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# COMMON BOUNDED UNIVERSAL FUNCTIONS FOR COMPOSITION OPERATORS

FRÉDÉRIC BAYART, SOPHIE GRIVAUX, AND RAYMOND MORTINI

ABSTRACT. Let  $\mathcal{A}$  be the set of automorphisms of the unit disk with 1 as attractive fixed point. We prove that there exists a *single* Blaschke product that is universal for *every* composition operator  $C_\phi$ ,  $\phi \in \mathcal{A}$ , acting on the unit ball of  $H^\infty(\mathbb{D})$ .

## 1. INTRODUCTION

This paper is devoted to the construction of common universal functions for some uncountable families of composition operators on the unit ball  $\mathcal{B}$  of  $H^\infty(\mathbb{D})$ . If  $\phi : \mathbb{D} \rightarrow \mathbb{D}$  is an analytic self-map of the unit disk  $\mathbb{D}$ , the composition operator  $C_\phi : f \mapsto f \circ \phi$  acts continuously on  $\mathcal{B}$  (note that  $\mathcal{B}$  will always be endowed with the topology of uniform convergence on compact sets). A function  $f \in \mathcal{B}$  is said to be  $\mathcal{B}$ -*universal* for  $C_\phi$ , (or just *universal*, if no ambiguities arise) if  $\mathcal{O}(f) = \{f \circ \phi^{[n]} ; n \geq 0\}$  is dense in  $\mathcal{B}$ , where  $\phi^{[n]} = \phi \circ \phi \circ \dots \circ \phi$  denotes the  $n$ -th iterate of  $\phi$ . The operator  $C_\phi$  is  $\mathcal{B}$ -universal if it admits a  $\mathcal{B}$ -universal function, and this happens ([3]) if and only if  $\phi$  is a hyperbolic or parabolic automorphism of the unit disk. In this case the universal function can be chosen to be a Blaschke product. Our aim in this paper is to construct common universal Blaschke products for some uncountable families of composition operators  $C_\phi$  acting on  $\mathcal{B}$ , the  $\phi$ 's being hyperbolic and parabolic automorphisms of  $\mathbb{D}$ .

Results on universal Blaschke products first appear in a paper by Heins [10]. A general theory of universal Blaschke products and their behaviour on the maximal ideal space of  $H^\infty$  was developed in [8] and [11]. Finally, these functions were the building blocks for studying  $B$ -universality for sequences of composition operators  $(C_{\phi_n})$  in [3].

Our study of universal Blaschke products in the present paper is motivated by previous results of common hypercyclicity of [1], [4] and [5]. Indeed the operators  $C_\phi$  act boundedly on different spaces, such as the space  $\mathcal{H}(\mathbb{D})$  of holomorphic functions on  $\mathbb{D}$ , or the Hardy spaces  $H^p(\mathbb{D})$ ,  $1 \leq p < +\infty$ , and

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when  $\phi$  is a hyperbolic or parabolic automorphism,  $C_\phi$  is hypercyclic on  $\mathcal{H}(\mathbb{D})$  (resp.  $H^p(\mathbb{D})$ ), i.e. there exists a function  $f \in \mathcal{H}(\mathbb{D})$  (resp.  $f \in H^p(\mathbb{D})$ ) such that  $\mathcal{O}(f)$  is dense in  $\mathcal{H}(\mathbb{D})$  (resp.  $H^p(\mathbb{D})$ ). It is then natural to ask about the existence of a function  $f$  which would be hypercyclic for *all* composition operators  $C_\phi$ . Since each function in  $H^p(\mathbb{D})$  has a radial limit almost everywhere on the unit circle  $\mathbb{T}$ , such a common hypercyclic function cannot exist on  $H^p(\mathbb{D})$ : if  $\mathcal{A}$  is a family of hyperbolic or parabolic automorphisms of  $\mathbb{D}$ , the fact that the family  $(C_\phi)_{\phi \in \mathcal{A}}$  has a common hypercyclic vector necessarily implies that the subset  $B$  of  $\mathbb{T}$  consisting of all the attractive fixed points of the automorphisms  $\phi \in \mathcal{A}$  has Lebesgue measure zero. Hence a natural family to consider is  $(C_\phi)_{\phi \in \mathcal{A}_0}$ , where  $\mathcal{A}_0$  is the class of hyperbolic or parabolic automorphisms of  $\mathbb{D}$  with 1 as attractive fixed point. Then this restricted family of composition operators acting on  $H^p(\mathbb{D})$  admits a common hypercyclic vector ([4] or [5]).

We deal here with the same question, but our underlying space is now the unit ball  $\mathcal{B}$  of  $H^\infty(\mathbb{D})$ . The main difficulty in this new setting lies in the fact that all the techniques of [1], [4] or [5] are “additive” and strongly use the linearity of the space, making it difficult to control the  $H^\infty$ -norm of the functions which are constructed. We have to use “multiplicative” techniques instead to prove the following theorem, which is the main result of this paper:

**Theorem 1.** *There exists a Blaschke product  $B$  which is universal for all composition operators  $C_\phi$  associated to hyperbolic or parabolic automorphisms of  $\mathbb{D}$  with 1 as attractive fixed point.*

The proof of this result uses an argument of Costakis and Sambarino ([6]). The hyperbolic and parabolic cases will be treated separately (in Sections 2 and 3 respectively), the hyperbolic case being as usual easier than the parabolic one, since we have a better control of the rate of convergence of the iterates to the attractive fixed point.

## 2. THE HYPERBOLIC CASE

We first consider for  $\lambda > 1$  the family of hyperbolic automorphisms

$$z \mapsto \frac{z + \frac{\lambda-1}{\lambda+1}}{1 + z \frac{\lambda-1}{\lambda+1}}$$

of  $\mathbb{D}$  with 1 as attractive fixed point and  $-1$  as repulsive fixed point. The action of such an automorphism is best understood when considered on the right half-plane  $\mathbb{C}_+ = \{w \in \mathbb{C} ; \Re w > 0\}$ : if  $\sigma : \mathbb{D} \rightarrow \mathbb{C}_+$  is the Cayley map defined by  $\sigma(z) = \frac{1+z}{1-z}$ , such an automorphism is conjugated via  $\sigma$  to a dilation  $\varphi_\lambda : w \mapsto \lambda w$ , where  $\lambda > 1$ . We will denote by  $\phi_\lambda$  the hyperbolic automorphism of  $\mathbb{D}$  such that  $\phi_\lambda = \sigma^{-1} \circ \varphi_\lambda \circ \sigma$ . A general hyperbolic automorphism with 1 as attractive fixed point has the form  $\phi_{\lambda,\beta} = \sigma^{-1} \circ \varphi_{\lambda,\beta} \circ \sigma$ ,

where  $\varphi_{\lambda,\beta}$  acts on  $\mathbb{C}_+$  as  $\varphi_{\lambda,\beta}(w) = \lambda(w - i\beta) + i\beta$ ,  $\lambda > 1$ ,  $\beta \in \mathbb{R}$ . We first show that the parameters  $\beta$  play essentially no role in this problem.

**Lemma 2.** *Let  $B$  be a Blaschke product which is universal for  $C_{\phi_\lambda}$ ,  $\lambda > 1$ . For any  $\beta \in \mathbb{R}$ ,  $B$  is universal for  $C_{\phi_{\lambda,\beta}}$ .*

*Proof.* Let  $f \in \mathcal{B}$  and let  $K$  be a compact subset of  $\mathbb{D}$ . For  $z \in K$ , we define

$$\begin{aligned} z_1(n) &= \sigma^{-1}(\lambda^n(\sigma(z) - i\beta) + i\beta) = \phi_{\lambda,\beta}^{[n]}(z) \\ z_2(n) &= \sigma^{-1}(\lambda^n(\sigma(z) - i\beta)) = \phi_\lambda^{[n]}(\sigma^{-1}(\sigma(z) - i\beta)). \end{aligned}$$

It is easy to show that there exists a constant  $C_1$  which depends only on  $K$  and  $\beta$  such that

$$|z_1(n) - z_2(n)| \leq \frac{C_1}{\lambda^{2n}}.$$

In fact, if  $w_1(n) = \lambda^n(\sigma(z) - i\beta) + i\beta$  and  $w_2(n) = \lambda^n(\sigma(z) - i\beta)$ , then

$$\begin{aligned} |z_1(n) - z_2(n)| &= \left| \int_{[w_1(n), w_2(n)]} \frac{2}{(1+w)^2} dw \right| \leq |\beta| \max_{w \in [w_1(n), w_2(n)]} \frac{2}{|1+w|^2} \\ &\leq |\beta| \frac{2}{\lambda^{2n} [\min\{\operatorname{Re} \sigma(z) : z \in K\}]^2} \leq \frac{C_1}{\lambda^{2n}}. \end{aligned}$$

On the other hand, there is another constant  $C_2$ , depending only on  $K$  and  $\beta$  such that

$$|z_1(n)| \leq 1 - \frac{C_2}{\lambda^n} \text{ and } |z_2(n)| \leq 1 - \frac{C_2}{\lambda^n}.$$

This can be seen in the following way:

$$|1 - z_j(n)| = 1 - \left| \frac{w_j(n) - 1}{w_j(n) + 1} \right| = 2/|w_j(n) + 1| \geq \frac{C_2}{\lambda^n}.$$

Since  $B$  belongs to  $H^\infty(\mathbb{D})$ , Cauchy's inequalities show that  $B(z_1(n)) - B(z_2(n))$  converges uniformly on  $K$  to 0. In fact

$$\begin{aligned} |B(z_1(n)) - B(z_2(n))| &\leq \int_{[z_1(n), z_2(n)]} \frac{|B'(\xi)|(1-|\xi|^2)}{1-|\xi|^2} |d\xi| \\ &\leq |z_1(n) - z_2(n)| \frac{1}{\min\{1 - |z_1(n)|^2, 1 - |z_2(n)|^2\}} \leq \frac{C_3}{\lambda^n}. \end{aligned}$$

On the other hand, since  $B$  is universal, there exists a sequence  $(n_k)$  such that  $B \circ \phi_\lambda^{[n_k]}(\sigma^{-1}(\sigma - i\beta))$  converges uniformly to  $f$  on  $K$  (the map  $z \mapsto \sigma^{-1}(\sigma(z) - i\beta)$  is an automorphism of  $\mathbb{D}$ ). We conclude that  $B \circ \phi_{\lambda,\beta}^{[n_k]}$  converges uniformly on  $K$  to  $f$ .  $\square$

In order to construct a common universal Blaschke product for all the  $C_{\phi_\lambda}$ 's, we will decompose  $]1, +\infty[$  as an increasing union of compact sub-intervals  $[a_k, b_k]$ . Following [6], we then decompose each interval  $[a, b]$  as  $[a, b] = \cup_{j=1}^{q-1} [\lambda_j, \lambda_{j+1}]$  where  $\lambda_1 = a$ ,  $\lambda_2 = \lambda_1 + \frac{\delta}{2N}$ , ...,  $\lambda_{j+1} = \lambda_j + \frac{\delta}{(j+1)N}$  if  $\lambda_j + \frac{\delta}{(j+1)N} \leq b$  and  $\lambda_{j+1} = b$  if  $\lambda_j + \frac{\delta}{(j+1)N} > b$ . Here  $N$  is a positive integer which will be chosen very large in the sequel, and  $\delta$  is a positive real number which will be chosen very small. The interval  $[a, b]$  has been divided into  $q$  successive sub-intervals ( $q$  depending on  $\delta$  and  $N$ , of course). The interest of such a decomposition of  $[a, b]$  in our context lies in the following. Recall that  $\|f\|_K$  denotes the supremum of the function  $f$  on the compact set  $K$ .

**Lemma 3.** *Let  $f$  be a finite Blaschke product such that  $f(1) = f(-1) = 1$ . For every compact subset  $K$  of  $\mathbb{D}$  and each interval  $[a, b] \subseteq ]1, \infty[$ , there exists a positive constant  $M$  depending on  $K$ ,  $f$  and  $a$  such that for every  $j = 1, \dots, q$  and every  $\lambda \in [\lambda_j, \lambda_{j+1}[$  the following assertions are true:*

- (1) for every  $l < j$ ,  $\left\| C_{\phi_\lambda}^{jN} C_{\phi_{\lambda_l}}^{-lN}(f) - 1 \right\|_K \leq M a^{-(j-l)N}$ ;
- (2) for every  $l > j$ ,  $\left\| C_{\phi_\lambda}^{jN} C_{\phi_{\lambda_l}}^{-lN}(f) - 1 \right\|_K \leq M a^{-(l-j)N}$ ;
- (3)  $\left\| C_{\phi_\lambda}^{jN} C_{\phi_{\lambda_j}}^{-jN}(f) - f \right\|_K \leq M \delta$ .

We will use repeatedly the following fact, which follows from the Schwarz-Pick estimates:

**Lemma 4.** *Let  $u \in \mathcal{B}$ . Then for every  $z \in \mathbb{D}$ ,*

$$|u(z) - 1| \leq \frac{1+|z|}{1-|z|} |u(0) - 1|.$$

*Proof.* We obviously have that  $\frac{u(z)-u(0)}{1-u(0)u(z)} = zg(z)$  for some  $g \in \mathcal{B}$ . Hence

$$\begin{aligned} |u(z) - 1| &\leq |u(z) - u(0)| + |u(0) - 1| \leq |z| |1 - \overline{u(0)}u(z)| + |u(0) - 1| \\ &\leq |z| |(1 - \overline{u(0)}) + \overline{u(0)}(1 - u(z))| + |u(0) - 1|. \end{aligned}$$

Therefore  $|u(z) - 1|(1 - |z|) \leq |1 - u(0)|(1 + |z|)$  for every  $z \in \mathbb{D}$ .  $\square$

Thus in order to prove assertions 1 and 2 above, for instance, it suffices to control in a suitable way the quantities  $f(\phi_{\lambda_l}^{-[lN]}(\phi_\lambda^{[jN]}(0)))$ .

*Proof of Lemma 3.* For every  $z \in \mathbb{D}$  we have

$$C_{\phi_\lambda}^{jN} C_{\phi_{\lambda_l}}^{-lN}(f)(z) = f\left(\sigma^{-1}\left(\frac{\lambda^{jN}}{\lambda_l^{lN}}\sigma(z)\right)\right).$$

Since  $f(1) = 1$  and  $f$  is Lipschitz with constant  $C$  up to the boundary of  $\mathbb{D}$ , we have

$$\left| f(\phi_{\lambda_l}^{-[lN]}(\phi_\lambda^{[jN]}(0))) - 1 \right| \leq C \left| \sigma^{-1}\left(\frac{\lambda^{jN}}{\lambda_l^{lN}}\right) - 1 \right| = \frac{2C}{\frac{\lambda^{jN}}{\lambda_l^{lN}} + 1}.$$

Assertion 1 follows from this estimate: since  $l < j$ ,

$$\frac{\lambda^{jN}}{\lambda_l^{lN}} \geq \frac{\lambda^{jN}}{\lambda_{j-1}^{lN}} \geq \lambda_{j-1}^{(j-l)N} \left(1 + \frac{\delta}{\lambda_{j-1} N j}\right)^{Nj} \geq \lambda_{j-1}^{(j-l)N} \geq a^{(j-l)N}.$$

By Lemma 4, there exists a positive constant  $M_1$  such that

$$\|C_{\phi_\lambda}^{jN} C_{\phi_{\lambda_l}}^{-lN}(f) - 1\|_K \leq \frac{M_1}{a^{(j-l)N}} \text{ for } l < j.$$

Assertion 2 is proved in the same fashion, using this time the fact that  $f(-1) = 1$ , so that

$$\left|f(\phi_{\lambda_l}^{-[lN]} \circ \phi_\lambda^{[jN]}(0)) - 1\right| \leq 2C \frac{\frac{\lambda^{jN}}{\lambda_l^{lN}}}{\frac{\lambda^{jN}}{\lambda_l^{lN}} + 1}$$

and that for  $l > j$ ,

$$\frac{\lambda^{jN}}{\lambda_l^{lN}} \leq \lambda_{j+1}^{(j-l)N} \leq a^{(j-l)N}.$$

As to assertion 3, we have for every  $z \in \mathbb{D}$

$$\begin{aligned} \left|C_{\phi_\lambda}^{jN} C_{\phi_{\lambda_j}}^{-jN}(f)(z) - f(z)\right| &\leq C \left|\sigma^{-1}\left(\frac{\lambda^{jN}}{\lambda_j^{jN}} \sigma(z)\right) - z\right| \\ &\leq C \left|\frac{\lambda^{jN}}{\lambda_j^{jN}} - 1\right| \cdot \frac{2|\sigma(z)|}{\left|\frac{\lambda^{jN}}{\lambda_j^{jN}} \sigma(z) + 1\right|^2}. \end{aligned}$$

Since  $\left|\frac{\lambda^{jN}}{\lambda_j^{jN}} \sigma(z) + 1\right|$  is bigger than its real part, which is bigger than 1, and since

$$0 \leq \left(\frac{\lambda}{\lambda_j}\right)^{jN} - 1 \leq \left(1 + \frac{\delta}{aNj}\right)^{Nj} - 1 \leq 2\delta/a \text{ when } \delta \text{ is small enough, we have}$$

$$\|C_{\phi_\lambda}^{jN} C_{\phi_{\lambda_j}}^{-jN}(f) - f\|_K \leq M_3 \delta$$

for some positive constant  $M_3$ . □

We need a last lemma.

**Lemma 5.** *The finite Blaschke products  $f$  such that  $f(1) = f(-1) = 1$  are dense in  $\mathcal{B}$  (for the topology of uniform convergence on compact sets).*

*Proof.* We use Carathéodory's theorem that the set of finite Blaschke products is dense in  $\mathcal{B}$  and a special case of an interpolation result given in [9, p. Lemma 2.10] that tells us that for every  $\varepsilon > 0$ , every compact subset  $K \subseteq \mathbb{D}$  and  $\alpha, \beta \in \mathbb{T}$  there exists a finite Blaschke product  $B_1$  satisfying  $B_1(1) = \alpha$ ,  $B_1(-1) = \beta$  and  $\|B_1 - 1\|_K < \varepsilon$ . Thus, given  $f \in \mathcal{B}$  and a finite Blaschke product  $B_0$  that is close to  $f$  on  $K$ , we solve the interpolation problem with  $\alpha = \overline{B_0(1)}$

and  $\beta = \overline{B_0(-1)}$  and set  $B = B_0B_1$ , in order to get the desired Blaschke product.  $\square$

With these two lemmas in hand, we prove the following proposition:

**Proposition 6.** *Let  $(f_k)_{k \geq 1}$  be a dense sequence of finite Blaschke products with  $f_k(1) = f_k(-1) = 1$ . Let  $(K_k)$  be an exhaustive sequence of compact subsets of  $\mathbb{D}$ , and  $([a_k, b_k])_{k \geq 1}$  an increasing sequence of compact intervals such that*

$$\bigcup_{k \geq 1} [a_k, b_k] = ]1, +\infty[.$$

There exist

- a sequence  $(B_n)_{n \geq 1}$  of finite Blaschke products;
- an increasing sequence  $(p_n)_{n \geq 1}$  of positive integers

such that the following are satisfied for every  $k \geq 1$ :

- (1)  $B_k(1) = 1$ ;
- (2)  $\|B_k - 1\|_{K_k} < 2^{-k}$ ;
- (3) for every  $\lambda \in [a_k, b_k]$ , there exists an integer  $n_k(\lambda) \leq p_k$  such that for every  $i \geq k$ ,

$$(1) \quad \left\| C_{\phi_\lambda}^{n_k(\lambda)}(B_1 \dots B_i) - f_k \right\|_{K_k} < 2^{-k}.$$

As a corollary, we obtain:

**Corollary 7.** *There exists a Blaschke product  $B$  which is universal for all the composition operators  $C_{\phi_{\lambda, \beta}}$ ,  $\lambda > 1$ ,  $\beta \in \mathbb{R}$ .*

*Proof.* Consider  $B = \prod_{n=1}^{\infty} B_n$ : this is a convergent Blaschke product by property 2, and going to the limit as  $i$  goes to infinity in equation (1) implies that for every  $\lambda \in [a, b]$  and  $k$  large enough ( $[a, b] \subseteq [a_k, b_k]$ ),

$$\left\| C_{\phi_\lambda}^{n_k(\lambda)}(B) - f_k \right\|_{K_k} \leq 2^{-k}.$$

Since the family  $(f_k)_{k \geq 1}$  is locally uniformly dense in  $\mathcal{B}$ , this proves the universality of  $B$  for  $C_{\phi_\lambda}$ , hence for  $C_{\phi_{\lambda, \beta}}$ .  $\square$

We turn now to the proof of Proposition 6:

*Proof.* The proof is done by induction on  $k$ . We consider a first partition  $a_1 = \lambda_1 < \lambda_2 < \dots < \lambda_{q_1} = b_1$  of  $[a_1, b_1]$  with parameters  $N_1$  and  $\delta_1$ , and the finite Blaschke product

$$B_1 = \prod_{l=1}^{q_1} C_{\phi_{\lambda_l}}^{-lN_1}(f_1).$$

We have  $B_1(1) = 1$ . Since  $f_1(-1) = 1$ ,  $C_{\phi_{\lambda_l}}^{-lN_1}(f_1)$  tends to 1 uniformly on compact sets as  $N_1$  tends to infinity, and if  $N_1$  is large enough,

$$(2) \quad \|B_1 - 1\|_{K_1} < 2^{-1}.$$

Since  $|\prod_{j=1}^s a_j - \prod_{j=1}^s b_j| \leq \sum_{j=1}^s |a_j - b_j|$  whenever  $a_j, b_j \in \mathbb{D}$ , for every  $j = 1, \dots, q_1$  and every  $\lambda \in [\lambda_j, \lambda_{j+1}[$ , we have for any compact subset  $K$  of  $\mathbb{D}$

$$\begin{aligned} \left\| C_{\phi_\lambda}^{jN_1}(B_1) - f_1 \right\|_K &\leq \sum_{l=1, l \neq j}^{q_1} \left\| C_{\phi_\lambda}^{jN_1} C_{\phi_{\lambda_l}}^{-lN_1}(f_1) - 1 \right\|_K \\ &\quad + \left\| C_{\phi_\lambda}^{jN_1} C_{\phi_{\lambda_j}}^{-jN_1}(f_1) - f_1 \right\|_K. \end{aligned}$$

But by Lemma 3, the quantity on the righthand side is less than

$$\sum_{l=1, l \neq j}^{q_1} \frac{M}{a_1^{|j-l|N_1}} + M \delta_1 \leq 2M \sum_{k=1}^{\infty} \frac{1}{a_1^{kN_1}} + M \delta_1.$$

Thus if  $N_1$  is large enough and  $\delta_1$  small enough

$$(3) \quad \left\| C_{\phi_\lambda}^{jN_1}(B_1) - f_1 \right\|_{K_1} < 2^{-1}.$$

We now fix  $N_1$  large enough and  $\delta_1$  small enough so that inequalities (2) and (3) are satisfied. It is easy to check that assertions 2 and 3 of Proposition 6 are satisfied with  $p_1 = q_1 N_1$  and  $n_1(\lambda) = j N_1$  for  $\lambda \in [\lambda_j, \lambda_{j+1}[$ . This terminates the first step of the construction.

If now the construction has been carried out until step  $k-1$ , we consider again a partition  $a_k = \lambda_1 < \dots < \lambda_{q_k} = b_k$  of  $[a_k, b_k]$  with parameters  $\delta_k$  and  $N_k$ , and set

$$B_k = \prod_{l=1}^{q_k} C_{\phi_{\lambda_l}}^{-lN_k}(f_k),$$

so that  $B_k$  is a finite Blaschke product with  $B_k(1) = 1$ . Just as above if  $N_k$  is large enough and  $\delta_k$  small enough, we have for every  $j \leq q_k$ , every  $\lambda \in [\lambda_j, \lambda_{j+1}[$

$$\left\| C_{\phi_\lambda}^{jN_k}(B_k) - f_k \right\|_{K_k} < 2^{-(k+1)}$$

and

$$\|B_k - 1\|_{K_k} < 2^{-k}.$$

Because  $B_1(1) = \dots = B_{k-1}(1) = 1$ , we can also choose simultaneously  $N_k$  large enough so that  $C_{\phi_\lambda}^{jN_k}(B_1 \dots B_{k-1})$  is very close to 1 on  $K_k$ . This gives (1) for  $i = k$ .

It remains to check that if  $r \leq k-1$ ,  $\lambda \in [a_r, b_r]$ ,

$$\left\| C_{\phi_\lambda}^{n_r(\lambda)}(B_1 \dots B_{k-1} B_k) - f_r \right\|_{K_r} < 2^{-r}.$$

We already know that

$$\left\| C_{\phi_\lambda}^{n_r(\lambda)}(B_1 \dots B_{k-1}) - f_r \right\|_{K_r} < 2^{-r},$$



and since  $B_k$  can be made arbitrarily close to 1 on any compact set if  $N_k$  is large enough, we also choose  $N_k$  so that  $\|B_k - 1\|_K$  is small enough, where

$$K = \bigcup_{r \leq k-1, \lambda \in [a_r, b_r]} \phi_\lambda^{n_r(\lambda)}(K_r),$$

and then assertions 2 and 3 are satisfied at step  $k$ .  $\square$

### 3. THE PARABOLIC CASE

We consider now the family of parabolic automorphisms of  $\mathbb{D}$  with 1 as attractive fixed point. If  $T_\lambda : \mathbb{C}_+ \rightarrow \mathbb{C}_+$  is the translation defined by  $w \mapsto w + i\lambda$ ,  $\lambda \in \mathbb{R} \setminus \{0\}$ , then such parabolic automorphisms have the form  $\psi_\lambda(z) = \sigma^{-1} \circ T_\lambda \circ \sigma$ . Our aim in this section is to construct a Blaschke product which is universal for all composition operators  $(C_{\psi_\lambda})$ ,  $\lambda > 0$ . This is more difficult than the hyperbolic case, because we have no suitable analog of Lemma 3: the estimate we get has the form

$$\left\| C_{\psi_\lambda}^{jN} C_{\psi_{\lambda_l}}^{-lN}(f) - 1 \right\|_K \leq \frac{M}{|j-l|N} \quad \text{for } l \neq j,$$

and the series on the righthand side is not convergent when we sum over all  $l \neq j$ .

In other words if  $K$  is any compact set, the sets  $\psi_\lambda^{[n]}(K)$  go towards the point 1 at a rate of  $1/n$ , which is too slow. This difficulty was tackled for the study of common hypercyclicity on the Hardy space  $H^2(\mathbb{D})$  by using either a fine analysis of properties of disjointness in [4] or probabilistic ideas in [5]. Here we use in a crucial way the tangential convergence of the sequence  $(\psi_\lambda^{[n]}(0))$  to the boundary. Indeed, the series  $\sum_n (1 - |\psi_n(0)|)$  is summable, whereas the series  $\sum_n |1 - \psi_n(0)|$  is not. The following lemma will play the same role as Lemma 3 in the hyperbolic case. We keep the notation of Section 2 and use the same kind of decomposition  $a = \lambda_1, \dots, \lambda_q = b$  of a compact sub-interval  $[a, b]$  of  $]0, +\infty[$ .

**Lemma 8.** *Let  $f$  be a finite Blaschke product such that  $f(1) = 1$ . For every compact subset  $K$  of  $\mathbb{D}$ , there exists a positive constant  $M$  depending on  $K$ ,  $f$  and  $a$  such that for every  $j = 1, \dots, q$  and every  $\lambda \in [\lambda_j, \lambda_{j+1}[$  the following assertions are true:*

- (1) for every  $l < j$ ,  $\left\| |C_{\psi_\lambda}^{jN} C_{\psi_{\lambda_l}}^{-lN}(f)| - 1 \right\|_K \leq \frac{M}{(j-l)^2 N^2}$ ;
- (2) for every  $l > j$ ,  $\left\| |C_{\psi_\lambda}^{jN} C_{\psi_{\lambda_l}}^{-lN}(f)| - 1 \right\|_K \leq \frac{M}{(l-j)^2 N^2}$ ;
- (3)  $\left\| C_{\psi_\lambda}^{jN} C_{\psi_{\lambda_j}}^{-jN}(f) - f \right\|_K \leq M \delta$ .

*Proof.* In order to prove assertions 1 and 2, it suffices to work at the point 0. Since the modulus of  $f$  is equal to 1 on  $\mathbb{T}$ , and since the operators commute, we have

$$\begin{aligned} \left| |f(\psi_{\lambda_l}^{-[lN]} \circ \psi_{\lambda}^{[jN]}(0))| - 1 \right| &= \left| |f(\psi_{\lambda}^{[jN]} \circ \psi_{\lambda_l}^{-[lN]}(0))| \right. \\ &\quad \left. - \left| f\left(\frac{\psi_{\lambda}^{[jN]} \circ \psi_{\lambda_l}^{-[lN]}(0)}{|\psi_{\lambda}^{[jN]} \circ \psi_{\lambda_l}^{-[lN]}(0)|}\right) \right| \right|. \end{aligned}$$

Since  $f$  is  $C$ -Lipschitz on  $\overline{\mathbb{D}}$  for some positive constant  $C$ , this quantity is less than

$$(4) \quad C \left| \psi_{\lambda}^{[jN]} \circ \psi_{\lambda_l}^{-[lN]}(0) - \frac{\psi_{\lambda}^{[jN]} \circ \psi_{\lambda_l}^{-[lN]}(0)}{|\psi_{\lambda}^{[jN]} \circ \psi_{\lambda_l}^{-[lN]}(0)|} \right| = C \left( 1 - |\psi_{\lambda}^{[jN]} \circ \psi_{\lambda_l}^{-[lN]}(0)| \right).$$

An easy computation shows that

$$\psi_{\lambda}^{[jN]} \circ \psi_{\lambda_l}^{-[lN]}(0) = \frac{iN(j\lambda - l\lambda_l)}{2 + iN(j\lambda - l\lambda_l)}.$$

Observe that  $|j\lambda - l\lambda_l| \geq |j - l|a$ . This gives

$$\begin{aligned} 1 - |\psi_{\lambda}^{[jN]} \circ \psi_{\lambda_l}^{-[lN]}(0)| &\leq 1 - |\psi_{\lambda}^{[jN]} \circ \psi_{\lambda_l}^{-[lN]}(0)|^2 \\ &\leq 1 - \frac{(j\lambda - l\lambda_l)^2 N^2}{4 + (j\lambda - l\lambda_l)^2 N^2} \\ &\leq \frac{C_1}{N^2(j-l)^2} \end{aligned}$$

for some positive constant  $C_1$  which does not depend on  $\lambda$ . Now equation (4) implies that for  $l \neq j$ ,

$$\left| |f(\psi_{\lambda}^{[jN]} \circ \psi_{\lambda_l}^{-[lN]}(0))| - 1 \right| \leq \frac{C_2}{N^2(j-l)^2}.$$

This proves assertions 1 and 2 of Lemma 8. Assertion 3 is proved in the same way as in Lemma 3, writing for  $\lambda \in [\lambda_j, \lambda_{j+1}[$

$$\left\| C_{\psi_{\lambda}}^{jN} C_{\psi_{\lambda_j}}^{-jN} f(z) - f(z) \right\| \leq CjN|\lambda_j - \lambda| \leq CjN \frac{\delta}{(j+1)N} \leq C'\delta.$$

□

The following proposition is the main ingredient of the proof:

**Proposition 9.** *Let  $f$  be a finite Blaschke product,  $[a, b] \subset ]0, +\infty[$ ,  $m_0$  a positive integer,  $K$  a compact subset of  $\mathbb{D}$  and  $\varepsilon > 0$ . There exist an integer  $m$ , integers  $(n(\lambda))_{\lambda \in [a, b]}$  with  $n(\lambda) \in [m_0, m]$  and a finite Blaschke product  $B$  such that*

- (1)  $B(1) = 1$ ;
- (2)  $\|B - 1\|_K < \varepsilon$ ;

- (3)  $C_{\psi_\lambda}^{n(\lambda)}(B) = u_\lambda v_\lambda$  where  $u_\lambda$  and  $v_\lambda$  belong to  $\mathcal{B}$ ,  $\|u_\lambda - f\|_K < \varepsilon$  and  $|v_\lambda(0)| > 1 - \varepsilon$ .

*Proof.* We use again the decomposition  $\lambda_1 = a$ ,  $\lambda_2 = a + \frac{\delta}{2N}, \dots, \lambda_q = b$ , where  $\delta > 0$  and  $N \geq m_0$ , will be fixed during the proof. Consider the Blaschke product

$$B_1 = \prod_{l=1}^q C_{\psi_{\lambda_l}}^{-lN}(f).$$

For  $\lambda \in [\lambda_j, \lambda_{j+1}[$ , we have

$$C_{\psi_\lambda}^{jN}(B) = C_{\psi_\lambda}^{jN} C_{\psi_{\lambda_j}}^{-jN}(f) \left( \prod_{l \neq j} C_{\psi_\lambda}^{jN} C_{\psi_{\lambda_l}}^{-lN}(f) \right) := u_{1,\lambda} v_{1,\lambda}$$

with  $u_{1,\lambda} = C_{\psi_\lambda}^{jN} C_{\psi_{\lambda_j}}^{-jN}(f)$  and  $v_{1,\lambda} = \prod_{l \neq j} C_{\psi_\lambda}^{jN} C_{\psi_{\lambda_l}}^{-lN}(f)$ . By assertion 3 of Lemma 8,  $\|u_{1,\lambda} - f\|_K \leq M\delta < \varepsilon$  if  $\delta$  is small enough. Moreover, still by Lemma 8,

$$\begin{aligned} 1 - |v_{1,\lambda}(0)| &= 1 - \prod_{l \neq j} |f(\psi_\lambda^{[jN]} \circ \psi_{\lambda_l}^{-[lN]}(0))| \\ &\leq \sum_{l \neq j} \left( 1 - |f(\psi_\lambda^{[jN]} \circ \psi_{\lambda_l}^{-[lN]}(0))| \right) \\ &\leq \frac{C'}{N^2} \end{aligned}$$

for some positive constant  $C'$ . Thus if  $N$  is large enough,  $|v_{1,\lambda}(0)| > 1 - \varepsilon$ . To conclude, it remains to observe that the same proof leads to

$$|B_1(0)| \geq 1 - \frac{C''}{N^2}$$

for some positive constant  $C''$ , so that using Lemma 4 and adjusting  $N$  large enough, there exists a real number  $\theta$  such that  $\|e^{i\theta} B_1 - 1\|_K < \varepsilon$ . If we set  $B_2 = e^{i\theta} B_1$ , then  $B_2$  satisfies the conclusions of the proposition (setting  $u_{2,\lambda} = u_{1,\lambda}$  and  $v_{2,\lambda} = e^{i\theta} v_{1,\lambda}$ , except that we are not sure that  $B_2(1) = 1$ ). To conclude, let  $F$  be a finite Blaschke product such that  $F$  is very close to 1 on a big compact set  $L \subset \mathbb{D}$  and  $F(1) = \overline{B_2(1)}$ . Then  $B = FB_2$  is the finite Blaschke product we are looking for. Indeed, setting  $u_\lambda = u_{2,\lambda}$  and  $v_\lambda = C_{\psi_\lambda}^{n(\lambda)}(F)v_{2,\lambda}$ , condition 3. is satisfied, provided  $L$  is big enough to contain each  $\psi_\lambda^{n(\lambda)}(0)$ . □

We can now proceed with the construction:

**Proposition 10.** *Let  $(f_k)_{k \geq 1}$  be a dense sequence of finite Blaschke products with  $f_k(1) = 1$ . Let  $(K_k)$  be an exhaustive sequence of compact subsets of  $\mathbb{D}$ , and  $([a_k, b_k])_{k \geq 1}$  an increasing sequence of compact intervals such that*

$$\bigcup_{k \geq 1} [a_k, b_k] = ]1, +\infty[.$$

*There exist finite Blaschke products  $(B_k)$ , integers  $(m_k)$ , and other integers  $(n_k(\lambda))_{\lambda \in [a_k, b_k]}$  with  $n_k(\lambda) \leq m_k$  and such that*

- (1)  $B_k(1) = 1$ ;
- (2)  $\|B_k - 1\|_{K_k} < 2^{-k}$ ;
- (3) for every  $j < k$ , every  $\lambda \in [a_k, b_k]$ ,  $|B_j \circ \psi_\lambda^{[n_k(\lambda)]}(0) - 1| < 2^{-k}$ ;
- (4) for every  $j < k$ , every  $\lambda \in [a_j, b_j]$ ,  $|B_k \circ \psi_\lambda^{[n_j(\lambda)]}(0) - 1| < 2^{-k}$ ;
- (5) for every  $\lambda \in [a_k, b_k]$ ,  $C_{\psi_\lambda}^{n_k(\lambda)}(B_k) = u_{k,\lambda} v_{k,\lambda}$  where

$$\|u_{k,\lambda} - f_k\|_{K_k} < 2^{-k} \text{ and } |v_{k,\lambda}(0)| > 1 - 2^{-k}.$$

*Proof.* The first step of the construction follows directly from Proposition 9. Now we assume that the construction has been done until step  $k - 1$  and show how to complete step  $k$ . By continuity at the point 1 of the functions  $(B_j)_{j < k}$ , we choose an integer  $m$  such that, for every  $\lambda \in [a_k, b_k]$ , for any  $n \geq m$ ,  $|B_j \circ \psi_\lambda^{[n]}(0) - 1| < 2^{-k}$ . We then set

$$K = K_k \cup \bigcup_{j < k, \lambda \in [a_j, b_j], n \leq m_j} \left\{ \psi_\lambda^{[n]}(0) \right\}.$$

The function  $B_k$  is then given immediately by Proposition 9.  $\square$

**Corollary 11.** *There exists a Blaschke product  $B$  which is universal for all the composition operators  $C_{\psi_\lambda}$ ,  $\lambda > 0$ .*

*Proof.* Set

$$B = \prod_{l \geq 1} B_l,$$

which is a convergent Blaschke product by assertion 2 of Proposition 10. We claim that  $B$  is  $\mathcal{B}$ -universal with respect to every composition operator  $C_{\psi_\lambda}$ . Indeed, fix  $\lambda > 0$  and  $k_0$  such that  $\lambda \in [a_{k_0}, b_{k_0}]$ . Let  $g$  be a universal function for *this* particular operator  $C_{\psi_\lambda}$ . Using the notation of Proposition 10, let  $(p_k)$  be an increasing sequence of integers such that  $f_{p_k}$  converges uniformly to  $g$  on compact subsets of  $\mathbb{D}$ . Now we decompose

$$C_{\psi_\lambda}^{n_{p_k}(\lambda)}(B) = C_{\psi_\lambda}^{n_{p_k}(\lambda)}(B_{p_k}) \left( \prod_{j \neq p_k} B_j \circ \psi_\lambda^{[n_{p_k}(\lambda)]} \right) := u_{p_k,\lambda} v_{p_k,\lambda} w_{p_k,\lambda}$$

where  $C_{\psi_\lambda}^{[n_{p_k}(\lambda)]}(B_{p_k}) = u_{p_k,\lambda} v_{p_k,\lambda}$  is the decomposition of Proposition 9. From assertions 3 and 4 of Proposition 10, we get that  $w_{p_k,\lambda}(0)$  tends to 1 (see [2])

for details), so that (cf. Fact 4)  $w_{p_k, \lambda}$  converges uniformly on compacta to 1. Taking a subsequence if necessary, we can assume that  $v_{p_k, \lambda}(0)$  converges to some unimodular number  $e^{i\theta}$ , and by Fact 4 again we have uniform convergence on compacta. Thus  $C_{\psi_\lambda}^{n_{p_k}(\lambda)}(B)$  converges uniformly to the function  $e^{i\theta}g$  on compacta. Since the function  $e^{i\theta}g$  is universal for  $C_{\psi_\lambda}$ , this implies that  $B$  is universal for  $C_{\psi_\lambda}$  too, and this terminates the proof of Corollary 11.  $\square$

The proof of Theorem 1 is now concluded by “intertwining” the two proofs of the hyperbolic and parabolic cases: the common universal Blaschke product has the form

$$B = \prod_{l \geq 1} B_l$$

where the  $B_l$ 's are finite Blaschke products satisfying a number of properties:  $B_1$  is constructed using Proposition 6, then  $B_2$  using Proposition 10, then  $B_3$  using Proposition 6 again, etc... taking care at each step not to destroy what has been done previously. Details are left to the reader.

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INSTITUT DE MATHÉMATIQUES BORDELAIS, UMR 5251, UNIVERSITÉ BORDEAUX 1, 351  
COURS DE LA LIBÉRATION, 33405 TALENCE CEDEX, FRANCE  
*E-mail address:* [bayart@math.u-bordeaux1.fr](mailto:bayart@math.u-bordeaux1.fr)

LABORATOIRE PAUL PAINLEVÉ, UMR 8524, UNIVERSITÉ DES SCIENCES ET TECHNOLOGIES  
DE LILLE, CITÉ SCIENTIFIQUE, 59655 VILLENEUVE D'ASCQ CEDEX, FRANCE  
*E-mail address:* [grivaux@math.univ-lille1.fr](mailto:grivaux@math.univ-lille1.fr)

LABORATOIRE DE MATHÉMATIQUES ET APPLICATIONS DE METZ, UMR 7122, UNIVERSITÉ  
PAUL VERLAINE, ILE DU SAULCY, 57045 METZ, FRANCE  
*E-mail address:* [mortini@math.univ-metz.fr](mailto:mortini@math.univ-metz.fr)