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COMMON BOUNDED UNIVERSAL FUNCTIONS FOR COMPOSITION OPERATORS

FRÉDÉRIC BAYART, SOPHIE GRIVAUX, AND RAYMOND MORTINI

ABSTRACT. Let \mathcal{A} be the set of automorphisms of the unit disk with 1 as attractive fixed point. We prove that there exists a *single* Blaschke product that is universal for *every* composition operator C_ϕ , $\phi \in \mathcal{A}$, acting on the unit ball of $H^\infty(\mathbb{D})$.

1. INTRODUCTION

This paper is devoted to the construction of common universal functions for some uncountable families of composition operators on the unit ball \mathcal{B} of $H^\infty(\mathbb{D})$. If $\phi : \mathbb{D} \rightarrow \mathbb{D}$ is an analytic self-map of the unit disk \mathbb{D} , the composition operator $C_\phi : f \mapsto f \circ \phi$ acts continuously on \mathcal{B} (note that \mathcal{B} will always be endowed with the topology of uniform convergence on compact sets). A function $f \in \mathcal{B}$ is said to be \mathcal{B} -*universal* for C_ϕ , (or just *universal*, if no ambiguities arise) if $\mathcal{O}(f) = \{f \circ \phi^{[n]} ; n \geq 0\}$ is dense in \mathcal{B} , where $\phi^{[n]} = \phi \circ \phi \circ \dots \circ \phi$ denotes the n -th iterate of ϕ . The operator C_ϕ is \mathcal{B} -universal if it admits a \mathcal{B} -universal function, and this happens ([3]) if and only if ϕ is a hyperbolic or parabolic automorphism of the unit disk. In this case the universal function can be chosen to be a Blaschke product. Our aim in this paper is to construct common universal Blaschke products for some uncountable families of composition operators C_ϕ acting on \mathcal{B} , the ϕ 's being hyperbolic and parabolic automorphisms of \mathbb{D} .

Results on universal Blaschke products first appear in a paper by Heins [10]. A general theory of universal Blaschke products and their behaviour on the maximal ideal space of H^∞ was developed in [8] and [11]. Finally, these functions were the building blocks for studying B -universality for sequences of composition operators (C_{ϕ_n}) in [3].

Our study of universal Blaschke products in the present paper is motivated by previous results of common hypercyclicity of [1], [4] and [5]. Indeed the operators C_ϕ act boundedly on different spaces, such as the space $\mathcal{H}(\mathbb{D})$ of holomorphic functions on \mathbb{D} , or the Hardy spaces $H^p(\mathbb{D})$, $1 \leq p < +\infty$, and

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when ϕ is a hyperbolic or parabolic automorphism, C_ϕ is hypercyclic on $\mathcal{H}(\mathbb{D})$ (resp. $H^p(\mathbb{D})$), i.e. there exists a function $f \in \mathcal{H}(\mathbb{D})$ (resp. $f \in H^p(\mathbb{D})$) such that $\mathcal{O}(f)$ is dense in $\mathcal{H}(\mathbb{D})$ (resp. $H^p(\mathbb{D})$). It is then natural to ask about the existence of a function f which would be hypercyclic for *all* composition operators C_ϕ . Since each function in $H^p(\mathbb{D})$ has a radial limit almost everywhere on the unit circle \mathbb{T} , such a common hypercyclic function cannot exist on $H^p(\mathbb{D})$: if \mathcal{A} is a family of hyperbolic or parabolic automorphisms of \mathbb{D} , the fact that the family $(C_\phi)_{\phi \in \mathcal{A}}$ has a common hypercyclic vector necessarily implies that the subset B of \mathbb{T} consisting of all the attractive fixed points of the automorphisms $\phi \in \mathcal{A}$ has Lebesgue measure zero. Hence a natural family to consider is $(C_\phi)_{\phi \in \mathcal{A}_0}$, where \mathcal{A}_0 is the class of hyperbolic or parabolic automorphisms of \mathbb{D} with 1 as attractive fixed point. Then this restricted family of composition operators acting on $H^p(\mathbb{D})$ admits a common hypercyclic vector ([4] or [5]).

We deal here with the same question, but our underlying space is now the unit ball \mathcal{B} of $H^\infty(\mathbb{D})$. The main difficulty in this new setting lies in the fact that all the techniques of [1], [4] or [5] are “additive” and strongly use the linearity of the space, making it difficult to control the H^∞ -norm of the functions which are constructed. We have to use “multiplicative” techniques instead to prove the following theorem, which is the main result of this paper:

Theorem 1. *There exists a Blaschke product B which is universal for all composition operators C_ϕ associated to hyperbolic or parabolic automorphisms of \mathbb{D} with 1 as attractive fixed point.*

The proof of this result uses an argument of Costakis and Sambarino ([6]). The hyperbolic and parabolic cases will be treated separately (in Sections 2 and 3 respectively), the hyperbolic case being as usual easier than the parabolic one, since we have a better control of the rate of convergence of the iterates to the attractive fixed point.

2. THE HYPERBOLIC CASE

We first consider for $\lambda > 1$ the family of hyperbolic automorphisms

$$z \mapsto \frac{z + \frac{\lambda-1}{\lambda+1}}{1 + z \frac{\lambda-1}{\lambda+1}}$$

of \mathbb{D} with 1 as attractive fixed point and -1 as repulsive fixed point. The action of such an automorphism is best understood when considered on the right half-plane $\mathbb{C}_+ = \{w \in \mathbb{C} ; \Re w > 0\}$: if $\sigma : \mathbb{D} \rightarrow \mathbb{C}_+$ is the Cayley map defined by $\sigma(z) = \frac{1+z}{1-z}$, such an automorphism is conjugated via σ to a dilation $\varphi_\lambda : w \mapsto \lambda w$, where $\lambda > 1$. We will denote by ϕ_λ the hyperbolic automorphism of \mathbb{D} such that $\phi_\lambda = \sigma^{-1} \circ \varphi_\lambda \circ \sigma$. A general hyperbolic automorphism with 1 as attractive fixed point has the form $\phi_{\lambda,\beta} = \sigma^{-1} \circ \varphi_{\lambda,\beta} \circ \sigma$,

where $\varphi_{\lambda,\beta}$ acts on \mathbb{C}_+ as $\varphi_{\lambda,\beta}(w) = \lambda(w - i\beta) + i\beta$, $\lambda > 1$, $\beta \in \mathbb{R}$. We first show that the parameters β play essentially no role in this problem.

Lemma 2. *Let B be a Blaschke product which is universal for C_{ϕ_λ} , $\lambda > 1$. For any $\beta \in \mathbb{R}$, B is universal for $C_{\phi_{\lambda,\beta}}$.*

Proof. Let $f \in \mathcal{B}$ and let K be a compact subset of \mathbb{D} . For $z \in K$, we define

$$\begin{aligned} z_1(n) &= \sigma^{-1}(\lambda^n(\sigma(z) - i\beta) + i\beta) = \phi_{\lambda,\beta}^{[n]}(z) \\ z_2(n) &= \sigma^{-1}(\lambda^n(\sigma(z) - i\beta)) = \phi_\lambda^{[n]}(\sigma^{-1}(\sigma(z) - i\beta)). \end{aligned}$$

It is easy to show that there exists a constant C_1 which depends only on K and β such that

$$|z_1(n) - z_2(n)| \leq \frac{C_1}{\lambda^{2n}}.$$

In fact, if $w_1(n) = \lambda^n(\sigma(z) - i\beta) + i\beta$ and $w_2(n) = \lambda^n(\sigma(z) - i\beta)$, then

$$\begin{aligned} |z_1(n) - z_2(n)| &= \left| \int_{[w_1(n), w_2(n)]} \frac{2}{(1+w)^2} dw \right| \leq |\beta| \max_{w \in [w_1(n), w_2(n)]} \frac{2}{|1+w|^2} \\ &\leq |\beta| \frac{2}{\lambda^{2n} [\min\{\operatorname{Re} \sigma(z) : z \in K\}]^2} \leq \frac{C_1}{\lambda^{2n}}. \end{aligned}$$

On the other hand, there is another constant C_2 , depending only on K and β such that

$$|z_1(n)| \leq 1 - \frac{C_2}{\lambda^n} \text{ and } |z_2(n)| \leq 1 - \frac{C_2}{\lambda^n}.$$

This can be seen in the following way:

$$|1 - z_j(n)| = 1 - \left| \frac{w_j(n) - 1}{w_j(n) + 1} \right| = 2/|w_j(n) + 1| \geq \frac{C_2}{\lambda^n}.$$

Since B belongs to $H^\infty(\mathbb{D})$, Cauchy's inequalities show that $B(z_1(n)) - B(z_2(n))$ converges uniformly on K to 0. In fact

$$\begin{aligned} |B(z_1(n)) - B(z_2(n))| &\leq \int_{[z_1(n), z_2(n)]} \frac{|B'(\xi)|(1 - |\xi|^2)}{1 - |\xi|^2} |d\xi| \\ &\leq |z_1(n) - z_2(n)| \frac{1}{\min\{1 - |z_1(n)|^2, 1 - |z_2(n)|^2\}} \leq \frac{C_3}{\lambda^n}. \end{aligned}$$

On the other hand, since B is universal, there exists a sequence (n_k) such that $B \circ \phi_\lambda^{[n_k]}(\sigma^{-1}(\sigma - i\beta))$ converges uniformly to f on K (the map $z \mapsto \sigma^{-1}(\sigma(z) - i\beta)$ is an automorphism of \mathbb{D}). We conclude that $B \circ \phi_{\lambda,\beta}^{[n_k]}$ converges uniformly on K to f . \square

In order to construct a common universal Blaschke product for all the C_{ϕ_λ} 's, we will decompose $]1, +\infty[$ as an increasing union of compact sub-intervals $[a_k, b_k]$. Following [6], we then decompose each interval $[a, b]$ as $[a, b] = \cup_{j=1}^{q-1} [\lambda_j, \lambda_{j+1}]$ where $\lambda_1 = a$, $\lambda_2 = \lambda_1 + \frac{\delta}{2N}$, ..., $\lambda_{j+1} = \lambda_j + \frac{\delta}{(j+1)N}$ if $\lambda_j + \frac{\delta}{(j+1)N} \leq b$ and $\lambda_{j+1} = b$ if $\lambda_j + \frac{\delta}{(j+1)N} > b$. Here N is a positive integer which will be chosen very large in the sequel, and δ is a positive real number which will be chosen very small. The interval $[a, b]$ has been divided into q successive sub-intervals (q depending on δ and N , of course). The interest of such a decomposition of $[a, b]$ in our context lies in the following. Recall that $\|f\|_K$ denotes the supremum of the function f on the compact set K .

Lemma 3. *Let f be a finite Blaschke product such that $f(1) = f(-1) = 1$. For every compact subset K of \mathbb{D} and each interval $[a, b] \subseteq]1, \infty[$, there exists a positive constant M depending on K , f and a such that for every $j = 1, \dots, q$ and every $\lambda \in [\lambda_j, \lambda_{j+1}[$ the following assertions are true:*

- (1) for every $l < j$, $\left\| C_{\phi_\lambda}^{jN} C_{\phi_{\lambda_l}}^{-lN}(f) - 1 \right\|_K \leq M a^{-(j-l)N}$;
- (2) for every $l > j$, $\left\| C_{\phi_\lambda}^{jN} C_{\phi_{\lambda_l}}^{-lN}(f) - 1 \right\|_K \leq M a^{-(l-j)N}$;
- (3) $\left\| C_{\phi_\lambda}^{jN} C_{\phi_{\lambda_j}}^{-jN}(f) - f \right\|_K \leq M \delta$.

We will use repeatedly the following fact, which follows from the Schwarz-Pick estimates:

Lemma 4. *Let $u \in \mathcal{B}$. Then for every $z \in \mathbb{D}$,*

$$|u(z) - 1| \leq \frac{1+|z|}{1-|z|} |u(0) - 1|.$$

Proof. We obviously have that $\frac{u(z)-u(0)}{1-u(0)u(z)} = zg(z)$ for some $g \in \mathcal{B}$. Hence

$$\begin{aligned} |u(z) - 1| &\leq |u(z) - u(0)| + |u(0) - 1| \leq |z| |1 - \overline{u(0)}u(z)| + |u(0) - 1| \\ &\leq |z| |(1 - \overline{u(0)}) + \overline{u(0)}(1 - u(z))| + |u(0) - 1|. \end{aligned}$$

Therefore $|u(z) - 1|(1 - |z|) \leq |1 - u(0)|(1 + |z|)$ for every $z \in \mathbb{D}$. \square

Thus in order to prove assertions 1 and 2 above, for instance, it suffices to control in a suitable way the quantities $f(\phi_{\lambda_l}^{-[lN]}(\phi_\lambda^{[jN]}(0)))$.

Proof of Lemma 3. For every $z \in \mathbb{D}$ we have

$$C_{\phi_\lambda}^{jN} C_{\phi_{\lambda_l}}^{-lN}(f)(z) = f\left(\sigma^{-1}\left(\frac{\lambda^{jN}}{\lambda_l^{lN}}\sigma(z)\right)\right).$$

Since $f(1) = 1$ and f is Lipschitz with constant C up to the boundary of \mathbb{D} , we have

$$\left| f(\phi_{\lambda_l}^{-[lN]}(\phi_\lambda^{[jN]}(0))) - 1 \right| \leq C \left| \sigma^{-1}\left(\frac{\lambda^{jN}}{\lambda_l^{lN}}\right) - 1 \right| = \frac{2C}{\frac{\lambda^{jN}}{\lambda_l^{lN}} + 1}.$$

Assertion 1 follows from this estimate: since $l < j$,

$$\frac{\lambda^{jN}}{\lambda_l^{lN}} \geq \frac{\lambda^{jN}}{\lambda_{j-1}^{lN}} \geq \lambda_{j-1}^{(j-l)N} \left(1 + \frac{\delta}{\lambda_{j-1} N j}\right)^{Nj} \geq \lambda_{j-1}^{(j-l)N} \geq a^{(j-l)N}.$$

By Lemma 4, there exists a positive constant M_1 such that

$$\|C_{\phi_\lambda}^{jN} C_{\phi_{\lambda_l}}^{-lN}(f) - 1\|_K \leq \frac{M_1}{a^{(j-l)N}} \text{ for } l < j.$$

Assertion 2 is proved in the same fashion, using this time the fact that $f(-1) = 1$, so that

$$\left|f(\phi_{\lambda_l}^{-[lN]} \circ \phi_\lambda^{[jN]}(0)) - 1\right| \leq 2C \frac{\frac{\lambda^{jN}}{\lambda_l^{lN}}}{\frac{\lambda^{jN}}{\lambda_l^{lN}} + 1}$$

and that for $l > j$,

$$\frac{\lambda^{jN}}{\lambda_l^{lN}} \leq \lambda_{j+1}^{(j-l)N} \leq a^{(j-l)N}.$$

As to assertion 3, we have for every $z \in \mathbb{D}$

$$\begin{aligned} \left|C_{\phi_\lambda}^{jN} C_{\phi_{\lambda_j}}^{-jN}(f)(z) - f(z)\right| &\leq C \left|\sigma^{-1}\left(\frac{\lambda^{jN}}{\lambda_j^{jN}} \sigma(z)\right) - z\right| \\ &\leq C \left|\frac{\lambda^{jN}}{\lambda_j^{jN}} - 1\right| \cdot \frac{2|\sigma(z)|}{\left|\frac{\lambda^{jN}}{\lambda_j^{jN}} \sigma(z) + 1\right|^2}. \end{aligned}$$

Since $\left|\frac{\lambda^{jN}}{\lambda_j^{jN}} \sigma(z) + 1\right|$ is bigger than its real part, which is bigger than 1, and since $0 \leq \left(\frac{\lambda}{\lambda_j}\right)^{jN} - 1 \leq \left(1 + \frac{\delta}{aNj}\right)^{Nj} - 1 \leq 2\delta/a$ when δ is small enough, we have

$$\|C_{\phi_\lambda}^{jN} C_{\phi_{\lambda_j}}^{-jN}(f) - f\|_K \leq M_3 \delta$$

for some positive constant M_3 . □

We need a last lemma.

Lemma 5. *The finite Blaschke products f such that $f(1) = f(-1) = 1$ are dense in \mathcal{B} (for the topology of uniform convergence on compact sets).*

Proof. We use Carathéodory's theorem that the set of finite Blaschke products is dense in \mathcal{B} and a special case of an interpolation result given in [9, p. Lemma 2.10] that tells us that for every $\varepsilon > 0$, every compact subset $K \subseteq \mathbb{D}$ and $\alpha, \beta \in \mathbb{T}$ there exists a finite Blaschke product B_1 satisfying $B_1(1) = \alpha$, $B_1(-1) = \beta$ and $\|B_1 - 1\|_K < \varepsilon$. Thus, given $f \in \mathcal{B}$ and a finite Blaschke product B_0 that is close to f on K , we solve the interpolation problem with $\alpha = \overline{B_0(1)}$

and $\beta = \overline{B_0(-1)}$ and set $B = B_0B_1$, in order to get the desired Blaschke product. \square

With these two lemmas in hand, we prove the following proposition:

Proposition 6. *Let $(f_k)_{k \geq 1}$ be a dense sequence of finite Blaschke products with $f_k(1) = f_k(-1) = 1$. Let (K_k) be an exhaustive sequence of compact subsets of \mathbb{D} , and $([a_k, b_k])_{k \geq 1}$ an increasing sequence of compact intervals such that*

$$\bigcup_{k \geq 1} [a_k, b_k] =]1, +\infty[.$$

There exist

- a sequence $(B_n)_{n \geq 1}$ of finite Blaschke products;
- an increasing sequence $(p_n)_{n \geq 1}$ of positive integers

such that the following are satisfied for every $k \geq 1$:

- (1) $B_k(1) = 1$;
- (2) $\|B_k - 1\|_{K_k} < 2^{-k}$;
- (3) for every $\lambda \in [a_k, b_k]$, there exists an integer $n_k(\lambda) \leq p_k$ such that for every $i \geq k$,

$$(1) \quad \left\| C_{\phi_\lambda}^{n_k(\lambda)}(B_1 \dots B_i) - f_k \right\|_{K_k} < 2^{-k}.$$

As a corollary, we obtain:

Corollary 7. *There exists a Blaschke product B which is universal for all the composition operators $C_{\phi_{\lambda, \beta}}$, $\lambda > 1$, $\beta \in \mathbb{R}$.*

Proof. Consider $B = \prod_{n=1}^{\infty} B_n$: this is a convergent Blaschke product by property 2, and going to the limit as i goes to infinity in equation (1) implies that for every $\lambda \in [a, b]$ and k large enough ($[a, b] \subseteq [a_k, b_k]$),

$$\left\| C_{\phi_\lambda}^{n_k(\lambda)}(B) - f_k \right\|_{K_k} \leq 2^{-k}.$$

Since the family $(f_k)_{k \geq 1}$ is locally uniformly dense in \mathcal{B} , this proves the universality of B for C_{ϕ_λ} , hence for $C_{\phi_{\lambda, \beta}}$. \square

We turn now to the proof of Proposition 6:

Proof. The proof is done by induction on k . We consider a first partition $a_1 = \lambda_1 < \lambda_2 < \dots < \lambda_{q_1} = b_1$ of $[a_1, b_1]$ with parameters N_1 and δ_1 , and the finite Blaschke product

$$B_1 = \prod_{l=1}^{q_1} C_{\phi_{\lambda_l}}^{-lN_1}(f_1).$$

We have $B_1(1) = 1$. Since $f_1(-1) = 1$, $C_{\phi_{\lambda_l}}^{-lN_1}(f_1)$ tends to 1 uniformly on compact sets as N_1 tends to infinity, and if N_1 is large enough,

$$(2) \quad \|B_1 - 1\|_{K_1} < 2^{-1}.$$

Since $|\prod_{j=1}^s a_j - \prod_{j=1}^s b_j| \leq \sum_{j=1}^s |a_j - b_j|$ whenever $a_j, b_j \in \mathbb{D}$, for every $j = 1, \dots, q_1$ and every $\lambda \in [\lambda_j, \lambda_{j+1}[$, we have for any compact subset K of \mathbb{D}

$$\begin{aligned} \left\| C_{\phi_\lambda}^{jN_1}(B_1) - f_1 \right\|_K &\leq \sum_{l=1, l \neq j}^{q_1} \left\| C_{\phi_\lambda}^{jN_1} C_{\phi_{\lambda_l}}^{-lN_1}(f_1) - 1 \right\|_K \\ &\quad + \left\| C_{\phi_\lambda}^{jN_1} C_{\phi_{\lambda_j}}^{-jN_1}(f_1) - f_1 \right\|_K. \end{aligned}$$

But by Lemma 3, the quantity on the righthand side is less than

$$\sum_{l=1, l \neq j}^{q_1} \frac{M}{a_1^{|j-l|N_1}} + M \delta_1 \leq 2M \sum_{k=1}^{\infty} \frac{1}{a_1^{kN_1}} + M \delta_1.$$

Thus if N_1 is large enough and δ_1 small enough

$$(3) \quad \left\| C_{\phi_\lambda}^{jN_1}(B_1) - f_1 \right\|_{K_1} < 2^{-1}.$$

We now fix N_1 large enough and δ_1 small enough so that inequalities (2) and (3) are satisfied. It is easy to check that assertions 2 and 3 of Proposition 6 are satisfied with $p_1 = q_1 N_1$ and $n_1(\lambda) = j N_1$ for $\lambda \in [\lambda_j, \lambda_{j+1}[$. This terminates the first step of the construction.

If now the construction has been carried out until step $k-1$, we consider again a partition $a_k = \lambda_1 < \dots < \lambda_{q_k} = b_k$ of $[a_k, b_k]$ with parameters δ_k and N_k , and set

$$B_k = \prod_{l=1}^{q_k} C_{\phi_{\lambda_l}}^{-lN_k}(f_k),$$

so that B_k is a finite Blaschke product with $B_k(1) = 1$. Just as above if N_k is large enough and δ_k small enough, we have for every $j \leq q_k$, every $\lambda \in [\lambda_j, \lambda_{j+1}[$

$$\left\| C_{\phi_\lambda}^{jN_k}(B_k) - f_k \right\|_{K_k} < 2^{-(k+1)}$$

and

$$\|B_k - 1\|_{K_k} < 2^{-k}.$$

Because $B_1(1) = \dots = B_{k-1}(1) = 1$, we can also choose simultaneously N_k large enough so that $C_{\phi_\lambda}^{jN_k}(B_1 \dots B_{k-1})$ is very close to 1 on K_k . This gives (1) for $i = k$.

It remains to check that if $r \leq k-1$, $\lambda \in [a_r, b_r]$,

$$\left\| C_{\phi_\lambda}^{n_r(\lambda)}(B_1 \dots B_{k-1} B_k) - f_r \right\|_{K_r} < 2^{-r}.$$

We already know that

$$\left\| C_{\phi_\lambda}^{n_r(\lambda)}(B_1 \dots B_{k-1}) - f_r \right\|_{K_r} < 2^{-r},$$

and since B_k can be made arbitrarily close to 1 on any compact set if N_k is large enough, we also choose N_k so that $\|B_k - 1\|_K$ is small enough, where

$$K = \bigcup_{r \leq k-1, \lambda \in [a_r, b_r]} \phi_\lambda^{n_r(\lambda)}(K_r),$$

and then assertions 2 and 3 are satisfied at step k . \square

3. THE PARABOLIC CASE

We consider now the family of parabolic automorphisms of \mathbb{D} with 1 as attractive fixed point. If $T_\lambda : \mathbb{C}_+ \rightarrow \mathbb{C}_+$ is the translation defined by $w \mapsto w + i\lambda$, $\lambda \in \mathbb{R} \setminus \{0\}$, then such parabolic automorphisms have the form $\psi_\lambda(z) = \sigma^{-1} \circ T_\lambda \circ \sigma$. Our aim in this section is to construct a Blaschke product which is universal for all composition operators (C_{ψ_λ}) , $\lambda > 0$. This is more difficult than the hyperbolic case, because we have no suitable analog of Lemma 3: the estimate we get has the form

$$\left\| C_{\psi_\lambda}^{jN} C_{\psi_{\lambda_l}}^{-lN}(f) - 1 \right\|_K \leq \frac{M}{|j-l|N} \quad \text{for } l \neq j,$$

and the series on the righthand side is not convergent when we sum over all $l \neq j$.

In other words if K is any compact set, the sets $\psi_\lambda^{[n]}(K)$ go towards the point 1 at a rate of $1/n$, which is too slow. This difficulty was tackled for the study of common hypercyclicity on the Hardy space $H^2(\mathbb{D})$ by using either a fine analysis of properties of disjointness in [4] or probabilistic ideas in [5]. Here we use in a crucial way the tangential convergence of the sequence $(\psi_\lambda^{[n]}(0))$ to the boundary. Indeed, the series $\sum_n (1 - |\psi_n(0)|)$ is summable, whereas the series $\sum_n |1 - \psi_n(0)|$ is not. The following lemma will play the same role as Lemma 3 in the hyperbolic case. We keep the notation of Section 2 and use the same kind of decomposition $a = \lambda_1, \dots, \lambda_q = b$ of a compact sub-interval $[a, b]$ of $]0, +\infty[$.

Lemma 8. *Let f be a finite Blaschke product such that $f(1) = 1$. For every compact subset K of \mathbb{D} , there exists a positive constant M depending on K , f and a such that for every $j = 1, \dots, q$ and every $\lambda \in [\lambda_j, \lambda_{j+1}[$ the following assertions are true:*

- (1) for every $l < j$, $\left\| |C_{\psi_\lambda}^{jN} C_{\psi_{\lambda_l}}^{-lN}(f)| - 1 \right\|_K \leq \frac{M}{(j-l)^2 N^2}$;
- (2) for every $l > j$, $\left\| |C_{\psi_\lambda}^{jN} C_{\psi_{\lambda_l}}^{-lN}(f)| - 1 \right\|_K \leq \frac{M}{(l-j)^2 N^2}$;
- (3) $\left\| C_{\psi_\lambda}^{jN} C_{\psi_{\lambda_j}}^{-jN}(f) - f \right\|_K \leq M \delta$.

Proof. In order to prove assertions 1 and 2, it suffices to work at the point 0. Since the modulus of f is equal to 1 on \mathbb{T} , and since the operators commute, we have

$$\begin{aligned} \left| |f(\psi_{\lambda_l}^{-[lN]} \circ \psi_{\lambda}^{[jN]}(0))| - 1 \right| &= \left| |f(\psi_{\lambda}^{[jN]} \circ \psi_{\lambda_l}^{-[lN]}(0))| \right. \\ &\quad \left. - \left| f\left(\frac{\psi_{\lambda}^{[jN]} \circ \psi_{\lambda_l}^{-[lN]}(0)}{|\psi_{\lambda}^{[jN]} \circ \psi_{\lambda_l}^{-[lN]}(0)|}\right) \right| \right|. \end{aligned}$$

Since f is C -Lipschitz on $\overline{\mathbb{D}}$ for some positive constant C , this quantity is less than

$$(4) \quad C \left| \psi_{\lambda}^{[jN]} \circ \psi_{\lambda_l}^{-[lN]}(0) - \frac{\psi_{\lambda}^{[jN]} \circ \psi_{\lambda_l}^{-[lN]}(0)}{|\psi_{\lambda}^{[jN]} \circ \psi_{\lambda_l}^{-[lN]}(0)|} \right| = C \left(1 - |\psi_{\lambda}^{[jN]} \circ \psi_{\lambda_l}^{-[lN]}(0)| \right).$$

An easy computation shows that

$$\psi_{\lambda}^{[jN]} \circ \psi_{\lambda_l}^{-[lN]}(0) = \frac{iN(j\lambda - l\lambda_l)}{2 + iN(j\lambda - l\lambda_l)}.$$

Observe that $|j\lambda - l\lambda_l| \geq |j - l|a$. This gives

$$\begin{aligned} 1 - |\psi_{\lambda}^{[jN]} \circ \psi_{\lambda_l}^{-[lN]}(0)| &\leq 1 - |\psi_{\lambda}^{[jN]} \circ \psi_{\lambda_l}^{-[lN]}(0)|^2 \\ &\leq 1 - \frac{(j\lambda - l\lambda_l)^2 N^2}{4 + (j\lambda - l\lambda_l)^2 N^2} \\ &\leq \frac{C_1}{N^2(j-l)^2} \end{aligned}$$

for some positive constant C_1 which does not depend on λ . Now equation (4) implies that for $l \neq j$,

$$\left| |f(\psi_{\lambda}^{[jN]} \circ \psi_{\lambda_l}^{-[lN]}(0))| - 1 \right| \leq \frac{C_2}{N^2(j-l)^2}.$$

This proves assertions 1 and 2 of Lemma 8. Assertion 3 is proved in the same way as in Lemma 3, writing for $\lambda \in [\lambda_j, \lambda_{j+1}[$

$$\left\| C_{\psi_{\lambda}}^{jN} C_{\psi_{\lambda_j}}^{-jN} f(z) - f(z) \right\| \leq CjN|\lambda_j - \lambda| \leq CjN \frac{\delta}{(j+1)N} \leq C'\delta.$$

□

The following proposition is the main ingredient of the proof:

Proposition 9. *Let f be a finite Blaschke product, $[a, b] \subset]0, +\infty[$, m_0 a positive integer, K a compact subset of \mathbb{D} and $\varepsilon > 0$. There exist an integer m , integers $(n(\lambda))_{\lambda \in [a, b]}$ with $n(\lambda) \in [m_0, m]$ and a finite Blaschke product B such that*

- (1) $B(1) = 1$;
- (2) $\|B - 1\|_K < \varepsilon$;

- (3) $C_{\psi_\lambda}^{n(\lambda)}(B) = u_\lambda v_\lambda$ where u_λ and v_λ belong to \mathcal{B} , $\|u_\lambda - f\|_K < \varepsilon$ and $|v_\lambda(0)| > 1 - \varepsilon$.

Proof. We use again the decomposition $\lambda_1 = a$, $\lambda_2 = a + \frac{\delta}{2N}, \dots, \lambda_q = b$, where $\delta > 0$ and $N \geq m_0$, will be fixed during the proof. Consider the Blaschke product

$$B_1 = \prod_{l=1}^q C_{\psi_{\lambda_l}}^{-lN}(f).$$

For $\lambda \in [\lambda_j, \lambda_{j+1}[$, we have

$$C_{\psi_\lambda}^{jN}(B) = C_{\psi_\lambda}^{jN} C_{\psi_{\lambda_j}}^{-jN}(f) \left(\prod_{l \neq j} C_{\psi_\lambda}^{jN} C_{\psi_{\lambda_l}}^{-lN}(f) \right) := u_{1,\lambda} v_{1,\lambda}$$

with $u_{1,\lambda} = C_{\psi_\lambda}^{jN} C_{\psi_{\lambda_j}}^{-jN}(f)$ and $v_{1,\lambda} = \prod_{l \neq j} C_{\psi_\lambda}^{jN} C_{\psi_{\lambda_l}}^{-lN}(f)$. By assertion 3 of Lemma 8, $\|u_{1,\lambda} - f\|_K \leq M\delta < \varepsilon$ if δ is small enough. Moreover, still by Lemma 8,

$$\begin{aligned} 1 - |v_{1,\lambda}(0)| &= 1 - \prod_{l \neq j} |f(\psi_\lambda^{[jN]} \circ \psi_{\lambda_l}^{-[lN]}(0))| \\ &\leq \sum_{l \neq j} \left(1 - |f(\psi_\lambda^{[jN]} \circ \psi_{\lambda_l}^{-[lN]}(0))| \right) \\ &\leq \frac{C'}{N^2} \end{aligned}$$

for some positive constant C' . Thus if N is large enough, $|v_{1,\lambda}(0)| > 1 - \varepsilon$. To conclude, it remains to observe that the same proof leads to

$$|B_1(0)| \geq 1 - \frac{C''}{N^2}$$

for some positive constant C'' , so that using Lemma 4 and adjusting N large enough, there exists a real number θ such that $\|e^{i\theta} B_1 - 1\|_K < \varepsilon$. If we set $B_2 = e^{i\theta} B_1$, then B_2 satisfies the conclusions of the proposition (setting $u_{2,\lambda} = u_{1,\lambda}$ and $v_{2,\lambda} = e^{i\theta} v_{1,\lambda}$, except that we are not sure that $B_2(1) = 1$). To conclude, let F be a finite Blaschke product such that F is very close to 1 on a big compact set $L \subset \mathbb{D}$ and $F(1) = \overline{B_2(1)}$. Then $B = FB_2$ is the finite Blaschke product we are looking for. Indeed, setting $u_\lambda = u_{2,\lambda}$ and $v_\lambda = C_{\psi_\lambda}^{n(\lambda)}(F)v_{2,\lambda}$, condition 3. is satisfied, provided L is big enough to contain each $\psi_\lambda^{n(\lambda)}(0)$. □

We can now proceed with the construction:

Proposition 10. *Let $(f_k)_{k \geq 1}$ be a dense sequence of finite Blaschke products with $f_k(1) = 1$. Let (K_k) be an exhaustive sequence of compact subsets of \mathbb{D} , and $([a_k, b_k])_{k \geq 1}$ an increasing sequence of compact intervals such that*

$$\bigcup_{k \geq 1} [a_k, b_k] =]1, +\infty[.$$

There exist finite Blaschke products (B_k) , integers (m_k) , and other integers $(n_k(\lambda))_{\lambda \in [a_k, b_k]}$ with $n_k(\lambda) \leq m_k$ and such that

- (1) $B_k(1) = 1$;
- (2) $\|B_k - 1\|_{K_k} < 2^{-k}$;
- (3) for every $j < k$, every $\lambda \in [a_k, b_k]$, $|B_j \circ \psi_\lambda^{[n_k(\lambda)]}(0) - 1| < 2^{-k}$;
- (4) for every $j < k$, every $\lambda \in [a_j, b_j]$, $|B_k \circ \psi_\lambda^{[n_j(\lambda)]}(0) - 1| < 2^{-k}$;
- (5) for every $\lambda \in [a_k, b_k]$, $C_{\psi_\lambda}^{n_k(\lambda)}(B_k) = u_{k,\lambda} v_{k,\lambda}$ where

$$\|u_{k,\lambda} - f_k\|_{K_k} < 2^{-k} \text{ and } |v_{k,\lambda}(0)| > 1 - 2^{-k}.$$

Proof. The first step of the construction follows directly from Proposition 9. Now we assume that the construction has been done until step $k - 1$ and show how to complete step k . By continuity at the point 1 of the functions $(B_j)_{j < k}$, we choose an integer m such that, for every $\lambda \in [a_k, b_k]$, for any $n \geq m$, $|B_j \circ \psi_\lambda^{[n]}(0) - 1| < 2^{-k}$. We then set

$$K = K_k \cup \bigcup_{j < k, \lambda \in [a_j, b_j], n \leq m_j} \left\{ \psi_\lambda^{[n]}(0) \right\}.$$

The function B_k is then given immediately by Proposition 9. \square

Corollary 11. *There exists a Blaschke product B which is universal for all the composition operators C_{ψ_λ} , $\lambda > 0$.*

Proof. Set

$$B = \prod_{l \geq 1} B_l,$$

which is a convergent Blaschke product by assertion 2 of Proposition 10. We claim that B is \mathcal{B} -universal with respect to every composition operator C_{ψ_λ} . Indeed, fix $\lambda > 0$ and k_0 such that $\lambda \in [a_{k_0}, b_{k_0}]$. Let g be a universal function for *this* particular operator C_{ψ_λ} . Using the notation of Proposition 10, let (p_k) be an increasing sequence of integers such that f_{p_k} converges uniformly to g on compact subsets of \mathbb{D} . Now we decompose

$$C_{\psi_\lambda}^{n_{p_k}(\lambda)}(B) = C_{\psi_\lambda}^{n_{p_k}(\lambda)}(B_{p_k}) \left(\prod_{j \neq p_k} B_j \circ \psi_\lambda^{[n_{p_k}(\lambda)]} \right) := u_{p_k,\lambda} v_{p_k,\lambda} w_{p_k,\lambda}$$

where $C_{\psi_\lambda}^{[n_{p_k}(\lambda)]}(B_{p_k}) = u_{p_k,\lambda} v_{p_k,\lambda}$ is the decomposition of Proposition 9. From assertions 3 and 4 of Proposition 10, we get that $w_{p_k,\lambda}(0)$ tends to 1 (see [2])

for details), so that (cf. Fact 4) $w_{p_k, \lambda}$ converges uniformly on compacta to 1. Taking a subsequence if necessary, we can assume that $v_{p_k, \lambda}(0)$ converges to some unimodular number $e^{i\theta}$, and by Fact 4 again we have uniform convergence on compacta. Thus $C_{\psi_\lambda}^{n_{p_k}(\lambda)}(B)$ converges uniformly to the function $e^{i\theta}g$ on compacta. Since the function $e^{i\theta}g$ is universal for C_{ψ_λ} , this implies that B is universal for C_{ψ_λ} too, and this terminates the proof of Corollary 11. \square

The proof of Theorem 1 is now concluded by “intertwining” the two proofs of the hyperbolic and parabolic cases: the common universal Blaschke product has the form

$$B = \prod_{l \geq 1} B_l$$

where the B_l 's are finite Blaschke products satisfying a number of properties: B_1 is constructed using Proposition 6, then B_2 using Proposition 10, then B_3 using Proposition 6 again, etc... taking care at each step not to destroy what has been done previously. Details are left to the reader.

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