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Upper oriented chromatic number of undirected graphs and oriented colorings of product graphs

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Abstract

The oriented chromatic number of an oriented graph $\overrightarrow{G}$ is the minimum order of an oriented graph $\overrightarrow{H}$ such that $\overrightarrow{G}$ admits a homomorphism to $\overrightarrow{H}$. The oriented chromatic number of an undirected graph G is then the greatest oriented chromatic number of its orientations.

In this paper, we introduce the new notion of the *upper oriented chromatic number* of an undirected graph G, defined as the minimum order of an oriented graph $\overrightarrow{U}$ such that *every* orientation $\overrightarrow{G}$ of G admits a homomorphism to $\overrightarrow{U}$. We give some properties of this parameter, derive some general upper bounds on the ordinary and upper oriented chromatic numbers of Cartesian, strong, direct and lexicographic products of graphs, and consider the particular case of products of paths.

1 Introduction

All the graphs we consider in this paper are simple, with no loops or multiple edges. An *oriented graph* $\overrightarrow{G} = (V(\overrightarrow{G}), E(\overrightarrow{G}))$ is an antisymmetric digraph obtained from an undirected graph $G = (V(G), E(G))$ having the same set of vertices, $V(G) = V(\overrightarrow{G})$, by giving to each edge $\{u, v\}$ in $E(G)$ one of its two possible orientations, (u, v) or (v, u). Such an oriented graph $\overrightarrow{G}$ is said to be an *orientation* of G.

Let G and H be two undirected graphs. A *homomorphim* from G to H is a mapping $h : V(G) \longrightarrow V(H)$ such that for every edge $\{u, v\}$ in $E(G)$, $\{h(u), h(v)\}$ is an edge in H. We write $G \to H$ whenever such a homomorphism exists. Homomorphisms of oriented graphs are defined similarly, by using arcs instead of edges.

A *proper k-coloring* of an undirected graph G is a mapping $c : V(G) \longrightarrow \{1, 2, \ldots, k\}$ such that $c(u) \neq c(v)$ for every edge $\{u, v\}$ in $E(G)$. Such a coloring can also be viewed as a partition of $V(G)$ into k disjoint independent sets $V_1, \ldots, V_k$. The *chromatic number* $\chi(G)$ of G is then the smallest k for which G admits a proper k-coloring. It is easy to see that $\chi(G)$ also corresponds to the smallest k for which $G \to K_k$, where K_k stands for the complete graph of order k.

An *oriented k-coloring* of an oriented graph $\overrightarrow{G}$ is a partition of $V(\overrightarrow{G})$ into k disjoint independent sets, such that all arcs linking any two of these sets *have the same direction.*

Such an oriented coloring is thus a mapping $\gamma : V(\overrightarrow{G}) \longrightarrow \{1, 2, \ldots, k\}$ such that $\gamma(u) \neq \gamma(v)$ for every arc (u, v) in $E(\overrightarrow{G})$ and $\gamma(u) \neq \gamma(x)$ whenever there exist two arcs (u, v) and (w, x) in $E(\overrightarrow{G})$ with $\gamma(v) = \gamma(w)$ (v and w are not necessarily distinct).

The *oriented chromatic number* $\chi_o(\overrightarrow{G})$ of an oriented graph $\overrightarrow{G}$ is then defined as the smallest k for which $\overrightarrow{G}$ admits an oriented k-coloring. As before, $\chi_o(\overrightarrow{G})$ also corresponds to the smallest order of an oriented graph $\overrightarrow{T}$ for which $\overrightarrow{G} \to \overrightarrow{T}$. If G is an *undirected* graph, the oriented chromatic number $\chi_o(G)$ of G is defined as the highest oriented chromatic number of its orientations:

$$\chi_o(G) = \max \{ \ \chi_o(\overrightarrow{G}), \ \overrightarrow{G} \text{ is an orientation of } G \ \}.$$

Most of the papers devoted to oriented colorings were concerned with upper bounds on the oriented chromatic number of special classes of graphs [1, 2, 3, 5, 7, 9, 11, 12, 14]. In many cases, such bounds were obtained by proving that every orientation of any graph in a given class admits a homomorphism to some specific oriented graph (sometimes a tournament). This observation motivates the introduction of a new parameter, that we call the *upper oriented chromatic number* of an undirected graph G, defined as the smallest order of an oriented graph $\overrightarrow{T}$ such that $\overrightarrow{G} \to \overrightarrow{T}$ for *every* orientation $\overrightarrow{G}$ of G.

The aim of this paper is to initiate the study of this new parameter. We shall give general bounds on the upper and ordinary oriented chromatic numbers of lexicographic, strong, Cartesian and direct products of undirected and oriented graphs, with a particular focus on products of paths.

This paper is organised as follows. In Section 2, we give the main definitions we shall use later and provide some elementary properties of the upper oriented chromatic number. The four following sections are respectively concerned with lexicographic, strong, Cartesian and direct products of graphs, and contain our main results. Some open problems and directions for future work are discussed in Section 7.

2 Definitions and notation

We denote by P_k the path on k vertices, by C_k the cycle on k vertices and by K_k the complete graph on k vertices. If G is an undirected graph, we denote by $\{u, v\}$ an edge linking vertices u and v in G. If $\overrightarrow{G}$ is an oriented graph, we denote by (u, v) an arc directed from vertex u to vertex v in $\overrightarrow{G}$. A *directed path* of length k in an oriented graph $\overrightarrow{G}$ is a sequence of vertices $x_1 \ldots x_k$ such that $(x_i, x_{i+1}) \in E(\overrightarrow{G})$ for every i, $1 \leq i < k$. We shall denote by $\overrightarrow{DP_k}$ the particular orientation of the path P_k corresponding to the directed path of length k, while $\overrightarrow{P_k}$ will denote any orientation of P_k. For any two sets A and B, the Cartesian product $A \times B$ denotes the set $\{ \ [a, b], \ a \in A, \ b \in B \ \}$.

An *oriented clique* $\overrightarrow{H}$ is an oriented graph in which every pair of vertices is joined by a directed path of length 1 or 2. From the definition of oriented colorings, it follows that $\chi_o(\overrightarrow{H}) = |V(\overrightarrow{H})|$, whenever $\overrightarrow{H}$ is an oriented clique.

We define the *upper oriented chromatic number* of an undirected graph G, denoted $\chi_o^+(G)$, as the smallest order of an oriented graph $\overrightarrow{T}$ such that $\overrightarrow{G} \to \overrightarrow{T}$ for every orientation $\overrightarrow{G}$ of G. The property of having upper oriented chromatic number at most k is *hereditary*, that is $\chi_o^+(H) \leq \chi_o^+(G)$ for every subgraph H of G.

From this definition, we clearly have:

Proposition 1 *For every undirected graph G, $\chi_o(G) \leq \chi_o^+(G)$.*

Consider for instance the cycle C_3 on three vertices. We have $\chi_o(C_3) = 3$ since $\chi_o(G) \leq |V(G)|$ for every undirected graph G and any two vertices in any orientation $\overrightarrow{C_3}$ of C_3 are linked by a directed path of length 1 or 2. However, $\chi_o^+(C_3) = 4$ since the smallest oriented graph contained both a directed 3-cycle and a transitive 3-cycle as subgraphs has four vertices.

More generally, for every $n \geq 3$, $\chi_o(K_n) = n$ and $\chi_o^+(K_n) = \varepsilon(n)$, where $\varepsilon(n)$ stands for the minimum size of an *n-universal* tournament, that is a tournament containing every tournament of order n as a subgraph. The following result is proved in [6, Chapter 17]:

Theorem 2 *For every $n \geq 1$,*

$$2^{\frac{n-1}{2}} \leq \varepsilon(n) \leq \begin{cases} n\,2^{\frac{n-1}{2}} & \text{if } n \text{ is odd,} \\ \frac{3}{2\sqrt{2}}\,n\,2^{\frac{n-1}{2}} & \text{if } n \text{ even.} \end{cases}$$

Hence, the difference between $\chi_o(G)$ and $\chi_o^+(G)$ can be arbitrarily large.

For every graph G, let $\omega(G)$ denote the *clique number* of G, that is the maximum order of a complete subgraph of G. From Theorem 2, we get the following general lower and upper bounds on the upper oriented chromatic number of any graph:

Corollary 3 *For every graph G of order n and clique number $\omega(G)$,*

$$\chi_o^+(G) \geq \chi_o^+(K_{\omega(G)}) \geq 2^{\frac{\omega(G)-1}{2}}$$

and

$$\chi_o^+(G) \leq \chi_o^+(K_n) \leq \begin{cases} n\,2^{\frac{n-1}{2}} & \text{if } n \text{ is odd,} \\ \frac{3}{2\sqrt{2}}\,n\,2^{\frac{n-1}{2}} & \text{if } n \text{ even.} \end{cases}$$

For complete bipartite graphs, we have the following:

Theorem 4 *For every m, n with $m \leq n$,*

$$\chi_o(K_{m,n}) = m + \min\{n, 2^m\}$$

and

$$\chi_o^+(K_{m,n}) \leq m + 2^m.$$

Proof. Let $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_n\}$ denote the two maximal independent sets of $K_{m,n}$. We first consider $\chi_o^+(K_{m,n})$. Let $\overrightarrow{T}$ be the oriented graph defined by

$$V(\overrightarrow{T}) = \{a_1, \ldots, a_m\} \cup \{b_S,\ S \subseteq \{1, 2, \ldots, m\}\}$$

and $(a_i, b_S) \in E(\overrightarrow{T})$ if and only if $i \in S$. Let now $\overrightarrow{K}_{m,n}$ be any orientation of $K_{m,n}$. For every j, $1 \leq j \leq n$, let $N^-(y_j) = \{x_i,\ (x_i, y_j) \in E(\overrightarrow{K}_{m,n})\}$. Clearly, the mapping $\varphi : V(\overrightarrow{K}_{m,n}) \longrightarrow V(\overrightarrow{T})$ defined by $\varphi(x_i) = a_i$ for every i, $1 \leq i \leq m$, and $\varphi(y_j) = b_{N^-(y_j)}$ for every j, $1 \leq j \leq n$, is a homomorphism and, therefore, $\chi_o^+(K_{m,n}) \leq m + 2^m$.

We now consider $\chi_o^+(K_{m,n})$. Suppose first that $n \geq 2^m$ and let $S_1, \ldots, S_{2^m}$ denote the 2^m subsets of $\{1, 2, \ldots, m\}$. Let now $\overrightarrow{K}_{m,n}$ be the orientation of $K_{m,n}$ defined by $(x_i, y_j) \in E(\overrightarrow{K}_{m,n})$ if and only if $j \leq 2^m$ and $i \in S_j$, and let $\overrightarrow{X}$ denote the subgraph of $\overrightarrow{K}_{m,n}$ induced by the set of vertices $\{x_1, \ldots, x_m\} \cup \{y_1, \ldots, y_{2^m}\}$. The subgraph $\overrightarrow{X}$ is clearly an oriented clique and, therefore, $\chi_o(K_{m,n}) \geq \chi_o(\overrightarrow{X}) = m + 2^m$. On the other hand, $\chi_o(K_{m,n}) \leq \chi_o^+(K_{m,n}) \leq m + 2^m$.

If $n < 2^m$, we get $\chi_o(K_{m,n}) \geq |V(K_{m,n})| = m + n$ by considering the orientation of $K_{m,n}$ corresponding to the subgraph of the above defined oriented graph $\overrightarrow{X}$ induced by the set of vertices $\{x_1, \ldots, x_m\} \cup \{y_1, \ldots, y_n\}$. Since $\chi_o(K_{m,n}) \leq |V(K_{m,n})|$, the result follows. $\square$

As said before, many upper bounds on the oriented chromatic number of classes of undirected graphs have been obtained by providing some oriented graph $\overrightarrow{T}$ such that every orientation of every undirected graph in the class admits a homomorphism to $\overrightarrow{T}$. Therefore, every such upper bound also holds for the upper oriented chromatic number of the same graphs. The following theorem summarizes some of these results. Let us recall that a graph is *outerplanar* if it has a planar drawing with all vertices lying on the outer face. A graph is *2-outerplanar* whenever it admits a planar drawing such that deleting all the vertices of the outer face produces an outerplanar graph. The *acyclic chromatic number* of an undirected graph G is the smallest number of colors needed in an *acyclic coloring* of G, that is a proper coloring of G in which every cycle uses at least three colors.

Theorem 5 *Let G be an undirected graph. We then have:*

1. *If G is a forest with at least three vertices, then $\chi_o^+(G) = 3$ [12],*
2. *If $G = C_k$ is a cycle on $k \geq 3$ vertices, then $\chi_o^+(G) = 4$ when $k \neq 5$, and $\chi_o^+(G) = 5$ when $k = 5$ [12],*
3. *If $\chi_a(G) \leq a$, then $\chi_o^+(G) \leq a2^{a-1}$ [11], and this bound is tight for $a \geq 3$ [9],*
4. *If G is an outerplanar graph, then $\chi_o^+(G) \leq 7$ and this bound is tight [12],*
5. *If G is a 2-outerplanar graph, then $\chi_o^+(G) \leq 67$ [2],*
6. *If G is a planar graph, then $\chi_o^+(G) \leq 80$ [11],*
7. *If G is a triangle-free planar graph, then $\chi_o^+(G) \leq 59$ [8],*
8. *If G has maximum degree $\Delta(G) = k$, then $\chi_o^+(G) \leq 2k^2 2^k$ [5].*

Let G be an undirected graph. The *square* of G is the graph G^2 defined by $V(G^2) = V(G)$ and $E(G^2) = \{\{u, v\},\ 1 \leq d_G(u, v) \leq 2\}$, where $d_G(u, v)$ denotes the distance between vertices u and v in G. The following theorem provides an upper bound on the upper oriented chromatic number of an undirected graph depending on the chromatic number of its square:

Proposition 6 *For every undirected graph G with $k = \chi(G^2)$, $\chi_o^+(G) \leq 2^k - 1$.*

Proof. Let σ be a proper k-coloring of G^2 and S be the set of $2^k - 1$ elements defined as

$$S = \{[a, b_1, \ldots, b_{a-1}],\ 1 \leq a \leq k,\ b_i \in \{0, 1\},\ 1 \leq i \leq a - 1\}\,.$$

Let now $\overrightarrow{G}$ be any orientation of G. We define a mapping φ from $V(G)$ to S as follows. For every vertex $u \in V(G)$, we set $\varphi(u) = [a(u), b_1(u), \ldots, b_{a(u)-1}(u)]$ where:

1. $a(u) = \sigma(u)$,

2. for every i, $1 \leq i \leq a(u) - 1$, if u has a neighbor v with $\sigma(v) = i$ such that (v, u) is an arc in $E(\overrightarrow{G})$ then $a_i(u) = 1$, otherwise $a_i(u) = 0$. (Since σ is a proper coloring of G^2, if such a vertex v exists it must be unique.)

We claim that φ is an oriented coloring of $\overrightarrow{G}$. Observe first that if u and v are adjacent vertices in $\overrightarrow{G}$, then $\varphi(u) \neq \varphi(v)$ since $\sigma(u) \neq \sigma(v)$. Suppose now that there exist two arcs (u, v) and (w, x) in $E(\overrightarrow{G})$ such that $\varphi(u) = \varphi(x)$ and $\varphi(v) = \varphi(w)$. If $u = x$, then $\sigma(v) \neq \sigma(w)$ since σ is a proper coloring of G^2, a contradiction. Otherwise, we may assume without loss of generality that $i = \sigma(u) = \sigma(x) < \sigma(v) = \sigma(w)$. Since $(u, v), (w, x) \in E(\overrightarrow{G})$, we get $b_i(v) = 1$ and $b_i(w) = 0$, again a contradiction.

The mapping φ is thus an oriented coloring of $\overrightarrow{G}$ using at most $2^k - 1$ colors and the result follows. $\square$

Since $\chi(G^2) \geq \Delta(G) + 1$ for every graph G, Proposition 6 does not give an interesting general bound for classes of graphs with unbounded degree.

Let G and H be two undirected graphs. The *Cartesian product* of G and H is the undirected graph $G \,\square\, H$ defined by $V(G \,\square\, H) = V(G) \times V(H)$ and $\{[u, v], [u', v']\}$ is an edge in $E(G \,\square\, H)$ if and only if either $u = u'$ and $\{v, v'\} \in E(H)$ or $v = v'$ and $\{u, u'\} \in E(G)$.

The *strong product* of G and H is the undirected graph $G \boxtimes H$ defined by $V(G \boxtimes H) = V(G) \times V(H)$ and $\{[u, v], [u', v']\}$ is an edge in $E(G \boxtimes H)$ if and only if either $u = u'$ and $\{v, v'\} \in E(H)$ or $v = v'$ and $\{u, u'\} \in E(G)$ or $\{u, u'\} \in E(G)$ and $\{v, v'\} \in E(H)$.

The *direct product* of G and H is the undirected graph $G \times H$ defined by $V(G \times H) = V(G) \times V(H)$ and $\{[u, v], [u', v']\}$ is an edge in $E(G \times H)$ if and only if $\{u, u'\} \in E(G)$ and $\{v, v'\} \in E(H)$.

The *lexicographic product* of G and H is the undirected graph $G[H]$ defined by $V(G[H]) = V(G) \times V(H)$ and $\{[u, v], [u', v']\}$ is an edge in $E(G[H])$ if and only if either $\{u, u'\} \in E(G)$ or $u = u'$ and $\{v, v'\} \in E(H)$.

Cartesian, strong, direct and lexicographic products of *oriented* graphs are defined similarly, by replacing edges by arcs in the above definitions.

It is not difficult to see that the Cartesian, strong and direct products are symmetric operations while $G[H]$ and $H[G]$ are generally not isomorphic graphs (this justifies our notation for the lexicographic product). However, these four products are associative. Moreover, $G \,\square\, H \subseteq G \boxtimes H$, $G \times H \subseteq G \boxtimes H$ and $G \boxtimes H \subseteq G[H]$ for every undirected or oriented graphs G and H. Therefore, every upper bound on the upper (or ordinary) oriented chromatic number of $G[H]$ (resp. $G \boxtimes H$) holds for $G \boxtimes H$ (resp. $G \,\square\, H$ and $G \times H$).

Following the reference book of Imrich and Klavžar [4], we shall denote by G_v (resp. H_u) the *v-layer* (resp. the *u-layer*) of $G \boxtimes H$, $G \,\square\, H$ or $G \times H$, that is the subgraph induced by $V(G) \times \{v\}$ (resp. $\{u\} \times V(H)$), for every $v \in V(H)$ (resp. $u \in V(G)$).

The study of the oriented chromatic number of Cartesian and strong products of graphs has been recently initiated by Natarajan, Narayanan and Subramanian [7]. In particular, they proved that for every undirected graph G, $\chi_o(G \,\square\, P_k) \leq (2k - 1)\chi_o(G)$ and $\chi_o(G \,\square\, C_k) \leq 2k\chi_o(G)$ for every $k \geq 3$. We shall improve these two bounds in Section 5.

3 Lexicographic products

Concerning the oriented chromatic number of the lexicographic product of oriented graphs, we have the following:

Theorem 7 *If $\overrightarrow{G}$, $\overrightarrow{H}$, $\overrightarrow{T}$ and $\overrightarrow{U}$ are oriented graphs such that $\overrightarrow{G} \to \overrightarrow{T}$ and $\overrightarrow{H} \to \overrightarrow{U}$, then $\overrightarrow{G}[\overrightarrow{H}] \to \overrightarrow{T}[\overrightarrow{U}]$. Therefore, for every oriented graphs $\overrightarrow{G}$ and $\overrightarrow{H}$,*

$$\chi_o(\overrightarrow{G}[\overrightarrow{H}]) \leq \chi_o(\overrightarrow{G})\chi_o(\overrightarrow{H}).$$

Proof. Let $\alpha : \overrightarrow{G} \to \overrightarrow{T}$ and $\beta : \overrightarrow{H} \to \overrightarrow{U}$ be two homomorphisms. Let now $\varphi : V(\overrightarrow{G}[\overrightarrow{H}]) \longrightarrow V(\overrightarrow{T}[\overrightarrow{U}])$ be the mapping defined by $\varphi([u,v]) = [\alpha(u), \beta(v)]$ for every vertex $[u,v]$ in $V(\overrightarrow{G}[\overrightarrow{H}])$.

We claim that φ is a homomorphism. To see this, let $([u,v],[u',v'])$ be an arc in $\overrightarrow{G}[\overrightarrow{H}]$. We then have $\varphi([u,v]) = [\alpha(u), \beta(v)]$ and $\varphi([u',v']) = [\alpha(u'), \beta(v')]$. If $(u,u') \in E(\overrightarrow{G})$, then $(\alpha(u), \alpha(u')) \in E(\overrightarrow{T})$. If $u = u'$ and $(v,v') \in E(\overrightarrow{H})$, then $\alpha(u) = \alpha(u')$ and $(\beta(v), \beta(v')) \in E(\overrightarrow{U})$. Therefore, every arc in $\overrightarrow{G}[\overrightarrow{H}]$ is mapped to an arc in $\overrightarrow{T}[\overrightarrow{U}]$ and φ is a homomorphism.

The inequality $\chi_o(\overrightarrow{G}[\overrightarrow{H}]) \leq \chi_o(\overrightarrow{G})\chi_o(\overrightarrow{H})$ directly follows from the definition of the oriented chromatic number. $\square$

For the lexicographic product of directed paths, we have the following:

Theorem 8 *For every $k, \ell \geq 3$, $\chi_o(\overrightarrow{DP_k}[\overrightarrow{DP_\ell}]) = 9$. Therefore, the bound given in Theorem 7 is tight.*

Proof. Since $\chi_o(\overrightarrow{P}) \leq 3$ for every oriented path $\overrightarrow{P}$, we have $\chi_o(\overrightarrow{DP_k}[\overrightarrow{DP_\ell}]) \leq 9$ by Theorem 7. Since any two vertices in $\overrightarrow{DP_3}[\overrightarrow{DP_3}]$ are linked by a directed path of length 1 or 2, we have $\chi_o(\overrightarrow{DP_k}[\overrightarrow{DP_\ell}]) \geq \chi_o(\overrightarrow{DP_3}[\overrightarrow{DP_3}]) = 9$ for every $k, \ell \geq 3$. $\square$

The following result provides a general upper bound on the upper oriented chromatic number of lexicographic products of undirected graphs.

Theorem 9 *Let G and H be two undirected graphs with $k = \chi(G^2)$ and $n = |V(H)|$. We then have*

$$\chi_o^+(G[H]) \leq k(n + 2^n)^{k-1}\chi_o^+(H).$$

Proof. Let α be a proper k-coloring of G^2 and $\ell = \chi_o^+(H)$. Let $\overrightarrow{U}$ be an oriented graph of order ℓ such that $\overrightarrow{H} \to \overrightarrow{U}$ for every orientation $\overrightarrow{H}$ of H and $\overrightarrow{T}$ be the oriented graph of order $n + 2^n$ such that $\overrightarrow{K}_{n,n} \to \overrightarrow{T}$ for every orientation $\overrightarrow{K}_{n,n}$ of $K_{n,n}$, as defined in the proof of Theorem 4 (we consider here the case $m = n$). We define the digraph $\overrightarrow{W}$ by

$$V(\overrightarrow{W}) = \{\ [a, b, c_1, \ldots, c_k],\ 1 \leq a \leq k,\ 1 \leq b \leq \ell,$$

$$c_a = 0,\ 1 \leq c_i \leq n + 2^n,\ 1 \leq i \leq k,\ i \neq a\ \}$$

and $([a,b,c_1,\ldots,c_k],[a',b',c'_1,\ldots,c'_k])$ is an arc in $E(\overrightarrow{W})$ if and only if either $a=a'$ and $(b,b')\in E(\overrightarrow{U})$, or $a\neq a'$ and $(c_{a'},c'_a)\in E(\overrightarrow{T})$. The digraph $\overrightarrow{W}$ is clearly an oriented graph of order $k\ell(n+2^n)^{k-1}$.

Let now $\overrightarrow{G[H]}$ be any orientation of $G[H]$, $\overrightarrow{H_u}$ be the oriented copy of H induced by the set of vertices $\{u\}\times V(H)$ and $\lambda_u:\overrightarrow{H_u}\to\overrightarrow{U}$ be a homomorphism. If $\{u,u'\}$ is an edge in G, let $\mu_{u,u'}$ be a homomorphism of the oriented subgraph of $\overrightarrow{G[H]}$ induced by the set of vertices $\{u,u'\}\times V(H)$ to $\overrightarrow{T}$. We shall now construct a mapping φ from $V(\overrightarrow{G[H]})$ to $V(\overrightarrow{W})$, and prove that this mapping is a homomorphism, which will give the desired result.

Let $[u,v]$ be any vertex in $V(\overrightarrow{G[H]})$ and $\varphi([u,v])=[a,b,c_1,\ldots,c_k]$ be defined as follows:

(i) $a=\alpha(u)$,
(ii) $b=\lambda_u(v)$,
(iii) if there is a neighbor u' of u in G with $\alpha(u')=a'$, then $c_{a'}=\mu_{u,u'}([u,v])$, otherwise $c_{a'}=0$.

Note that if such a neighbor u' of u exists in item (iii), then it must be unique since α is a proper coloring of G^2.

Let now $([u,v],[u',v'])$ be any arc in $E(\overrightarrow{G[H]})$, $\varphi([u,v])=[a,b,c_1,\ldots,c_k]$ and $\varphi([u',v'])=[a',b',c'_1,\ldots,c'_k]$. If $u=u'$, then $a=\alpha(u)=\alpha(u')=a'$ and $(b,b')=(\lambda_u(v),\lambda_u(v'))\in E(\overrightarrow{U})$. If $u\neq u'$, then $\{u,u'\}\in E(G)$, $c_{a'}=\mu_{u,u'}([u,v])$, $c'_a=\mu_{u,u'}([u',v'])$ and, therefore, $(c_{a'},c'_a)\in E(\overrightarrow{T})$. Every arc of $\overrightarrow{G[H]}$ is thus mapped to an arc of $\overrightarrow{W}$ and φ is a homomorphism. $\square$

Since $\chi(P^2)\leq 3$ and $\chi_o^+(P)\leq 3$ for every path P, Theorem 9 implies that for every $k,\ell\geq 3$, $\chi_o^+(P_k[P_\ell])\leq 9(\ell+2^\ell)^2$.

4 Strong products

Since $\overrightarrow{G}\boxtimes\overrightarrow{H}\subseteq\overrightarrow{G}[\overrightarrow{H}]$ for every two oriented graphs $\overrightarrow{G}$ and $\overrightarrow{H}$, Theorem 7 implies the following:

Corollary 10 *For every oriented graphs $\overrightarrow{G}$ and $\overrightarrow{H}$,*

$$\chi_o(\overrightarrow{G}\boxtimes\overrightarrow{H})\leq\chi_o(\overrightarrow{G})\chi_o(\overrightarrow{H}).$$

This result can be strengthtened as follows:

Theorem 11 *If $\overrightarrow{G}$, $\overrightarrow{H}$, $\overrightarrow{T}$ and $\overrightarrow{U}$ are oriented graphs such that $\overrightarrow{G}\to\overrightarrow{T}$ and $\overrightarrow{H}\to\overrightarrow{U}$, then $\overrightarrow{G}\boxtimes\overrightarrow{H}\to\overrightarrow{T}\boxtimes\overrightarrow{U}$.*

Proof. Let $\alpha:\overrightarrow{G}\to\overrightarrow{T}$ and $\beta:\overrightarrow{H}\to\overrightarrow{U}$ be two homomorphisms. Let now $\varphi:V(\overrightarrow{G}\boxtimes\overrightarrow{H})\longrightarrow V(\overrightarrow{T}\boxtimes\overrightarrow{U})$ be the mapping defined by $\varphi([u,v])=[\alpha(u),\beta(v)]$ for every vertex $[u,v]$ in $V(\overrightarrow{G}\boxtimes\overrightarrow{H})$.

We claim that φ is a homomorphism. To see this, let $([u,v],[u',v'])$ be an arc in $\overrightarrow{G} \boxtimes \overrightarrow{H}$. We then have $\varphi([u,v]) = [\alpha(u),\beta(v)]$ and $\varphi([u',v']) = [\alpha(u'),\beta(v')]$. If $u = u'$ and $(v,v') \in E(\overrightarrow{H})$, then $\alpha(u) = \alpha(u')$ and $(\beta(v),\beta(v')) \in E(\overrightarrow{U})$. Similarly, if $v = v'$ and $(u,u') \in E(\overrightarrow{G})$, then $\beta(v) = \beta(v')$ and $(\alpha(u),\alpha(u')) \in E(\overrightarrow{T})$. Finally, if $(u,u') \in E(\overrightarrow{G})$ and $(v,v') \in E(\overrightarrow{H})$, then $(\alpha(u),\alpha(u')) \in E(\overrightarrow{T})$ and $(\beta(v),\beta(v')) \in E(\overrightarrow{U})$. Therefore, every arc in $\overrightarrow{G} \boxtimes \overrightarrow{H}$ is mapped to an arc in $\overrightarrow{T} \boxtimes \overrightarrow{U}$ and φ is a homomorphism. $\square$

Since $\chi_o(\overrightarrow{P}) = 3$ for every oriented path $\overrightarrow{P}$, Corollary 10 gives that the oriented chromatic number of the strong product of any two oriented paths is at most 9. We can decrease this bound to 7 for directed paths and show that this new bound is tight:

Theorem 12 *For every $k,\ell \geq 3$, $\chi_o(\overrightarrow{DP_k} \boxtimes \overrightarrow{DP_\ell}) = 7$.*

Proof. Let $\overrightarrow{DP_k} = x_1 \dots x_k$ and $\overrightarrow{DP_\ell} = y_1 \dots y_\ell$. All arcs in $E(\overrightarrow{DP_k} \boxtimes \overrightarrow{DP_\ell})$ are either of the form $([x_i,y_j],[x_{i+1}y_j])$, or $([x_i,y_j],[x_iy_{j+1}])$, or $([x_i,y_j],[x_{i+1}y_{j+1}])$. Let $\overrightarrow{T}_7 = \overrightarrow{T}(7;1,2,3)$ be the circulant tournament defined by $V(\overrightarrow{T}_7) = \{0,1,\dots,6\}$ and $(i,j) \in E(\overrightarrow{T}_7)$ if and only if $(j-i) \mod 7 \in \{1,2,3\}$. We will show that $\overrightarrow{DP_k} \boxtimes \overrightarrow{DP_\ell}$ admits a homomorphism to $\overrightarrow{T}_7$, which proves $\chi_o(\overrightarrow{DP_k} \boxtimes \overrightarrow{DP_\ell}) \leq 7$.

Let $\varphi : V(\overrightarrow{DP_k} \boxtimes \overrightarrow{DP_\ell}) \longrightarrow V(\overrightarrow{T}_7)$ be the mapping defined by $\varphi([x_i,y_j]) = 2j+i \pmod 7$, for every i,j, $1 \leq i \leq k$, $1 \leq j \leq \ell$. For every arc (u,v) in $\overrightarrow{DP_k} \boxtimes \overrightarrow{DP_\ell}$, we claim that $\varphi(v) - \varphi(u) \in \{1,2,3\}$. If (u,v) is of the form $([x_i,y_j],[x_{i+1}y_j])$, then $\varphi(v) - \varphi(u) = 2j+i+1-2j-i = 1$. If (u,v) is of the form $([x_i,y_j],[x_iy_{j+1}])$, then $\varphi(v)-\varphi(u) = 2(j+1)+i-2j-i = 2$. Finally, if (u,v) is of the form $([x_i,y_j],[x_{i+1}y_{j+1}])$, then $\varphi(v)-\varphi(u)) = 2(j+1)+i+1-2j-i = 3$. Every arc of $\overrightarrow{DP_k} \boxtimes \overrightarrow{DP_\ell}$ is thus mapped to an arc of $\overrightarrow{T}_7$ and φ is a homomorphism.

To see that this bound is tight, it is enough to observe that any two vertices in the subgraph $\overrightarrow{X}$ of $\overrightarrow{DP_k} \boxtimes \overrightarrow{DP_\ell}$ induced by the set of vertices

$$\{\ [x_1,y_1],\ [x_2,y_1],\ [x_2,y_2],\ [x_2,y_3],\ [x_3,y_1],\ [x_3,y_2],\ [x_3,y_3]\ \}$$

are linked by a directed path of length 1 or 2. Therefore, $\chi_o(\overrightarrow{DP_k} \boxtimes \overrightarrow{DP_\ell}) \geq \chi_o(\overrightarrow{X}) = 7$ and the result follows. $\square$

The following result provides a general upper bound on the upper oriented chromatic number of strong products of undirected graphs.

Theorem 13 *For every undirected graphs G and H,*

$$\chi_o^+(G \boxtimes H) \leq (2^{\chi(H^2)} - 1)\chi(G)\chi_o^+(G)\chi_o^+(H).$$

Proof. Let $k = \chi(H^2)$, $\ell = \chi(G)$, $m = \chi_o^+(G)$ and $n = \chi_o^+(H)$. Moreover, let $\overrightarrow{T}$ be an oriented graph of order m such that $\overrightarrow{G} \to \overrightarrow{T}$ for every orientation $\overrightarrow{G}$ of G and $\overrightarrow{U}$ be an oriented graph of order n such that $\overrightarrow{H} \to \overrightarrow{U}$ for every orientation $\overrightarrow{H}$ of H.

Let now $\overrightarrow{W}$ be the digraph defined by

$$V(\overrightarrow{W}) = \{[\alpha,\beta,\mu,\lambda,c_1,\dots,c_{\beta-1}],\ \alpha \in \{1,2,\dots,\ell\},\ \beta \in \{1,2,\dots,k\},$$
$$\mu \in \{1,2,\dots,m\},\ \lambda \in \{1,2,\dots,n\},\ c_i \in \{0,1\},\ 1 \leq i \leq \beta-1\}$$

and $([\alpha,\beta,\mu,\lambda,c_1,\dots,c_k],[\alpha',\beta',\mu',\lambda',c'_1,\dots,c'_k])$ is an arc in $E(\overrightarrow{W})$ if and only if one of the following holds:

(i) $\alpha = \alpha'$ and $(\lambda, \lambda') \in E(\overrightarrow{U})$,
(ii) $\alpha \neq \alpha'$, $\beta = \beta'$ and $(\mu, \mu') \in E(\overrightarrow{T})$,
(iii) $\alpha \neq \alpha'$, $\beta < \beta'$, and $c'_\beta = 1$,
(iv) $\alpha \neq \alpha'$, $\beta > \beta'$, and $c_{\beta'} = 0$.

The graph $\overrightarrow{W}$ is clearly an oriented graph (with no opposite arcs) of order $(2^k-1)\ell mn$.

Let now $\overrightarrow{G \boxtimes H}$ be any orientation of $G \boxtimes H$, γ be a proper ℓ-coloring of G, and h be a proper k-coloring of H^2. For every vertex $u \in V(G)$, let $\mu_u : \overrightarrow{H_u} \to \overrightarrow{U}$ be a homomorphism. Similarly, for every vertex $v \in V(H)$, let $\lambda_v : \overrightarrow{G_v} \to \overrightarrow{T}$ be a homomorphism.

We shall now construct a mapping φ from $V(\overrightarrow{G \boxtimes H})$ to $V(\overrightarrow{W})$, and prove that this mapping is a homomorphism, which will give the desired result.

Let $[u, v]$ be any vertex in $V(\overrightarrow{G \boxtimes H})$ and $\varphi([u, v]) = [\alpha, \beta, \mu, \lambda, c_1, \ldots, c_k]$ be defined as follows:

(i) $\alpha = \gamma(u)$,
(ii) $\beta = h(v)$,
(iii) $\mu = \mu_v(u)$,
(iv) $\lambda = \lambda_u(v)$,
(v) if there is an arc $([w, x], [u, v])$ in $E(\overrightarrow{G \boxtimes H})$ such that $u \neq w$ and $h(x) < h(v)$, then $c_{h(x)} = 1$,
(vi) if there is an arc $([u, v], [w, x])$ in $E(\overrightarrow{G \boxtimes H})$ such that $u \neq w$ and $h(x) < h(v)$, then $c_{h(x)} = 0$,
(vii) every c_i that has not been set in (v) or (vi) is set to 0.

Note that in items (v) and (vi) above, if such an arc exists, then it must be unique since h is a proper coloring of H^2.

Let now $([u, v], [u', v'])$ be any arc in $E(\overrightarrow{G \boxtimes H})$, $\varphi([u, v]) = [\alpha, \beta, \mu, \lambda, c_1, \ldots, c_k]$, and $\varphi([u', v']) = [\alpha', \beta', \mu', \lambda', c'_1, \ldots, c'_k]$. If $u = u'$ then $\alpha = \alpha'$ and $(\lambda, \lambda') = (\lambda_u(v), \lambda_u(v')) \in E(\overrightarrow{U})$. Similarly, if $v = v'$ then $\beta = \beta'$ and $(\mu, \mu') = (\mu_v(u), \mu_v(u')) \in E(\overrightarrow{T})$. Now, if $u \neq u'$ and $v \neq v'$, then $\alpha \neq \alpha'$, $\beta \neq \beta'$, and either $\beta < \beta'$, in which case $c'_\beta = 1$, or $\beta > \beta'$, in which case $c_{\beta'} = 0$. Every arc of $\overrightarrow{G \boxtimes H}$ is thus mapped to an arc of $\overrightarrow{W}$ and φ is a homomorphism. $\square$

Since $\chi(P) = 2$ and $\chi(P^2) = \chi_o^+(P) = 3$ for every path P, we get the following:

Corollary 14 *For every $k, \ell \geq 3$, $\chi_o^+(P_k \boxtimes P_\ell) \leq (2^3 - 1).2.3.3 = 126$.*

For $k = 2$ and $k = 3$, Natarajan, Narayanan and Subramanian [7] obtained better bounds by (implicitely) proving that $\chi_o^+(P_2 \boxtimes P_\ell) \leq 11$ and $\chi_o^+(P_3 \boxtimes P_\ell) \leq 67$, for every $\ell \geq 3$.

5 Cartesian products

Since $\overrightarrow{G} \,\square\, \overrightarrow{H} \subseteq \overrightarrow{G}[\overrightarrow{H}]$ for every two oriented graphs $\overrightarrow{G}$ and $\overrightarrow{H}$, Theorem 7 also implies the following:

Corollary 15 *For every oriented graphs $\overrightarrow{G}$ and $\overrightarrow{H}$,*

$$\chi_o(\overrightarrow{G} \,\square\, \overrightarrow{H}) \leq \chi_o(\overrightarrow{G})\chi_o(\overrightarrow{H}).$$

As before, this result can be strengthtened as follows:

Theorem 16 *If $\overrightarrow{G}$, $\overrightarrow{H}$, $\overrightarrow{T}$ and $\overrightarrow{U}$ are oriented graphs such that $\overrightarrow{G} \to \overrightarrow{T}$ and $\overrightarrow{H} \to \overrightarrow{U}$, then $\overrightarrow{G} \,\square\, \overrightarrow{H} \to \overrightarrow{T} \,\square\, \overrightarrow{U}$.*

Proof. The proof is similar to the proof of Theorem 11. Let $\alpha : \overrightarrow{G} \to \overrightarrow{T}$ and $\beta : \overrightarrow{H} \to \overrightarrow{U}$ be two homomorphisms. Let now $\varphi : V(\overrightarrow{G} \,\square\, \overrightarrow{H}) \longrightarrow V(\overrightarrow{T} \,\square\, \overrightarrow{U})$ be the mapping defined by $\varphi([u,v]) = [\alpha(u), \beta(v)]$ for every vertex $[u,v]$ in $V(\overrightarrow{G} \,\square\, \overrightarrow{H})$.

We claim that φ is a homomorphism. To see this, let $([u,v],[u',v'])$ be an arc in $\overrightarrow{G} \,\square\, \overrightarrow{H}$. We then have $\varphi([u,v]) = [\alpha(u), \beta(v)]$ and $\varphi([u',v']) = [\alpha(u'), \beta(v')]$. If $u = u'$ and $(v,v') \in E(\overrightarrow{H})$, then $\alpha(u) = \alpha(u')$ and $(\beta(v), \beta(v')) \in E(\overrightarrow{U})$. Similarly, if $v = v'$ and $(u,u') \in E(\overrightarrow{G})$, then $\beta(v) = \beta(v')$ and $(\alpha(u), \alpha(u')) \in E(\overrightarrow{T})$. Therefore, every arc in $\overrightarrow{G} \,\square\, \overrightarrow{H}$ is mapped to an arc in $\overrightarrow{T} \,\square\, \overrightarrow{U}$ and φ is a homomorphism. $\square$

Corollary 15 implies that the oriented chromatic number of the Cartesian product of two oriented paths is at most 9. From Theorem 12, we get that the oriented chromatic number of the Cartesian product of any two *directed* paths is at most 7. These two bounds can be improved as follows (this result also follows from a result of Natarajan *et al.* [7]):

Theorem 17 *For every oriented paths $\overrightarrow{P_k}$ and $\overrightarrow{P_\ell}$, $k, \ell \geq 1$, $\chi_o(\overrightarrow{P_k} \,\square\, \overrightarrow{P_\ell}) \leq 3$.*

Proof. Let $\overrightarrow{P_k} = x_1 \ldots x_k$ and $\overrightarrow{P_\ell} = y_1 \ldots y_\ell$. Let $\overrightarrow{C_3}$ be the directed cycle on three vertices given by $V(\overrightarrow{C_3}) = \{0,1,2\}$ and $E(\overrightarrow{C_3}) = \{(0,1),(1,2),(2,0)\}$. We inductively define a mapping $\varphi : V(\overrightarrow{P_k} \,\square\, \overrightarrow{P_\ell}) \longrightarrow V(\overrightarrow{C_3})$ as follows:

(i) $\varphi[x_1, y_1] = 0$,

(ii) for every j, $2 \leq j \leq \ell$, $\varphi([x_1, y_j]) = \varphi([x_1, y_{j-1}]) + 1 \pmod 3$ if $(y_{j-1}, y_j) \in E(\overrightarrow{P_\ell})$, and $\varphi([x_1, y_j]) = \varphi([x_1, y_{j-1}]) - 1 \pmod 3$ otherwise.

(iii) for every i, $2 \leq i \leq k$, and for every j, $1 \leq j \leq \ell$, $\varphi([x_i, y_j]) = \varphi([x_{i-1}, y_j]) + 1 \pmod 3$ if $(x_{i-1}, x_i) \in E(\overrightarrow{P_k})$, and $\varphi([x_i, y_j]) = \varphi([x_{i-1}, y_j]) - 1 \pmod 3$ otherwise.

It is then routine to check that the mapping φ is a homomorphism and thus $\chi_o(\overrightarrow{P_k} \,\square\, \overrightarrow{P_\ell}) \leq 3$. $\square$

We now consider the upper oriented chromatic number of Cartesian products of undirected graphs.

Theorem 18 *If G and H are two undirected graphs with $k = min\{\chi(G), \chi(H)\}$, then*

$$\chi_o^+(G \,\square\, H) \leq k \chi_o^+(G) \chi_o^+(H).$$

Proof. Assume without loss of generality that $k = \chi(H)$ and let λ be a proper k-coloring of H. Let $\overrightarrow{T}$ and $\overrightarrow{U}$ be two oriented graphs of order $\chi_o^+(G)$ and $\chi_o^+(H)$, respectively, such that $\overrightarrow{G} \to \overrightarrow{T}$ for every orientation $\overrightarrow{G}$ of G and $\overrightarrow{H} \to \overrightarrow{U}$ for every orientation $\overrightarrow{H}$ of H.

Let now $\overrightarrow{W}$ be the oriented graph defined by

$$V(\overrightarrow{W}) = \{[\ell, a, b],\ 1 \leq \ell \leq \chi(H),\ a \in V(\overrightarrow{T}),\ b \in V(\overrightarrow{U})\}$$

and $([\ell, a, b], [\ell', a', b']) \in E(\overrightarrow{W})$ if and only if either $\ell = \ell'$ and $(a, a') \in E(\overrightarrow{T})$ or $\ell \neq \ell'$ and $(b, b') \in E(\overrightarrow{U})$. We shall prove that any orientation of $G \,\square\, H$ admits a homomorphism to $\overrightarrow{W}$, which gives the desired result.

Fix any orientation $\overrightarrow{G \,\square\, H}$ of $G \,\square\, H$. For every $v \in V(H)$, let $\alpha_v : \overrightarrow{G_v} \to \overrightarrow{T}$ be a homomorphism. Similarly, for every $u \in V(G)$, let $\beta_u : \overrightarrow{H_u} \to \overrightarrow{U}$ be a homomorphism.

Let $\varphi : V(\overrightarrow{G \,\square\, H}) \longrightarrow V(\overrightarrow{W})$ be the mapping defined by $\varphi([u, v]) = [\lambda(u), \alpha_v(u), \beta_u(v)]$ for every $[u, v] \in V(\overrightarrow{G \,\square\, H})$. We claim that φ is a homomorphism. To see this, let $([u, v], [u', v'])$ be an arc in $\overrightarrow{G \,\square\, H}$. We then have $\varphi([u, v]) = [\lambda(u), \alpha_v(u), \beta_u(v)]$ and $\varphi([u', v']) = [\lambda(u'), \alpha_{v'}(u), \beta_{u'}(v)]$. If $u = u'$ and $(v, v') \in E(\overrightarrow{H})$, then $\lambda(u) = \lambda(u')$, $\beta_u = \beta_{u'}$ and, therefore, $([\lambda(u), \alpha_v(u), \beta_u(v)], [\lambda(u'), \alpha_{v'}(u), \beta_{u'}(v)]) \in E(\overrightarrow{W})$. Similarly, if $v = v'$ and $(u, u') \in E(\overrightarrow{G})$, then $\lambda(u) \neq \lambda(u')$, $\alpha_v = \alpha_{v'}$ and, therefore, $([\lambda(u), \alpha_v(u), \beta_u(v)], [\lambda(u'), \alpha_{v'}(u), \beta_{u'}(v)]) \in E(\overrightarrow{W})$.

Every arc in $\overrightarrow{G \,\square\, H}$ is thus mapped to an arc in $\overrightarrow{W}$ and φ is a homomorphism. $\square$

Since $\chi(P) \leq 2$ and $\chi_o^+(P) \leq 3$ for every path P, Theorem 18 implies that $\chi_o^+(P_k \,\square\, P_\ell) \leq 2.3.3 = 18$ for every $k, \ell \geq 3$. In [3], Fertin, Raspaud and Roychowdhury (implicitely) proved that the upper oriented chromatic number of the Cartesian product of any two paths is at most 11.

Moreover, since $\chi(T) \leq 2$ and $\chi_o^+(T) \leq 3$ for every tree T, we get the following:

Corollary 19 *Let T be a tree. For every undirected graph G, $\chi_o^+(G \,\square\, T) \leq 6\chi_o^+(G)$.*

In the same vein, since the cycle C_5 has upper oriented chromatic number 5 and every cycle except C_5 has upper oriented chromatic number 4, we obtain:

Corollary 20 *Let C_k be the cycle on k vertices. For every undirected graph G, $\chi_o^+(G \,\square\, C_5) \leq 15\chi_o^+(G)$, and $\chi_o^+(G \,\square\, C_k) \leq 12\chi_o^+(G)$ for every $k \geq 3$, $k \neq 5$.*

These two corollaries improve the results of Natarajan, Narayanan and Subramanian [7], who proved that $\chi_o(G \,\square\, P_k) \leq (2k-1)\chi_o(G)$ and $\chi_o(G \,\square\, C_k) \leq 2k\chi_o(G)$ for every graph G.

6 Direct products

Since $\overrightarrow{G} \times \overrightarrow{H} \subseteq \overrightarrow{G}[\overrightarrow{H}]$ for every two oriented graphs $\overrightarrow{G}$ and $\overrightarrow{H}$, we get from Theorem 7 that $\chi_o(\overrightarrow{G} \times \overrightarrow{H}) \leq \chi_o(\overrightarrow{G})\chi_o(\overrightarrow{H})$. This bound can be improved as follows:

Theorem 21 *For every oriented graphs $\overrightarrow{G}$ and $\overrightarrow{H}$,*

$$\chi_o(\overrightarrow{G} \times \overrightarrow{H}) \leq \min\{\chi_o(\overrightarrow{G}), \chi_o(\overrightarrow{H})\}.$$

Proof. This result directly follows from the fact that $\overrightarrow{G} \times \overrightarrow{H} \to \overrightarrow{G}$ and $\overrightarrow{G} \times \overrightarrow{H} \to \overrightarrow{H}$ since the mapping $\gamma : V(\overrightarrow{G} \times \overrightarrow{H}) \longrightarrow V(\overrightarrow{G})$ (resp. $\gamma : V(\overrightarrow{G} \times \overrightarrow{H}) \longrightarrow V(\overrightarrow{H})$) defined by $\gamma([u, v]) = u$ (resp. $\gamma([u, v]) = v$) is clearly a homomorphism from $\overrightarrow{G} \times \overrightarrow{H}$ to $\overrightarrow{G}$ (resp. to $\overrightarrow{H}$). $\square$

We now consider the upper oriented chromatic number of direct products of undirected graphs.

Theorem 22 *For every undirected graphs G and H with $k = \chi(G^2)$ and $\ell = \chi(H^2)$,*

$$\chi_o^+(G \times H) \leq \frac{2^{k(\ell-1)} - 1}{2^{\ell-1} - 1}.$$

Proof. Let $\overrightarrow{W}$ be the digraph defined by

$$V(\overrightarrow{W}) = \{\ [\alpha, \beta, c_{1,1}, \ldots, c_{1,\ell}, \ldots, c_{\alpha-1,1}, \ldots, c_{\alpha-1,\ell}],\ 1 \leq \alpha \leq k,\ 1 \leq \beta \leq \ell,$$
$$c_{i,j} \in \{0,1\},\ c_{i,\beta} = 0,\ 1 \leq i \leq \alpha - 1,\ 1 \leq j \leq \ell\ \}$$

and

$$([\alpha, \beta, c_{1,1}, \ldots, c_{1,\ell}, \ldots, c_{\alpha-1,1}, \ldots, c_{\alpha-1,\ell}], [\alpha', \beta', c'_{1,1}, \ldots, c'_{1,\ell}, \ldots, c'_{\alpha'-1,1}, \ldots, c'_{\alpha'-1,\ell}])$$

is an arc in $E(\overrightarrow{W})$ if and only if one of the following holds:

(i) $\alpha < \alpha'$, $\beta < \beta'$ and $c'_{\alpha,\beta} = 1$,
(ii) $\alpha < \alpha'$, $\beta > \beta'$ and $c'_{\alpha,\beta} = 0$,
(iii) $\alpha > \alpha'$, $\beta < \beta'$ and $c_{\alpha',\beta} = 1$,
(iv) $\alpha > \alpha'$, $\beta > \beta'$ and $c_{\alpha',\beta} = 0$.

The digraph $\overrightarrow{W}$ is clearly an oriented graph of order $1+2^{\ell-1}+\cdots+2^{(k-1)(\ell-1)} = \frac{2^{k(\ell-1)}-1}{2^{\ell-1}-1}$.

Let now $\overrightarrow{G \times H}$ be any orientation of $G \times H$, a be any proper k-coloring of G^2 and b be any proper ℓ-coloring of H^2. We shall now construct a mapping φ from $V(\overrightarrow{G \times H})$ to $V(\overrightarrow{W})$ and prove that this mapping is a homomorphism, which will give the desired result.

Let $[u, v]$ be any vertex in $V(\overrightarrow{G \times H})$ and

$$\varphi([u, v]) = [\alpha, \beta, c_{1,1}, \ldots, c_{1,\ell}, \ldots, c_{\alpha-1,1}, \ldots, c_{\alpha-1,\ell}]$$

be defined as follows:

(i) $\alpha = a(u)$,
(ii) $\beta = b(v)$,
(iii) if there is an arc $([u, v], [w, x])$ in $E(\overrightarrow{G \times H})$ such that $a(u) > a(w)$ and $b(v) < b(x)$, then $c_{a(w),b(x)} = 1$,
(iv) if there is an arc $([u, v], [w, x])$ in $E(\overrightarrow{G \times H})$ such that $a(u) > a(w)$ and $b(v) > b(x)$, then $c_{a(w),b(x)} = 0$,
(v) if there is an arc $([w, x], [u, v])$ in $E(\overrightarrow{G \times H})$ such that $a(u) > a(w)$ and $b(v) < b(x)$, then $c_{a(w),b(x)} = 0$,
(vi) if there is an arc $([w, x], [u, v])$ in $E(\overrightarrow{G \times H})$ such that $a(u) > a(w)$ and $b(v) > b(x)$, then $c_{a(w),b(x)} = 1$,
(vii) every c_i that has not been set in (iii), (iv), (v) or (vi) is set to 0.

Note that in items (iii) to (vi) above, if such an arc exists, then it must be unique since a and b are proper colorings of G^2 and H^2, respectively.

Let now $([u, v], [u', v'])$ be any arc in $E(\overrightarrow{G \times H})$, with

$$\varphi([u, v]) = [\alpha, \beta, c_{1,1}, \ldots, c_{1,\ell}, \ldots, c_{\alpha-1,1}, \ldots, c_{\alpha-1,\ell}]$$

and

$$\varphi([u',v']) = [\alpha', \beta', c'_{1,1}, \ldots, c'_{1,\ell}, \ldots, c'_{\alpha'-1,1}, \ldots, c'_{\alpha'-1,\ell}].$$

We thus have $\{u,u'\} \in E(G)$ and $\{v,v'\} \in E(H)$.

Suppose first that $\alpha = a(u) < a'(u) = \alpha'$. In that case, if $\beta = b(v) < \beta' = b(v')$ then $c'_{\alpha,b(v)} = 1$, otherwise $c'_{\alpha,b(v')} = 0$. Suppose now that $\alpha' = a(u') < a(u) = \alpha$. In that case, if $\beta = b(v) < \beta' = b(v')$ then $c_{\alpha',b(v)} = 1$, otherwise $c_{\alpha',b(v')} = 0$. Therefore, in all cases, φ maps the arc $([u,v],[u',v'])$ to some arc in $E(\overrightarrow{W})$ and is thus a homomorphism. $\square$

Since $\chi(P^2) = 3$ for every path P, we get that the direct product of any two paths has upper oriented chromatic number at most $\frac{2^{3.2}-1}{2^2-1} = \frac{63}{3} = 21$. However, every such product is the disjoint union of two components that are almost grid graphs and it has been (implicitely) shown in [3] that the upper oriented chromatic number of any such component is at most 11.

7 Discussion

In this paper, we introduced and initiated the study of a new parameter called the upper oriented chromatic number of undirected graphs which, from our point of view, arises "naturally" in the framework of oriented colorings. We gave some general upper bounds on the upper oriented chromatic number of several types of product graphs and derived some upper bounds on the ordinary oriented chromatic number of such graphs.

We hope that this new parameter will motivate further research. In particular, most of our upper bounds for product graphs can be significantly improved when considering specific graph classes (a first step in this direction was our results on products of paths). For instance, it would be interesting to determine the upper and ordinary oriented chromatic numbers of products of trees, outerplanar graphs or, more generally, of partial k-trees.

In [12], we conjectured that the oriented chromatic number of every *connected* cubic graph is at most 7 and, to our knowledge, this conjecture is still open. A related question is thus the following:

Question 1 *Determine the upper oriented chromatic number of cubic graphs.*

It has been proved in [13] that every orientation of any cubic graph admits a homomorphism to the Paley tournament of order 11 and, therefore, $\chi_o^+(G) \leq 11$ for every cubic graph G. This upper bound can probably be improved. On the other hand, we know that there exists cubic graphs with oriented chromatic number 7 [12].

Every tree has upper oriented chromatic number at most 3 and we know that there are trees with oriented chromatic number 3. Every such tree T thus satisfies the equality $\chi_o^+(T) = \chi_o(T)$. This property also holds for every outerplanar graph (or, more generally, for every partial 2-tree) with oriented chromatic number 7. These observations lead to the following question:

Question 2 *Give necessary or sufficient conditions for a graph G to satisfy the equality $\chi_o^+(G) = \chi_o(G)$.*

The notion of the *oriented chromatic index* of an oriented graph $\overrightarrow{G}$ has been introduced in [10], and is defined as the smallest order of an oriented graph $\overrightarrow{T}$ such that the *line digraph* $LD(\overrightarrow{G})$ of $\overrightarrow{G}$ admits a homomorphism to $\overrightarrow{T}$. (Recall that $LD(\overrightarrow{G})$ is given by

$V(LD(\overrightarrow{G})) = E(\overrightarrow{G})$ and for every two arcs (u, v) and (u', v') in $E(\overrightarrow{G})$, $((u, v), (u', v'))$ is an arc in $E(LD(\overrightarrow{G}))$ if and only if $v = u'$.) In the same vein, we can thus define the *upper oriented chromatic index* of an undirected graph G as the smallest order of an oriented graph $\overrightarrow{U}$ such that the line digraph of every orientation of G admits a homomorphism to $\overrightarrow{U}$. Here again, many upper bounds obtained for the oriented chromatic index of specific graph classes implicitely deal with the upper oriented chromatic index of these classes. We thus propose to study this other parameter as well.

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