpha$ . In Section 5, we show how the crossings mean number function converges, and in which sense, to the one of a Gaussian process when the intensity  $\lambda$  goes to infinity. We give rates of this convergence. Finally, in Section 6, we study in detail the case of a Gaussian kernel of width  $\sigma$ . We are mainly interested in the mean number of local extrema of this process, as a function of  $\sigma$ . Thanks to the heat equation, and also to scaling properties between  $\sigma$  and  $\lambda$ , we prove that the mean number of local extrema is a decreasing function of  $\sigma$ , and give its asymptotics as  $\sigma$  is small or large.

## 2. GENERAL PROPERTIES

**2.1. Elementary properties.** The shot noise process given by the formal sum (1) can also be written as the stochastic integral

$$(2) \quad X(t) = \int_{\mathbb{R} \times \mathbb{R}} z g(t-s) N(ds, dz),$$

where  $N$  is a Poisson random measure of intensity  $\lambda\nu(ds)F(dz)$ , where  $F$  is the common law of  $\beta$ . We recall the basic facts (see [18] chapter 10 for instance) that for measurable sets  $A \subset \mathbb{R} \times \mathbb{R}$  the random variable  $N(A)$  is Poisson distributed with mean  $\lambda\nu \otimes F(A)$  and if  $A_1, \dots, A_n$  are disjoint then  $N(A_1), \dots, N(A_n)$  are independent. We also recall that for measurable functions  $k : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ , the stochastic integral  $\int k(s, z) N(ds, dz)$  of  $k$  with respect to  $N$  exists a.s. if and only if

$$(3) \quad \int_{\mathbb{R} \times \mathbb{R}} \min(|k(s, z)|, 1) \lambda\nu(ds)F(dz) < \infty.$$

We focus in this paper on stationary shot noise processes obtained when  $\nu(ds) = ds$  is the Lebesgue measure. Such processes are obtained as the almost sure limit of truncated shot noise processes defined for  $\nu_T(ds) = \mathbf{1}_{[-T, T]}(s)ds$ , as  $T$  tends to infinity. Therefore, from now on and in all the paper, we make the following assumption.

**Assumption 1.** *The measure  $\nu$  is absolutely continuous with respect to the Lebesgue measure and its Radon Nikodym derivative is bounded by 1 almost everywhere.*

Then, assuming that the random impulse  $\beta$  is an integrable random variable of  $L^1(\Omega)$  and that the kernel function  $g$  is an integrable function of  $L^1(\mathbb{R})$ , is enough to check (3) and to ensure the almost sure convergence of the infinite sum (see also Campbell Theorem and [16]). We will work under this assumption in the whole paper.

**Finite dimensional distribution.** The main tool to study the law of  $X$  is its characteristic function. From Lemma 10.2 of [18], using (2), the characteristic function of  $X(t)$ , is given by

$$\begin{aligned} \forall u \in \mathbb{R}, \quad \psi_{X(t)}(u) = \mathbb{E}(e^{iuX(t)}) &= \exp\left(\int_{\mathbb{R} \times \mathbb{R}} [e^{iuzg(t-s)} - 1] \lambda \nu(ds) F(dz)\right) \\ &= \exp\left(\lambda \int_{\mathbb{R}} [\widehat{F}(ug(t-s)) - 1] \nu(ds)\right), \end{aligned}$$

where  $\widehat{F}(u) = \mathbb{E}(e^{iu\beta})$  is the characteristic function of  $\beta$  and Fourier transform of  $F$ . This formula is also well-known using the series representation (1) [23, 15, 28]. More generally, the finite dimensional distributions of the process  $X$  are characterized by

$$(4) \quad \mathbb{E} \left( \exp \left( i \sum_{j=1}^k u_j X(t_j) \right) \right) = \exp \left( \lambda \int_{\mathbb{R}} [\widehat{F} \left( \sum_{j=1}^k u_j g(t_j - s) \right) - 1] \nu(ds) \right),$$

for any  $k \geq 1$  and  $(t_1, \dots, t_k), (u_1, \dots, u_k) \in \mathbb{R}^k$ . Note that when  $\beta = 1$  almost surely, we have  $F = \delta_1$  (the Dirac mass at point 1) and  $\widehat{F}(u) = e^{iu}$ .

Stationary case: when  $\nu(ds) = ds$ , it is clear that  $\mathbb{E} \left( \exp \left( i \sum_{j=1}^k u_j X(t_0 + t_j) \right) \right) = \mathbb{E} \left( \exp \left( i \sum_{j=1}^k u_j X(t_j) \right) \right)$ , for any  $t_0 \in \mathbb{R}$ , by translation invariance of the Lebesgue measure, which means that  $X$  is a strictly stationary process.

**Moments.** Since  $g \in L^1(\mathbb{R})$  and  $\beta \in L^1(\Omega)$ ,  $X$  is an integrable process with

$$(5) \quad \mathbb{E}X(t) = \int_{\mathbb{R} \times \mathbb{R}} zg(t-s) \lambda \nu(ds) F(dz) = \lambda \mathbb{E}(\beta) \int_{\mathbb{R}} g(t-s) \nu(ds).$$

If moreover  $g \in L^2(\mathbb{R})$  and  $\beta \in L^2(\Omega)$ , then  $X$  defines a second order process with covariance function given by

$$\forall t, t' \in \mathbb{R}, \quad \text{Cov}(X(t), X(t')) = \int_{\mathbb{R} \times \mathbb{R}} z^2 g(t-s) g(t'-s) \lambda \nu(ds) F(dz) = \lambda \mathbb{E}(\beta^2) \int_{\mathbb{R}} g(t-s) g(t'-s) \nu(ds).$$

In particular, for all  $t \in \mathbb{R}$ ,

$$\text{Var}(X(t)) = \lambda \mathbb{E}(\beta^2) \int_{\mathbb{R}} g(t-s)^2 \nu(ds).$$

Stationary case: when  $\nu(ds) = ds$ , for  $t, t' \in \mathbb{R}$ ,  $\text{Cov}(X(t), X(t')) = S(t - t')$ , with

$$(6) \quad S(t) = g * \check{g}(t) = \lambda \mathbb{E}(\beta^2) \int_{\mathbb{R}} g(t-s) g(-s) ds, \quad \forall t \in \mathbb{R},$$

where  $\check{g}(t) = g(-t)$ . In particular, according to Fourier inverse theorem, the strictly stationary second order process  $X$  admits  $\frac{\lambda \mathbb{E}(\beta^2)}{2\pi} |\widehat{g}|^2$  for spectral density (see [11] p.522 for definition).

More generally, when for  $n \geq 2$ ,  $g \in L^1(\mathbb{R}) \cap \dots \cap L^n(\mathbb{R})$  and  $\mathbb{E}(|\beta|^n) < +\infty$ , according to [5], the  $n$ -th moment of  $X(t)$  exists and is given by

$$(7) \quad \mathbb{E}(X(t)^n) = \sum_{(r_1, \dots, r_n) \in I(n)} K_n(r_1, \dots, r_n) \prod_{k=1}^n \left( \lambda \mathbb{E}(\beta^k) \int_{\mathbb{R}} g(t-s)^k \nu(ds) \right)^{r_k},$$

where  $I(n) = \left\{ (r_1, \dots, r_n) \in \mathbb{N}^n; \sum_{k=1}^n k r_k = n \right\}$  and  $K_n(r_1, \dots, r_n) = \frac{n!}{(1!)^{r_1} (2!)^{r_2} \dots (n!)^{r_n} r_1! \dots r_n!}$ . Note also that in this case the  $n$ -th order cumulant of  $X(t)$  is finite and equals to

$$C_n = \frac{1}{i^n} \frac{d^n}{du^n} \log(\psi_{X(t)}(u))|_{u=0} = \lambda \mathbb{E}(\beta^n) \int_{\mathbb{R}} g(t-s)^n \nu(ds).$$

**Infinite divisibility property.** The process  $X$  is infinitely divisible. Actually let us choose  $(X_{\lambda/n}^{(j)})_{1 \leq j \leq n}$  i.i.d. shot noise processes with intensity  $(\lambda/n)\nu(ds)$  for  $n \geq 1$ . Let us denote  $X_\lambda$  a shot noise process with intensity  $\lambda\nu(ds)$ . From (4), it is clear that

$$X_\lambda \stackrel{fdd}{=} X_{\lambda/n}^{(1)} + \dots + X_{\lambda/n}^{(n)},$$

which proves the infinite divisibility according to [18] p.243. Here, as usual,  $\stackrel{fdd}{=}$  stands for the equality in finite dimensional distributions.

Stationary case: when  $\nu(ds) = ds$  and  $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$  and  $\beta \in L^2(\Omega)$ , the strictly stationary process  $X$  is mixing and thus ergodic. Actually, this comes from the fact that the codifference

$$\tau(t) := \log \mathbb{E}(e^{i(X(0)-X(t))}) - \log \mathbb{E}(e^{iX(0)}) - \log \mathbb{E}(e^{-iX(t)}) = \int_{\mathbb{R} \times \mathbb{R}} \left( e^{izg(-s)} - 1 \right) \left( e^{-izg(t-s)} - 1 \right) ds F(dz)$$

satisfies

$$|\tau(t)| \leq \mathbb{E}(\beta^2) |g| * |\check{g}|(t) \xrightarrow{|t| \rightarrow +\infty} 0,$$

which gives the result according to Proposition 4 and Remark 5 of [26].

**2.2. Simulation procedure in the stationary case.** We assume in this section that  $\nu$  is the Lebesgue measure on  $\mathbb{R}$  such that  $X$  is stationary. In some examples, the kernel function  $g$  of the shot noise process (1) will not have a bounded support. For instance this will be the case of a Gaussian kernel. In order to get a sample of  $t \rightarrow X(t)$  for  $t \in [-1, 1]$ , using a software like MatLab for instance, we need to truncate the sum in (1). Let  $T > 1$ , we write

$$\begin{aligned} X(t) = \sum_{\tau_i \in \mathbb{R}} \beta_i g(t - \tau_i) &= \sum_{|\tau_i| \leq T} \beta_i g(t - \tau_i) + \sum_{|\tau_i| > T} \beta_i g(t - \tau_i) \\ (8) \qquad \qquad \qquad &= X_T(t) + R_T(t), \end{aligned}$$

such that  $\lim_{T \rightarrow +\infty} X_T(t) = X(t)$  almost surely. Actually, when  $g$  is continuous and satisfies

$$\int_{\mathbb{R}} \sup_{t \in [-1, 1]} |g(t - s)| ds < +\infty$$

almost surely the convergence holds uniformly in  $t \in [-1, 1]$ .

Now, let us remark that  $X_T(\cdot) \stackrel{fdd}{=} \int_{\mathbb{R} \times \mathbb{R}} z g(\cdot - s) N_T(ds, dz)$ , where  $N_T$  is a Poisson random measure of intensity given by  $\lambda \nu_T(ds) F(dz)$ , with  $\nu_T(ds) = \mathbf{1}_{[-T, T]}(s) ds$  a finite measure. Therefore,

$$X_T(\cdot) \stackrel{fdd}{=} \sum_{i=1}^{\gamma_T} \beta_i g(\cdot - U_T^{(i)}),$$

where  $\gamma_T$  is a Poisson random variable of intensity  $\lambda \nu_T(\mathbb{R}) = 2\lambda T$ ,  $\{\beta_i\}$  are i.i.d. with common law  $F$ ,  $\{U_T^{(i)}\}$  are i.i.d. with uniform law on  $[-T, T]$ . Here and in the sequel the convention is that  $\sum_{i=1}^0 = 0$ .

The simulation algorithm to synthesize a sample of  $t \rightarrow X_T(t)$  on  $[-1, 1]$  is then the following:

1. Choose  $T > 1$ ,
2. Let  $n$  be sampled from the Poisson distribution of parameter  $2\lambda T$ ,
3. Let  $U_T^{(1)}, \dots, U_T^{(n)}$  be  $n$  points sampled independently and uniformly on  $[-T, T]$ ,
4. Let  $\beta_1, \dots, \beta_n$  be  $n$  independent samples of  $\beta$ ,
5. Finally for  $t \in [-1, 1]$  compute  $\sum_{i=1}^n \beta_i g(t - U_T^{(i)})$ .

Now, what is a “good” choice for  $T$ ? Ideally  $T$  should be as large as possible. If it is taken too small, it will clearly create a “bias” on the distribution and on the stationarity of the sample, since  $X_T$  is obviously not stationary.

Example: when  $g = g_\sigma$  is a Gaussian kernel of width  $\sigma$ , we can compute the “bias” on  $X(1)$ : it is given by

$$\mathbb{E}R_T(1) = \lambda \mathbb{E}(\beta) \int_{|s|>T} g_\sigma(1-s) ds = \lambda \mathbb{E}(\beta) \left( \int_{-\infty}^{-T+1} g_\sigma(s) ds + \int_{T+1}^{+\infty} g_\sigma(s) ds \right).$$

In this example, we see that taking  $T = 1$  will create a large “bias” since the first of the two above integral will be equal to 0.5. In order to make both integrals negligible, one has to take  $T - 1$  large compared to  $\sigma$  - say for instance  $T - 1 = 10\sigma$ , as illustrated in Figure 1.

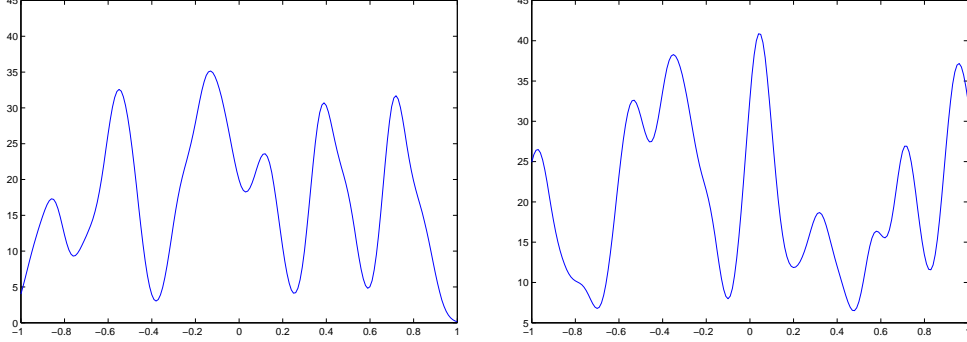


FIGURE 1. Importance of the choice of the interval on which the Poisson points are sampled. Samples of a shot noise process on the interval  $[-1, 1]$  when:  $g_\sigma$  is the Gaussian kernel of width  $\sigma = 0.05$ ,  $\lambda = 20$  and  $\beta = 1$ . On the left:  $T = 1$  and clearly there are some boundary effects. On the right, the simulation is obtained when taking  $T = 1.5 = 1 + 10\sigma$ . The boundary effects are no more noticeable.

**2.3. Regularity.** In this section, we focus on the sample path regularity of the shot noise process  $X$  given by (1). The process inherits regularity from the chosen kernel function as soon as sufficient integrability assumptions hold. We deal with two notions of regularity: a mean square one, which is valid for second order process and needs  $L^2$  assumptions and an almost sure one, which holds under uniform  $L^1$  assumptions. We refer to [25] who studies convergence of generalized shot noise series in general Banach spaces.

**Proposition 1.** *Let  $\beta \in L^2(\Omega)$  and  $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ . Assume that  $g \in C^1(\mathbb{R})$  with  $g' \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ . Then the shot noise process  $X$  given by (1) admits a mean square derivative.*

*Proof.* Let us consider  $S(t, t') = \text{Cov}(X(t), X(t')) = \lambda \mathbb{E}(\beta^2) \int_{\mathbb{R}} g(t-s)g(t'-s)\nu(ds)$ , the covariance function of the process  $X$ . According to Theorem 2.2.2 of [1] it is sufficient to remark that assumptions on  $g$  ensure that  $\frac{\partial^2 S}{\partial t \partial t'}$  exists and is finite at point  $(t, t) \in \mathbb{R}^2$  with

$$(9) \quad \frac{\partial^2 S}{\partial t \partial t'}(t, t) = \lambda \mathbb{E}(\beta^2) \int_{\mathbb{R}} g'(t-s)g'(t-s)\nu(ds).$$

Therefore for all  $t \in \mathbb{R}$  the limit

$$X'(t) = \lim_{h \rightarrow 0} \frac{X(t+h) - X(t)}{h},$$

exists in  $L^2(\Omega)$  and the covariance function of the second-order process  $(X'(t))_{t \in \mathbb{R}}$  is given by  $(t, t') \mapsto \lambda \mathbb{E}(\beta^2) \int_{\mathbb{R}} g'(t-s)g'(t'-s)\nu(ds)$ .  $\square$

This result only gives second order properties of the derivative and one can not deduce the law of  $X'$ . However, under uniform  $L^1$  assumptions, we can differentiate  $X$  under the sum as stated in the next proposition.

**Proposition 2.** *Let  $\beta \in L^1(\Omega)$ . Let  $g \in \mathcal{C}^1(\mathbb{R}) \cap L^1(\mathbb{R})$ . Assume that there exists  $\varepsilon > 0$  such that*

$$\int_{\mathbb{R}} \sup_{|t| \leq \varepsilon} |g'(t-s)| ds < +\infty,$$

*then almost surely the series  $X(\cdot) = \sum_i \beta_i g(\cdot - \tau_i)$  converges uniformly on any compact set of  $\mathbb{R}$ .*

*Moreover  $X$  is almost surely continuously differentiable on  $\mathbb{R}$  with*

$$X'(t) = \sum_i \beta_i g'(t - \tau_i), \forall t \in \mathbb{R}.$$

*Proof.* Let  $A > 0$  and remark that for any  $s \in \mathbb{R}$  and  $|t| \leq A$

$$|g(t-s)| = \left| \int_0^t g'(u-s) du + g(-s) \right| \leq \int_{-A}^A |g'(s-u)| du + |g(-s)|,$$

such that by Fubini's theorem and Assumption 1

$$\int_{\mathbb{R}} \sup_{t \in [-A, A]} |g(t-s)| \nu(ds) \leq 2A \int_{\mathbb{R}} |g'(s)| ds + \int_{\mathbb{R}} |g(s)| ds < +\infty.$$

Therefore, since  $\beta \in L^1(\Omega)$ , the series  $\sum_i \beta_i \sup_{t \in [-A, A]} |g(t - \tau_i)|$  converges almost surely which means that  $\sum_i \beta_i g(\cdot - \tau_i)$  converges uniformly on  $[-A, A]$  almost surely. This implies that the sample paths of  $X$  are almost surely continuous on  $\mathbb{R}$ . Similarly, for all  $t_0 \in \mathbb{R}$ , almost surely the series  $\sum_i \beta_i g'(\cdot - \tau_i)$  converges uniformly on  $[t_0 - \varepsilon, t_0 + \varepsilon]$  and therefore  $X$  is continuously differentiable on  $[t_0 - \varepsilon, t_0 + \varepsilon]$  with  $X'(t) = \sum_i \beta_i g'(t - \tau_i)$  for all  $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$ . This concludes the proof using the fact that  $\mathbb{R} = \cup_{t_0 \in \mathbb{Q}} [t_0 - \varepsilon, t_0 + \varepsilon]$ .  $\square$

Sufficient conditions to ensure both kinds of regularity are given in the following corollary.

**Corollary 1.** *Let  $\beta \in L^2(\Omega)$ . Let  $g \in \mathcal{C}^2(\mathbb{R})$  such that  $g, g', g'' \in L^1(\mathbb{R})$ . Then  $X$  is almost surely and mean square continuously differentiable on  $\mathbb{R}$  with*

$$X'(t) = \sum_i \beta_i g'(t - \tau_i), \forall t \in \mathbb{R}.$$

*Proof.* It is sufficient to check assumptions of Propositions 1 and 2. Note that  $g, g' \in L^1(\mathbb{R})$  imply that  $g \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}) \subset L^2(\mathbb{R})$ . Similarly we also have  $g' \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ . Moreover since  $g', g'' \in L^1(\mathbb{R})$ , for any  $\varepsilon > 0$

$$\int_{\mathbb{R}} \sup_{|t| \leq \varepsilon} |g'(t-s)| ds < +\infty.$$

$\square$

Iterating this result one can obtain higher order smoothness properties. In particular it is straightforward to obtain the following result for Gaussian kernels.

**Example (Gaussian kernel):** Let  $\beta \in L^2(\Omega)$ ,  $g(t) = g_1(t) = \frac{1}{\sqrt{2\pi}} \exp(-t^2/2)$  and  $X$  given by (1). Then, the process  $X$  is almost surely and mean square smooth on  $\mathbb{R}$ . Moreover, for any  $n \in \mathbb{N}$ ,

$$\forall t \in \mathbb{R}, \quad X^{(n)}(t) = \sum_i \beta_i g_1^{(n)}(t - \tau_i) = \sum_i \beta_i (-1)^n H_n(t - \tau_i) g_1(t - \tau_i),$$

where  $H_n$  is the Hermite polynomial of order  $n$ .

## 3. EXISTENCE OF A PROBABILITY DENSITY FOR SHOT NOISE SERIES

We focus in this section on the question of the existence of a probability density for a shot noise series. This question arises naturally when studying regularity in variation of the law [6] or level sets [19] of shot noise series. In particular the crossing theory for processes usually assumes the existence of a bounded density for the random vector  $(X(t), X'(t))$  for  $t \in \mathbb{R}$ . For sake of generality we consider here a  $\mathbb{R}^d$ -valued shot noise process given on  $\mathbb{R}$  by

$$(10) \quad Y(t) = \sum_i \beta_i h(t - \tau_i),$$

where  $h : \mathbb{R} \mapsto \mathbb{R}^d$  is a given (deterministic) measurable vectorial function in  $L^1(\mathbb{R})$ . In this setting one can recover  $X$  given by (1) with  $d = 1$  and  $h = g$ , or recover  $(X, X')$ -if exists- with  $d = 2$  and  $h = (g, g')$ .

The question of the existence of a probability density for the random vector  $Y(t)$  for some  $t \in \mathbb{R}$  can be addressed from different points of view. One can exploit formula (10), where  $Y$  is written as a sum of random variables. It allows for instance to establish an integral equation to compute or approximate the density in some examples [22, 21, 15]. We adopt this point of view in the first part of this section to derive a sufficient condition for the existence of a density for  $Y$ . However it does not imply any continuity or boundedness of the distribution. Therefore, in the second part we deal with integrability property for the characteristic function, which implies both existence, continuity and boundedness of the density.

**3.1. Sufficient condition for the existence of a density in the stationary case.** In the following we assume that  $\nu$  is the Lebesgue measure. Then, the  $\mathbb{R}^d$ -valued process  $Y$  given by (10) is stationary such that it is sufficient to study the law of the random vector  $Y(0)$ . We introduce a truncated process in a similar way as in Section 2.2. We will use the same notations: let  $T > 0$  and write  $Y(0) = Y_T(0) + R_T(0)$  with  $Y_T(0) = \sum_{\{i: |\tau_i| \leq T\}} \beta_i h(-\tau_i)$  independent from  $R_T(0)$ . Note that, as

in the one dimensional case

$$Y_T(0) \stackrel{d}{=} \sum_{i=1}^{\gamma_T} \beta_i h(U_T^{(i)}),$$

where  $\gamma_T$  is a Poisson random variable of parameter  $\lambda \nu_T(\mathbb{R}) = 2\lambda T$ ,  $\{\beta_i\}$  are i.i.d. with common law  $F$ ,  $\{U_T^{(i)}\}$  are i.i.d. with uniform law on  $[-T, T]$ . Here  $\stackrel{d}{=}$  stands for the equality in law and we recall

the convention that  $\sum_{i=1}^0 = 0$ .

**Proposition 3.** *If there exists  $m \geq 1$  such that for any  $T > 0$  large enough, conditionally on  $\{\gamma_T = m\}$ , the random variable  $Y_T(0)$  admits a density, then  $Y(0)$  admits a density.*

*Proof.* Let  $T > 0$  sufficiently large. First, let us remark that conditionally on  $\{\gamma_T = m\}$ ,  $Y_T(0) = \sum_{i=1}^m \beta_i h(U_T^{(i)})$ . Next, note that if a random vector  $V$  in  $\mathbb{R}^d$  admits a density  $f_V$  then, for  $U_T$  with uniform law on  $[-T, T]$  and  $\beta$  with law  $F$ , independent of  $V$ , the random vector  $W = V + \beta h(U_T)$  admits  $w \in \mathbb{R}^d \mapsto \frac{1}{2T} \int_{\mathbb{R}} \int_{-T}^T f_V(w - zh(t)) dt F(dz)$  for density. Therefore, by induction the assumption implies that  $\sum_{i=1}^n \beta_i h(U_T^{(i)})$  has a density, for any  $n \geq m$ . Then, we follow the same lines as [3], proof of Proposition A.2. Let  $A \subset \mathbb{R}^d$  be a Borel set with Lebesgue measure 0, since  $Y_T(0)$  and  $R_T(0)$  are independent

$$\mathbb{P}(Y(0) \in A) = \mathbb{P}(Y_T(0) + R_T(0) \in A) = \int_{\mathbb{R}^d} \mathbb{P}(Y_T(0) \in A - y) \mu_T(dy).$$



with  $\mu_T$  the law of  $R_T(0)$ . But for any  $y \in \mathbb{R}^d$ ,

$$\begin{aligned} \mathbb{P}(Y_T(0) \in A - y) &= \mathbb{P}\left(\sum_{i=1}^{\gamma_T} \beta_i h(U_T^{(i)}) \in A - y\right) \\ &= \sum_{n=0}^{+\infty} \mathbb{P}\left(\sum_{i=1}^{\gamma_T} \beta_i h(U_T^{(i)}) \in A - y \mid \gamma_T = n\right) \mathbb{P}(\gamma_T = n) \\ &= \sum_{n=0}^{m-1} \mathbb{P}\left(\sum_{i=1}^n \beta_i h(U_T^{(i)}) \in A - y\right) \mathbb{P}(\gamma_T = n), \end{aligned}$$

since  $A - y$  has Lebesgue measure 0 and  $\sum_{i=1}^n \beta_i h(U_T^{(i)})$  has a density for any  $n \geq m$ . Hence, for any  $T > 0$  large enough,

$$\mathbb{P}(Y(0) \in A) \leq \mathbb{P}(\gamma_T \leq m - 1).$$

Letting  $T \rightarrow +\infty$  we conclude that  $\mathbb{P}(Y(0) \in A) = 0$  such that  $Y(0)$  admits a density.  $\square$

Let us remark that, under the assumptions of Proposition 3, considering  $Y_T(0)$  instead of  $Y(0)$  or equivalently  $\nu_T(ds) = \mathbf{1}_{[-T, T]}(s)ds$  instead of the Lebesgue measure, we would obtain that conditionally to  $\{\gamma_T \geq m\}$  the random variable  $Y_T(0)$  admits a density. Let us emphasize that  $Y_T(0)$  does not admit a density since  $\mathbb{P}(Y_T(0) = 0) \geq \mathbb{P}(\gamma_T = 0) > 0$ .

Let us mention that Breton [6] gives a similar assumption for real-valued shot noise series in Proposition 2.1. In particular his Corollary 2.1. can be adapted in our multivalued setting.

**Corollary 2.** *Let  $h : \mathbb{R} \mapsto \mathbb{R}^d$  be an integrable function and  $\beta = 1$  a.s. Let us define  $h_d : \mathbb{R}^d \mapsto \mathbb{R}^d$  by  $h_d(x) = h(x_1) + \dots + h(x_d)$ , for  $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ . If the  $h_d$  image measure of the  $d$ -dimensional Lebesgue measure is absolutely continuous with respect to the  $d$ -dimensional Lebesgue measure then the random vector  $Y(0)$ , given by (10), admits a density.*

*Proof.* Let  $A \subset \mathbb{R}^d$  a Borel set with Lebesgue measure 0 then assumptions ensure that  $\int_{\mathbb{R}^d} \mathbf{1}_{h_d(x) \in A} dx = 0$ . Therefore, for any  $T > 0$ , using notations of Proposition 3,

$$\mathbb{P}\left(\sum_{i=1}^d h(U_T^{(i)}) \in A\right) = \frac{1}{(2T)^d} \int_{[-T, T]^d} \mathbf{1}_{h_d(x) \in A} dx = 0.$$

Hence  $\sum_{i=1}^d h(U_T^{(i)})$  admits a density and Proposition 3 gives the conclusion.  $\square$

**Example (Gaussian kernel):** let  $g(t) = \frac{1}{\sqrt{2\pi}} \exp(-t^2/2)$ ,  $\beta = 1$  a.s. and  $X$  given by (1). Let us consider  $h = (g, g')$  and  $h_2 : (x_1, x_2) \in \mathbb{R}^2 \mapsto h(x_1) + h(x_2)$ . The Jacobian of  $h_2$  is

$$J(h_2)(x_1, x_2) = \frac{1}{2\pi} P(x_1, x_2) \exp(-(x_1^2 + x_2^2)/2)$$

with  $P(x_1, x_2) = (1 + x_1 x_2)(x_1 - x_2)$ . Hence, the  $h_2$  image measure of the 2-dimensional Lebesgue measure is absolutely continuous with respect to the 2-dimensional Lebesgue measure. Then, for any  $t \in \mathbb{R}$ , the random vector  $(X(t), X'(t))$  is absolutely continuous with respect to the Lebesgue measure. Note that in particular this implies the existence of a density for  $X(t)$ .

Most of results known on crossing theory for stationary processes (see for instance [2]) are based on the assumptions that for any  $t \in \mathbb{R}$ , the random vector  $(X(t), X'(t))$  admits a continuous bounded density. One way to get this assumption true is to check the integrability of the characteristic function of this vector.

### 3.2. Is the characteristic function integrable ? Existence and boundedness of a density.

Let  $d \geq 1$  and consider a  $d$ -dimensional shot noise process  $Y(t)$  defined by Equation (10) with a kernel function  $h$  and impulse  $\beta = 1$  a.s. When  $t \in \mathbb{R}$  is fixed the characteristic function of  $Y(t)$  is defined by  $\psi_{Y(t)}(u) = \mathbb{E}(\exp(iu \cdot Y(t)))$ , for  $u \in \mathbb{R}^d$ . Let us remark that  $u \cdot Y$  is a 1d shot noise process with kernel  $u \cdot h$  such that we get  $\psi_{Y(t)}(u) = \exp(\lambda \int_{\mathbb{R}} (e^{iu \cdot h(s)} - 1) \nu(ds))$ . Now, let us assume that

$$\int_{\mathbb{R}^d} \exp(-\lambda \int_{\mathbb{R}} (1 - \cos(u \cdot h(s))) \nu(ds)) du < +\infty.$$

This implies that  $\psi_{Y(t)}$  belongs to  $L^1(\mathbb{R}^d)$  and thanks to Fourier inversion theorem,  $Y(t)$  has a bounded continuous density (see for instance [13] p.482).

**Examples:** for  $d = 1$  and  $\nu(ds) = ds$ . We consider  $X$  the stationary shot noise process given by (1) and denote by  $\psi$  the characteristic function of  $X(t)$ ,  $t \in \mathbb{R}$ , which does not depend on  $t$ .

- Power-law shot noise: if there exists  $A > 0$  and  $\alpha > 1/2$  such that  $g(t) = 1/t^\alpha$  for  $t > A$ , then  $|\psi|$  is integrable. Actually, for any  $u \neq 0$ ,

$$\begin{aligned} \log |\psi(u)| &\leq -\lambda \int_A^{+\infty} (1 - \cos(us^{-\alpha})) ds \\ &= -\lambda \frac{1}{\alpha} |u|^{1/\alpha} \int_0^{|u|A^{-\alpha}} \frac{1 - \cos t}{t^{1+1/\alpha}} dt. \end{aligned}$$

Since the last integral has a finite positive limit as  $u$  goes to infinity, and since  $\exp(-|u|^{1/\alpha})$  is integrable, this shows that  $|\psi|$  is integrable. Note that when  $g$  is causal *i.e.*  $g(t) = 0$  for  $t < 0$ , one can define the shot noise process for  $A = 0$  and  $\alpha > 1$ . In this case,  $X(t)$  is a Levy stable random variable with stability index  $1/\alpha$  as proved in [21].

- Exponential kernel: if  $g(t) = e^{-t} \mathbf{1}_{\{t > 0\}}$ , then  $|\psi|$  is integrable iff  $\lambda > 1$ . Actually, for  $|u| > 1$ ,

$$\begin{aligned} \log |\psi(u)| &= -\lambda \int_0^{+\infty} (1 - \cos(ue^{-s})) ds = -\lambda \int_0^{|u|} \frac{1 - \cos(t)}{t} dt \\ &= -\lambda \left( \int_0^1 \frac{1 - \cos(t)}{t} dt + \log |u| - \int_1^{|u|} \frac{\cos(t)}{t} dt \right). \end{aligned}$$

Since the last term in this sum has a finite limit as  $|u|$  goes to  $+\infty$ , it proves that  $\psi(u)$  is integrable iff  $1/|u|^\lambda$  is, that is iff  $\lambda > 1$ .

- Compactly supported kernel: if  $g$  has compact support, then  $|\psi|$  is not integrable. Moreover  $X(t)$  does not admit a density. Actually, there exists  $A > 0$  such that  $g(s) = 0$  for  $|s| > A$ . Then  $\int_{\mathbb{R}} (1 - \cos(ug(s))) ds \leq 2A$ , and thus  $|\psi(u)| \geq \exp(-2A\lambda)$ , which shows that it can not be integrable. Another way to see that  $|\psi|$  is not integrable in this case is to look at the probability of  $\{X(t) = 0\}$ . Indeed,  $X(t)$  is 0 as soon as there are no point of the Poisson process in the interval  $[-A, A]$ , and such an event has a strictly positive probability. Thus  $\mathbb{P}(X(t) = 0) > 0$ , which proves that  $X(t)$  doesn't have a density, and consequently,  $|\psi|$  is not integrable.

It seems that the general picture is this: if  $g$  has “heavy tails”, then  $|\psi|$  is integrable, whereas when  $g(s)$  goes to 0 faster than  $\exp(-|s|)$ , then  $|\psi|$  is not integrable. A hint to understand this is the following idea: when  $g$  goes fast to 0, then  $\mathbb{P}(|X(t)| \leq \varepsilon)$  is “large” compared to  $\varepsilon$ , and thus the density of  $X(t)$ , when it exists, is not bounded in a neighborhood of 0. And in particular, it implies that  $|\psi|$  is not integrable. All these statements are formalized in the following proposition.

**Proposition 4.** *Let  $X$  be a stationary 1d shot noise process defined by (1) with kernel function  $g$  and  $\nu(ds) = ds$ . Assume for sake of simplicity that  $\beta = 1$  almost surely. Then:*

- (1) *If  $g$  is such that there exist  $\alpha > 1$  and  $A > 0$  such that  $\forall |s| > A$ ,  $|g(s)| \leq e^{-|s|^\alpha}$ , then  $\exists \varepsilon_0 > 0$  such that  $\forall 0 < \varepsilon < \varepsilon_0$ :*

$$\mathbb{P}(|X(t)| \leq \varepsilon) \geq \frac{1}{2} e^{-2\lambda T_\varepsilon} \text{ where } T_\varepsilon \text{ is defined by } T_\varepsilon = (-\log \varepsilon)^{1/\alpha}.$$

- (2) If  $g$  is such that there exists  $A > 0$  such that  $\forall |s| > A$ ,  $|g(s)| \leq e^{-|s|}$  and if  $\lambda < 1/4$  then  $\exists \varepsilon_0 > 0$  such that  $\forall 0 < \varepsilon < \varepsilon_0$ :

$$\mathbb{P}(|X(t)| \leq \varepsilon) \geq \left(1 - \frac{\lambda}{(1 - 2\lambda)^2}\right) e^{-2\lambda T_\varepsilon} \text{ where } T_\varepsilon \text{ is defined by } T_\varepsilon = -\log \varepsilon.$$

This implies in both cases that  $\mathbb{P}(|X(t)| \leq \varepsilon)/\varepsilon$  goes to  $+\infty$  as  $\varepsilon$  goes to 0, and thus the density of  $X(t)$  -if it exists- is not bounded in a neighborhood of 0.

*Proof.* We start with the first case. Let  $\varepsilon > 0$  and let  $T_\varepsilon = (-\log \varepsilon)^{1/\alpha}$ . Assume that  $\varepsilon$  is small enough to have  $T_\varepsilon > A$ . We have by definition  $X(t) \stackrel{d}{=} X(0) \stackrel{d}{=} \sum_i g(\tau_i)$ . If we denote  $X_{T_\varepsilon}(0) = \sum_{|\tau_i| \leq T_\varepsilon} g(\tau_i)$

and  $R_{T_\varepsilon}(0) = \sum_{|\tau_i| > T_\varepsilon} g(\tau_i)$ , then  $X_{T_\varepsilon}(0)$  and  $R_{T_\varepsilon}(0)$  are independent and  $X(0) = X_{T_\varepsilon}(0) + R_{T_\varepsilon}(0)$ .

We also have:  $\mathbb{P}(|X(0)| \leq \varepsilon) \geq \mathbb{P}(|X_{T_\varepsilon}(0)| = 0 \text{ and } |R_{T_\varepsilon}(0)| \leq \varepsilon) = \mathbb{P}(|X_{T_\varepsilon}(0)| = 0) \times \mathbb{P}(|R_{T_\varepsilon}(0)| \leq \varepsilon)$ . Now on the one hand we have:  $\mathbb{P}(|X_{T_\varepsilon}(0)| = 0) \geq \mathbb{P}(\text{there are no } \tau_i \text{ in } [-T_\varepsilon, T_\varepsilon]) = e^{-2\lambda T_\varepsilon}$ . On the other hand, the first moments of the random variable  $R_{T_\varepsilon}(0)$  are given by:  $\mathbb{E}(R_{T_\varepsilon}(0)) = \lambda \int_{|s| > T_\varepsilon}^{+\infty} g(s) ds$  and  $\text{Var}(R_{T_\varepsilon}(0)) = \lambda \int_{|s| > T_\varepsilon}^{+\infty} g^2(s) ds$ . Now, we use the following inequality on the tail of  $\int e^{-s^\alpha}$ :

$$\forall T > 0, e^{-T^\alpha} = \int_T^{+\infty} \alpha s^{\alpha-1} e^{-s^\alpha} ds \geq \alpha T^{\alpha-1} \int_T^{+\infty} e^{-s^\alpha} ds.$$

Thus, we obtain bounds for the tail of  $\int g$  and of  $\int g^2$ :

$$\int_T^{+\infty} e^{-s^\alpha} ds \leq \frac{e^{-T^\alpha}}{\alpha T^{\alpha-1}} \text{ and } \int_T^{+\infty} (e^{-s^\alpha})^2 ds \leq \frac{e^{-2T^\alpha}}{2\alpha T^{\alpha-1}}.$$

Back to the moments of  $R_{T_\varepsilon}(0)$ , since  $T_\varepsilon = (-\log \varepsilon)^{1/\alpha}$  we have:

$$|\mathbb{E}(R_{T_\varepsilon}(0))| \leq \frac{2\lambda\varepsilon}{\alpha T_\varepsilon^{\alpha-1}} \text{ and } \text{Var}(R_{T_\varepsilon}(0)) \leq \frac{\lambda\varepsilon^2}{\alpha T_\varepsilon^{\alpha-1}}.$$

We can take  $\varepsilon$  small enough in such a way that we can assume that  $|\mathbb{E}(R_{T_\varepsilon}(0))| < \varepsilon$ . Then, using Chebyshev's inequality, we have

$$\begin{aligned} \mathbb{P}(|R_{T_\varepsilon}(0)| \leq \varepsilon) &= \mathbb{P}(-\varepsilon - \mathbb{E}(R_{T_\varepsilon}(0)) \leq R_{T_\varepsilon}(0) - \mathbb{E}(R_{T_\varepsilon}(0)) \leq \varepsilon - \mathbb{E}(R_{T_\varepsilon}(0))) \\ &\geq 1 - \mathbb{P}(|R_{T_\varepsilon}(0) - \mathbb{E}(R_{T_\varepsilon}(0))| \geq \varepsilon - |\mathbb{E}(R_{T_\varepsilon}(0))|) \\ &\geq 1 - \frac{\text{Var}(R_{T_\varepsilon}(0))}{(\varepsilon - |\mathbb{E}(R_{T_\varepsilon}(0))|)^2} \geq 1 - \frac{\lambda}{\alpha T_\varepsilon^{\alpha-1} (1 - 2\lambda/\alpha T_\varepsilon^{\alpha-1})^2}, \end{aligned}$$

which is larger than  $1/2$  for  $T_\varepsilon$  large enough (i.e. for  $\varepsilon$  small enough).

For the second case, we can make exactly the same computations by setting  $\alpha = 1$ , and get  $\mathbb{P}(|R_{T_\varepsilon}(0)| \leq \varepsilon) \geq 1 - \lambda/(1 - 2\lambda)^2$ , which is  $> 0$  when  $\lambda < 1/4$ .  $\square$

**Example (Gaussian kernel):** Let  $g(t) = \frac{1}{\sqrt{2\pi}} \exp(-t^2/2)$ ,  $\nu(ds) = ds$ ,  $\beta = 1$  a.s. and  $X$  given by (1). Then, for any  $t \in \mathbb{R}$ , the random variable  $X(t)$  admits a density that is not bounded in a neighborhood of 0.

Such a feature is particularly bothersome when considering crossings of these processes since most of known results are based on the existence of bounded density for each marginal of the process. However such a behavior is extremely linked to the number of points of the Poisson process  $\{\tau_i\}$  that are thrown in the interval of study. The density -if it exists- will be more regular as this number increases. This can be settled considering the integrability of the characteristic function conditionally on a certain number of points in the interval. Then, the main tool is the classical stationary phase estimate for oscillatory integrals (see [29] for example). We will moreover need such results in the framework of two variables  $(u, v)$ , when studying crossing functions.

**Proposition 5** (Stationary phase estimate for oscillatory integrals). *Let  $a < b$  and let  $\varphi$  be a function of class  $\mathcal{C}^2$  defined on  $[a, b]$ . Assume that  $\varphi'$  and  $\varphi''$  cannot simultaneously vanish on  $[a, b]$  and denote  $m = \min_{s \in [a, b]} \sqrt{\varphi'(s)^2 + \varphi''(s)^2} > 0$ . Let us also assume that  $n_0 = \#\{s \in [a, b] \text{ s. t. } \varphi''(s) = 0\} < +\infty$ . Then*

$$\forall u \in \mathbb{R} \text{ s.t. } |u| > \frac{1}{m}, \quad \left| \int_a^b e^{iu\varphi(s)} ds \right| \leq \frac{8\sqrt{2}(2n_0 + 1)}{\sqrt{m|u|}}.$$

Now, let  $\varphi_1$  and  $\varphi_2$  be two functions of class  $\mathcal{C}^3$  defined on  $[a, b]$ . Assume that the derivatives of these functions are linearly independent, in the sense that for all  $s \in [a, b]$ , the matrix  $\Phi(s) = \begin{pmatrix} \varphi_1'(s) & \varphi_2'(s) \\ \varphi_1''(s) & \varphi_2''(s) \end{pmatrix}$  is invertible. Denote  $m = \min_{s \in [a, b]} \|\Phi(s)^{-1}\|^{-1} > 0$ , where  $\|\cdot\|$  is the matricial norm induced by the Euclidean one. Assume moreover that there exists  $n_0 < +\infty$  such that  $\#\{s \in [a, b] \text{ s.t. } \det(\Phi'(s)) = 0\} \leq n_0$ , where  $\Phi'(s) = \begin{pmatrix} \varphi_1'''(s) & \varphi_2'''(s) \\ \varphi_1^{(4)}(s) & \varphi_2^{(4)}(s) \end{pmatrix}$ . Then

$$\forall (u, v) \in \mathbb{R}^2 \text{ s.t. } \sqrt{u^2 + v^2} > \frac{1}{m}, \quad \left| \int_a^b e^{iu\varphi_1(s) + iv\varphi_2(s)} ds \right| \leq \frac{8\sqrt{2}(2n_0 + 3)}{\sqrt{m\sqrt{u^2 + v^2}}}.$$

*Proof.* For the first part of the proposition, by assumption,  $[a, b]$  is the union of the three compact sets

$$\{s \in [a, b]; |\varphi''| \geq m/2\}, \{s \in [a, b]; |\varphi'| \geq m/2 \text{ and } \varphi'' \geq 0\} \text{ and } \{s \in [a, b]; |\varphi'| \geq m/2 \text{ and } \varphi'' \leq 0\}.$$

Therefore there exists  $1 \leq n \leq 2n_0 + 1$  and a subdivision  $(a_i)_{0 \leq i \leq n}$  of  $[a, b]$  such that  $[a_{i-1}, a_i]$  is included in one of the previous subsets for any  $1 \leq i \leq n$ . If  $[a_{i-1}, a_i] \subset \{s \in [a, b]; |\varphi''(s)| \geq m/2\}$ , according to Proposition 2 p.332 of [29]

$$\left| \int_{a_{i-1}}^{a_i} e^{iu\varphi(s)} ds \right| = \left| \int_{a_{i-1}}^{a_i} e^{iu(m/2)(2\varphi(s)/m)} ds \right| \leq 8 \frac{\sqrt{2}}{\sqrt{m|u|}};$$

otherwise,

$$\left| \int_{a_{i-1}}^{a_i} e^{iu\varphi(s)} ds \right| \leq \frac{6}{m|u|}$$

The result follows from summing up these  $n$  integrals.

For the second part of the proposition, we use polar coordinates, and write  $(u, v) = (r \cos \theta, r \sin \theta)$ . For  $\theta \in [0, 2\pi)$ , let  $\varphi_\theta$  be the function defined on  $[a, b]$  by  $\varphi_\theta(s) = \varphi_1(s) \cos \theta + \varphi_2(s) \sin \theta$ . Then  $\begin{pmatrix} \varphi_\theta'(s) \\ \varphi_\theta''(s) \end{pmatrix} = \Phi(s) \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$ , and thus  $1 = \|\Phi(s)^{-1} \begin{pmatrix} \varphi_\theta'(s) \\ \varphi_\theta''(s) \end{pmatrix}\|$ . This implies that for all  $s \in [a, b]$ ,  $\sqrt{\varphi_\theta'(s)^2 + \varphi_\theta''(s)^2} \geq 1/\|\Phi(s)^{-1}\| \geq m$ . Moreover, thanks to Rolle's Theorem, the number of points  $s \in [a, b]$  such that  $\varphi_\theta''(s) = 0$  is bounded by one plus the number of  $s \in [a, b]$  such that  $\varphi_1''(s)\varphi_2'''(s) - \varphi_1'''(s)\varphi_2''(s) = 0$ , that is by  $1 + n_0$ . Thus, we can apply the result of the first part of the proposition to each function  $\varphi_\theta$  and the obtained bound will depend only on  $m$ ,  $n_0$  and  $r = \sqrt{u^2 + v^2}$ .  $\square$

**Proposition 6.** *Let  $X$  be a stationary 1d shot noise process defined by (1) with kernel function  $g \in L^1(\mathbb{R})$  and  $\nu(ds) = ds$ . Assume for sake of simplicity that  $\beta = 1$  almost surely. Let  $T > 0$  and  $\gamma_T = \#\{i; \tau_i \in [-T, T]\}$ . Let  $a < b$  and assume that  $g$  is a function of class  $\mathcal{C}^2$  on  $[-T + a, T + b]$  such that*

$$m = \min_{s \in [-T+a, T+b]} \sqrt{g'(s)^2 + g''(s)^2} > 0 \text{ and } n_0 = \#\{s \in [-T + a, T + b] \text{ s. t. } g''(s) = 0\} < +\infty.$$

*Then, conditionally on  $\{\gamma_T \geq k_0\}$  with  $k_0 \geq 3$ , for all  $t \in [a, b]$ , the law of  $X(t)$  admits a continuous bounded density.*

*Proof.* Actually, we will prove that conditionally on  $\{\gamma_T \geq k_0\}$ , the law of the truncated process  $X_T(t) = \sum_{|\tau_i| \leq T} g(t - \tau_i)$  admits a continuous bounded density for  $t \in [a, b]$ . The result will follow, using the fact that  $X(t) = X_T(t) + R_T(t)$ , with  $R_T(t)$  independent from  $X_T(t)$ , as given by (8). Therefore let us denote  $\psi_{t,k_0}^T$  the characteristic function of  $X_T(t)$  conditionally on  $\{\gamma_T \geq k_0\}$ . Then, for all  $u \in \mathbb{R}$ , we get

$$\begin{aligned} \psi_{t,k_0}^T(u) &= \frac{1}{\mathbb{P}(\gamma_T \geq k_0)} \sum_{k \geq k_0} \mathbb{E} \left( e^{iuX_T(t)} | \gamma_T = k \right) \mathbb{P}(\gamma_T = k) \\ &= \frac{1}{\mathbb{P}(\gamma_T \geq k_0)} \sum_{k \geq k_0} \left( \frac{1}{2T} \int_{-T}^T e^{iug(t-s)} ds \right)^k e^{-2\lambda T} \frac{(2\lambda T)^k}{k!} \end{aligned}$$

Therefore,

$$|\psi_{t,k_0}^T(u)| \leq (2T)^{-k_0} \left| \int_{-T+t}^{T+t} e^{iug(s)} ds \right|^{k_0}.$$

Hence, using Proposition 5 on  $[-T+t, T+t] \subset [-T+a, T+b]$ , one can find  $C$  a positive constant that depends on  $T, k_0, \lambda, m$  and  $n_0$  such that for any  $|u| > 1/m$

$$|\psi_{t,k_0}^T(u)| \leq C|u|^{-k_0/2}.$$

Then  $\psi_{t,k_0}^T$  is integrable on  $\mathbb{R}$ , since  $k_0 \geq 3$ , and thanks to Fourier inverse Theorem it is the characteristic function of a bounded continuous density.  $\square$

#### 4. CROSSINGS

The goal of this section is to investigate crossings for smooth shot noise processes. This is a very different situation from the one studied in [22, 4, 17] where shot noise processes are non differentiable. However crossings of smooth processes have been extensively studied especially in the Gaussian processes realm (see [2] for instance). Then, most of known results are based on assumptions on density probabilities, which are not well-adapted in our setting, as seen in the previous section. In the next subsection, we revisit these results with a more adapted point of view based on characteristic functions.

**4.1. General formula.** When  $X$  is an almost surely continuously differentiable process on  $\mathbb{R}$ , we can consider its multiplicity function on an interval  $[a, b]$  defined by

$$(11) \quad \forall \alpha \in \mathbb{R}, \quad N_X(\alpha, [a, b]) = \#\{t \in [a, b]; X(t) = \alpha\}.$$

This defines a positive random process taking integer values. Let briefly recall some points of “vocabulary”. For a given level  $\alpha \in \mathbb{R}$ , a point  $t \in [a, b]$  such that  $X(t) = \alpha$  is called “crossing” of the level  $\alpha$ . Then  $N_X(\alpha, [a, b])$  counts the number of crossings of the level  $\alpha$  in the interval  $[a, b]$ . Now we have to distinguish three different types of crossings (see for instance [9]): the up-crossings are points for which  $X(t) = \alpha$  and  $X'(t) > 0$ , the down-crossings are points for which  $X(t) = \alpha$  and  $X'(t) < 0$  and the tangencies are points for which  $X(t) = \alpha$  and  $X'(t) = 0$ . The following proposition gives a simple criterion which ensures that the number of tangencies is 0 almost surely.

**Proposition 7.** *Let  $a, b \in \mathbb{R}$  with  $a \leq b$ . Let  $X$  be a real valued random process almost surely  $\mathcal{C}^2$  on  $[a, b]$ . Let us assume that there exists  $\phi \in L^1(\mathbb{R})$  and  $C_{a,b} > 0$  such that*

$$\forall t \in [a, b], \quad \left| \mathbb{E} \left( e^{iuX(t)} \right) \right| \leq C_{a,b} \phi(u).$$

Then,

$$\forall \alpha \in \mathbb{R}, \quad \mathbb{P}(\exists t \in [a, b], X(t) = \alpha \text{ and } X'(t) = 0) = 0.$$

*Proof.* Let  $M > 0$  and let denote  $A_M$  the event corresponding to

$$\max_{t \in [a, b]} |X'(t)| \leq M \quad \text{and} \quad \max_{t \in [a, b]} |X''(t)| \leq M$$

such that

$$\mathbb{P}(\exists t \in [a, b], X(t) = \alpha, X'(t) = 0) = \lim_{M \rightarrow +\infty} \mathbb{P}(\exists t \in [a, b], X(t) = \alpha, X'(t) = 0, A_M).$$

Note that on  $A_M$ , for  $t, s \in [a, b]$ , we have by the mean value theorem

$$|X(t) - X(s)| \leq M|t - s| \quad \text{and} \quad |X'(t) - X'(s)| \leq M|t - s|.$$

Let us assume that there exists  $t \in [a, b]$  such that  $X(t) = \alpha$  and  $X'(t) = 0$ . Then for any  $n \in \mathbb{N}$  there exists  $k_n \in [2^n a, 2^n b] \cap \mathbb{Z}$  such that  $|t - 2^{-n} k_n| \leq 2^{-n}$  with, by Taylor formula,

$$|X(2^{-n} k_n) - \alpha| \leq 2^{-2n} M \quad \text{and} \quad |X'(2^{-n} k_n)| \leq 2^{-n} M.$$

Therefore, let us denote

$$B_n = \bigcup_{k_n \in [2^n a, 2^n b] \cap \mathbb{Z}} \{|X(2^{-n} k_n) - \alpha| \leq 2^{-2n} M \text{ and } |X'(2^{-n} k_n)| \leq 2^{-n} M\}.$$

Since  $(B_n \cap A_M)_{n \in \mathbb{N}}$  is a decreasing sequence we get

$$\mathbb{P}(\exists t \in [a, b]; X(t) = \alpha, X'(t) = 0, A_M) \leq \lim_{n \rightarrow +\infty} \mathbb{P}(B_n \cap A_M).$$

But, according to assumption, for any  $n \in \mathbb{N}$  the random variable  $X(2^{-n} k_n)$  admits a uniformly bounded density function. Therefore, there exists  $c_{a,b}$  such that

$$\mathbb{P}(|X(2^{-n} k_n) - \alpha| \leq 2^{-2n} M, |X'(2^{-n} k_n)| \leq 2^{-n} M) \leq c_{a,b} 2^{-2n} M.$$

Hence

$$\mathbb{P}(B_n \cap A_M) \leq (b - a + 1) c_{a,b} 2^{-n} M,$$

which yields the result.  $\square$

In particular, assumptions of Proposition 7 allow us to use Kac's counting formula (see Lemma 3.1 [2]), which we recall in the following proposition.

**Proposition 8** (Kac's counting formula). *Let  $a, b \in \mathbb{R}$  with  $a < b$ . Let  $X$  be a real valued random process defined on  $\mathbb{R}$  almost surely continuously differentiable on  $[a, b]$ . Let  $\alpha \in \mathbb{R}$ . Assume that*

$$(12) \quad \mathbb{P}(\exists t \in [a, b] \text{ s.t. } X(t) = \alpha \text{ and } X'(t) = 0) = 0 \quad \text{and} \quad \mathbb{P}(X(a) = \alpha) = \mathbb{P}(X(b) = \alpha) = 0.$$

*Then, almost surely*

$$N_X(\alpha, [a, b]) = \lim_{\delta \rightarrow 0} \frac{1}{2\delta} \int_a^b \mathbf{1}_{|X(t) - \alpha| < \delta} |X'(t)| dt.$$

To study crossings, one can also use the *co-area formula* which is valid in the framework of bounded variations functions (see for instance [12]), but we won't need here such a general framework. When  $X$  is an almost surely continuously differentiable process on  $[a, b]$ , for any bounded and continuous function  $h$  on  $\mathbb{R}$ , we have:

$$(13) \quad \int_a^b h(X(t)) |X'(t)| dt = \int_{\mathbb{R}} h(\alpha) N_X(\alpha, [a, b]) d\alpha \quad \text{a.s.}$$

In particular when  $h = 1$  this shows that  $\alpha \mapsto N_X(\alpha, [a, b])$  is integrable on  $\mathbb{R}$  and  $\int_{\mathbb{R}} N_X(\alpha, [a, b]) d\alpha = \int_a^b |X'(t)| dt$  is the total variation of  $X$  on  $[a, b]$ .

Let us also recall that when there are no tangencies of  $X'$  for the zero-level, then the number of local extrema for  $X$  is given by  $N_{X'}(0, [a, b])$ , which corresponds to the sum of the number of local minima (up zero-crossings of  $X'$ ) and of local maxima (down zero-crossings of  $X'$ ). Moreover, according to Rolle's theorem, whatever the level  $\alpha$  is,  $N_X(\alpha, [a, b]) \leq N_{X'}(0, [a, b]) + 1$  a.s.

Dealing with random processes one may be more interested in the mean number of crossings. We will denote by  $C_X(\alpha, [a, b])$  the mean number of crossings of the level  $\alpha$  by the process  $X$  in  $[a, b]$ :

$$(14) \quad C_X(\alpha, [a, b]) = \mathbb{E}(N_X(\alpha, [a, b])) = \mathbb{E}(\#\{t \in [a, b] \text{ such that } X(t) = \alpha\}).$$

Let us emphasize that this function is no more with integer values and can be continuous with respect to  $\alpha$ . When moreover  $X$  is a stationary process, by the additivity of means, we get  $C_X(\alpha, [a, b]) = (b - a)C_X(\alpha, [0, 1])$  for  $\alpha \in \mathbb{R}$ . In this case  $C_X(\alpha, [0, 1])$  corresponds to the mean number of crossings of the level  $\alpha$  per unit length. Let us also recall that when  $X$  is a strictly stationary ergodic process, the ergodic theorem states that  $(2T)^{-1}N_X(\alpha, [-T, T]) \xrightarrow{T \rightarrow +\infty} C_X(\alpha, [0, 1])$  a.s. (see [9] for instance).

A straightforward result can be derived from the co-area formula (13).

**Proposition 9.** *Let  $a, b \in \mathbb{R}$  with  $a < b$ . Let  $X$  be an almost surely and mean square continuously differentiable process on  $[a, b]$ . Then  $\alpha \mapsto C_X(\alpha, [a, b]) \in L^1(\mathbb{R})$ . Moreover, for any bounded continuous function  $h$ :*

$$(15) \quad \int_a^b \mathbb{E}(h(X(t))|X'(t)|)dt = \int_{\mathbb{R}} h(\alpha)C_X(\alpha, [a, b])d\alpha.$$

*Proof.* Taking the expected values in (13) for  $h = 1$  on the interval  $[a, b]$  we get by Fubini's theorem that

$$\int_a^b \mathbb{E}(|X'(t)|)dt = \int_{\mathbb{R}} C_X(\alpha, [a, b])d\alpha.$$

Since  $t \mapsto \mathbb{E}(|X'(t)|)$  is continuous on  $[a, b]$  by mean square continuity of  $X'$ , the total variation of  $X$  on  $[a, b]$  has finite expectation, which concludes the proof.  $\square$

Let us emphasize that this result implies in particular that  $C_X(\alpha, [a, b]) < +\infty$  for almost every level  $\alpha \in \mathbb{R}$  but one cannot conclude for a fixed given level.

One should expect to have a similar formula for  $C_X$  than the Kac's formula obtain for  $N_X$ . However, when  $X$  is a process satisfying the assumptions of Proposition 8, Kac's formula only gives an upper bound on  $C_X(\alpha, [a, b])$ , according to Fatou's Lemma:

$$C_X(\alpha, [a, b]) \leq \liminf_{\delta \rightarrow 0} \frac{1}{2\delta} \int_a^b \mathbb{E}(\mathbf{1}_{|X(t)-\alpha|<\delta}|X'(t)|)dt.$$

This upper bound is not very tractable without assumptions on the existence of a bounded joint density for the law of  $(X(t), X'(t))$ . As far as shot noise processes are concerned, one can exploit the infinite divisibility property by considering the mean crossing function of the sum of independent processes. The next proposition gives an upper bound in this setting. Another application of this proposition will be seen in Section 6 where we will decompose a shot noise process into the sum of two independent processes (for which crossings are easy to compute) by partitioning the set of points of the Poisson process.

**Proposition 10** (Crossings of a sum of independent processes). *Let  $a, b \in \mathbb{R}$  with  $a < b$ . Let  $n \geq 2$  and  $X_j$  be independent real-valued processes almost surely and mean square two times continuously differentiable on  $[a, b]$  for  $1 \leq j \leq n$ . Assume that there exist constants  $C_j$  and probability measures  $d\mu_j$  on  $\mathbb{R}$  such that if  $dP_{X_j(t)}$  denotes the probability measure of  $X_j(t)$ , then*

$$\forall t \in [a, b], \quad dP_{X_j(t)} \leq C_j d\mu_j, \quad \text{for } 1 \leq j \leq n.$$

*Let  $X$  be the process obtained by  $X = \sum_{j=1}^n X_j$  and assume that  $X$  satisfies (12) for  $\alpha \in \mathbb{R}$ . Then*

$$(16) \quad C_X(\alpha, [a, b]) \leq \sum_{j=1}^n \left( \prod_{i \neq j} C_i \right) (C_{X'_j}(0, [a, b]) + 1).$$

*Moreover, in the case where all the  $X_j$  are stationary on  $\mathbb{R}$ :*

$$C_X(\alpha, [a, b]) \leq \sum_{j=1}^n C_{X'_j}(0, [a, b]).$$

*Proof.* We first need an elementary result. Let  $f$  be a  $C^1$  function on  $[a, b]$ , then for all  $\delta > 0$ , and for all  $x \in \mathbb{R}$ , we have:

$$(17) \quad \frac{1}{2\delta} \int_a^b \mathbf{1}_{|f(t)-x| \leq \delta} |f'(t)| dt \leq N_{f'}(0, [a, b]) + 1.$$

This result (that can be found as an exercise at the end of Chapter 3 of [2]) can be proved this way: let  $a_1 < \dots < a_n$  denote the points at which  $f'(t) = 0$  in  $[a, b]$ . On each interval  $[a, a_1], [a_1, a_2], \dots, [a_n, b]$ ,  $f$  is monotonic and thus  $\int_{a_i}^{a_{i+1}} \mathbf{1}_{|f(t)-x| \leq \delta} |f'(t)| dt \leq 2\delta$ . Summing up these integrals, we have the announced result.

For the process  $X$ , since it satisfies the conditions of Kac's formula (12), by Proposition 8 and Fatou's Lemma,

$$C_X(\alpha, [a, b]) \leq \liminf_{\delta \rightarrow 0} \frac{1}{2\delta} \int_a^b \mathbb{E}(\mathbf{1}_{|X(t)-\alpha| \leq \delta} |X'(t)|) dt.$$

Now, for each  $\delta > 0$ , we have  $\mathbb{E}(\mathbf{1}_{|X(t)-\alpha| \leq \delta} |X'(t)|) \leq \sum_{j=1}^n \mathbb{E}(\mathbf{1}_{|X_1(t)+\dots+X_n(t)-\alpha| \leq \delta} |X'_j(t)|)$ . Then, thanks to the independence of  $X_1, \dots, X_n$  and to the bound on the laws of  $X_j(t)$ , we get:

$$\begin{aligned} & \int_a^b \mathbb{E}(\mathbf{1}_{|X_1(t)+\dots+X_n(t)-\alpha| \leq \delta} |X'_1(t)|) dt \\ &= \int_a^b \int_{\mathbb{R}^{n-1}} \mathbb{E}(\mathbf{1}_{|X_1(t)+x_2+\dots+x_n-\alpha| \leq \delta} |X'_1(t)| \mid X_2(t) = x_2, \dots, X_n(t) = x_n) dP_{X_2(t)}(x_2) \dots dP_{X_n(t)}(x_n) dt \\ &\leq \left( \prod_{j=2}^n C_j \right) \int_{\mathbb{R}^{n-1}} \int_a^b \mathbb{E}(\mathbf{1}_{|X_1(t)+x_2+\dots+x_n-\alpha| \leq \delta} |X'_1(t)|) dt d\mu_2(x_2) \dots d\mu_n(x_n). \end{aligned}$$

Now, (17) holds almost surely for  $X_1$ , taking expectation we get

$$\frac{1}{2\delta} \int_a^b \mathbb{E}(\mathbf{1}_{|X_1(t)+x_2+\dots+x_n-\alpha| \leq \delta} |X'_1(t)|) dt \leq C_{X'_1}(0, [a, b]) + 1.$$

Using the fact the  $d\mu_j$  are probability measures we get

$$\frac{1}{2\delta} \int_a^b \mathbb{E}(\mathbf{1}_{|X_1(t)+\dots+X_n(t)-\alpha| \leq \delta} |X'_1(t)|) dt \leq \left( \prod_{j=2}^n C_j \right) (C_{X'_1}(0, [a, b]) + 1).$$

We obtain similar bounds for the other terms. Since this holds for all  $\delta > 0$ , we have the bound (16) on the expected number of crossings of the level  $\alpha$  by the process  $X$ .

When the  $X_j$  are stationary, things become simpler: we can take  $C_j = 1$  for any  $1 \leq j \leq n$ , and also by stationarity we have that for all  $p \geq 1$  integer:  $C_X(\alpha, [a, b+p(b-a)]) = (p+1)C_X(\alpha, [a, b])$ . Now using (16) for all  $p$ , then dividing by  $(p+1)$ , we have that for all  $p$ :  $C_X(\alpha, [a, b]) \leq \sum_{j=1}^n C_{X'_j}(0, [a, b]) + \frac{n}{p+1}$ . Finally, letting  $p$  goes to infinity, we have the result.  $\square$

As previously seen, taking the expectation in Kac's formula only allows us to get an upper bound for  $C_X$ . However, under stronger assumptions, one can justify the interversion of the limit and the expectation. This is known as Rice's formula. We recall it here under original assumptions on the characteristic functions of the process, which are more tractable than densities, in the setting of shot noise processes.

**Proposition 11** (Rice's Formula). *Let  $a, b \in \mathbb{R}$  with  $a \leq b$ . Let  $X$  be a real valued random process almost surely continuously differentiable on  $[a, b]$ . Let us denote  $\psi_{t,\varepsilon}$ , respectively  $\psi_{t,0} := \psi_t$ , the characteristic function of  $(X(t), (X(t+\varepsilon) - X(t))/\varepsilon)$ , respectively  $(X(t), X'(t))$ , for  $t \in [a, b]$  and*



$\varepsilon > 0$ . Assume that for all  $t \in [a, b]$  and  $\varepsilon$  sufficiently small, and for all  $0 \leq k \leq 3$ , the partial derivatives  $\frac{\partial^k}{\partial v^k} \psi_{t,\varepsilon}(u, v)$  exist and satisfy

$$(18) \quad \left| \frac{\partial^k}{\partial v^k} \psi_{t,\varepsilon}(u, v) \right| \leq C(1 + \sqrt{u^2 + v^2})^{-l},$$

for  $l > 2$  and  $C$  a positive constant. Then, the crossings mean number function  $\alpha \mapsto C_X(\alpha, [a, b])$  is continuous on  $\mathbb{R}$  and given by

$$(19) \quad C_X(\alpha, [a, b]) = \int_a^b \int_{\mathbb{R}} |z| p_t(\alpha, z) dz dt < +\infty,$$

where  $p_t(\alpha, z) = \frac{1}{4\pi^2} \hat{\tilde{\psi}}_t(\alpha, z)$  is the joint distribution density of  $(X(t), X'(t))$ .

*Proof.* It is enough to check assumptions i)–iii) of Theorem 2 of [19] p.262. Assumptions for  $k = 0$  ensure that  $\psi_{t,\varepsilon} \in L^1(\mathbb{R}^2)$ , respectively  $\psi_t \in L^1(\mathbb{R}^2)$ , such that  $(X(t), (X(t+\varepsilon) - X(t))/\varepsilon)$ , respectively  $(X(t), X'(t))$ , admits  $p_{t,\varepsilon} = \frac{1}{4\pi^2} \hat{\tilde{\psi}}_{t,\varepsilon}$ , respectively  $p_t = \frac{1}{4\pi^2} \hat{\tilde{\psi}}_t$ , as density.

i)  $p_{t,\varepsilon}(x, z)$  is continuous in  $(t, x)$  for each  $z, \varepsilon$ , according to Lebesgue's dominated convergence theorem using the fact that  $X$  is almost surely continuous on  $\mathbb{R}$ .

ii) Since  $X$  is almost surely continuously differentiable on  $\mathbb{R}$  we clearly have for any  $(u, v) \in \mathbb{R}^2$ ,  $\psi_{t,\varepsilon}(u, v) \rightarrow \psi_t(u, v)$  as  $\varepsilon \rightarrow 0$ . Then by Lebesgue's dominated convergence theorem  $p_{t,\varepsilon}(x, z) \rightarrow p_t(x, z)$  as  $\varepsilon \rightarrow 0$ , uniformly in  $(t, x)$  for each  $z \in \mathbb{R}$ .

iii) For any  $z \neq 0$ , integrating by parts we get

$$p_{t,\varepsilon}(x, z) = \frac{i}{4\pi^2 z^3} \int_{\mathbb{R}^2} e^{-ixu - izv} \frac{\partial^3}{\partial v^3} \psi_{t,\varepsilon}(u, v) du dv,$$

such that  $p_{t,\varepsilon}(x, z) \leq Ch(z)$  for all  $t, \varepsilon, x$  with  $h(z) = (1 + |z|^3)^{-1}$  satisfying  $\int_{\mathbb{R}} |z| h(z) dz < +\infty$  and  $C$  a positive constant.  $\square$

Let us mention that Rice's formula (19) can be obtained under weaker assumption when considering  $X$  strictly stationary and mean square differentiable on  $[a, b]$ . Actually, according to Theorem 2 iii) of [14], if  $\mathbb{P}(X'(0) = 0) = 0$  then the law of  $X(0)$  has a density denoted by  $f_{X(0)}(x)dx$ , and Rice's formula (19) holds for  $f_{X(0)}(x)dx$ -almost every  $\alpha \in \mathbb{R}$ . If the joint distribution density does not exist the integral on  $\mathbb{R}$  is replaced by  $f_{X(0)}(\alpha) \mathbb{E}(|X'(0)| | X(0) = \alpha)$ .

However, let us emphasize that Proposition 11 states also that the crossings mean number function  $\alpha \mapsto C_X(\alpha, [a, b])$ , which was already known to be integrable, is also continuous on  $\mathbb{R}$ .

We have not been able to obtain this property by a different way with weaker assumptions. We use it to state the next theorem concerning shot noise processes.

**Theorem 1.** *Let  $X$  be a stationary 1d shot noise process defined by (1) with kernel function  $g$  and  $\nu(ds) = ds$ . Assume for sake of simplicity that  $\beta = 1$  almost surely. Let us assume that  $g$  is a function of class  $\mathcal{C}^4$  on  $\mathbb{R}$  with  $g, g', g'' \in L^1(\mathbb{R})$ . Let  $T > 0$ ,  $a \leq b$ , and assume that for all  $s \in [-T + a, T + b]$ , the matrices  $\Phi(s) = \begin{pmatrix} g'(s) & g''(s) \\ g''(s) & g^{(3)}(s) \end{pmatrix}$  and  $\Phi'(s) = \begin{pmatrix} g''(s) & g^{(3)}(s) \\ g^{(3)}(s) & g^{(4)}(s) \end{pmatrix}$  are invertible. Let  $\gamma_T = \#\{i; \tau_i \in [-T, T]\}$ . Then, conditionally on  $\{\gamma_T \geq k_0\}$  with  $k_0 \geq 8$ , the crossings mean number function  $\alpha \mapsto \mathbb{E}(N_X(\alpha, [a, b]) | \gamma_T \geq k_0)$  is continuous on  $\mathbb{R}$ .*

*Proof.* Let  $t \in [a, b]$ . We write  $X(t) = X_T(t) + R_T(t)$  as given by (8). Let us write for  $\varepsilon$  small enough

$$\psi_{t,\varepsilon,k_0} = \psi_{t,\varepsilon,k_0}^T \psi_{t,\varepsilon}^{R_T}$$

with  $\psi_{t,\varepsilon,k_0}$ , respectively  $\psi_{t,\varepsilon,k_0}^T$ , the characteristic function of  $(X(t), (X(t+\varepsilon) - X(t))/\varepsilon)$ , respectively  $(X_T(t), (X_T(t+\varepsilon) - X_T(t))/\varepsilon)$ , conditionally on  $\{\gamma_T \geq k_0\}$ . Note that,  $R_T$  is independent from  $\gamma_T$  such that  $\psi_{t,\varepsilon}^{R_T}$  is just the characteristic function of  $(R_T(t), (R_T(t+\varepsilon) - R_T(t))/\varepsilon)$ . According to Rice's formula the result will follow from (18). By Leibnitz formula, for  $0 \leq k \leq 3$ , one has

$$(20) \quad \frac{\partial^k}{\partial v^k} \psi_{t,\varepsilon,k_0}(u, v) = \sum_{l=0}^k \binom{k}{l} \frac{\partial^l}{\partial v^l} \psi_{t,\varepsilon,k_0}^T(u, v) \frac{\partial^{k-l}}{\partial v^{k-l}} \psi_{t,\varepsilon}^{R_T}(u, v).$$

On the one hand

$$\left| \frac{\partial^{k-l}}{\partial v^{k-l}} \psi_{t,\varepsilon}^{R_T}(u, v) \right| \leq \mathbb{E} \left( \left| \frac{R_T(t+\varepsilon) - R_T(t)}{\varepsilon} \right|^{k-l} \right),$$

with

$$\left| \frac{R_T(t+\varepsilon) - R_T(t)}{\varepsilon} \right| \leq \sum_{|\tau_i| > T} |g_\varepsilon(t - \tau_i)| \text{ for } g_\varepsilon(s) = \frac{1}{\varepsilon} \int_0^\varepsilon g'(s+x) dx.$$

Let us remark that since  $g', g'' \in L^1(\mathbb{R})$  one has  $g, g' \in L^\infty(\mathbb{R})$ . Then for any  $\varepsilon > 0$ ,  $g_\varepsilon \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$  with  $\|g_\varepsilon\|_\infty \leq \|g'\|_\infty$  and  $\|g_\varepsilon\|_1 \leq \|g'\|_1$ . Then using (7), one can find  $c > 0$  such that for all  $0 \leq k \leq 3$ , with  $(k-1)_+ = \max(0, k-1)$ ,

$$(21) \quad \left| \frac{\partial^k}{\partial v^k} \psi_{t,\varepsilon}^{R_T}(u, v) \right| \leq c \max(1, \|g'\|_\infty)^{(k-1)_+} \max(1, \lambda \|g'\|_1)^k.$$

On the other hand,

$$\begin{aligned} \mathbb{P}(\gamma_T \geq k_0) \psi_{t,\varepsilon,k_0}^T(u, v) &= \sum_{k \geq k_0} \mathbb{E} \left( e^{iuX_T(t) + iv(X_T(t+\varepsilon) - X_T(t))/\varepsilon} | \gamma_T = k \right) \mathbb{P}(\gamma_T = k) \\ &= \sum_{k \geq k_0} \chi_{t,\varepsilon}^T(u, v)^k \mathbb{P}(\gamma_T = k) \end{aligned}$$

where  $\chi_{t,\varepsilon}^T(u, v) = (2T)^{-1} \int_{-T+t}^{T+t} e^{iug(s) + ivg_\varepsilon(s)} ds$ , is the characteristic function of  $(g(t-U_T), g_\varepsilon(t-U_T))$ , with  $U_T$  a uniform random variable on  $[-T, T]$ . It follows that  $|\chi_{t,\varepsilon}^T(u, v)| \leq 1$ , so that one can find  $c > 0$  such that for all  $0 \leq k \leq 3$ ,

$$\left| \frac{\partial^k}{\partial v^k} \psi_{t,\varepsilon,k_0}^T(u, v) \right| \leq c \max(1, \|g'\|_\infty)^{(k-1)_+} \max(1, \lambda \|g'\|_1)^k \frac{\mathbb{P}(\gamma_T \geq k_0 - k)}{\mathbb{P}(\gamma_T \geq k_0)} |\chi_{t,\varepsilon}^T(u, v)|^{k_0-k}.$$

This, together with (21) and (20), implies that one can find  $c > 0$  such that for all  $0 \leq k \leq 3$ ,

$$(22) \quad \left| \frac{\partial^k}{\partial v^k} \psi_{t,\varepsilon,k_0}^T(u, v) \right| \leq c \max(1, \|g'\|_\infty)^{(k-1)_+} \max(1, \lambda \|g'\|_1)^k \frac{\mathbb{P}(\gamma_T \geq k_0 - k)}{\mathbb{P}(\gamma_T \geq k_0)} |\chi_{t,\varepsilon}^T(u, v)|^{k_0-k}.$$

Moreover, let  $\Phi_\varepsilon(s) = \begin{pmatrix} g'(s) & g'_\varepsilon(s) \\ g''(s) & g''_\varepsilon(s) \end{pmatrix}$  and  $\Phi'_\varepsilon(s) = \begin{pmatrix} g''(s) & g''_\varepsilon(s) \\ g^{(3)}(s) & g_\varepsilon^{(3)}(s) \end{pmatrix}$ . Then  $\det \Phi_\varepsilon(s)$  converges to  $\det \Phi(s)$  as  $\varepsilon \rightarrow 0$ , uniformly in  $s \in [-T-a, T+b]$ . The assumption on  $\Phi$  ensures that one can find  $\varepsilon_0$  such that for  $\varepsilon \leq \varepsilon_0$ , the matrix  $\Phi_\varepsilon(s)$  is invertible for all  $s \in [-T-a, T+b]$ . The same holds true for  $\Phi'_\varepsilon(s)$ . Denote  $m = \min_{s \in [-T-a, T+b], \varepsilon \leq \varepsilon_0} \|\Phi_\varepsilon(s)^{-1}\|^{-1} > 0$ , where  $\|\cdot\|$  is the matricial norm induced by the Euclidean one. According to Proposition 5 with  $n_0 = 0$ ,

$$\forall (u, v) \in \mathbb{R}^2 \text{ s. t. } \sqrt{u^2 + v^2} > \frac{1}{m}, \quad |\chi_{t,\varepsilon}^T(u, v)| = (2T)^{-1} \left| \int_{-T+t}^{T+t} e^{iug(s) + ivg_\varepsilon(s)} ds \right| \leq \frac{24\sqrt{2}}{\sqrt{m\sqrt{u^2 + v^2}}}.$$

Therefore, one can find a constant  $c_{k_0} > 0$  such that, for all  $0 \leq k \leq 3$ ,  $\left| \frac{\partial^k}{\partial v^k} \psi_{t,\varepsilon,k_0}^T(u, v) \right|$  is less than

$$c_{k_0} (2T)^{-k_0+3} \max(1, \|g'\|_\infty)^{(k-1)_+} \max(1, \lambda \|g'\|_1)^k \frac{\mathbb{P}(\gamma_T \geq k_0 - k)}{\mathbb{P}(\gamma_T \geq k_0)} \left(1 + \sqrt{u^2 + v^2}\right)^{-(k_0-3)/2},$$

with  $(k_0 - 3)/2 > 2$  when  $k_0 > 7$  so that (18) holds, which concludes the proof.  $\square$

As previously seen, the classical Rice's formula for the mean crossings number function, when it holds, involves the joint probability density of  $(X(t), X'(t))$ . Such a formula is not tractable in our context where the existence of a density is even a question in itself. Considering the crossings mean number as an integrable function, it is natural to consider its Fourier transform as we do in the next section.

**4.2. Fourier transform of crossings mean number.** In the following proposition we obtain a closed formula for the Fourier transform of the crossings mean number function, which only involves characteristic functions of the process. This can be helpful, when considering shot noise processes, whose characteristic functions are well-known.

**Proposition 12.** *Let  $a, b \in \mathbb{R}$  with  $a < b$ . Let  $X$  be an almost surely and mean square continuously differentiable process on  $[a, b]$ . Then  $\alpha \mapsto C_X(\alpha, [a, b]) \in L^1(\mathbb{R})$  and its Fourier transform  $u \mapsto \widehat{C_X}(u, [a, b])$  is given by*

$$(23) \quad \widehat{C_X}(u, [a, b]) = \int_a^b \mathbb{E} \left( e^{iuX(t)} |X'(t)| \right) dt.$$

Moreover, if  $\psi_t$  denotes the joint characteristic function of  $(X(t), X'(t))$ , then  $\widehat{C_X}(u, [a, b])$  can be computed by

$$\begin{aligned} \widehat{C_X}(u, [a, b]) &= -\frac{1}{\pi} \int_a^b \int_0^{+\infty} \frac{1}{v} \left( \frac{\partial \psi_t}{\partial v}(u, v) - \frac{\partial \psi_t}{\partial v}(u, -v) \right) dv dt \\ &= -\frac{1}{\pi} \int_a^b \int_0^{+\infty} \frac{1}{v^2} (\psi_t(u, v) + \psi_t(u, -v) - 2\psi_t(u, 0)) dv dt. \end{aligned}$$

*Proof.* According to Proposition 9 we can choose  $h$  in Equation (15) of the form  $h(x) = \exp(iux)$  for any  $u$  real. This shows that  $\widehat{C}(u, [a, b]) = \int_a^b \mathbb{E} (e^{iuX(t)} |X'(t)|) dt$ . Let us now identify the right-hand term. Let  $\mu_t(dx, dy)$  denote the distribution of  $(X(t), X'(t))$ . Then the joint characteristic function  $\psi_t(u, v)$  of  $(X(t), X'(t))$  is

$$\psi_t(u, v) = \mathbb{E} (\exp(iuX(t) + ivX'(t))) = \int_{\mathbb{R}^2} e^{iux+ivy} \mu_t(dx, dy).$$

Since the random vector  $(X(t), X'(t))$  has moment of order two, then  $\psi_t$  is twice continuously differentiable on  $\mathbb{R}^2$ . Now, let us consider the integral

$$\begin{aligned} I_A &= \int_0^A \frac{1}{v} \left( \frac{\partial \psi_t}{\partial v}(u, v) - \frac{\partial \psi_t}{\partial v}(u, -v) \right) dv = \int_{v=0}^A \int_{x,y \in \mathbb{R}^2} \frac{iy e^{iux+ivy} - iy e^{iux-ivy}}{v} \mu_t(dx, dy) dv \\ &= -2 \int_{v=0}^A \int_{\mathbb{R}^2} y e^{iux} \frac{\sin(vy)}{v} \mu_t(dx, dy) dv = -2 \int_{\mathbb{R}^2} y e^{iux} \int_{v=0}^A \frac{\sin(v)}{v} dv \mu_t(dx, dy) \end{aligned}$$

The order of integration has been reversed thanks to Fubini's Theorem ( $|y e^{iux} \frac{\sin(vy)}{v}| \leq y^2$  which is integrable on  $[0, A] \times \mathbb{R}^2$  with respect to  $dv \times \mu_t(dx, dy)$ , since  $X'(t)$  is a second order random variable). As  $A$  goes to  $+\infty$ , then  $\int_{v=0}^A \frac{\sin(v)}{v} dv$  goes to  $\frac{\pi}{2} \text{sign}(y)$ , and moreover for all  $A, x$  and  $y$ , we have  $|y e^{iux} \int_{v=0}^A \frac{\sin(v)}{v} dv| \leq 3|y|$ , thus by Lebesgue's dominated convergence theorem, the limit of  $-\frac{1}{\pi} I_A$  exists as  $A$  goes to infinity and its value is:

$$\lim_{A \rightarrow +\infty} -\frac{1}{\pi} \int_0^A \frac{1}{v} \left( \frac{\partial \psi_t}{\partial v}(u, v) - \frac{\partial \psi_t}{\partial v}(u, -v) \right) dv = \int_{\mathbb{R}^2} |y| e^{iux} \mu_t(dx, dy) = \mathbb{E} (e^{iuX(t)} |X'(t)|).$$

The second expression in the proposition is simply obtained by integration by parts in the above formula.  $\square$

It is then natural to invert the Fourier transform to get an almost everywhere expression for  $C_X(\alpha, [a, b])$  itself.

**Proposition 13.** *Let  $a, b \in \mathbb{R}$  with  $a \leq b$ . Let  $X$  be a real valued random process almost surely continuously differentiable on  $[a, b]$ . Let us denote  $\psi_t$  the characteristic function of  $(X(t), X'(t))$ , for  $t \in [a, b]$ . Assume that for all  $t \in [a, b]$ , for  $1 \leq k \leq 2$ , the partial derivatives  $\frac{\partial^k}{\partial v^k} \psi_t(u, v)$  exist and satisfy*

$$(24) \quad \left| \frac{\partial^k}{\partial v^k} \psi_t(u, v) \right| \leq C(1 + \sqrt{u^2 + v^2})^{-l},$$

for  $l > 1$  and  $C$  a positive constant. Then  $\widehat{C_X}(u, [a, b]) \in L^1(\mathbb{R})$  and for almost every  $\alpha \in \mathbb{R}$

$$(25) \quad C_X(\alpha, [a, b]) = -\frac{1}{2\pi^2} \int_a^b \int_{\mathbb{R}} \int_0^{+\infty} \frac{e^{-i\alpha v}}{v} \left( \frac{\partial \psi_t}{\partial v}(u, v) - \frac{\partial \psi_t}{\partial v}(u, -v) \right) dv du dt.$$

*Proof.* Let  $u \in \mathbb{R}$  and  $t \in [a, b]$ , according to Proposition 12 one has

$$\mathbb{E} \left( e^{iuX(t)} |X'(t)| \right) = -\frac{1}{\pi} \int_0^{+\infty} \frac{1}{v} \left( \frac{\partial \psi_t}{\partial v}(u, v) - \frac{\partial \psi_t}{\partial v}(u, -v) \right) dv$$

It is sufficient to remark that integrating by parts, for any  $\varepsilon \in (0, 1)$ ,

$$\begin{aligned} & \int_{\varepsilon}^1 \frac{1}{v} \left( \frac{\partial \psi_t}{\partial v}(u, v) - \frac{\partial \psi_t}{\partial v}(u, -v) \right) dv \\ &= -\ln(\varepsilon) \left( \frac{\partial \psi_t}{\partial v}(u, \varepsilon) - \frac{\partial \psi_t}{\partial v}(u, -\varepsilon) \right) - \int_{\varepsilon}^1 \ln(v) \left( \frac{\partial^2 \psi_t}{\partial v^2}(u, v) - \frac{\partial^2 \psi_t}{\partial v^2}(u, -v) \right) dv. \end{aligned}$$

Therefore, as  $\varepsilon$  goes to zero, one gets

$$(26) \quad \int_0^1 \frac{1}{v} \left( \frac{\partial \psi_t}{\partial v}(u, v) - \frac{\partial \psi_t}{\partial v}(u, -v) \right) dv = - \int_0^1 \ln(v) \left( \frac{\partial^2 \psi_t}{\partial v^2}(u, v) - \frac{\partial^2 \psi_t}{\partial v^2}(u, -v) \right) dv.$$

Then, according to (24) for  $k = 2$ ,

$$\left| \int_0^1 \frac{1}{v} \left( \frac{\partial \psi_t}{\partial v}(u, v) - \frac{\partial \psi_t}{\partial v}(u, -v) \right) dv \right| \leq 2C(1 + |u|)^{-l}.$$

On the other hand, according to (24) for  $k = 1$ ,

$$\left| \int_1^{+\infty} \frac{1}{v} \left( \frac{\partial \psi_t}{\partial v}(u, v) - \frac{\partial \psi_t}{\partial v}(u, -v) \right) dv \right| \leq C'(1 + |u|)^{-l} \ln(2 + |u|).$$

Then  $\widehat{C_X}(u, [a, b]) \in L^1(\mathbb{R})$  and its inverse Fourier transform given by (25) is equal to  $C_X(\cdot, [a, b])$  almost everywhere.  $\square$

Let us emphasize that the assumptions of Proposition 13 are weaker than those of Proposition 11 but the result holds almost everywhere. Actually (25) will hold everywhere as soon as the crossings mean number function  $C_X(\cdot, [a, b])$  is continuous on  $\mathbb{R}$ , which is not implied by Proposition 13 alone. However such a weak result can still be used in practice as explained in [27].

**Remark.** The last expression considerably simplifies when  $X$  is a stationary Gaussian process almost surely and mean square continuously differentiable on  $\mathbb{R}$ . By independence of  $X(t)$  and  $X'(t)$  we get  $\psi_t(u, v) = \phi_X(u)\phi_{X'}(v)$  where  $\phi_X$ , respectively  $\phi_{X'}$ , denotes the characteristic function of  $X(t)$ , resp.  $X'(t)$  (independent of  $t$  by stationarity). Then,  $\psi_t$  satisfies the assumptions of Proposition 13. Moreover one can check that  $X$  also satisfies the assumptions of Proposition 11 such that the crossings mean number function is continuous on  $\mathbb{R}$ . Its Fourier transform is given by

$$\widehat{C_X}(u, [a, b]) = -\frac{b-a}{\pi} \phi_X(u) \int_{\mathbb{R}} \frac{1}{v} \frac{\partial \phi_{X'}}{\partial v}(v) dv.$$

By the inverse Fourier transform and continuity of  $C_X(\cdot, [a, b])$  we get Rice's formula

$$(27) \quad C_X(\alpha, [a, b]) = \frac{b-a}{\pi} \left( \frac{m_2}{m_0} \right)^{1/2} e^{-(\alpha - \mathbb{E}(X(0)))^2 / 2m_0}, \quad \forall \alpha \in \mathbb{R},$$

where  $m_0 = \text{Var}(X(t))$  and  $m_2 = \text{Var}(X'(t))$ . Let us quote that in fact Rice's formula holds as soon as  $X$  is a.s. continuous (see Exercise 3.2 of [2]) in the sense that  $C_X(\alpha, [a, b]) = +\infty$  if  $m_2 = +\infty$ .

#### 4.3. Application: convergence of crossings mean number.

**Proposition 14.** *Let  $a, b \in \mathbb{R}$  with  $a < b$ . Let  $(X_n)_{n \in \mathbb{N}}$  and  $X$  be almost surely and mean square continuously differentiable processes on  $[a, b]$  such that*

$$\widehat{C_{X_n}}(u, [a, b]) \xrightarrow{n \rightarrow +\infty} \widehat{C_X}(u, [a, b]), \forall u \in \mathbb{R}.$$

*Then the sequence of crossings mean number functions  $C_{X_n}(\cdot, [a, b])$  converges weakly to  $C_X(\cdot, [a, b])$ , denoted by  $C_{X_n}(\cdot, [a, b]) \xrightarrow{n \rightarrow \infty} C_X(\cdot, [a, b])$ , which means:*

$$\forall h \text{ bounded continuous function on } \mathbb{R}, \quad \int h(\alpha) C_{X_n}(\alpha, [a, b]) d\alpha \xrightarrow{n \rightarrow \infty} \int h(\alpha) C(\alpha, [a, b]) d\alpha.$$

*Proof.* This result simply comes from the fact that we can see  $\alpha \mapsto C_{X_n}(\alpha)/\widehat{C_{X_n}}(0)$  as a probability density function on  $\mathbb{R}$ , and then apply classical results which relate convergence of characteristic functions to weak convergence.  $\square$

Using Plancherel equality, a stronger convergence can be obtained in  $L^2(\mathbb{R})$  in the following setting.

**Proposition 15.** *Let  $a, b \in \mathbb{R}$  with  $a < b$ . Let  $(X_n)_{n \in \mathbb{N}}$  and  $X$  be almost surely and mean square twice continuously differentiable processes on  $[a, b]$  such that*

$$\widehat{C_{X_n}}(u, [a, b]) \xrightarrow{n \rightarrow +\infty} \widehat{C_X}(u, [a, b]), \forall u \in \mathbb{R}.$$

*Assume moreover that the mean total variations and the mean numbers of local extrema are uniformly bounded: there exists  $M > 0$  such that and*

$$i) \sup_{n \in \mathbb{N}} \int_a^b \mathbb{E}(|X'_n(t)|) \leq M;$$

$$ii) C_{X'}(0, [a, b]) \leq M \text{ and } C_{X'_n}(0, [a, b]) \leq M, \forall n \in \mathbb{N}.$$

*Then the sequence of crossings mean number functions  $C_{X_n}(\cdot, [a, b])$  converges to  $C_X(\cdot, [a, b])$  in  $L^2(\mathbb{R})$ .*

*Proof.* Note that the condition *ii)* implies that  $C_{X_n}(\cdot, [a, b])$ , respectively  $C_X(\cdot, [a, b])$ , is bounded by  $M + 1$  such that  $C_{X_n}(\cdot, [a, b])$ , respectively  $C_X(\cdot, [a, b])$ , is in  $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \subset L^2(\mathbb{R})$ . Let us prove that  $\left( \left| \widehat{C_{X_n}}(\cdot, [a, b]) \right|^2 \right)_{n \in \mathbb{N}}$  is uniformly integrable. First, by Plancherel Theorem we get

$$\|\widehat{C_{X_n}}(\cdot, [a, b])\|_{L^2(\mathbb{R})} = \sqrt{2\pi} \|C_{X_n}(\cdot, [a, b])\|_{L^2(\mathbb{R})} \leq \sqrt{2\pi} \|C_{X_n}(\cdot, [a, b])\|_{L^\infty(\mathbb{R})}^{1/2} \|C_{X_n}(\cdot, [a, b])\|_{L^1(\mathbb{R})}^{1/2}.$$

But, according to *ii)*,

$$\|C_{X_n}(\cdot, [a, b])\|_{L^1(\mathbb{R})} = \int_a^b \mathbb{E}(|X'_n(t)|) dt \leq M.$$

Therefore,  $\sup_{n \in \mathbb{N}} \|\widehat{C_{X_n}}(\cdot, [a, b])\|_{L^2(\mathbb{R})} < +\infty$ . Secondly, remark that  $\left| \widehat{C_{X_n}}(u, [a, b]) \right| \leq \|C_{X_n}(\cdot, [a, b])\|_{L^1(\mathbb{R})}$  such that for any Borel set  $A$ ,

$$\forall n \in \mathbb{N}, \quad \int_A \left| \widehat{C_{X_n}}(u, [a, b]) \right|^2 du \leq M^2 |A|,$$

which finishes to prove that  $\left( \left| \widehat{C_{X_n}}(\cdot, [a, b]) \right|^2 \right)_{n \in \mathbb{N}}$  is uniformly integrable. Therefore  $\widehat{C_{X_n}}(\cdot, [a, b])$  converges to  $\widehat{C_X}(\cdot, [a, b])$  in  $L^2(\mathbb{R})$ , which concludes the proof using Plancherel Theorem.  $\square$

**Proposition 16.** *Let  $\beta^{(n)}$  be a sequence of random variables in  $L^2(\Omega)$ . Let  $(g_n)_{n \in \mathbb{N}}$  be a sequence of functions such that  $g_n \in \mathcal{C}^2(\mathbb{R})$  with  $g_n, g'_n, g''_n \in L^1(\mathbb{R})$ . Assume that there exist  $\beta \in L^2(\Omega)$  and  $g \in \mathcal{C}^2(\mathbb{R})$  with  $g, g', g'' \in L^1(\mathbb{R})$  such that*

- (1)  $g_n^{(k)} \xrightarrow{n \rightarrow \infty} g^{(k)}$  in  $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$  for  $k = 0, 1$ ;
- (2)  $\beta^{(n)} \xrightarrow{n \rightarrow \infty} \beta$  in  $L^2(\Omega)$ .

Let us consider the shot noise processes  $X_n$  and  $X$  defined by  $X_n(t) = \sum_i \beta_i^{(n)} g_n(t - \tau_i)$  and  $X(t) = \sum_i \beta_i g(t - \tau_i)$ , where  $\{\tau_i\}$  is a Poisson point process of intensity  $\lambda$  on  $\mathbb{R}$ . Then for all  $a < b$ ,

$$\widehat{C_{X_n}}(u, [a, b]) \xrightarrow{n \rightarrow \infty} \widehat{C_X}(u, [a, b]), \forall u \in \mathbb{R}.$$

*Proof.* We use Equation (23) to compute the Fourier transform of the crossings mean number functions and get

$$\widehat{C_{X_n}}(u, [a, b]) - \widehat{C_X}(u, [a, b]) = \int_a^b \mathbb{E}(e^{iuX_n(t)} |X'_n(t)|) - \mathbb{E}(e^{iuX(t)} |X'(t)|) dt,$$

with

$$\mathbb{E}(e^{iuX_n(t)} |X'_n(t)|) - \mathbb{E}(e^{iuX(t)} |X'(t)|) = \mathbb{E}((e^{iuX_n(t)} - e^{iuX(t)}) |X'(t)|) + \mathbb{E}(e^{iuX_n(t)} (|X'_n(t)| - |X'(t)|)).$$

Thus

$$|\mathbb{E}(e^{iuX_n(t)} |X'_n(t)|) - \mathbb{E}(e^{iuX(t)} |X'(t)|)| \leq |u| \mathbb{E}(|X_n(t) - X(t)| \cdot |X'(t)|) + \mathbb{E}(|X'_n(t) - X'(t)|).$$

Let us introduce a sequence of auxiliary shot noise processes  $Y_n$  with impulses  $\beta^{(n)}$  and kernel function  $g$  such that

$$X_n(t) - Y_n(t) = \sum_i \beta_i^{(n)} (g_n - g)(t - \tau_i) \text{ and } Y_n(t) - X(t) = \sum_i (\beta_i^{(n)} - \beta_i) g(t - \tau_i),$$

with  $\{\tau_i\}$  the points of a Poisson process of intensity  $\lambda$  on the line, and  $\{\beta_i^{(n)}\}$  and  $\{\beta_i\}$  independent samples of respectively  $\beta^{(n)}$  and  $\beta$ . Then

$$\begin{aligned} \mathbb{E}(|X_n(t) - X(t)| \cdot |X'(t)|) &\leq \mathbb{E}(|X_n(t) - Y_n(t)| \cdot |X'(t)|) + \mathbb{E}(|Y_n(t) - X(t)| \cdot |X'(t)|) \\ &\leq \left( \sqrt{\mathbb{E}(|X_n(t) - Y_n(t)|^2)} + \sqrt{\mathbb{E}(|Y_n(t) - X(t)|^2)} \right) \sqrt{\mathbb{E}(|X'(t)|^2)}. \end{aligned}$$

According to the elementary properties of Section 2.1, we get

$$\mathbb{E}((X_n(t) - Y_n(t))^2) = \lambda \mathbb{E}((\beta^{(n)})^2) \int_{\mathbb{R}} (g_n - g)^2(s) ds + \lambda^2 \mathbb{E}(\beta^{(n)})^2 \left( \int_{\mathbb{R}} (g_n - g)(s) ds \right)^2.$$

And in a similar way:

$$\mathbb{E}((Y_n(t) - X(t))^2) = \lambda \mathbb{E}((\beta^{(n)} - \beta)^2) \int_{\mathbb{R}} g^2(s) ds + \lambda^2 \mathbb{E}(\beta^{(n)} - \beta)^2 \left( \int_{\mathbb{R}} g(s) ds \right)^2.$$

On the other hand, we have

$$\begin{aligned} \mathbb{E}(|X'_n(t) - X'(t)|) &\leq \mathbb{E}(|X'_n(t) - Y'_n(t)|) + \mathbb{E}(|Y'_n(t) - X'(t)|) \\ &\leq \lambda \left( \mathbb{E}(|\beta^{(n)}|) \int_{\mathbb{R}} |g'_n - g'| (s) ds + \mathbb{E}(|\beta^{(n)} - \beta|) \int_{\mathbb{R}} |g'| (s) ds \right). \end{aligned}$$

Since  $\int |g_n - g|$ ,  $\int (g_n - g)^2$ ,  $\int |g'_n - g'|$  and  $\mathbb{E}(|\beta^{(n)} - \beta|^2)$  all go to 0 as  $n$  goes to infinity, we obtained the announced result.  $\square$

In particular this implies the weak convergence of the mean crossings number function. Let us remark that assumption *i*) of Proposition 15 is clearly satisfied and therefore, under assumption *ii*), the convergence holds also in  $L^2(\mathbb{R})$ .

As the intensity  $\lambda$  of the shot noise process tends to infinity, due to its infinitely divisible property and since it is of second order, we obtain, after renormalization, a Gaussian process at the limit. This behavior is studied in detail in the next section.

## 5. HIGH INTENSITY AND GAUSSIAN FIELD

**5.1. General feature.** It is well-known that, as the intensity  $\lambda$  of the Poisson process goes to infinity, the shot noise process converges to a normal process. Precise bounds on the distance between the law of a  $X(t)$  and the normal distribution are given in the paper of A. Papoulis “High density shot noise and Gaussianity” [23]. Moreover, the paper of L. Heinrich and V. Schmidt [16] gives conditions of normal convergence for a wide class of shot noise processes (not restricted to 1d, nor to Poisson processes). In this section we obtain a stronger result for smooth stationary shot noise processes by considering convergence in law in the space of continuous functions. In all this section we make the following assumption

$$\mathbf{H}_s : \begin{cases} \nu \text{ is the Lebesgue measure on } \mathbb{R}; \\ g \in \mathcal{C}^2(\mathbb{R}) \text{ with } g, g', g'' \in L^1(\mathbb{R}); \\ \beta \in L^2(\Omega); \end{cases}$$

and we will denote  $X_\lambda$  the strictly stationary shot noise process given by (1) with intensity  $\lambda > 0$ .

**Theorem 2.** *Let us assume  $\mathbf{H}_s$  and define the normalized shot noise process  $Z_\lambda(t) = \frac{1}{\sqrt{\lambda}} (X_\lambda(t) - \mathbb{E}(X_\lambda(t)))$ ,  $t \in \mathbb{R}$ . Then,*

$$Y_\lambda = \begin{pmatrix} Z_\lambda \\ Z'_\lambda \end{pmatrix} \xrightarrow[\lambda \rightarrow +\infty]{\text{fdd}} \sqrt{\mathbb{E}(\beta^2)} \begin{pmatrix} B \\ B' \end{pmatrix},$$

where  $B$  is a stationary centered Gaussian process almost surely and mean square continuously differentiable, with covariance function

$$\text{Cov}(B(t), B(t')) = \int_{\mathbb{R}} g(t-s)g(t'-s)ds = g * \check{g}(t-t').$$

When, moreover  $g'' \in L^p(\mathbb{R})$  for  $p > 1$ , the convergence holds in distribution on the space of continuous functions on compact sets endowed with the topology of the uniform convergence.

*Proof.* We begin with the proof of finite dimensional distributions convergence. Let  $k$  be an integer with  $k \geq 1$  and let  $t_1, \dots, t_k \in \mathbb{R}$  and  $w_1 = (u_1, v_1), \dots, w_k = (u_k, v_k) \in \mathbb{R}^2$ .

Let us write

$$\sum_{j=1}^k Y_\lambda(t_j) \cdot w_j = \frac{1}{\sqrt{\lambda}} \left( \sum_i \beta_i \tilde{g}(\tau_i) - \mathbb{E} \left( \sum_i \beta_i \tilde{g}(\tau_i) \right) \right),$$

for  $\tilde{g}(s) = \sum_{j=1}^k (u_j g(t_j - s) + v_j g'(t_j - s))$ . Therefore

$$\log \mathbb{E} \left( e^{i \sum_{j=1}^k Y_\lambda(t_j) \cdot w_j} \right) = \lambda \int_{\mathbb{R} \times \mathbb{R}} \left( e^{iz \left( \frac{\tilde{g}(s)}{\sqrt{\lambda}} \right)} - 1 - iz \frac{\tilde{g}(s)}{\sqrt{\lambda}} \right) ds F(dz).$$

Note that as  $\lambda \rightarrow +\infty$ ,

$$\lambda \left( e^{iz \left( \frac{\tilde{g}(s)}{\sqrt{\lambda}} \right)} - 1 - iz \frac{\tilde{g}(s)}{\sqrt{\lambda}} \right) \rightarrow -\frac{1}{2} z^2 \tilde{g}(s)^2,$$

with for all  $\lambda > 0$

$$\left| \lambda \exp \left( iz \left( \frac{\tilde{g}(s)}{\sqrt{\lambda}} \right) \right) - 1 - iz \frac{\tilde{g}(s)}{\sqrt{\lambda}} \right| \leq \frac{1}{2} z^2 \tilde{g}(s)^2.$$

By the dominated convergence theorem, since  $\tilde{g} \in L^2(\mathbb{R})$  and  $\beta \in L^2(\Omega)$  we get that as  $\lambda \rightarrow +\infty$

$$\mathbb{E} \left( \exp \left( i \sum_{j=1}^k Y_\lambda(t_j) \cdot w_j \right) \right) \rightarrow \exp \left( -\frac{1}{2} \mathbb{E}(\beta^2) \int_{\mathbb{R}} \tilde{g}(s)^2 ds \right).$$

Let us identify the limiting process. Let us recall that  $X_\lambda$  is a second order process with covariance function given by (6), namely  $\text{Cov}(X_\lambda(t), X_\lambda(t')) = \lambda \mathbb{E}(\beta^2) S(t-t')$  with  $S(t) = g * \check{g}(t)$ . Hence one can define  $B$  to be a stationary Gaussian centered process with  $(t, t') \mapsto S(t-t')$  as covariance function. The assumptions on  $g$  ensure that the function  $S$  is twice differentiable. Therefore  $B$  is mean square

differentiable with  $B'$  a stationary Gaussian centered process with  $(t, t') \mapsto -S''(t - t') = g' * \check{g}'(t - t')$  as covariance function. Moreover

$$\mathbb{E}((B'(t) - B'(t'))^2) = 2(S''(0) - S''(t - t')) \leq 2\|g'\|_\infty \|g''\|_1 |t - t'|,$$

such that by Theorem 3.4.1 of [1] the process  $B'$  is almost surely continuous on  $\mathbb{R}$ . Therefore as in [11] p. 536, one can check that almost surely  $B(t) = B(0) + \int_0^t B'(s)ds$ , such that  $B$  is almost surely continuously differentiable. We conclude for the fdd convergence by noticing that

$$\int_{\mathbb{R}} \tilde{g}(s)^2 ds = \text{Var} \left( \sum_{j=1}^k u_j B(t_j) + v_j B'(t_j) \right).$$

Let us prove the convergence in distribution on the space of continuous functions on compact sets endowed with the topology of the uniform convergence. It is enough to prove the tightness of the sequence  $(Y_\lambda)_\lambda$  according to Lemma 14.2 and Theorem 14.3 of [18]. Let  $t, s \in \mathbb{R}$  and remark that for any  $q \geq 1$ , on the one hand

$$\mathbb{E}((Z_\lambda(t) - Z_\lambda(t'))^2) = \mathbb{E}(\beta^2) \int_{\mathbb{R}} (g(t-s) - g(t'-s))^2 ds \leq \mathbb{E}(\beta^2) \|g'\|_q \|g'\|_1 |t - t'|^{2-1/q}.$$

On the other hand,

$$\mathbb{E}((Z'_\lambda(t) - Z'_\lambda(t'))^2) = \mathbb{E}(\beta^2) \int_{\mathbb{R}} (g'(t-s) - g'(t'-s))^2 ds \leq \mathbb{E}(\beta^2) \|g''\|_q \|g''\|_1 |t - t'|^{2-1/q}.$$

Note that assuming that  $g'' \in L^p(\mathbb{R})$  allows us to choose  $q = p > 1$  in the second upper bound such that  $2 - 1/q > 1$ . Moreover assumption  $\mathbf{H}_s$  implies that  $g' \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}) \subset L^p(\mathbb{R})$  such that one can also choose  $q = p$  in the first upper bound. Then,  $(Y_\lambda)_\lambda$  satisfies a Kolmogorov-Chentsov criterion which implies its tightness according to Corollary 14.9 of [18].  $\square$

In particular, when  $a < b$ , the functional  $(f, g) \mapsto \int_a^b h(f(t))|g(t)|dt$  is clearly continuous and bounded on  $\mathcal{C}([a, b], \mathbb{R}) \times \mathcal{C}([a, b], \mathbb{R})$  for any continuous bounded function  $h$  on  $\mathbb{R}$ . Then Theorem 2 implies that

$$\int_a^b \mathbb{E}(h(Z_\lambda(t)) |Z'_\lambda(t)|) dt \xrightarrow{\lambda \rightarrow +\infty} \int_a^b \mathbb{E}(h(B(t)) |B'(t)|) dt.$$

By the co-area formula (13), this means the weak convergence of crossings mean number, i.e.

$$C_{Z_\lambda}(\cdot, [a, b]) \xrightarrow{\lambda \rightarrow +\infty} C_B(\cdot, [a, b]).$$

This implies also the pointwise convergence of Fourier transforms. Such a result can be compared to the classical central limit theorem. Numerous of improved results can be obtained under stronger assumptions than the classical ones. This is the case for instance for the rate of convergence derived by Berry-Esseen Theorem or the convergence of the densities distributions. We refer to [13] chapter 15 and 16. Adapting the technical proofs allows us to get similar results for crossings in the next section.

**5.2. High intensity: rate of convergence for crossings mean number.** Let us remark that only  $\mathbb{E}(\beta^2)$  appears in the limit field. For sake of simplicity we may assume that  $\beta = 1$  a.s. Note that, according to Rice's formula [9], as recalled in Equation (27), since the limit Gaussian field is stationary,  $C_B(\alpha, [a, b]) = (b - a)C_B(\alpha, [0, 1])$  with

$$C_B(\alpha, [0, 1]) = \frac{1}{\pi} \left( \frac{m_2}{m_0} \right)^{1/2} e^{-\alpha^2/2m_0}, \forall \alpha \in \mathbb{R},$$

where  $m_0 = \text{Var}(B(t)) = g * \check{g}(0)$  and  $m_2 = \text{Var}(B'(t)) = g' * \check{g}'(0)$ . Moreover its Fourier transform is given by  $\widehat{C_B}(u, [0, 1]) = \sqrt{\frac{2m_2}{\pi}} e^{-m_0 u^2/2}$ . We obtain the following rate of convergence, for which proof is postponed to the Appendix.



**Proposition 17.** *Under  $\mathbf{H}_s$  with  $\beta = 1$  a.s., there exist three constants  $a_1$ ,  $a_2$  and  $a_3$  (depending only on  $g$  and its derivative) such that*

$$\forall \lambda > 0, \forall u \in \mathbb{R} \text{ such that } |u| < a_1 \sqrt{\lambda} \text{ then } |\widehat{C_{Z_\lambda}}(u, [0, 1]) - \sqrt{\frac{2m_2}{\pi}} e^{-m_0 u^2/2}| \leq \frac{a_2 + a_3 |u|}{\sqrt{\lambda}},$$

where  $m_0 = g * \check{g}(0)$  and  $m_2 = g' * \check{g}'(0)$ .

Let us emphasize that this implies the uniform convergence of the Fourier transform of the mean crossing functions on any fixed interval. Moreover, taking  $u = 0$ , the previous upper bound may be a bit refined such that the following corollary is in force.

**Corollary 3.** *Under  $\mathbf{H}_s$  with  $\beta = 1$  a.s., the mean total variation of the process satisfies:*

$$\forall \lambda > 0, \left| \frac{\mathbb{E}(|X'_\lambda(t)|)}{\sqrt{\lambda}} - \sqrt{\frac{2m_2}{\pi}} \right| \leq \frac{14m_3}{3\pi m_2 \sqrt{\lambda}},$$

where  $m_2 = g' * \check{g}'(0)$  and  $m_3 = \int_{\mathbb{R}} |g'(s)|^3 ds$ .

Then, according to Proposition 15,

$$C_{Z_\lambda}(\alpha, [0, 1]) \xrightarrow{\lambda \rightarrow +\infty} C_B(\alpha, [0, 1]) = \frac{1}{\pi} \left( \frac{m_2}{m_0} \right)^{1/2} e^{-\alpha^2/2m_0},$$

in  $L^2(\mathbb{R})$  when  $C_{Z'_\lambda}(0, [0, 1]) = C_{X'_\lambda}(0, [0, 1])$  is uniformly bounded on  $\lambda$  and  $g'' * \check{g}''(0) = \int_{\mathbb{R}} g''(s)^2 ds < +\infty$ , which implies that  $C_{B'_\lambda}(0, [0, 1]) < +\infty$ . This means that the convergence holds for almost every  $\alpha \in \mathbb{R}$ . Under additional assumptions we obtain the following uniform convergence. The proof is inspired by Theorem 2 p.516 of [13] concerning central limit theorem for densities.

**Theorem 3.** *Let us assume  $\mathbf{H}_s$  with  $\beta = 1$  a.s. Let us assume moreover that  $g$  is a function of class  $\mathcal{C}^4$  on  $\mathbb{R}$  such that for all  $s \in [-1, 2]$ ,  $\Phi(s) = \begin{pmatrix} g'(s) & g''(s) \\ g''(s) & g^{(3)}(s) \end{pmatrix}$  and  $\Phi'(s) = \begin{pmatrix} g''(s) & g^{(3)}(s) \\ g^{(3)}(s) & g^{(4)}(s) \end{pmatrix}$  are invertible.*

*Let  $\gamma_\lambda = \#\{i; \tau_{\lambda,i} \in [-1, 1]\}$  with  $\{\tau_{\lambda,i}\}_i$  the points of a Poisson point process with intensity  $\lambda > 0$ . Then, for all  $\alpha \in \mathbb{R}$ , if  $Z_\lambda = (X_\lambda - \lambda \int_{\mathbb{R}} g)/\sqrt{\lambda}$ , we have*

$$C_{Z_\lambda}(\alpha, [0, 1] | \gamma_\lambda \geq \lambda) \xrightarrow{\lambda \rightarrow +\infty} C_B(\alpha, [0, 1]) = \frac{1}{\pi} \left( \frac{m_2}{m_0} \right)^{1/2} e^{-\alpha^2/2m_0}, \text{ uniformly in } \alpha \in \mathbb{R},$$

where  $m_0 = g * \check{g}(0)$  and  $m_2 = g' * \check{g}'(0)$ .

*Proof.* Let  $\lambda > 7$ . Since  $\lambda > 7$ , according to Proposition 12,  $\widehat{C_{Z_\lambda}}(u, [0, 1] | \gamma_\lambda \geq \lambda)$  and  $\widehat{C_B}(u, [0, 1])$  are integrable such that  $C_{Z_\lambda}(\alpha, [0, 1] | \gamma_\lambda \geq \lambda)$  and  $C_B(\alpha, [0, 1])$  are bounded continuous functions with, for any  $\alpha \in \mathbb{R}$ ,

$$|C_{Z_\lambda}(\alpha, [0, 1] | \gamma_\lambda \geq \lambda) - C_B(\alpha, [0, 1])| \leq \frac{1}{2\pi} \int_{\mathbb{R}} \left| \widehat{C_{Z_\lambda}}(u, [0, 1] | \gamma_\lambda \geq \lambda) - \widehat{C_B}(u, [0, 1]) \right| du.$$

Let  $u \in \mathbb{R}$ , then

$$\widehat{C_{Z_\lambda}}(u, [0, 1]) - \widehat{C_{Z_\lambda}}(u, [0, 1] | \gamma_\lambda \geq \lambda) = \frac{1}{\mathbb{P}(\gamma_\lambda \geq \lambda)} \mathbb{E} \left( e^{iuZ_\lambda(0)} |Z'_\lambda(0)| \mathbf{1}_{\gamma_\lambda < \lambda} \right) - \frac{\mathbb{P}(\gamma_\lambda < \lambda)}{\mathbb{P}(\gamma_\lambda \geq \lambda)} \widehat{C_{Z_\lambda}}(u, [0, 1])$$

Note that  $\left| \widehat{C_{Z_\lambda}}(u, [0, 1]) \right| \leq \mathbb{E}(|Z'_\lambda(0)|)$ , which is bounded according to Corollary 3, while by Cauchy-Schwarz inequality,

$$\left| \mathbb{E} \left( e^{iuZ_\lambda(0)} |Z'_\lambda(0)| \mathbf{1}_{\gamma_\lambda < \lambda} \right) \right| \leq \mathbb{E} (Z'_\lambda(0)^2)^{1/2} \mathbb{P}(\gamma_\lambda < \lambda)^{1/2},$$

with  $\mathbb{E} (Z'_\lambda(0)^2) = \text{Var}(Z'_\lambda(0)) \leq \max(1, \|g'\|_\infty) \|g'\|_1$ . Therefore, one can find  $c_1 > 0$  such that

$$\left| \widehat{C_{Z_\lambda}}(u, [0, 1]) - \widehat{C_{Z_\lambda}}(u, [0, 1] | \gamma_\lambda \geq \lambda) \right| \leq c_1 \frac{\mathbb{P}(\gamma_\lambda < \lambda)^{1/2}}{\mathbb{P}(\gamma_\lambda \geq \lambda)}.$$

According to Markov's inequality,

$$\mathbb{P}(\gamma_\lambda < \lambda) = \mathbb{P}\left(e^{-\ln(2)\gamma_\lambda} > e^{-\ln(2)\lambda}\right) \leq \mathbb{E}\left(e^{-\ln(2)(\gamma_\lambda - \lambda)}\right) = \exp(-(1 - \ln(2))\lambda).$$

Choosing  $\lambda$  large enough such that in particular  $\frac{\mathbb{P}(\gamma_\lambda < \lambda)^{1/2}}{\mathbb{P}(\gamma_\lambda \geq \lambda)} \leq \frac{1}{\sqrt{\lambda}}$ , according to Proposition 17, one can find  $c_2$  such that for all  $|u| < \lambda^{1/8}$ ,

$$|\widehat{C_{Z_\lambda}}(u, [0, 1] | \gamma_\lambda) - \widehat{C_B}(u, [0, 1])| \leq c_2 \lambda^{-3/8}.$$

Thus we may conclude that

$$\int_{|u| \leq \lambda^{1/8}} \left| \widehat{C_{Z_\lambda}}(u, [0, 1] | \gamma_\lambda \geq \lambda) - \widehat{C_B}(u, [0, 1]) \right| du \xrightarrow{\lambda \rightarrow +\infty} 0.$$

Now, let us be concerned with the remaining integral for  $|u| \geq \lambda^{1/8}$ . According to Proposition 12,

$$\widehat{C_{Z_\lambda}}(u, [0, 1] | \gamma_\lambda \geq \lambda) = \frac{e^{-iu\sqrt{\lambda} \int_{\mathbb{R}} g}}{\sqrt{\lambda}} \widehat{C_{X_\lambda}}\left(\frac{u}{\sqrt{\lambda}}, [0, 1] | \gamma_\lambda \geq \lambda\right),$$

with  $\widehat{C_{X_\lambda}}\left(\frac{u}{\sqrt{\lambda}}, [0, 1] | \gamma_\lambda \geq \lambda\right) = \int_0^1 \mathbb{E}\left(e^{i\frac{u}{\sqrt{\lambda}} X_\lambda(t)} | X'_\lambda(t) | | \gamma_\lambda \geq \lambda\right) dt$  and

$$\mathbb{E}\left(e^{i\frac{u}{\sqrt{\lambda}} X_\lambda(t)} | X'_\lambda(t) | | \gamma_\lambda \geq \lambda\right) = -\frac{1}{\pi} \int_0^{+\infty} \frac{1}{v} \left( \frac{\partial \psi_{t,\lambda}}{\partial v} \left( \frac{u}{\sqrt{\lambda}}, \frac{v}{\sqrt{\lambda}} \right) - \frac{\partial \psi_{t,\lambda}}{\partial v} \left( \frac{u}{\sqrt{\lambda}}, -\frac{v}{\sqrt{\lambda}} \right) \right) dv,$$

where  $\psi_{t,\lambda}$  is the characteristic function of  $(X_\lambda(t), X'_\lambda(t))$ , conditionally on  $\{\gamma_\lambda \geq \lambda\}$ . According to (22) and (26), one can find a positive constant  $c_3 > 0$  such that

$$\begin{aligned} \left| \mathbb{E}\left(e^{i\frac{u}{\sqrt{\lambda}} X_\lambda(t)} | X'_\lambda(t) | | \gamma_\lambda \geq \lambda\right) \right| &\leq c_3 \lambda^2 \frac{\mathbb{P}(\gamma_\lambda \geq \lambda - 2)}{\mathbb{P}(\gamma_\lambda \geq \lambda)} \\ &\quad \times \int_{\mathbb{R}} \left| \chi_t \left( \frac{u}{\sqrt{\lambda}}, \frac{v}{\sqrt{\lambda}} \right) \right|^{\lambda-2} \left( \frac{1}{\sqrt{\lambda}} |\ln(|v|)| \mathbf{1}_{0 \leq |v| \leq 1} + |v|^{-1} \mathbf{1}_{|v| \geq 1} \right) dv, \end{aligned}$$

where  $\chi_{t,\lambda}(u, v) = \frac{1}{2} \int_{-1+t}^{1+t} e^{iug(s) + ivg'(s)} ds$ , is the characteristic function of  $(g(t-U), g'(t-U))$ , with  $U$  a uniform random variable on  $[-1, 1]$ . Then,

$$\begin{aligned} &\int_{|u| \geq \lambda^{1/8}} \left| \widehat{C_{Z_\lambda}}(u, [0, 1] | \gamma_\lambda \geq \lambda) - \widehat{C_B}(u, [0, 1]) \right| du \\ &\leq \int_{|u| \geq \lambda^{1/8}} \left| \widehat{C_{Z_\lambda}}(u, [0, 1] | \gamma_\lambda \geq \lambda) \right| du + \int_{|u| \geq \lambda^{1/8}} \left| \widehat{C_B}(u, [0, 1]) \right| du \\ &= I_1(\lambda) + I_2(\lambda). \end{aligned}$$

Now, for  $\theta \in [0, 2\pi]$ , let us consider the random variable  $V_{t,\theta} = \cos(\theta)g(t-U) + \sin(\theta)g'(t-U)$  such that for any  $r > 0$ ,  $\chi_t(r \cos(\theta), r \sin(\theta)) = \mathbb{E}(e^{irV_{t,\theta}}) := \varphi_{t,\theta}(r)$ . By a change of variables in polar coordinates, since  $\lambda > 1$ , we get

$$I_1(\lambda) \leq c_4(\lambda) \int_{\lambda^{1/8}}^{+\infty} \int_0^{2\pi} \left| \varphi_{t,\theta} \left( \frac{r}{\sqrt{\lambda}} \right) \right|^\lambda r |\ln(r |\sin(\theta)|)| d\theta dr,$$

with  $c_4(\lambda) = c_3 \lambda^{3/2} \frac{\mathbb{P}(\gamma_\lambda \geq \lambda - 2)}{\mathbb{P}(\gamma_\lambda \geq \lambda)}$ . Since  $\det \Phi(s) \neq 0$  for any  $s \in [-1+t, 1+t]$ , we have the following property (see [13] p.516): there exists  $\delta > 0$  such that

$$|\varphi_{t,\theta}(r)| \leq e^{-\frac{\kappa(t)}{4} r^2}, \quad \forall r \in (0, \delta], \forall \theta \in [0, 2\pi], \quad \text{and} \quad \eta = \sup_{r > \delta, \theta \in [0, 2\pi]} |\varphi_{t,\theta}(r)| < 1,$$

with  $\kappa(t) = \min_{\theta \in [0, 2\pi]} \text{Var}(V_{t,\theta}) > 0$ . Note also that according to Proposition 5,  $|\varphi_{t,\theta}(r)| \leq 24\sqrt{\frac{2}{m}} r^{-1/2}$  for any  $r > m$  with  $m = \min_{s \in [-1, 2]} \|\Phi(s)^{-1}\|^{-1}$ , which may be assumed to be greater than  $\delta$ . Then, for

$\lambda$  large enough such that  $\lambda^{1/8} \in (e, \delta\sqrt{\lambda})$ ,

$$I_1(\lambda) \leq c_5(\lambda) \left( \int_{\lambda^{1/8}}^{\delta\sqrt{\lambda}} e^{-\frac{\kappa(t)}{4}\lambda^{1/4}} r \ln(r) dr + \int_{\delta\sqrt{\lambda}}^{m\sqrt{\lambda}} \eta^\lambda r \ln(r) dr + \left(24\sqrt{\frac{2}{m}}\right)^5 \int_{m\sqrt{\lambda}}^{+\infty} \eta^{\lambda-5} r^{-3/2} \ln(r) dr \right)$$

with  $c_5(\lambda) = c_4(\lambda) \left( \int_0^{2\pi} |\ln(\theta)| d\theta \right)$ . This enables us to conclude that  $I_1(\lambda) \xrightarrow{\lambda \rightarrow +\infty} 0$ . This conclude the proof since clearly  $I_2(\lambda) \xrightarrow{\lambda \rightarrow +\infty} 0$ .  $\square$

Notice that to obtain the convergence in Theorem 3 without the conditioning on  $\{\gamma_\lambda \geq \lambda\}$  (which is an event of probability going to 1 exponentially fast as  $\lambda$  goes to infinity), one simply needs to have an upper-bound polynomial in  $\lambda$  on the second moment of the crossings number  $N_{Z_\lambda}(\alpha, [0, 1])$ .

## 6. THE GAUSSIAN KERNEL

In this section we will be interested in the particular case where a Gaussian kernel is used in the shot noise process  $X$  defined by Equation (1), and where  $\beta = 1$  a.s. In the following, we will denote by  $g_\sigma$  the Gaussian kernel of width  $\sigma$  defined for all  $t \in \mathbb{R}$  by

$$g_\sigma(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-t^2/2\sigma^2}.$$

And we will write  $X_\sigma$  for the shot noise process given by

$$X_\sigma(t) = \sum_i \frac{1}{\sigma\sqrt{2\pi}} e^{-(t-\tau_i)^2/2\sigma^2}, t \in \mathbb{R},$$

where the  $\{\tau_i\}$  are the points of a Poisson process of a fixed intensity  $\lambda > 0$  on the line.

**6.1. Moments, regularity and scaling properties.** According to the elementary properties of Section 2.1,  $X_\sigma(t)$  has moments of any order  $k$ . The first moments of  $X_\sigma(t)$  are

$$\mathbb{E}X_\sigma(t) = \lambda \int_{\mathbb{R}} g_\sigma(s) ds = \lambda \text{ and } \text{Var}X_\sigma(t) = \lambda \int_{\mathbb{R}} g_\sigma(s)^2 ds = \frac{\lambda}{2\sigma\sqrt{\pi}}.$$

And the covariance is:

$$\text{Cov}(X_\sigma(t), X_\sigma(t')) = \lambda \int_{\mathbb{R}} g_\sigma(t-s) g_\sigma(t'-s) ds = \frac{\lambda}{2\sigma\sqrt{\pi}} e^{-(t-t')^2/4\sigma^2}.$$

For  $k \geq 1$ , the  $k$ -th order cumulant of  $X_\sigma(t)$  is equal to

$$C_k = \lambda \int_{\mathbb{R}} g_\sigma(s)^k ds = \frac{\lambda}{\sqrt{k}\sqrt{2\pi}^{k-1} \sigma^{k-1}}.$$

Moreover, applying Corollary 1, the process  $X_\sigma$  is almost surely smooth on  $\mathbb{R}$ . For any  $n \in \mathbb{N}$ , its derivative of order  $n$  is the shot noise process  $X_\sigma^{(n)}$  given by

$$X_\sigma^{(n)}(t) = \frac{d^n X_\sigma}{dt^n}(t) = \sum_i g_\sigma^{(n)}(t - \tau_i)$$

and it has also moments of any order  $k$ .

One of the main features of the shot noise process  $X_\sigma$  is that it can be seen in a “dynamic” way, which means that we can study how it evolves as the width  $\sigma$  of the Gaussian kernel changes and consider it as a random field indexed by the variable  $(\sigma, t)$ . Then, the main tool is the heat equation which is satisfied by the Gaussian kernel (this is the reason for which it is also called the heat kernel):

$$(28) \quad \forall \sigma > 0, \forall t \in \mathbb{R}, \quad \frac{\partial g_\sigma}{\partial \sigma}(t) = \sigma g_\sigma''(t) \text{ and also consequently } \frac{\partial g_\sigma'}{\partial \sigma}(t) = \sigma g_\sigma'''(t).$$

Since the Gaussian kernel  $g_\sigma$  is a very smooth function, both in  $\sigma > 0$  and  $t \in \mathbb{R}$ , by the same type of proof as the ones in Propositions 1 and 2, we have that  $(\sigma, t) \mapsto X_\sigma(t)$  is almost surely and mean square continuously twice differentiable on  $(0, +\infty) \times \mathbb{R}$  and that

$$(29) \quad \frac{\partial X_\sigma}{\partial \sigma}(t) = \sum_i \frac{\partial g_\sigma}{\partial \sigma}(t - \tau_i) = \sigma X''_\sigma(t).$$

We will see in the following that this equation will be of great interest to study the crossings of  $X_\sigma$ .

Another interesting property of the shot noise process with Gaussian kernel is that we have two scale parameters: the width  $\sigma$  of the Gaussian kernel, but also the intensity  $\lambda$  of the Poisson process. These two parameters are linked in the sense that changing one of them amounts to change the other one in an appropriate way. This is described more precisely in the following lemma.

**Lemma 1.** *For the purpose of this lemma, we write the shot noise  $X_\sigma^{[\lambda]}$  to recall the value  $\lambda$  of the intensity of the underlying Poisson point process. We then have the three following scaling properties:*

- (1) *Changing  $\sigma$  and  $\lambda$  in a proportional way: for all  $\alpha > 0$ ,*

$$\{X_{\alpha\sigma}^{[\lambda/\alpha]}(t); t \in \mathbb{R}\} \stackrel{fdd}{=} \left\{\frac{1}{\alpha}X_\sigma^{[\lambda]}(\frac{t}{\alpha}); t \in \mathbb{R}\right\}.$$

- (2) *Increasing the width of the Gaussian kernel: for all  $\sigma_1$  and  $\sigma_2$ , we have*

$$\{X_{\sqrt{\sigma_1^2 + \sigma_2^2}}^{[\lambda]}(t); t \in \mathbb{R}\} \stackrel{a.s.}{=} \{(X_{\sigma_1}^{[\lambda]} * g_{\sigma_2})(t); t \in \mathbb{R}\}.$$

- (3) *Increasing the intensity of the Poisson process: for all  $\alpha > 0$ , we have*

$$\{X_\sigma^{[\lambda\sqrt{1+\alpha^2}]}(t); t \in \mathbb{R}\} \stackrel{fdd}{=} \{\sqrt{1+\alpha^2} \cdot (X_\sigma^{[\lambda]} * g_{\alpha\sigma})(t\sqrt{1+\alpha^2}); t \in \mathbb{R}\}.$$

*Proof.* For the first property, let  $\{\tau_i\}$  be a Poisson point process of intensity  $\lambda/\alpha$  on the line. Then

$$X_{\alpha\sigma}^{[\lambda/\alpha]}(t) = \sum_i \frac{1}{\alpha\sigma\sqrt{2\pi}} e^{-(t-\tau_i)^2/2\alpha^2\sigma^2} = \frac{1}{\alpha} \sum_i g_\sigma\left(\frac{t}{\alpha} - \frac{\tau_i}{\alpha}\right).$$

Since the points  $\{\tau_i/\alpha\}$  are now the points of a Poisson process on intensity  $\lambda$  on the line, we obtain the first scaling property. The second property comes simply from the fact that if  $g_{\sigma_1}$  and  $g_{\sigma_2}$  are two Gaussian kernels of respective width  $\sigma_1$  and  $\sigma_2$ , then their convolution is the Gaussian kernel of width  $\sqrt{\sigma_1^2 + \sigma_2^2}$ . The third property is just a consequence of combining the first and second properties.  $\square$

These scaling properties are illustrated on Figure 2, where we show two different ways to obtain a sample of  $t \mapsto X_1^{[1]}(t)$  on an interval  $[-T, T]$  with  $T > 0$ . The first way (top right figure) is to directly sample a Poisson point process of intensity  $\lambda = 1$  on the line according to Section 2.2. The second way (bottom figure) consists in considering a sample of  $X_1^{[0.5]}$  on  $[-2T, 2T]$  (thin line), first convolving it with  $g_{\sqrt{3}}$ , then multiplying it by a factor 2 (thick line), and finally scaling time by a factor 2. Notice also that thanks to the divisibility property of the Poisson point process, we could also obtain a sample of  $t \mapsto X_1^{[1]}(t)$  by adding to independent samples of  $t \mapsto X_1^{[0.5]}(t)$ .

**6.2. Crossings and their properties.** In the following, we assume that the intensity  $\lambda$  of the Poisson point process and the width  $\sigma > 0$  are fixed. The shot noise with kernel  $g_\sigma$  is still denoted by  $X_\sigma$ .

In this section, we will be interested in the crossings and in the local extrema of both  $X_\sigma$ , and the truncated shot noise process  $X_{\sigma,M}$  defined for all real  $M > 0$  by

$$X_{\sigma,M}(t) = \sum_{\tau_i \in [-M, M]} g_\sigma(t - \tau_i), t \in \mathbb{R},$$

where, as before, the  $\{\tau_i\}$  are the points of a Poisson point process of intensity  $\lambda$  on  $\mathbb{R}$ . Notice that this truncated shot noise process is not stationary anymore. It can be seen as a shot noise process

with intensity  $\lambda \mathbf{1}_{[-M,M]}(s)ds$  on  $\mathbb{R}$ . Moreover, we have that  $X_{\sigma,M}$  converges almost surely uniformly to  $X_\sigma$  on any interval  $[a,b]$  as  $M$  goes to infinity. Indeed, we have for  $M > \max(|a|, |b|)$ :

$$\forall t \in [a,b], \quad |X_{\sigma,M}(t) - X_\sigma(t)| \leq \frac{1}{\sigma\sqrt{2\pi}} \sum_{\tau_i > M} e^{-(\tau_i - b)^2/2\sigma^2} + \frac{1}{\sigma\sqrt{2\pi}} \sum_{\tau_i < -M} e^{-(\tau_i - a)^2/2\sigma^2}.$$

The right-hand term is a random variable that converges almost surely to 0 as  $M$  goes to infinity, which proves the almost sure uniform convergence of  $X_{\sigma,M}$  to  $X_\sigma$  on  $[a,b]$ . Notice that the same kind of inequality can also be obtained for the convergence of the derivatives of  $X_{\sigma,M}$  to the ones of  $X_\sigma$ . The interest of the truncated process  $X_{\sigma,M}$  is twofold: *i)* Since the number of terms in the sum defining  $X_{\sigma,M}$  is almost surely finite, it is easier to compute explicit bounds on its number of crossings. *ii)* The truncated process is the one that is used in simulations (see also Section 2.2 on the simulation procedure), and also in practical applications.

Since the Gaussian kernel  $g_\sigma$ , and its derivatives are smooth functions which belong to  $L^\infty$ ,  $L^1$  and  $L^2$ , many results of the previous sections about crossings can be applied here. In particular, we have:

- The function  $\alpha \mapsto C_{X_\sigma}(\alpha, [a,b])$  belongs to  $L^1(\mathbb{R})$  (by Proposition 9).

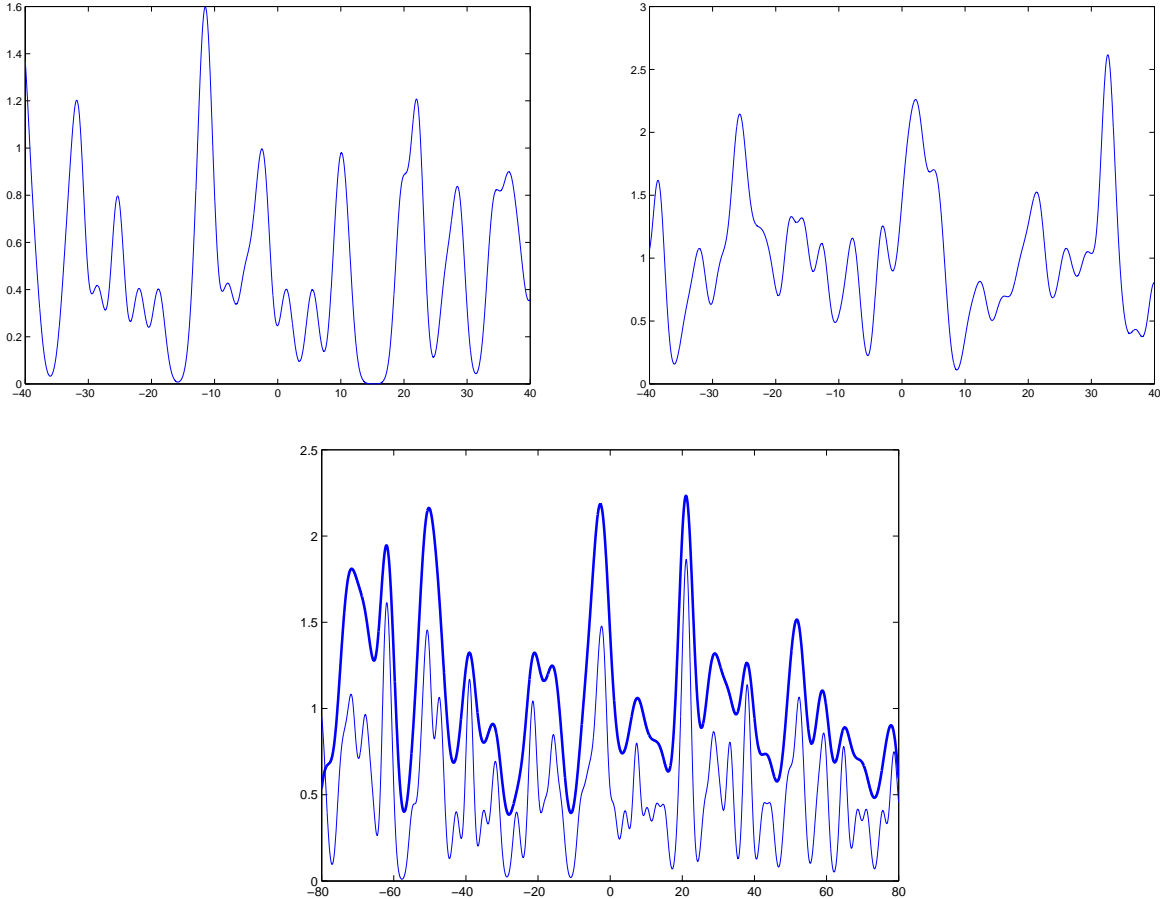


FIGURE 2. Illustration of the scaling properties. Top: two samples of a shot noise process on the interval  $[-40, 40]$  obtained with a Gaussian kernel of width  $\sigma = 1$  and intensity of the Poisson point process  $\lambda = 0.5$  on the left and  $\lambda = 1$  on the right. Bottom: according to the third scaling property (used here with  $\alpha = \sqrt{3}$ , and thus  $\sqrt{1 + \alpha^2} = 2$ ), a sample of  $X_1^{[1]}$  on an interval  $[-T, T]$  can be obtained from a sample of  $X_1^{[0.5]}$  on  $[-2T, 2T]$  (thin line) by first convolving it with  $g_{\sqrt{3}}$ , then multiplying it by a factor 2 (thick line), and finally scaling time by a factor 2.

- For any  $T > 0$ , the function  $\alpha \mapsto C_{X_\sigma}(\alpha, [a, b] | \gamma_T \geq 8)$  is continuous (by Theorem 1), with  $\gamma_T = \#\{\tau_i \in [-T, T]\}$ .
- If  $(\sigma_n)_{n \in \mathbb{N}}$  is a sequence that converges to  $\sigma > 0$ , then  $\widehat{C_{X_{\sigma_n}}}(u, [a, b])$  converges uniformly to  $\widehat{C_{X_\sigma}}(u, [a, b])$  as  $n$  goes to infinity (by Proposition 16). And consequently, by Proposition 14,  $C_{X_{\sigma_n}}(\cdot, [a, b])$  converges weakly to  $C_{X_\sigma}(\cdot, [a, b])$ .

The second point comes from the fact that the Gaussian kernel satisfies the hypothesis of Theorem 1. Indeed, the derivatives of  $g_\sigma$  are given by  $g_\sigma^{(k)}(s) = \frac{1}{\sigma\sqrt{2\pi}} e^{-s^2/2\sigma^2} \cdot \frac{(-1)^k}{\sigma^k} H_k(\frac{s}{\sigma})$ , where the  $H_k$ 's are the Hermite polynomials ( $H_1(x) = x$ ;  $H_2(x) = x^2 - 1$ ;  $H_3(x) = x^3 - 3x$  and  $H_4(x) = x^4 - 6x^2 + 3$ ). Thus, if we denote  $\Phi(s) = \begin{pmatrix} g'_\sigma(s) & g''_\sigma(s) \\ g''_\sigma(s) & g^{(3)}_\sigma(s) \end{pmatrix}$  and  $\Phi'(s) = \begin{pmatrix} g''_\sigma(s) & g^{(3)}_\sigma(s) \\ g^{(3)}_\sigma(s) & g^{(4)}_\sigma(s) \end{pmatrix}$ , we get  $\det \Phi(s) = \left(\frac{1}{\sigma\sqrt{2\pi}} e^{-s^2/2\sigma^2}\right)^2 \cdot \frac{-1}{\sigma^4} \left(\frac{s^2}{\sigma^2} + 1\right) < 0$  and  $\det \Phi'(s) = \left(\frac{1}{\sigma\sqrt{2\pi}} e^{-s^2/2\sigma^2}\right)^2 \cdot \frac{-1}{\sigma^6} \left(\frac{s^4}{\sigma^4} + 3\right) < 0$ . These two matrices are thus invertible for all  $s \in \mathbb{R}$ . The third point is a consequence of the convergence of  $g_{\sigma_n}$  to  $g_\sigma$ , and of  $g'_{\sigma_n}$  to  $g'_\sigma$  both in  $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ . Note also that these statements hold true for the truncated shot noise  $X_{\sigma, M}$ , except that the second point must be considered for  $T \leq M$ . In particular, for almost every  $\alpha \in \mathbb{R}$ , the expected numbers of crossings of the level  $\alpha$  by  $X_\sigma$  or  $X_{\sigma, M}$  are finite. But clearly  $\{\gamma_M = 0\} \subset \{\forall t \in [a, b], X_{\sigma, M}(t) = 0\}$ , such that  $C_{X_{\sigma, M}}(0, [a, b]) = +\infty$ . However, we will prove in the next section that, for every  $\alpha \in \mathbb{R}$ ,  $C_{X_\sigma}(\alpha, [a, b]) < +\infty$ , by considering the 0-crossings of the derivative  $X'_\sigma$ . Actually, according to Rolle's Theorem

$$N_{X_\sigma}(\alpha, [a, b]) \leq N_{X'_\sigma}(0, [a, b]) + 1.$$

Considering  $X_\sigma$  and  $X_{\sigma, M}$ , we will denote  $\rho(\sigma, [a, b])$  (respectively  $\rho_M(\sigma, [a, b] | \gamma_M \geq k)$ ) the mean number of local extrema of  $X_\sigma(t)$  (resp. of  $X_{\sigma, M}(t)$  conditionally on  $\gamma_M \geq k$  for  $k \geq 1$ ) in the interval  $[a, b]$ . Finally, when  $[a, b]$  is simply the interval  $[0, 1]$ , we simply denote  $\rho(\sigma)$  instead of  $\rho(\sigma, [0, 1])$ . It is the mean number of local extrema per unit length. The aim of Proposition 18 below will be to show that local extrema are exactly the points where the derivative vanishes, such that

$$\rho(\sigma, [a, b]) = \mathbb{E}(N_{X'_\sigma}(0, [a, b])) \text{ and } \rho_M(\sigma, [a, b] | \gamma_M \geq k) = \mathbb{E}(N_{X'_{\sigma, M}}(0, [a, b] | \gamma_M \geq k)).$$

In order to obtain this result, we would like to use a property similar to the one of Proposition 7. Now, as already noticed in Section 3.2, the characteristic function of the shot noise process  $X'_\sigma(t)$  is not integrable. However, the result can be obtained by conditioning.

**Proposition 18.** *Let  $a \leq b$ . For  $M > 1$ , let  $\gamma_M = \#\{\tau_i \in [-M, M]\}$ . Then,*

- $\mathbb{P}(\exists t \in [0, 1] \text{ such that } X'_\sigma(t) = 0 \text{ and } X''_\sigma(t) = 0) = 0.$
- $\mathbb{P}(\exists t \in [0, 1] \text{ such that } X'_{\sigma, M}(t) = 0 \text{ and } X''_{\sigma, M}(t) = 0 | \gamma_M \geq 1) = 0.$

*Proof.* We use Proposition 6 with the kernel function  $h = g'_\sigma$  on the interval  $[-T+1, T]$  for  $T > 0$ . For this function we can compute  $h'(s) = \frac{1}{\sigma^3\sqrt{2\pi}}(-1 + \frac{s^2}{\sigma^2})e^{-s^2/2\sigma^2}$  and  $h''(s) = \frac{1}{\sigma^4\sqrt{2\pi}}(3\frac{s}{\sigma} - \frac{s^3}{\sigma^3})e^{-s^2/2\sigma^2}$ , and thus  $n_0 = 3$  and  $m(\sigma, T) = \min_{s \in [-T, T+1]} \sqrt{h'(s)^2 + h''(s)^2} > 0$  (we don't need to have an exact value for it but notice that it is of the order of  $e^{-T^2/2\sigma^2}$  when  $T$  is large). Finally, as in the proof of Proposition 6, we get that there is a constant  $c(T, \sigma)$  which depends continuously on  $\sigma$  and  $T$  such that

$$|\mathbb{E}(e^{iuX'_\sigma(t)} | \gamma_T \geq 3)| \leq \frac{1}{\mathbb{P}(\gamma_T \geq 3)} \frac{\lambda^3 c(T, \sigma)^3}{(1 + \sqrt{|u|})^3},$$

with  $\gamma_T = \#\{\tau_i \in [-T, T]\}$ . We can now use Proposition 7 and we get that for all  $T > 1$ :

$$\mathbb{P}(\exists t \in [0, 1] \text{ such that } X'_\sigma(t) = 0 \text{ and } X''_\sigma(t) = 0 | \gamma_T \geq 3) = 0.$$

Since the events  $\{\gamma_T \geq 3\}$  are an increasing sequence of events and that  $\mathbb{P}(\gamma_T \geq 3)$  goes to 1 as  $T$  goes to infinity, we obtain that:

$$\mathbb{P}(\exists t \in [0, 1] \text{ such that } X'_\sigma(t) = 0 \text{ and } X''_\sigma(t) = 0) = 0.$$

For the second part of the proposition with the process  $X'_{\sigma,M}$ , we already have, thanks to the bound on the characteristic function of the truncated process, that

$$\mathbb{P}(\exists t \in [0, 1] \text{ such that } X'_{\sigma,M}(t) = 0 \text{ and } X''_{\sigma,M}(t) = 0 \mid \gamma_M \geq 3) = 0$$

Thus, we just need to see what happens when  $\gamma_M = 1$  and  $\gamma_M = 2$ .

When  $\gamma_M = 1$ , we just have  $X'_{\sigma,M}(t) = g'_\sigma(t - \tau_1)$  where  $\tau_1$  is a point uniformly distributed on  $[-M, M]$ . Thus, since  $g'_\sigma(s) = \frac{-s}{\sigma^3\sqrt{2\pi}}e^{-s^2/2\sigma^2}$  and  $g''_\sigma(s) = \frac{s^2 - \sigma^2}{\sigma^5\sqrt{2\pi}}e^{-s^2/2\sigma^2}$ , we have  $X'_{\sigma,M}(t) = 0$  implies  $t = \tau_1$ , and consequently  $X''_{\sigma,M}(t) = g''_\sigma(t - \tau_1) \neq 0$ .

When  $\gamma_M = 2$ , we have  $X'_{\sigma,M}(t) = g'_\sigma(t - \tau_1) + g'_\sigma(t - \tau_2)$  where  $\tau_1$  and  $\tau_2$  are two independent points uniformly distributed on  $[-M, M]$ . We can again explicitly compute that the event  $\{X'_{\sigma,M}(t) = 0 \text{ and } X''_{\sigma,M}(t) = 0\}$  is an event of probability zero. We thus have proved that

$$\mathbb{P}(\exists t \in [0, 1] \text{ such that } X'_{\sigma,M}(t) = 0 \text{ and } X''_{\sigma,M}(t) = 0 \mid \gamma_M \geq 1) = 0.$$

Note that  $X_{\sigma,M} = 0$  when  $\gamma_M = 0$  such that  $\mathbb{P}(\exists t \in [0, 1] \text{ such that } X'_{\sigma,M}(t) = 0 \text{ and } X''_{\sigma,M}(t) = 0 \mid \gamma_M = 0) = 1$ . □

**6.3. Bounds on the number of local extrema.** To obtain bounds on the number of local extrema, we will first start with the truncated process, which is, even if it is not stationary, easier to handle with since it is a finite sum. Indeed, we have the following result:

**Proposition 19** (Local extrema of the truncated process). *Let  $\gamma_M = \#\{\tau_i \in [-M, M]\}$ . Assume the event  $\gamma_M \geq 1$  holds, then the local extrema of  $X_\sigma^{(M)}$  are all in the interval  $[-M, M]$ , and moreover the random variable  $N_{X'_{\sigma,M}}(0, [-M, M])$  which counts the number of these local extrema is bounded from above:*

$$N_{X'_{\sigma,M}}(0, [-M, M]) \leq 2\gamma_M - 1 \text{ when } \gamma_M \geq 1 \text{ holds.}$$

Consequently, for all interval  $[a, b] \subset \mathbb{R}$ , we have

$$\rho_M(\sigma, [a, b] \mid \gamma_M \geq 1) \leq \rho_M(\sigma, [-M, M] \mid \gamma_M \geq 1) \leq 2\mathbb{E}(\gamma_M \mid \gamma_M \geq 1) - 1,$$

with  $\mathbb{E}(\gamma_M \mid \gamma_M \geq 1) = \frac{2\lambda M}{1 - e^{-2\lambda M}}$ .

*Proof.* If there are no points of the Poisson point process in  $[-M, M]$ , then the process  $X_{\sigma,M}$  is 0 everywhere - and talking about its local extrema is not interesting. This is why we assume that there is at least one point of the Poisson process in  $[-M, M]$  (which means that  $A_{M,1}$  holds). Then we have for all  $t \in \mathbb{R}$ ,  $X'_{\sigma,M}(t) = \frac{-1}{\sigma^3\sqrt{2\pi}} \sum_{\tau_i \in [-M, M]} (t - \tau_i)e^{-(t-\tau_i)^2/2\sigma^2}$ . This shows in particular that  $X'_{\sigma,M}(t) < 0$  for all  $t > M$  and  $X'_{\sigma,M}(t) > 0$  for all  $t < -M$ . And thus all the local extrema of  $X'_{\sigma,M}$  are in the interval  $[-M, M]$ .

Notice that we can write for all  $t \in \mathbb{R}$ ,

$$X'_{\sigma,M}(t) = \frac{-1}{\sigma^3\sqrt{2\pi}} \sum_{\tau_i \in [-M, M]} (t - \tau_i)e^{-(t-\tau_i)^2/2\sigma^2} = \frac{-e^{-t^2/2\sigma^2}}{\sigma^3\sqrt{2\pi}} \sum_{\tau_i \in [-M, M]} e^{-\tau_i^2/2\sigma^2} (t - \tau_i)e^{\tau_i t/\sigma^2}.$$

Then, we use the following lemma:

**Lemma 2.** *Let  $n \geq 1$  be an integer. Let  $P_1, \dots, P_n$  be  $n$  real non-zero polynomials and let  $a_1, \dots, a_n$  be  $n$  real numbers, then*

$$\#\{t \in \mathbb{R} \text{ such that } \sum_{i=1}^n P_i(t)e^{a_i t} = 0\} \leq \sum_{i=1}^n \deg(P_i) + n - 1.$$

This elementary result can be proved by induction on  $n$ . For  $n = 1$ , it is obviously true. Assume the result holds for  $n \geq 1$ , then we prove it for  $n + 1$  in the following way. For  $t \in \mathbb{R}$ ,  $\sum_{i=1}^{n+1} P_i(t)e^{a_i t} = 0 \iff f(t) := P_{n+1}(t) + \sum_{i=1}^n P_i(t)e^{(a_i - a_{n+1})t} = 0$ . Let  $k$  denote the degree of  $P_{n+1}$ . Thanks to Rolle's Theorem, we have that  $N_f(0, \mathbb{R}) \leq N_{f'}(0, \mathbb{R}) + 1 \leq N_{f''}(0, \mathbb{R}) + 2 \leq \dots \leq N_{f^{(k+1)}}(0, \mathbb{R}) + k + 1$ .

But  $f^{(k+1)}$  can be written as  $f^{(k+1)}(t) = \sum_{i=1}^n Q_i(t)e^{(a_i - a_{n+1})t}$ , where the  $Q_i$  are polynomials of degree  $\deg(Q_i) \leq \deg(P_i)$ . Thus by induction  $N_{f^{(k+1)}}(0, \mathbb{R}) \leq \sum_{i=1}^n \deg(P_i) + n - 1$ , and then  $N_f(0, \mathbb{R}) \leq \sum_{i=1}^n \deg(P_i) + n - 1 + k + 1 \leq \sum_{i=1}^{n+1} \deg(P_i) + n$ . This proves the result for  $n + 1$ .

Back to our process  $X'_{\sigma, M}$ , using the above lemma with polynomials of degree 1, we get  $N_{X'_{\sigma, M}}(0, [-M, M]) \leq 2\gamma_M - 1$  (still assume that  $\gamma_M \geq 1$ ). Taking the conditional expectation concludes the proof.  $\square$

Having bounds on the number of local extrema of the truncated process will now allow us to obtain bounds for the shot noise process itself:

**Proposition 20.** *The mean number of local extrema of  $X_\sigma$  per unit length is finite, and more precisely we have the following bounds:*

$$\forall \alpha \in \mathbb{R}, \quad C_{X_\sigma}(\alpha, [0, 1]) \leq \rho(\sigma) \leq (3\lambda(2 + 2\sigma) + 1)e^\lambda.$$

*Proof.* The left-hand inequality is simply a consequence of the stationarity of  $X_\sigma$  and of Rolle's Theorem which states that between any two crossings of a level  $\alpha$  by a real-valued smooth function, there exist a zero of its derivative.

To obtain the right-hand inequality (the bound on  $\rho(\sigma)$ ), we will apply Proposition 10 to the process  $X'_\sigma$  for the crossings of the level 0 on the interval  $[0, 1]$ . We already know by Proposition 18 and Corollary 2 that the condition (12) is satisfied by  $X'_\sigma$ . Then we write for all  $t \in [0, 1]$ :

$$X'_\sigma(t) = \sum_{\tau_i \in \mathbb{R}} g'_\sigma(t - \tau_i) = \frac{1}{\sigma\sqrt{2\pi}} \sum_{\tau_i \in [-\sigma, 1+\sigma]} \frac{-(t - \tau_i)}{\sigma^2} e^{-(t - \tau_i)^2/2\sigma^2} + \frac{1}{\sigma\sqrt{2\pi}} \sum_{\tau_i \in \mathbb{R} \setminus [-\sigma, 1+\sigma]} \frac{-(t - \tau_i)}{\sigma^2} e^{-(t - \tau_i)^2/2\sigma^2}$$

Let  $Y_1(t)$  (resp.  $Y_2(t)$ ) denote the first (resp. second) term. We then have

$$Y_2'(t) = \frac{1}{\sigma\sqrt{2\pi}} \sum_{\tau_i \in \mathbb{R} \setminus [-\sigma, 1+\sigma]} \left( \frac{(t - \tau_i)^2}{\sigma^4} - \frac{1}{\sigma^2} \right) e^{-(t - \tau_i)^2/2\sigma^2}.$$

Since  $(t - \tau_i)^2 > \sigma^2$  for all  $t \in [0, 1]$  and all  $\tau_i \in \mathbb{R} \setminus [-\sigma, 1 + \sigma]$ , we get  $Y_2'(t) > 0$  on  $[0, 1]$  and thus  $N_{Y_2'}(0, [0, 1]) = 0$  a.s. Note that when the event  $\#\{\tau_i \in [-\sigma, 1 + \sigma]\} = 0$  holds, then  $X'_\sigma = Y_2$  such that  $N_{X'_\sigma}(0, [0, 1]) \leq 1$ . On the other hand, let us work conditionally on  $\#\{\tau_i \in [-\sigma, 1 + \sigma]\} \geq 1$ . The probability of this event is  $1 - e^{-\lambda(1+2\sigma)}$ . Thanks to Lemma 2, we know that  $N_{Y_1'}(0, [0, 1]) \leq 3\#\{\tau_i \in [-\sigma, 1 + \sigma]\} - 1$  such that  $\mathbb{E}(N_{Y_1'}(0, [0, 1]) | \#\{\tau_i \in [-\sigma, 1 + \sigma]\} \geq 1) \leq 3\lambda(1 + 2\sigma)/(1 - e^{-\lambda(1+2\sigma)}) - 1$ . To use Proposition 10, we need to obtain uniform bounds on the law of  $Y_1(t)$  and of  $Y_2(t)$  when  $t \in [0, 1]$ . As in the notations of the proposition, we will denote these constants by  $C_1$  and  $C_2$ . Let us start with  $Y_1$ . Let  $U$  be a random variable following the uniform distribution on  $[-1 - \sigma, 1 + \sigma]$ . For  $t \in [0, 1]$ , we can write  $U$  as  $U = \eta_t U_t + (1 - \eta_t) V_t$ , where  $U_t$  is uniform on  $[-1 - \sigma + t, \sigma + t]$ ,  $V_t$  is uniform on  $[-1 - \sigma, -1 - \sigma + t] \cup [\sigma + t, \sigma + 1]$  and  $\eta_t$  is an independent Bernoulli random variable with parameter  $\frac{1+2\sigma}{2+2\sigma}$ . We then have  $g'_\sigma(U) = \eta_t g'_\sigma(U_t) + (1 - \eta_t) g'_\sigma(V_t)$ . Thus the law of  $g'_\sigma(U)$  is the mixture of the law of  $g'_\sigma(U_t)$  and of the one of  $g'_\sigma(V_t)$ , with respective weights  $\frac{1+2\sigma}{2+2\sigma}$  and  $1 - \frac{1+2\sigma}{2+2\sigma}$ . Consequently

$$\forall t \in [0, 1], \forall x \in \mathbb{R}, \quad dP_{g'_\sigma(U_t)}(x) \leq \frac{2 + 2\sigma}{1 + 2\sigma} dP_{g'_\sigma(U)}(x).$$

The law of  $Y_1(t)$  conditionally on  $\#\{\tau_i \in [-\sigma, 1 + \sigma]\} \geq 1$  can be written as

$$dP_{Y_1(t)}(x) = \frac{1}{1 - e^{-\lambda(1+2\sigma)}} \sum_{k=1}^{+\infty} e^{-\lambda(1+2\sigma)} \frac{(\lambda(1 + 2\sigma))^k}{k!} (dP_{g'_\sigma(U_t)} * \dots * dP_{g'_\sigma(U_t)})(x).$$

Thus, if we write  $f_0 = dP_{g'_\sigma(U)}$ , we get

$$dP_{Y_1(t)}(x) \leq \frac{1}{1 - e^{-\lambda(1+2\sigma)}} \sum_{k=1}^{+\infty} e^{-\lambda(1+2\sigma)} \frac{(\lambda(1 + 2\sigma))^k}{k!} \left( \frac{2 + 2\sigma}{1 + 2\sigma} \right)^k (f_0 * \dots * f_0)(x) = e^\lambda \frac{1 - e^{-\lambda(2+2\sigma)}}{1 - e^{-\lambda(1+2\sigma)}} \tilde{f}_0(x),$$

where  $\tilde{f}_0$  is a probability measure on  $\mathbb{R}$ . This shows that we can take  $C_1 = e^\lambda \frac{1 - e^{-\lambda(2+2\sigma)}}{1 - e^{-\lambda(1+2\sigma)}}$ .



For  $Y_2(t)$ , we first notice that  $Y_2(t)$  can be decomposed as the sum of two independent random variables in the following way:

$$Y_2(t) = \sum_{\tau_i \in (-\infty, -1-\sigma+t] \cup [1+\sigma+t, +\infty)} g'_\sigma(t - \tau_i) + \sum_{\tau_i \in (-\sigma-1+t, -\sigma) \cup (1+\sigma, 1+\sigma+t)} g'_\sigma(t - \tau_i).$$

The first random variable in the sum above has a law that does not depend on  $t$ . For the second random variable, using the same trick as above (*i.e.* decompose here a uniform random variable on the interval  $(-1-\sigma, -\sigma) \cup (\sigma, 1+\sigma)$  as a mixture with weights  $1/2$  and  $1/2$  of two uniform random variables: one on  $(-1-\sigma, -1-\sigma+t) \cup (t+\sigma, 1+\sigma)$ , and the other one on the rest), we obtain that  $C_2 = e^\lambda$ .

And finally the bound on the expectation of the number of local extrema is

$$\begin{aligned} \rho(\sigma) &\leq \left( C_1 \frac{3\lambda(1+2\sigma)}{1 - e^{-\lambda(1+2\sigma)}} + C_2 \right) (1 - e^{-\lambda(1+2\sigma)}) + e^{-\lambda(1+2\sigma)} \\ &\leq e^\lambda \frac{2+2\sigma}{1+2\sigma} (3\lambda(1+2\sigma)) + e^\lambda = (3\lambda(2+2\sigma) + 1)e^\lambda. \end{aligned}$$

□

We already saw that the truncated process uniformly converges almost surely on any fixed interval  $[a, b]$  to the shot noise process. We will now show that there is also convergence for the number of crossings, more precisely we have:

**Proposition 21.** *For any interval  $[a, b]$ , the crossings of  $X_{\sigma, M}$  on  $[a, b]$  converges almost surely to the ones of  $X_\sigma$ :*

$$\forall a < b, \forall \alpha \in \mathbb{R}, \quad N_{X_{\sigma, M}}(\alpha, [a, b]) \xrightarrow{a.s.} N_{X_\sigma}(\alpha, [a, b]) \text{ when } M \rightarrow \infty.$$

Moreover, there is also convergence of the mean of the number of crossings or of local extrema, conditionally on  $\gamma_T \geq 8$  for any  $T > 0$ :  $\forall a < b, \forall \alpha \in \mathbb{R}$ ,

$$C_{X_{\sigma, M}}(\alpha, [a, b] | \gamma_T \geq 8) \xrightarrow{M \rightarrow \infty} C_{X_\sigma}(\alpha, [a, b] | \gamma_T \geq 8) \text{ and } \rho_M(\sigma, [a, b] | \gamma_T \geq 8) \xrightarrow{M \rightarrow \infty} \rho(\sigma, [a, b] | \gamma_T \geq 8).$$

*Proof.* The first part of the proposition is a consequence of the fact that  $X_{\sigma, M}$  and its derivative  $X'_{\sigma, M}$  both converge uniformly on  $[a, b]$  to respectively  $X_\sigma$  and  $X'_\sigma$  almost surely. Thus, for a given  $\alpha \in \mathbb{R}$ , if we denote  $t_1, \dots, t_k$  the points in  $[a, b]$  such that  $X_\sigma(t) = \alpha$ , we may assume that for all  $j$ ,  $t_j \in (a, b)$  and  $X'_\sigma(t_j) \neq 0$  (this comes from the fact that  $X_\sigma(t)$  has a density, and that according to Proposition 18 there are no tangencies). Let  $\varepsilon > 0$  be small enough in such a way that: for all  $j$ ,  $|X'_\sigma(t)| > 0$  on  $[t_j - \varepsilon, t_j + \varepsilon]$ . Then let  $\eta > 0$  be such that  $|X'_\sigma(t)| \geq \eta$  for  $t \in \cup [t_j - \varepsilon, t_j + \varepsilon]$ , and  $|X_\sigma(t) - \alpha| \geq \eta$  for  $t \notin \cup [t_j - \varepsilon, t_j + \varepsilon]$ . Then, using the uniform convergence of  $X_{\sigma, M}$  and of its derivative  $X'_{\sigma, M}$ , there exists  $M_0$  such that for all  $M \geq M_0$ , for all  $t \in [a, b]$ ,  $|X_{\sigma, M}(t) - X_\sigma(t)| \leq \eta/2$  and  $|X'_{\sigma, M}(t) - X'_\sigma(t)| \leq \eta/2$ . Finally, for all  $M \geq M_0$ , we get that there exists exactly one crossing of the level  $\alpha$  in each interval  $[t_j - \varepsilon, t_j + \varepsilon]$ , and  $|X_{\sigma, M}(t) - \alpha| \geq \eta/2 > 0$  for  $t \notin \cup [t_j - \varepsilon, t_j + \varepsilon]$ . This proves that for  $M \geq M_0$ ,  $N_{X_{\sigma, M}}(\alpha, [a, b]) = k = N_{X_\sigma}(\alpha, [a, b])$ .

For the second part of the proposition, a first idea would be to use the first part and try to apply dominated convergence to obtain the convergence of means. Unfortunately, we have no bound independent of  $M$  on  $N_{X_{\sigma, M}}$ . But instead we can use dominated convergence in Rice's formula. Indeed, the characteristic function of  $(X_{\sigma, M}, X'_{\sigma, M})$  converges pointwise to the one of  $(X_\sigma, X'_\sigma)$  as  $M$  goes to infinity, and moreover, conditionally on  $\{\gamma_T \geq 8\}$ , following the proofs of Proposition 11 and of Theorem 1, we can bound them (and their partial derivatives) by a same integrable function independent of  $M$ . Thus, by Rice's formula (19) and dominated convergence, we get that for any  $\alpha$ ,  $C_{X_{\sigma, M}}(\alpha, [a, b] | \gamma_T \geq 8)$  converges to  $C_{X_\sigma}(\alpha, [a, b] | \gamma_T \geq 8)$  as  $M$  goes to infinity. The same kind of computation can be made for  $X'_{\sigma, M}$  and  $X'_\sigma$ , which provides the convergence of the mean number of local extrema conditionally on  $\{\gamma_T \geq 8\}$ .

□

Let us now be concerned with the influence of  $\sigma$  and consider  $X_\sigma$  as a random field of variables  $(\sigma, t) \in (0, +\infty) \times \mathbb{R}$ .

**6.4. Heat equation and local extrema.** The convolution of a 1d function with a Gaussian kernel of increasing width  $\sigma$  (which amounts to apply the heat equation) is a very common smoothing technique in signal processing. One of its main property is generally formulated by the wide-spread idea that “Gaussian convolution in 1d can not create new extrema” (and it is in some sense the only kernel that has this property - see [30]). This has been studied (together with its extension in higher dimension) for applications in image processing by Lindeberg in [20], and also by other authors (for instance to study mixtures of Gaussian distributions as in [8] and [7]). However, in most cases, the correct mathematical framework for the validity of this property is not exactly stated. Thus we start this section with a proposition giving the conditions under which one can obtain properties for the zero-crossings of a function solution of the heat equation.

The following proposition is stated under a general form for a function  $h$  in the two variables  $\sigma$  and  $t$ . But we have to keep in mind that we will want to apply this to  $h(\sigma, t) = X'_\sigma(t)$  or  $h(\sigma, t) = X'_{\sigma, M}(t)$  to follow the local extrema of the truncated -or not- shot noise process when  $\sigma$  evolves.

**Proposition 22.** *Let  $\sigma_0 > 0$  and  $(\sigma, t) \mapsto h(\sigma, t)$  be a  $C^2$  function defined on  $(0, \sigma_0] \times [a, b]$ , which satisfies the heat equation:*

$$\forall (\sigma, t) \in (0, \sigma_0] \times \mathbb{R}, \quad \frac{\partial h}{\partial \sigma}(\sigma, t) = \sigma \frac{\partial^2 h}{\partial t^2}(\sigma, t).$$

We assume that

- (1) *There are no  $t \in [a, b]$  such that  $h(\sigma_0, t) = 0$  and  $\frac{\partial h}{\partial t}(\sigma_0, t) = 0$ ,*
- (2) *There are no  $(\sigma, t) \in (0, \sigma_0] \times [a, b]$  such that  $h(\sigma, t) = 0$  and  $\nabla h(\sigma, t) = 0$ .*

Then we have the following properties for the zero-crossings of  $h$ :

- *Global curves: If  $t_0 \in (a, b)$  is such that  $h(\sigma_0, t_0) = 0$ , there exists  $\sigma_0^- < \sigma_0$  and a maximal continuous path  $\sigma \mapsto \Gamma_{t_0}(\sigma)$  defined on  $(\sigma_0^-, \sigma_0]$  such that  $\Gamma_{t_0}(\sigma_0) = t_0$  and for all  $\sigma \in (\sigma_0^-, \sigma_0]$  we have  $h(\sigma, \Gamma_{t_0}(\sigma)) = 0$ . Moreover, if  $\Gamma_{t_0}(\sigma)$  stays within some compact set of  $\mathbb{R}$  for all  $\sigma$ , then  $\sigma_0^- = 0$ .*
- *Non-intersecting curves: If  $\tilde{t}_0 \neq t_0$  is another point in  $(a, b)$  such that  $h(\sigma_0, \tilde{t}_0) = 0$ , then for all  $\sigma \in (0, \sigma_0]$  we have  $\Gamma_{t_0}(\sigma) \neq \Gamma_{\tilde{t}_0}(\sigma)$ .*
- *Local description of the curves: If  $(\sigma_1, t_1) \in (0, \sigma_0] \times \mathbb{R}$  is such that  $h(\sigma_1, t_1) = 0$  then there exist: else a  $C^1$  function  $\eta$  defined on a neighborhood of  $\sigma_1$  and such that  $h(\sigma, \eta(\sigma)) = 0$  in this neighborhood of  $\sigma_1$ ; or a  $C^1$  function  $\xi$  defined on a neighborhood of  $t_1$  and such that  $h(\xi(t), t) = 0$  in this neighborhood of  $t_1$ , and moreover if  $\xi'(t_1) = 0$  then  $\xi''(t_1) < 0$  (it is a local maximum).*

*Proof.* Let  $(\sigma_0, t_0)$  be a point such that  $h(\sigma_0, t_0) = 0$ . By Assumption (1), we have that  $\frac{\partial h}{\partial t}(\sigma_0, t_0) \neq 0$ . Then, thanks to the implicit function theorem, there exist two open intervals  $I = (\sigma_0^-, \sigma_0^+)$  and  $J = (t_0^-, t_0^+)$  containing respectively  $\sigma_0$  and  $t_0$ , and a  $C^1$  function  $\eta : I \rightarrow J$  such that  $\eta(\sigma_0) = t_0$  and  $\forall (\sigma, t) \in I \times J$ ,  $h(\sigma, t) = 0 \Leftrightarrow t = \eta(\sigma)$ . Let us now denote  $\eta = \Gamma_{t_0}$ . We need to prove that we can take  $\sigma_0^- = 0$  when  $\Gamma_{t_0}$  remains bounded. Assume we cannot: the maximal interval on which  $\Gamma_{t_0}$  is defined is  $(\sigma_0^-, \sigma_0^+)$  with  $\sigma_0^- > 0$ . By assumption, there is an  $M_0 > 0$  such that for all  $\sigma \in (\sigma_0^-, \sigma_0^+)$ , then  $|\Gamma_{t_0}(\sigma)| \leq M_0$ . We can thus find a sub-sequence  $(\sigma_k)$  converging to  $\sigma_0^-$  as  $k$  goes to infinity and a point  $t_1 \in [-M_0, M_0]$  such that  $\Gamma_{t_0}(\sigma_k)$  goes to  $t_1$  as  $k$  goes to infinity. By continuity of  $h$ , we have  $h(\sigma_0^-, t_1) = 0$ . Now, we also have  $\frac{\partial h}{\partial t}(\sigma_0^-, t_1) = 0$ . Indeed, if it were  $\neq 0$ , we could again apply the implicit function theorem in the same way at the point  $(\sigma_0^-, t_1)$ , and get a contradiction with the maximality of  $I = (\sigma_0^-, \sigma_0^+)$ . Then, by Assumption (2), we have  $\frac{\partial h}{\partial \sigma}(\sigma_0^-, t_1) \neq 0$ . We can again apply the implicit function theorem, and we thus obtain that there exist two open intervals  $I_1 = (\sigma_1^-, \sigma_1^+)$  and  $J_1 = (t_1^-, t_1^+)$  containing respectively  $\sigma_0^-$  and  $t_1$ , and a  $C^1$  function  $\xi : J_1 \rightarrow I_1$  such that  $\xi(t_1) = \sigma_0^-$  and  $\forall (\sigma, t) \in I_1 \times J_1$ ,  $h(\sigma, t) = 0 \Leftrightarrow \sigma = \xi(t)$ . Moreover we can compute the derivatives of  $\xi$  at  $t_1$ . We start from the implicit definition of  $\xi$ :  $h(\xi(t), t) = 0$ . By differentiation, we get  $\xi'(t) \frac{\partial h}{\partial \sigma}(\xi(t), t) + \frac{\partial h}{\partial t}(\xi(t), t) = 0$ . Taking the value at  $t = t_1$ , we get  $\xi'(t_1) = 0$ . We can again differentiate, and find  $\xi''(t) \frac{\partial h}{\partial \sigma}(\xi(t), t) + \xi'(t)^2 \frac{\partial^2 h}{\partial \sigma^2}(\xi(t), t) + 2\xi'(t) \frac{\partial^2 h}{\partial \sigma \partial t}(\xi(t), t) + \frac{\partial^2 h}{\partial t^2}(\xi(t), t) = 0$ . Taking

again the value at  $t = t_1$ , we get

$$\xi''(t_1) = -\frac{1}{\xi(t_1)} = -\frac{1}{\sigma_0^-} < 0.$$

Thus it shows that  $\xi$  has a strict local maximum at  $t_1$ : there exist a neighborhood  $U_1$  of  $\sigma_0^- = \xi(t_1)$  and a neighborhood  $V_1$  of  $t_1$  such that for all points in  $U_1 \times V_1$ , then  $h(\sigma, t) = 0$  implies  $\sigma = \xi(t) \leq \xi(t_1) = \sigma_0^-$ , which is in contradiction with the definition of  $\Gamma_{t_0}$  on  $(\sigma_0^-, \sigma_0^+)$ .

Now assume that  $t_0$  and  $\tilde{t}_0$  are two points such that  $h(\sigma_0, t_0) = h(\sigma_0, \tilde{t}_0) = 0$  and such that there exists  $\sigma_1 < \sigma_0$  such that  $\Gamma_{t_0}(\sigma_1) = \Gamma_{\tilde{t}_0}(\sigma_1) = t_1$ . Then, if  $\frac{\partial h}{\partial t}(\sigma_1, t_1) \neq 0$ , the implicit function theorem implies that  $\Gamma_{t_0}(\sigma) = \Gamma_{\tilde{t}_0}(\sigma)$  for all  $\sigma \in [\sigma_1, \sigma_0]$  and in particular  $t_0 = \tilde{t}_0$ . But now, if  $\frac{\partial h}{\partial t}(\sigma_1, t_1) = 0$ , then as above this implies that  $\frac{\partial h}{\partial \sigma}(\sigma_1, t_1) \neq 0$  and using again the implicit function theorem, this would be in contradiction with the fact  $\Gamma_{t_0}(\sigma)$  is defined for  $\sigma \in [\sigma_1, \sigma_0]$ .  $\square$

The properties stated in Proposition 22 are illustrated on Figure 3, where the different types of curves formed by the set of points  $\{(t, \sigma) \in \mathbb{R}^2; h(\sigma, t) = 0\}$  are shown.

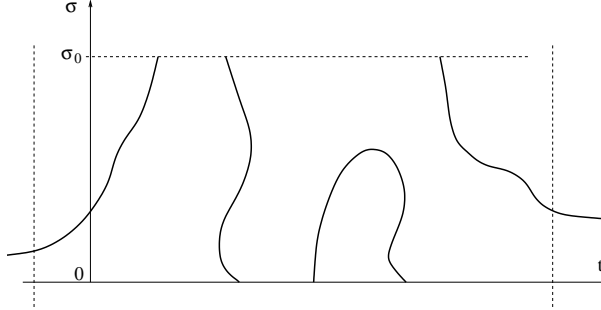


FIGURE 3. Curves of  $h(\sigma, t) = 0$  in the  $(t, \sigma)$  domain: here  $t$  is along the horizontal axis and  $\sigma$  is along the vertical one. According to Proposition 22, the zeros-crossings of  $h$  are a set of non-intersecting curves, that are locally else functions of  $\sigma$  or functions of  $t$  with no local minima.

Let us consider the Gaussian shot noise processes  $X'_\sigma$  and  $X'_{\sigma, M}$ . Both of them satisfies almost surely the heat equation. Then, we have to check the assumptions (1) and (2) of Proposition 22. Assumption (1) is given by Proposition 18. Assumption (2) is given by the following proposition, where  $\nabla X'_\sigma(t) := \nabla h(\sigma, t)$  for  $h(\sigma, t) = X'_\sigma(t)$  and similarly for  $\nabla X'_{\sigma, M}(t)$ .

**Proposition 23.** *Let  $\lambda > 0$ ,  $\sigma_0 > 0$  and  $a \leq b$ . Then,*

$$\mathbb{P}(\exists(\sigma, t) \in (0, \sigma_0) \times [a, b] \text{ such that } X'_\sigma(t) = 0 \text{ and } \nabla X'_\sigma(t) = 0) = 0.$$

For  $M > 1$ , let  $\gamma_M = \#\{\tau_i \in [-M, M]\}$ . Then we have

$$\mathbb{P}(\exists(\sigma, t) \in (0, \sigma_0) \times [a, b] \text{ such that } X'_{\sigma, M}(t) = 0 \text{ and } \nabla X'_{\sigma, M}(t) = 0 \mid \gamma_M \geq 3) = 0.$$

*Proof.* It is sufficient to remark that when  $Y(\sigma, t)$  satisfies the heat equation

$$\{Y(\sigma, t) = 0 \text{ and } \nabla Y(\sigma, t) = 0\} = \{Y(\sigma, t) = 0 \text{ and } Y'(\sigma, t) = 0 \text{ and } Y''(\sigma, t) = 0\}.$$

Then a slight modification of the proof of Proposition 7 allows us to conclude that  $\mathbb{P}(\exists(\sigma, t) \in (0, \sigma_0) \times [a, b] \text{ such that } Y(\sigma, t) = 0 \text{ and } \nabla Y(\sigma, t) = 0) = 0$ , under the same assumption of uniform integrability for the characteristic function of  $Y(\sigma, t)$ . Conditioning by  $\gamma_M \geq 3$  allows us to check the assumption for  $X'_{\sigma, M}$ , following Proposition 6. Then we let  $M \rightarrow +\infty$  to get the first statement.  $\square$

Let us now see how this applies for the truncated shot noise  $X'_{\sigma, M}$  before considering  $X'_\sigma$ .

**Proposition 24.** *For all  $M > 0$ , the function  $\sigma \mapsto \rho_M(\sigma, \mathbb{R} \mid \gamma_M \geq 3) = \rho_M(\sigma, [-M, M] \mid \gamma_M \geq 3)$  is decreasing. Moreover, it has the limit  $2\mathbb{E}(\gamma_M \mid \gamma_M \geq 3) - 1$  as  $\sigma$  goes to 0.*

*Proof.* Let  $M > 0$  and assume that  $\gamma_M \geq 3$  holds: there is at least three points of the Poisson point process of intensity  $\lambda > 0$  lying in the interval  $[-M, M]$ . We have:

$$\forall t \in \mathbb{R}, \forall \sigma > 0, \quad X'_{\sigma, M}(t) = \frac{1}{\sigma\sqrt{2\pi}} \sum_{\tau_i \in [-M, M]} \frac{-(t - \tau_i)}{\sigma^2} e^{-(t - \tau_i)^2 / 2\sigma^2}.$$

This shows that if we set  $Y(\sigma, t) = X'_{\sigma, M}(t)$ , then it cannot vanish when  $t$  is outside  $[-M, M]$ , and moreover all the conditions of Proposition 22 are satisfied. Thus, if we have  $\sigma_1 < \sigma_0$ , we can follow the zeros of  $Y(\sigma_0, \cdot)$  in an injective way from  $\sigma_0$  to  $\sigma_1$ . Thus  $\#\{t \in [-M, M] \text{ s. t. } Y(\sigma_0, t) = 0\} \leq \#\{t \in [-M, M] \text{ s. t. } Y(\sigma_1, t) = 0\}$ . Taking the expected value on both sides knowing that  $A_{M,3}$  holds proves the first part of the Proposition. For the second part: we first notice that for  $\sigma < \frac{1}{2} \min\{|\tau_i - \tau_j|; \tau_i, \tau_j \in [-M, M] \text{ and } i \neq j\}$ , then  $X''_{\sigma, M}(t) > 0$  for  $t \in [-M, M] \setminus \cup_i [\tau_i - \sigma, \tau_i + \sigma]$ . Moreover for  $\sigma$  small enough,  $X'_{\sigma, M}(t) \geq 1$  on each  $[\tau_i - \sigma, \tau_i - \sigma/2]$ ,  $X'_{\sigma, M}(t) \leq -1$  on each  $[\tau_i + \sigma/2, \tau_i + \sigma]$  and  $X''_{\sigma, M}(t) < 0$  on each  $[\tau_i - \sigma/2, \tau_i + \sigma/2]$ . Which shows that  $N_{X'_{\sigma, M}}(0, [-M, M])$  goes to  $2\gamma_M - 1$  as  $\sigma$  goes to 0. By Proposition 19, we also have that this limit is an upper-bound for  $N_{X'_{\sigma, M}}(0, [-M, M])$ . Thus by dominated convergence, this implies that  $\rho_M(\sigma, [-M, M] | \gamma_M \geq 3)$  goes to  $2\mathbb{E}(\gamma_M | \gamma_M \geq 3) - 1$  as  $\sigma$  goes to 0.  $\square$

We now give the main result for the number of local extrema of  $X_\sigma$  as a function of  $\sigma$ .

**Theorem 4.** Assume that for all  $0 < \sigma_1 < \sigma_0$

$$\mathbb{E}(\#\{\sigma \in [\sigma_1, \sigma_0] \text{ such that } X'_\sigma(0) = 0\}) < +\infty.$$

Then, the function  $\sigma \mapsto \rho(\sigma)$ , which gives the mean number of local extrema of  $X_\sigma$  per unit length, is decreasing. Moreover it has the limit  $2\lambda$  as  $\sigma$  goes to 0.

*Proof.* Let  $\lambda > 0$  and  $0 < \sigma_1 < \sigma_0$  be fixed. By assumption  $\mathbb{E}(\#\{\sigma \in [\sigma_1, \sigma_0] \text{ such that } X'_\sigma(0) = 0\}) < +\infty$ . Notice that by stationarity this expected value is independent of the value of  $t$  (taken as 0 above). Let  $T > 0$  and let us consider the zeros of  $Y(\sigma, t) = X'_\sigma(t)$  for  $(\sigma, t) \in [\sigma_1, \sigma_0] \times [0, T]$ . Let  $t_0 \in [0, T]$  be such that  $Y(\sigma_0, t_0) = 0$ . By Proposition 22, there is a continuous path  $\sigma \mapsto \Gamma_{t_0}(\sigma)$  that will: else “cross the left or right boundary of the domain”, i.e. be such that there exists  $\sigma \in$  such that  $\Gamma_{t_0}(\sigma) = 0$  or  $T$ , or will be defined until  $\sigma_1$  and such that  $\Gamma_{t_0}(\sigma_1) \in [0, T]$ . We thus have:

$$\rho(\sigma_0, [0, T]) \leq 2\mathbb{E}(\#\{\sigma \in [\sigma_1, \sigma_0] \text{ such that } X'_\sigma(0) = 0\}) + \rho(\sigma_1, [0, T]).$$

Dividing both sides by  $T$  and letting  $T$  go to infinity then shows that  $\rho(\sigma_0) \leq \rho(\sigma_1)$ .

For the second part of the Proposition, instead of looking at the local extrema of  $X_\sigma$  in  $[0, 1]$ , we will only look at the local maxima (which are the down-crossings of 0 by the derivative) in  $[0, 1]$ . Let  $D_{X'_\sigma}(0, [0, 1])$  be the random variable which counts these local maxima, and let  $\rho^-(\sigma) = \mathbb{E}(D_{X'_\sigma}(0, [0, 1]))$ . By stationarity of  $X_\sigma(t)$  and because between any two local maxima, there is a local minima, we have that  $\rho^-(\sigma) = \frac{1}{2}\rho(\sigma)$ . Now, we introduce “barriers” in the following way: let  $E_{\sigma_0}$  be the event “there are no points of the Poisson point process in the intervals  $[-2\sigma_0, 2\sigma_0]$  and  $[1-2\sigma_0, 1+2\sigma_0]$ ”. If we assume that  $E_{\sigma_0}$  holds, then  $X''_\sigma(t) > 0$  for all  $t$  in  $[-\sigma_0, \sigma_0] \cup [1-\sigma_0, 1+\sigma_0]$  and all  $\sigma \leq \sigma_0$ , and therefore there are no local maxima of  $X_\sigma$  in these intervals. Then by Proposition 22, we can follow all the local maxima of  $X_\sigma$  in  $[0, 1]$  from  $\sigma = \sigma_0$  down to  $\sigma = 0$ . Thus  $\sigma \mapsto D_{X'_\sigma}(0, [0, 1])\mathbf{1}_{E_{\sigma_0}}$  is decreasing function of  $\sigma$  for  $\sigma \leq \sigma_0$ . Moreover, by the same reasoning as in the proof of the previous Proposition (in the case of the truncated process), we also have that  $D_{X'_\sigma}(0, [0, 1])$  goes to  $\#\{\tau_i \in [0, 1]\}$  as  $\sigma$  goes to 0. Thus by monotone convergence, it implies that  $\rho^-(\sigma | E_{\sigma_0})$  goes to  $\mathbb{E}(\#\{\tau_i \in [0, 1]\} | E_{\sigma_0})$ . Since the sequence of events  $E_{\sigma_0}$  is an increasing sequence of events as  $\sigma_0$  decreases to 0, we finally get:

$$\lim_{\sigma \rightarrow 0} \rho^-(\sigma) = \lim_{\sigma_0 \rightarrow 0} \mathbb{E}(\#\{\tau_i \in [0, 1]\} | E_{\sigma_0}) = \lambda.$$

$\square$

The properties of the local extrema of  $X_\sigma$  are illustrated on Figures 4 and 5.

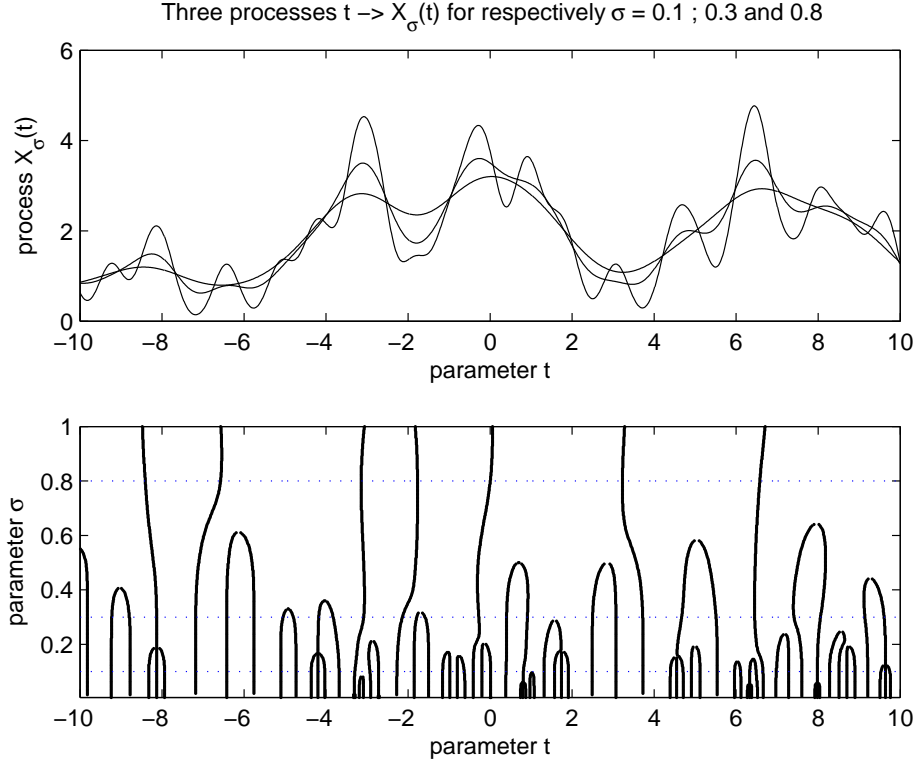


FIGURE 4. Top: three processes  $t \mapsto X_\sigma(t)$  obtained from the same Poisson point process of intensity  $\lambda = 2$  and for a Gaussian kernel of respective width  $\sigma = 0.1; 0.3$  and  $0.8$ . Bottom: evolution of the local extrema of  $t \mapsto X_\sigma(t)$  as  $\sigma$  goes from 0 to 1. The three values  $\sigma = 0.1; 0.3$  and  $0.8$  are plotted as dotted line. They indicate the local extrema of the three processes above.

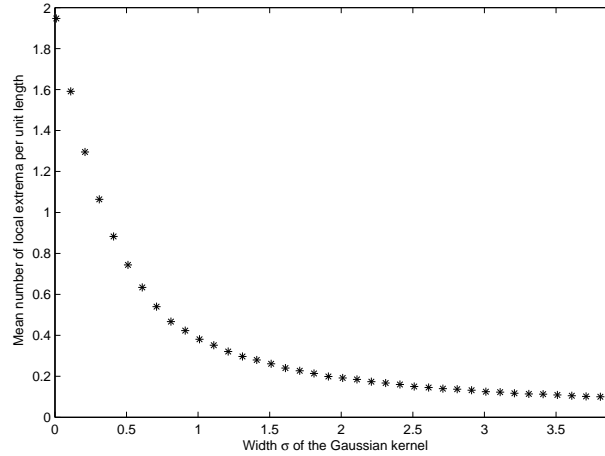


FIGURE 5. Empirical mean number of local extrema per unit length of  $X_\sigma$  as a function of  $\sigma$  (here  $\lambda = 1$  and we have taken the mean value from 10 samples on the interval  $[-100, 100]$ ).

**6.5. What about the intensity  $\lambda$  ?** For the purpose of this section, we introduce again the intensity  $\lambda$  of the underlying Poisson point process in the notations. In particular we will denote the mean number of local extrema of  $X_\sigma^{[\lambda]}$  per unit length by  $\rho(\sigma, \lambda)$  (instead of simply  $\rho(\sigma)$  as in the previous

section).

Thanks to the scaling relations of  $X_\sigma^{[\lambda]}$ , we also obtain scaling relations for  $\rho(\sigma, \lambda)$ :

**Proposition 25** (Scaling relations for  $\rho(\sigma, \lambda)$ ). *The mean number  $\rho(\sigma, \lambda)$  of local extrema of  $X_\sigma^{[\lambda]}$  per unit length satisfies the following scaling relation:*

$$\forall \alpha > 0, \quad \alpha \rho(\alpha \sigma, \lambda) = \rho(\sigma, \alpha \lambda).$$

*Proof.* We have

$$X_{\alpha\sigma}^{[\lambda]}(t) = \frac{1}{\alpha\sigma\sqrt{2\pi}} \sum_{\tau_i} \frac{-(t - \tau_i)}{\alpha^2\sigma^2} e^{-(t - \tau_i)^2/2\alpha^2\sigma^2} = \frac{1}{\alpha^2\sigma\sqrt{2\pi}} \sum_{\tau_i} \frac{-(t/\alpha - \tau_i/\alpha)}{\sigma^2} e^{-(t/\alpha - \tau_i/\alpha)^2/2\sigma^2},$$

where the  $\{\tau_i\}$  are the points of a Poisson point process of intensity  $\lambda$  on  $\mathbb{R}$ . Then, since the  $\{\tau_i/\alpha\}$  are now the points of a Poisson point process of intensity  $\alpha\lambda$  on  $\mathbb{R}$ , we have that the expected number of points  $t \in [0, \alpha]$  such that  $X_{\alpha\sigma}^{[\lambda]}(t) = 0$  - which by definition equals  $\alpha\rho(\alpha\sigma, \lambda)$ , also equals the expected number of points  $t \in [0, 1]$  such that  $X_\sigma^{[\alpha\lambda]}(t) = 0$  (which is  $\rho(\sigma, \alpha\lambda)$ ).  $\square$

In particular, using the fact that  $\rho(\sigma, \lambda) = \lambda\rho(\sigma\lambda, 1)$  the upper bound of Proposition 20 gives a polynomial bound in  $\lambda$ :

$$\forall \alpha \in \mathbb{R}, \quad \rho(\sigma, \lambda) \leq \lambda(3(2 + 2\sigma\lambda) + 1)e.$$

Notice that, under the assumption that  $\mathbb{E}(\#\{\sigma \in [\sigma_1, \sigma_0] \text{ such that } X'_\sigma(0) = 0\}) < +\infty$  for all  $0 < \sigma_1 < \sigma_0$ , the fact that the function  $\sigma \mapsto \rho(\sigma, \lambda)$  is a decreasing function with limit  $2\lambda$  when  $\sigma \rightarrow 0$  (Theorem 4) can be translated into the following properties on  $\lambda \mapsto \rho(\sigma, \lambda)$  thanks to the above scaling relation:

$$\forall \alpha \geq 1, \quad \rho(\sigma, \alpha\lambda) \leq \alpha\rho(\sigma, \lambda); \quad \rho(\sigma, \lambda) \leq 2\lambda \quad \text{and} \quad \frac{\rho(\sigma, \lambda)}{2\lambda} \xrightarrow{\lambda \rightarrow 0} 1.$$

On the other hand, using Theorem 3, if the second moment of  $N_{X_\sigma^{[\lambda]}}(0)$  is bounded by a polynomial in  $\lambda$ , then we will get

$$\rho(\sigma, \lambda) \xrightarrow{\lambda \rightarrow +\infty} \frac{1}{\sigma\pi} \sqrt{\frac{3}{2}}.$$

And this also will imply that  $\rho(\sigma, \lambda)$  is equivalent to  $\frac{1}{\sigma\pi} \sqrt{\frac{3}{2}}$  as  $\sigma$  goes to  $+\infty$ .

All these properties are empirically checked and illustrated on Figures 5 and 6.

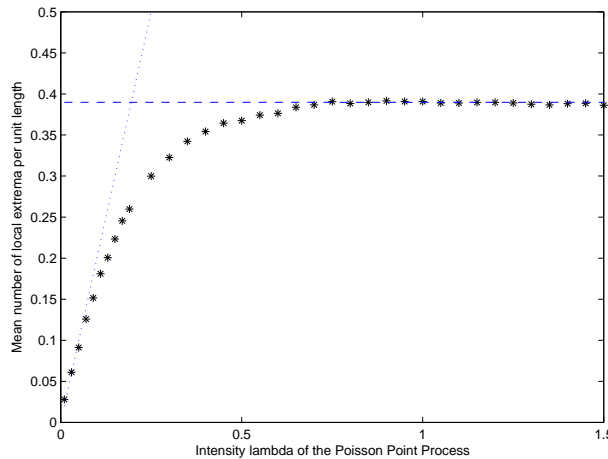


FIGURE 6. Empirical mean number of local extrema per unit length of  $X_\sigma^{[\lambda]}$  as a function of  $\lambda$  (here  $\sigma = 1$  and we have taken the mean value from 50 samples on the interval  $[-100, 100]$ ). The horizontal dashed line is the constant  $\frac{1}{\pi} \sqrt{\frac{3}{2}}$  and the dotted line is the map  $\lambda \mapsto 2\lambda$ .

## 7. APPENDIX

*Proof of Proposition 17.* For  $k \geq 0$  and  $l \geq 0$  integers, let us denote  $m_{kl} = \int |g(s)|^k |g'(s)|^l ds$ . We will also simply denote  $m_0 = m_{20} = \int g(s)^2 ds$  and  $m_2 = m_{02} = \int g'(s)^2 ds$ .

Let  $\psi_\lambda(u, v)$  denote the joint characteristic function of  $(Z_\lambda(t), Z'_\lambda(t))$ , then

$$\psi_\lambda(u, v) = \mathbb{E}(e^{i\frac{u}{\sqrt{\lambda}}X_\lambda + i\frac{v}{\sqrt{\lambda}}X'_\lambda})e^{-iu\sqrt{\lambda}f}g = \exp\left(\lambda \int_{\mathbb{R}} (e^{i\frac{u}{\sqrt{\lambda}}g(s) + i\frac{v}{\sqrt{\lambda}}g'(s)} - 1 - i\frac{u}{\sqrt{\lambda}}g(s)) ds\right).$$

We now use the fact  $\int g' = 0$ , and we thus have  $\psi_\lambda(u, v) = \exp(H_\lambda(u, v))$  where

$$H_\lambda(u, v) = \lambda \int_{\mathbb{R}} \left( e^{i\frac{u}{\sqrt{\lambda}}g(s) + i\frac{v}{\sqrt{\lambda}}g'(s)} - 1 - i\frac{u}{\sqrt{\lambda}}g(s) - i\frac{v}{\sqrt{\lambda}}g'(s) \right) ds.$$

We need to notice that

$$\forall (u, v) \in \mathbb{R}^2, |\psi_\lambda(u, v)| = |\exp(H_\lambda(u, v))| = |\mathbb{E}(e^{iuZ_\lambda + ivZ'_\lambda})| \leq 1.$$

In the following, we will also need these simple bounds:

$$(30) \quad \forall x \in \mathbb{R}, |e^{ix} - 1 - ix + \frac{x^2}{2}| \leq \frac{|x|^3}{3!} \quad \text{and} \quad \forall z \in \mathbb{C}, |e^z - 1| \leq |z|e^{|z|}.$$

We first estimate  $H_\lambda(u, 0)$ . We have

$$H_\lambda(u, 0) = \lambda \int (e^{i\frac{u}{\sqrt{\lambda}}g(s)} - 1 - i\frac{u}{\sqrt{\lambda}}g(s)) ds = -\frac{1}{2}u^2m_0 + K_\lambda(u),$$

where  $K_\lambda(u) = \lambda \int (e^{i\frac{u}{\sqrt{\lambda}}g(s)} - 1 - i\frac{u}{\sqrt{\lambda}}g(s) + \frac{1}{2}\frac{u^2}{\lambda}g^2(s)) ds$ . Then, thanks to the simple bounds (30), we get

$$|K_\lambda(u)| \leq \frac{|u|^3m_{30}}{6\sqrt{\lambda}} \quad \text{and consequently} \quad |e^{H_\lambda(u,0)} - e^{-\frac{1}{2}u^2m_0}| \leq \frac{|u|^3m_{30}}{6\sqrt{\lambda}}e^{-\frac{1}{2}u^2m_0}e^{\frac{|u|^3m_{30}}{6\sqrt{\lambda}}}.$$

We then estimate  $H_\lambda(u, v) - H_\lambda(u, 0)$ :

$$\begin{aligned} H_\lambda(u, v) - H_\lambda(u, 0) &= \lambda \int (e^{i\frac{u}{\sqrt{\lambda}}g(s) + i\frac{v}{\sqrt{\lambda}}g'(s)} - e^{i\frac{u}{\sqrt{\lambda}}g(s)}) ds \\ &= \lambda \int e^{i\frac{u}{\sqrt{\lambda}}g(s)} (e^{i\frac{v}{\sqrt{\lambda}}g'(s)} - 1 - i\frac{v}{\sqrt{\lambda}}g'(s)) ds \\ &= -\frac{v^2}{2} \int g'(s)^2 e^{i\frac{u}{\sqrt{\lambda}}g(s)} ds + F_\lambda(u, v), \end{aligned}$$

where  $F_\lambda(u, v) = \lambda \int e^{i\frac{u}{\sqrt{\lambda}}g(s)} (e^{i\frac{v}{\sqrt{\lambda}}g'(s)} - 1 - i\frac{v}{\sqrt{\lambda}}g'(s) + \frac{v^2}{2\lambda}g'(s)^2) ds$ . And again, thanks to the simple bounds (30), we get:  $|F_\lambda(u, v)| \leq \frac{|v|^3m_{03}}{6\sqrt{\lambda}}$ . This implies that

$$\begin{aligned} \left| e^{H_\lambda(u,v) - H_\lambda(u,0)} - e^{-\frac{v^2}{2} \int g'(s)^2 e^{i\frac{u}{\sqrt{\lambda}}g(s)} ds} \right| &\leq \left| e^{-\frac{v^2}{2} \int g'(s)^2 e^{i\frac{u}{\sqrt{\lambda}}g(s)} ds} \right| \cdot \left| e^{F_\lambda(u,v)} - 1 \right| \\ &\leq \frac{|v|^3m_{03}}{6\sqrt{\lambda}} e^{-\frac{v^2}{2} \int g'(s)^2 \cos(\frac{u}{\sqrt{\lambda}}g(s)) ds} + \frac{|v|^3m_{03}}{6\sqrt{\lambda}}. \end{aligned}$$

Let us now compute  $\widehat{C_{Z_\lambda}}(u, [0, 1])$ . By Proposition 12, we know that

$$-\pi \widehat{C_{Z_\lambda}}(u, [0, 1]) = \int_0^{+\infty} \frac{1}{v^2} (\psi_\lambda(u, v) + \psi_\lambda(u, -v) - 2\psi_\lambda(u, 0)) dv.$$

Let  $V > 0$  be a real number. We split the integral above in two parts, and write it as the sum of the integral between 0 and  $V$ , and of the integral between  $V$  and  $+\infty$ . Since for all  $(u, v)$ , we have  $|\psi_\lambda(u, v)| \leq 1$ , we get

$$\left| \int_V^{+\infty} \frac{1}{v^2} (\psi_\lambda(u, v) + \psi_\lambda(u, -v) - 2\psi_\lambda(u, 0)) dv \right| \leq 4 \int_V^{+\infty} \frac{1}{v^2} dv = \frac{4}{V}.$$

On the other hand, let  $I_V(u)$  denote the integral between 0 and  $V$ . We have

$$I_V(u) = \int_0^V \frac{1}{v^2} e^{H_\lambda(u,0)} \left( e^{H_\lambda(u,v)-H_\lambda(u,0)} + e^{H_\lambda(u,-v)-H_\lambda(u,0)} - 2 \right) dv.$$

We then decompose this into:

$$\begin{aligned} I_V(u) &= \int_0^V \frac{1}{v^2} e^{H_\lambda(u,0)} \left( e^{H_\lambda(u,v)-H_\lambda(u,0)} + e^{H_\lambda(u,-v)-H_\lambda(u,0)} - 2e^{-\frac{v^2}{2} \int g'(s)^2 e^{i\frac{u}{\sqrt{\lambda}}g(s)} ds} \right) dv \\ &\quad + \int_0^V \frac{1}{v^2} e^{H_\lambda(u,0)} \left( 2e^{-\frac{v^2}{2} \int g'(s)^2 e^{i\frac{u}{\sqrt{\lambda}}g(s)} ds} - 2e^{-\frac{v^2}{2}m_2} \right) dv \\ &\quad + \int_0^V \frac{1}{v^2} (e^{H_\lambda(u,0)} - e^{-\frac{1}{2}u^2m_0} + e^{-\frac{1}{2}u^2m_0}) \left( 2e^{-\frac{v^2}{2}m_2} - 2 \right) dv. \end{aligned}$$

Using the bounds we computed above, we get that

$$\begin{aligned} \left| I_V(u) - 2e^{-\frac{1}{2}u^2m_0} \int_0^V \frac{e^{-\frac{v^2}{2}m_2} - 1}{v^2} dv \right| &\leq 2 \int_0^V \frac{vm_{03}}{6\sqrt{\lambda}} e^{-\frac{v^2}{2} \int g'(s)^2 \cos(\frac{u}{\sqrt{\lambda}}g(s)) ds + \frac{|v|^3m_{03}}{6\sqrt{\lambda}}} dv \\ &\quad + 2 \int_0^V \frac{1}{v^2} \left| e^{-\frac{v^2}{2} \int g'(s)^2 e^{i\frac{u}{\sqrt{\lambda}}g(s)} ds} - 2e^{-\frac{v^2}{2}m_2} \right| dv \\ &\quad + 2 \left| e^{H_\lambda(u,0)} - e^{-\frac{1}{2}u^2m_0} \right| \int_0^V \frac{1 - e^{-\frac{v^2}{2}m_2}}{v^2} dv. \end{aligned}$$

Let  $J_V^{(n)}(u)$ , for  $n = 1, 2, 3$  respectively denote the three terms above. To give an upper bound for  $J_V^{(1)}(u)$ , we will need the following basic inequality:  $\forall x \in \mathbb{R}$ ,  $\cos(x) \geq 1 - \frac{x^2}{2}$ . This gives us the bound:

$$J_V^{(1)}(u) \leq 2 \int_0^V \frac{vm_{03}}{6\sqrt{\lambda}} e^{-\frac{v^2m_2}{2} + \frac{v^2}{2} \frac{u^2m_{22}}{2\lambda} + \frac{|v|^3m_{03}}{6\sqrt{\lambda}}} dv.$$

For the second term, we use

$$\begin{aligned} \left| e^{-\frac{v^2}{2} \int g'(s)^2 e^{i\frac{u}{\sqrt{\lambda}}g(s)} ds} - 2e^{-\frac{v^2}{2}m_2} \right| &\leq e^{-\frac{v^2}{2}m_2} \left| e^{-\frac{v^2}{2} \int g'(s)^2 (e^{i\frac{u}{\sqrt{\lambda}}g(s)} - 1) ds} - 1 \right| \\ &\leq e^{-\frac{v^2}{2}m_2} \left| \frac{v^2}{2} \int g'(s)^2 (e^{i\frac{u}{\sqrt{\lambda}}g(s)} - 1) ds \right| e^{\left| \frac{v^2}{2} \int g'(s)^2 (e^{i\frac{u}{\sqrt{\lambda}}g(s)} - 1) ds \right|}. \end{aligned}$$

But  $\left| \int g'(s)^2 (e^{i\frac{u}{\sqrt{\lambda}}g(s)} - 1) ds \right| \leq \int g'(s)^2 \frac{|u|}{\sqrt{\lambda}} g(s) ds = \frac{|u|}{\sqrt{\lambda}} m_{12}$  and thus

$$J_V^{(2)}(u) \leq \frac{|u|}{\sqrt{\lambda}} m_{12} \int_0^V e^{-\frac{v^2}{2}m_2 + \frac{v^2}{2} \frac{|u|}{\sqrt{\lambda}} m_{12}} dv.$$

For the third term, we use an integration by parts to obtain that

$$\int_0^V \frac{1 - e^{-\frac{v^2}{2}m_2}}{v^2} dv = \frac{e^{-\frac{V^2}{2}m_2} - 1}{V} + \int_0^V m_2 e^{-\frac{v^2}{2}m_2} dv \leq \frac{1}{2} \sqrt{2\pi m_2},$$

which gives

$$J_V^{(3)}(u) \leq \sqrt{2\pi m_2} \frac{|u|^3 m_{30}}{6\sqrt{\lambda}} e^{-\frac{1}{2}u^2m_0 + \frac{|u|^3 m_{30}}{6\sqrt{\lambda}}}.$$

Moreover we also have

$$\left| 2 \int_0^V \frac{1 - e^{-\frac{v^2}{2}m_2}}{v^2} dv - \sqrt{2\pi m_2} \right| \leq \frac{1 - e^{-\frac{V^2}{2}m_2}}{V} + \int_V^{+\infty} m_2 e^{-\frac{v^2}{2}m_2} dv \leq \frac{2}{V}.$$



The partial conclusion of all these estimates is that

$$\left| \widehat{\pi C_{Z_\lambda}}(u, [0, 1]) - \sqrt{2\pi m_2} e^{-m_0 u^2/2} \right| \leq \frac{4}{V} + \frac{2e^{-m_0 u^2/2}}{V} + J_V^{(1)}(u) + J_V^{(2)}(u) + J_V^{(3)}(u).$$

We now have to choose  $V$  in an appropriate way. The choice of  $V$  will be given by the bound on  $J_V^{(1)}(u)$ . Assume in the following that  $u$  satisfies the condition (U1) given by:  $\frac{u^2 m_{22}}{2\lambda} \leq \frac{m_2}{4}$ , and let us set

$$V = \frac{3\sqrt{\lambda} m_2}{4m_{03}}.$$

Then for all  $v \in [0, V]$ ,  $-\frac{v^2 m_2}{2} + \frac{v^2}{2} \frac{u^2 m_{22}}{2\lambda} + \frac{|v|^3 m_{03}}{6\sqrt{\lambda}} \leq -\frac{v^2 m_2}{4}$ , and thus

$$J_V^{(1)}(u) \leq \frac{m_{03}}{3\sqrt{\lambda}} \int_0^V v e^{-\frac{v^2 m_2}{4}} dv \leq \frac{2m_{03}}{3m_2 \sqrt{\lambda}}.$$

For the term  $J_V^{(2)}(u)$ , we notice that if  $u$  satisfies the condition (U2) given by:  $\frac{|u|}{\sqrt{\lambda}} m_{12} \leq \frac{m_2}{2}$ , then for all  $V > 0$ , we can bound  $J_V^{(2)}(u)$  by:

$$J_V^{(2)}(u) \leq \frac{|u|}{\sqrt{\lambda}} m_{12} \int_0^V e^{-\frac{v^2}{4} m_2} dv \leq \frac{|u|}{\sqrt{\lambda}} m_{12} \sqrt{\frac{\pi}{m_2}}.$$

Finally, for the third term, we have that if  $u$  satisfies the condition (U3) given by:  $\frac{|u| m_{30}}{3\sqrt{\lambda}} \leq \frac{1}{2} m_0$ , then  $J_V^{(3)}(u)$  can be bounded, independently of  $V$ , by

$$J_V^{(3)}(u) \leq \sqrt{2\pi m_2} \frac{|u|^3 m_{30}}{6\sqrt{\lambda}} e^{-\frac{1}{4} u^2 m_0} \leq \sqrt{2\pi m_2} \frac{2|u| m_{30}}{3m_0 \sqrt{\lambda}} e^{-1},$$

because of the fact that for all  $x \geq 0$ , then  $x e^{-x} \leq e^{-1}$ .

The final conclusion of all these computations is that if we set  $a_1 = \min(\sqrt{\frac{m_2}{2m_{22}}}, \frac{m_2}{2m_{12}}, \frac{3m_0}{2m_{30}})$ , then for all  $u$  and  $\lambda > 0$  we have

$$|u| \leq a_1 \sqrt{\lambda} \implies \left| \widehat{\pi C_{Z_\lambda}}(u, [0, 1]) - \sqrt{2\pi m_2} e^{-m_0 u^2/2} \right| \leq \frac{a_2}{\sqrt{\lambda}} + \frac{a_3 |u|}{\sqrt{\lambda}},$$

where  $a_2 = \frac{24m_{30} + 2m_{03}}{3m_2}$  and  $a_3 = m_{12} \sqrt{\frac{\pi}{m_2}} + \frac{2\sqrt{2\pi m_2} m_{30} e^{-1}}{3m_0}$ . □

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