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FAST DIFFUSION EQUATIONS: MATCHING LARGE TIME ASYMPTOTICS BY RELATIVE ENTROPY METHODS

JEAN DOLBEAULT AND GIUSEPPE TOSCANI

ABSTRACT. A non self-similar change of coordinates provides improved matching asymptotics of the solutions of the fast diffusion equation for large times, compared to already known results, in the range for which Barenblatt solutions have a finite second moment. The method is based on relative entropy estimates and a time-dependent change of variables which is determined by second moments, and not by the scaling corresponding to the self-similar Barenblatt solutions, as it is usually done.

1. INTRODUCTION AND MAIN RESULTS

Consider on \mathbb{R}^d the fast diffusion equation

$$(1) \quad \frac{\partial v}{\partial \tau} + \nabla \cdot (v \nabla v^{m-1}) = 0$$

for some $m \in (m_c, 1)$ with $m_c := (d-2)/d$. Assume that the initial data is a given nonnegative function v_0 in $L^1(\mathbb{R}^d)$. It is well known (see for instance [2]) that the large time behavior of the solution is captured by the *Barenblatt solutions* given for any $(\tau, y) \in \mathbb{R}^+ \times \mathbb{R}^d$ by

$$\mathbf{B}(\tau, y) := (1 + \tau)^{-\frac{1}{m-m_c}} \left(D + \frac{1}{2d(m-m_c)} (1 + \tau)^{-\frac{2}{d(m-m_c)}} |y|^2 \right)_+^{-\frac{1}{1-m}}$$

where $D > 0$ is determined by the condition $\|v_0\|_{L^1(\mathbb{R}^d)} = \|\mathbf{B}(\tau, \cdot)\|_{L^1(\mathbb{R}^d)}$. Using entropy methods, it has been established in [5] how a linearized problem involving the relative entropy and the relative Fisher information determines the best rate of convergence towards the Barenblatt solution. The note [6] is devoted to a refinement of the estimates in which the dependence on v_0 is clarified and the precise value of the best possible rate of convergence is computed in terms of a spectral gap of the linearized operator associated to the relative entropy and the relative Fisher information for all values of $m < 1$. By taking advantage of the translation invariance, it is moreover possible to impose that the solution evolves in the orthogonal of

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the eigenspace associated to the first non-zero eigenvalue of the linearized operator for $m \in (m_1, 1)$ with $m_1 := (d-1)/d$, thus providing an improved rate of convergence. The corresponding conserved quantity is the center of mass, while the generators of the eigenspace are the derivatives of the Barenblatt solution with respect to each of the coordinates. There is no other conserved quantity known, so that further improvements cannot be achieved directly by this method.

One may however notice that the eigenspace corresponding to the second non-zero eigenvalue in the range $m \in (m_1, 1)$ is generated by the infinitesimal dilation of the Barenblatt solution. It is therefore natural to try to adjust the Barenblatt solution by a scaling. This can be done by taking a time-dependent change of variables where the scale is determined by the solution itself and not anymore by its asymptotic, self-similar behavior, thus providing improved convergence rates. Asymptotically, we will recover the self-similar profile, but with a better matching. There is a price to pay: the rescaled equation has a time-dependent coefficient, which converges to a constant. From the point of view of the entropy – entropy production inequality, however, nothing is changed, which is the main observation of this paper.

Let $\tilde{m}_1 := d/(d+2)$. In the range $m \in (\tilde{m}_1, m_1)$, the infinitesimal dilation of the Barenblatt solution generates the eigenspace corresponding to the first non-zero eigenvalue. Our time-dependent change of variables therefore improves on the rate of convergence for any $m \in (\tilde{m}_1, m_1)$, and also for any $m \in (m_1, 1)$ if the center of mass is chosen at the origin.

The reader interested in understanding the heuristics of our approach is invited to go directly to Section 2. In the remainder of this section, we will give a precise statement of our main result and some additional references.

Define the mass and the center of mass of v_0 respectively by

$$M := \int_{\mathbb{R}^d} v_0 \, dy \quad \text{and} \quad x_0 := \frac{1}{M} \int_{\mathbb{R}^d} y v_0 \, dy .$$

Consider the family of the Barenblatt profiles

$$(2) \quad B_\sigma(x) := \sigma^{-\frac{d}{2}} \left(C_M + \frac{1}{\sigma} |x|^2 \right)^{\frac{1}{m-1}} \quad \forall x \in \mathbb{R}^d$$

where σ is a positive real parameter and

$$C_M := \left(\frac{M}{M_*} \right)^{-\frac{2(1-m)}{d(m-m_c)}}, \quad M_* := \int_{\mathbb{R}^d} (1 + |x|^2)^{\frac{1}{m-1}} \, dx = \pi^{\frac{d}{2}} \frac{\Gamma\left(\frac{d(m-m_c)}{2(1-m)}\right)}{\Gamma\left(\frac{1}{1-m}\right)} .$$

Notice that $\|B_\sigma\|_{L^1(\mathbb{R}^d)} = M$ for any $\sigma > 0$ and B_σ is a solution of

$$\nabla \cdot \left[B \left(\sigma^{\frac{d}{2}(m_c-m)} \nabla B^{m-1} - 2x \right) \right] = 0 .$$

Let us recall the definition of m_c , m_1 , \tilde{m}_1 and introduce the exponents m_2 and \tilde{m}_2 , for later use:

$$m_c = \frac{d-2}{d}, \quad m_1 = \frac{d-1}{d}, \quad \tilde{m}_1 = \frac{d}{d+2},$$

$$m_2 := \frac{d+1}{d+2} \quad \text{and} \quad \tilde{m}_2 := \frac{d+4}{d+6},$$

which are such that $m_c < m_1 < m_2 < 1$, $m_c < \tilde{m}_1 < \tilde{m}_2$ and, if $d \geq 2$, $\tilde{m}_1 \leq m_1$ and $\tilde{m}_2 \leq m_2$. For later purpose, it is also convenient to define

$$(3) \quad K_M := \int_{\mathbb{R}^d} |x|^2 B_1 dx = \frac{(1-m)\tilde{m}_1}{m-\tilde{m}_1} M C_M$$

for any $m \in (\tilde{m}_1, 1)$. Notice that K_M is finite if $m > \tilde{m}_1$.

If v is a solution of (1), consider the time-dependent scale $R(\tau)$ defined by

$$(4) \quad \frac{d \log R}{d\tau} = 2 \left(\frac{K_M}{\int_{\mathbb{R}^d} |y-x_0|^2 v(\tau, y) dy} \right)^{\frac{d}{2}(m-m_c)}, \quad R(0) = 1.$$

The justification of such a choice for $\tau \mapsto R(\tau)$ will be made clear in Section 2. Note that, in view of the fact that the quantity $\int_{\mathbb{R}^d} |y-x_0|^2 v(\tau, y) dy$ is non-decreasing in time along the solution of the fast diffusion equation (1) and because of its asymptotic behaviour, the time-dependent scale $R(\tau)$ is such that $\log R(\tau)$ is increasing from zero to infinity. We also define σ as a function of τ by the condition

$$\sigma(\tau) := \frac{1}{K_M R(\tau)^2} \int_{\mathbb{R}^d} |y-x_0|^2 v(\tau, y) dy.$$

As a consequence, the equation for R can be rewritten as

$$(5) \quad 2 \sigma^{-\frac{d}{2}(m-m_c)} = R^{1-d(1-m)} \frac{dR}{d\tau}$$

and we can define for any $x = y/R(t) \in \mathbb{R}^d$ the Barenblatt type solution \mathcal{B} by

$$\mathcal{B}(\tau, y) := \frac{1}{R(\tau)^d} \mathfrak{B} \left(\tau, \frac{y}{R(\tau)} \right) \quad \text{where} \quad \mathfrak{B}(\tau, x) := B_{\sigma(\tau)}(x).$$

The difference of \mathcal{B} with the Barenblatt solution \mathbf{B} is that \mathcal{B} depends on $\int_{\mathbb{R}^d} |y|^2 v(\tau, y) dy$, so that they are only asymptotically equivalent, as we shall see later. The point is that \mathcal{B} provides a better asymptotic matching than \mathbf{B} . Our goal is indeed to measure the rate of convergence of v towards \mathcal{B} . For this purpose, it is convenient to change variables and study the rate of convergence of u defined by

$$u(\tau, x) = R(\tau)^d v(\tau, x_0 + R(\tau)x)$$

towards \mathfrak{B} . Let us consider the *relative entropy* of J. Ralston and W.I. Newmann defined in [22, 24] by

$$\mathcal{E}(\tau) := \frac{1}{m-1} \int_{\mathbb{R}^d} [u^m - \mathfrak{B}^m - m \mathfrak{B}^{m-1} (u - \mathfrak{B})] dx .$$

Theorem 1. *Assume that $m \in (\tilde{m}_1, 1)$, $d \geq 2$. Let v be a solution of (1) with initial datum $v_0 \in L^1_+(\mathbb{R}^d)$ such that v_0^m and $y \mapsto |y|^2 v_0(y)$ are integrable. With the above notations, we have*

$$\limsup_{\tau \rightarrow \infty} R(\tau)^{\gamma(m)} \mathcal{E}(\tau) < \infty ,$$

where $R(\tau) \sim \tau^{\frac{1}{d(m-m_c)}}$ as $\tau \rightarrow \infty$ and

$$\gamma(m) = \begin{cases} \frac{((d-2)m-(d-4))^2}{4(1-m)} & \text{if } m \in (\tilde{m}_1, \tilde{m}_2] , \\ 4(d+2)m - 4d & \text{if } m \in [\tilde{m}_2, m_2] , \\ 4 & \text{if } m \in [m_2, 1) . \end{cases}$$

Once a relative entropy estimate is known, it is possible to control the decay rate of $u - \mathfrak{B}$ in various norms, for instance in $L^q(\mathbb{R}^d, dx)$ for

$$q \geq \max \left\{ 1, \frac{2d(1-m)}{2(2-m)+d(1-m)} \right\} ,$$

or in C^k , by interpolation. Up to a change of variables, this also allows to prove decay rates of $v - \mathcal{B}$. See [5] for more details.

Compared to the results of [6], an improvement for any $m > \tilde{m}_1$ has been obtained. The values obtained in [6] for $\gamma(m)$ are indeed

$$\gamma(m) = \begin{cases} 2dm - 2(d-2) & \text{if } m \in (\tilde{m}_1, m_1] , \\ 2 & \text{if } m \in [m_1, 1) , \end{cases}$$

except that in [6] the scale $R(\tau)$ is determined by the self-similar Barenblatt solutions (both scales are anyway equivalent as $\tau \rightarrow \infty$: see Lemma 8). Also see Figure 2 at the end of this paper for more details on $m \mapsto \gamma(m)$ in the setting of [6] compared to the results of Theorem 1.

Compared to other methods, it may look surprising that the scale $R(\tau)$ and, as a consequence, the coefficient σ both depend on the solution v of (1). Asymptotically, as $\tau \rightarrow \infty$, $R(\tau)$ is equivalent to the scale given by the self-similar change of variables, but what has been gained is a better matching with the closest Barenblatt solution. The family of the Barenblatt solutions is globally invariant under scaling and, among all such solutions, there is one which is closer to our solution of the evolution equation: the one with the same second moment.

Convergence results of a suitably rescaled flow associated to (1) towards an asymptotic profile has been established in [19] for $m > m_c$ (also see for instance [26]) and in [5, 15] for $m \leq m_c$. Getting rates of convergence beyond a simple interpolation between mass and uniform estimates has required the use of the relative entropies introduced by J. Ralston and W.I.

Newman in [22, 24]. First results in this direction have been achieved in [13] using the entropy / entropy-production method of D. Bakry and M. Emery (also see [11] for general diffusions and [1] for an overview) and in [16] using sharp Gagliardo-Nirenberg interpolation inequalities. F. Otto made the link with gradient flows with respect to the Wasserstein distance in [23], and D. Cordero-Erausquin, B. Nazaret and C. Villani gave a proof of the corresponding Gagliardo-Nirenberg inequalities using mass transportation techniques in [14]. The condition $m \geq m_1$ was a strong limitation to these first approaches. Gagliardo-Nirenberg inequalities indeed degenerate into a critical Sobolev inequality for $m = m_1$, while the displacement convexity condition also requires $m \geq m_1$. For $m < m_1$, various limitations appear. To work with Wasserstein's distance, it is crucial to have second moments bounded, which amounts to request $m > \tilde{m}_1$ for the Barenblatt profiles; see for instance [17, 18]. Linearization of entropy estimates around the Barenblatt profiles has been considered in [12, 20] for $m \in (m_c, 1)$. In a certain sense, this is also the strategy in [17, 18]. Integrability of the Barenblatt profiles means $m > m_c$. This condition has been removed in a series of recent papers (see [4, 5, 6, 7]) together with a clarification of the strategy of linearization of the relative entropies, at least from the point of view of functional inequalities. In this paper, we shall however restrict m to the interval $(\tilde{m}_1, 1)$, for spectral reasons that are explained in Section 3 and for the second moment to be well defined. For $m \leq \tilde{m}_1$, even with an appropriate definition a relative second moment, our method gives no improvement on the convergence rates because of the presence of the continuous spectrum.

Rescalings and convergence towards Barenblatt solutions, or *intermediate asymptotics*, has not been the only issue of large time asymptotics. We can for instance quote [25] for a study (in the porous media case) of the time evolution of the second moment, and [10, 18, 21, 9] for the search of improved convergence rates when moment conditions are imposed in the framework of Wasserstein's or other Fourier based distances. The question of improved rates has been precisely formulated in [18], and solved in [6] in the weighted L^2 framework that we shall use in this paper, as far as the first moment (position of the center of mass) is concerned. The main contribution of this work is to explain how improvements based on the second moment can also be achieved.

This paper is organized as follows. In Section 2, we explain how faster convergence results can be achieved by introducing an appropriate time-dependent rescaling, which is given by (4) and not by the explicit dependence of the Barenblatt solutions. Improved Hardy-Poincaré inequalities are established in Section 3, using the spectral results of [17, 18] and the spectral equivalence found in [6]. The large time behaviour of the solution is studied in Section 4. The proof of Theorem 1 is then completed in Section 5. Further considerations on the case $d = 1$ and the limiting regime as $m \rightarrow 1_-$ are presented in the last section.

2. THE RELATIVE ENTROPY APPROACH

The result of Theorem 1 is easy to understand using a time-dependent rescaling and the relative entropy formalism. Define the function u such that

$$(6) \quad v(\tau, y + x_0) = R^{-d} u(t, x), \quad R = R(\tau), \quad t = \frac{1}{2} \log R, \quad x = \frac{y}{R}$$

where v is a solution of (1) with initial datum $v_0 \in L^1_+(\mathbb{R}^d)$. A simple computation shows that u has to be a solution of

$$(7) \quad \frac{\partial u}{\partial t} + \nabla \cdot \left[u \left(\sigma^{\frac{d}{2}(m-m_c)} \nabla u^{m-1} - 2x \right) \right] = 0 \quad t > 0, \quad x \in \mathbb{R}^d,$$

with initial datum $u_0 = v_0$ (we assume that R is chosen such that $R(0) = 1$) and σ given by

$$2\sigma^{-\frac{d}{2}(m-m_c)} = R^{1-d(1-m)} \frac{dR}{d\tau},$$

which is nothing else than (5). By virtue of the definition of $R(\tau)$, the new time $t = \frac{1}{2} \log R(\tau)$ increases monotonically from 0 to $+\infty$. Consequently, the old and new times can be uniquely related, and τ can be expressed in terms of t through the inverse function of R , so that $\tau = R^{-1}(e^{2t})$. Using this transformation and with a slight abuse of notations, we shall consider from now on σ as a function of t . It is important to notice that, as long as $\frac{d\sigma}{dt} \neq 0$, the Barenblatt profile B_σ is *not* a solution of (7), but we may still consider the relative entropy

$$\mathcal{F}_\sigma[u] := \frac{1}{m-1} \int_{\mathbb{R}^d} [u^m - B_\sigma^m - m B_\sigma^{m-1} (u - B_\sigma)] dx.$$

Let us briefly sketch the strategy of our method before giving all details.

If we consider a solution of (7) and compute the time derivative of the relative entropy, we find that

$$(8) \quad \frac{d}{dt} \mathcal{F}_{\sigma(t)}[u(t, \cdot)] = \frac{d\sigma}{dt} \left(\frac{d}{d\sigma} \mathcal{F}_\sigma[u] \right) \Big|_{\sigma=\sigma(t)} + \frac{m}{m-1} \int_{\mathbb{R}^d} (u^{m-1} - B_{\sigma(t)}^{m-1}) \frac{\partial u}{\partial t} dx.$$

Here comes the main difference with previous works. As we shall see below in the proof of Lemma 2 (also see Remark 1), the first term of the right hand side in (8) involves

$$\left(\frac{d}{d\sigma} \mathcal{F}_\sigma[u] \right) \Big|_{\sigma=\sigma(t)} = \frac{m D(\sigma)}{1-m} \left(\int_{\mathbb{R}^d} |x|^2 B_\sigma dx - \int_{\mathbb{R}^d} |x|^2 u dx \right) \Big|_{\sigma=\sigma(t)}$$

where $D(\sigma) := \frac{d}{d\sigma} (\sigma^{d(m_c-m)/2})$. When taking a time-dependent rescaling based on the self-similar variables, one finds that σ is constant in t , so that $\frac{d\sigma}{dt} = 0$ and the term $\frac{d\sigma}{dt} \frac{d}{d\sigma} (\mathcal{F}_\sigma[u])$ does not contribute. In our approach,

σ depends on t but can still be chosen so that this term does not show up either. It is indeed enough to require that

$$(9) \quad \int_{\mathbb{R}^d} |x|^2 B_\sigma dx = \int_{\mathbb{R}^d} |x|^2 u dx ,$$

which amounts to ask that R solves the ordinary differential equation (4), to obtain that $\frac{d}{d\sigma} (\mathcal{F}_\sigma[u]) = 0$. This will be justified in the first step of our method, below (see Lemma 2).

In a second step, we shall use the fact that

$$(10) \quad \frac{d}{dt} \mathcal{F}_{\sigma(t)}[u(t, \cdot)] = -\frac{m \sigma(t)^{\frac{d}{2}(m-m_c)}}{1-m} \int_{\mathbb{R}^d} u \left| \nabla \left[u^{m-1} - B_{\sigma(t)}^{m-1} \right] \right|^2 dx .$$

From there on, the computation goes essentially as in [5, 6]. For completeness, we will briefly reproduce it. However, with our choice of σ , we gain an additional orthogonality condition which will be explicitly stated in the third step of the method: see Lemma 3. This orthogonality condition is the crucial point (see Corollary 7) for improving the rates in Theorem 1, compared to the results of [6]. Now let us give further details.

First step: choice of the scaling parameter. For a while, we do not need to take into account the dependence of σ in t . The main idea of this paper is indeed to choose σ in terms of u by minimizing $\sigma \mapsto \mathcal{F}_\sigma[u]$, so that

$$\frac{d}{d\sigma} \mathcal{F}_\sigma[u] = 0 .$$

Lemma 2. *For any given $u \in L^1_+(\mathbb{R}^d)$ such that u^m and $|x|^2 u$ are both integrable, if $m \in (\tilde{m}_1, 1)$, there is a unique $\sigma = \sigma^* > 0$ which minimizes $\sigma \mapsto \mathcal{F}_\sigma[u]$, and it is explicitly given by*

$$(11) \quad \sigma^* = \frac{1}{K_M} \int_{\mathbb{R}^d} |x|^2 u dx .$$

For $\sigma = \sigma^*$, the Barenblatt profile B_{σ^*} satisfies (9).

In (11), K_M is the constant which has been defined in (3). The condition $m > \tilde{m}_1$ guarantees that B_σ^m is integrable and $K_M = \int_{\mathbb{R}^d} |x|^2 B_1 dx$ is finite.

Proof. We have to minimize

$$e(\sigma) := (1-m) \int_{\mathbb{R}^d} B_\sigma^m dx + m \int_{\mathbb{R}^d} B_\sigma^{m-1} u dx ,$$

which corresponds to the two σ -dependent terms in the expression of $\mathcal{F}_\sigma[u]$. Using the fact that $B_\sigma^{m-1}(x) = \sigma^{d(1-m)/2} (C_M + |x|^2/\sigma)$ and $\int_{\mathbb{R}^d} u dx = M$, it is easy to see that

$$\begin{aligned} e(\sigma) &= (1-m) \sigma^{\frac{d}{2}(1-m)} [M C_M + K_M] \\ &\quad + m \sigma^{\frac{d}{2}(1-m)} \left(M C_M + \int_{\mathbb{R}^d} \frac{|x|^2}{\sigma} u dx \right) \end{aligned}$$

Collecting terms, we get

$$e(\sigma) = \sigma^{\frac{d}{2}(1-m)} [M C_M + (1-m) K_M] + m \sigma^{\frac{d}{2}(1-m)-1} \int_{\mathbb{R}^d} |x|^2 u \, dx .$$

Hence, by optimizing on $\sigma > 0$, we find that $\sigma = \sigma^*$ is given by

$$\sigma^* = \frac{m(m-m_c)}{(1-m)[M C_M + (1-m) K_M]} \int_{\mathbb{R}^d} |x|^2 u \, dx .$$

Using (3), we observe that

$$\frac{m(m-m_c)}{(1-m)[M C_M + (1-m) K_M]} = \frac{1}{K_M} ,$$

which proves (9) and therefore (11). \square

Remark 1. To prove (9) directly, we may notice that σ^* is determined by the condition

$$0 = e'(\sigma) = m(1-m) \int_{\mathbb{R}^d} B_\sigma^{m-2} \frac{dB_\sigma}{d\sigma} (B_\sigma - u) \, dx .$$

Hence we obtain

$$0 = m \int_{\mathbb{R}^d} (B_\sigma - u) \frac{dB_\sigma^{m-1}}{d\sigma} \, dx = m \int_{\mathbb{R}^d} (B_\sigma - u) \left(C(\sigma) C_M + D(\sigma) |x|^2 \right) \, dx ,$$

where $C(\sigma) = \frac{d}{d\sigma} (\sigma^{d(1-m)/2})$, and $D(\sigma) = \frac{d}{d\sigma} (\sigma^{d(1-m)/2-1})$. Taking into account that both B_σ and u have the same mass M , we get (9).

The dependence of σ in t when u is a solution of (7) has not been taken into account yet. The choice of Lemma 2 determines an ordinary differential equation for R in terms of $\int_{\mathbb{R}^d} |x|^2 u(t, x) \, dx$. Undoing the time-dependent rescaling (6), this equation is exactly (4). With the choice $R(0) = 1$, we recall that $u_0 = v_0$.

As already mentioned, the choice of σ in Lemma 2 has a major interest. If we consider a solution of (7) and compute the time derivative of the relative entropy, we find that the first term of the right hand side in (8) drops so that

$$\frac{d}{dt} \mathcal{F}_{\sigma(t)}[u(t, \cdot)] = \frac{m}{m-1} \int_{\mathbb{R}^d} \left(u^{m-1} - B_{\sigma(t)}^{m-1} \right) \frac{\partial u}{\partial t} \, dx$$

and we are back to the usual computations in self-similar variables.

Second step: the entropy / entropy production estimate. According to the definition of B_σ , we know that

$$2x = \sigma^{\frac{d}{2}(m-m_c)} \nabla B_\sigma^{m-1} .$$

Using (7), we obtain

$$\begin{aligned} \frac{m}{m-1} \int_{\mathbb{R}^d} \left(u^{m-1} - B_{\sigma(t)}^{m-1} \right) \frac{\partial u}{\partial t} dx \\ = - \frac{m \sigma(t)^{\frac{d}{2}(m-m_c)}}{1-m} \int_{\mathbb{R}^d} u \left| \nabla \left[u^{m-1} - B_{\sigma(t)}^{m-1} \right] \right|^2 dx , \end{aligned}$$

thus proving (10). Let $w := u/B_\sigma$ and observe that the relative entropy can be written as

$$\mathcal{F}_\sigma[u] = \frac{m}{1-m} \int_{\mathbb{R}^d} \left[w - 1 - \frac{1}{m}(w^m - 1) \right] B_\sigma^m dx .$$

Define the *relative Fisher information* by

$$\mathcal{I}_\sigma[u] := \int_{\mathbb{R}^d} \left| \frac{1}{m-1} \nabla \left[(w^{m-1} - 1) B_\sigma^{m-1} \right] \right|^2 B_\sigma w dx .$$

For a solution u of (7), we find that

$$\frac{d}{dt} \mathcal{F}_{\sigma(t)}[u(t, \cdot)] = -m(1-m) \sigma(t)^{\frac{d}{2}(m-m_c)} \mathcal{I}_{\sigma(t)}[u(t, \cdot)] \quad \forall t > 0 .$$

As in [5, Lemma 3] (also see [6]) we can estimate from below and above the entropy $\mathcal{F}_\sigma[u]$ by

$$(12) \quad h^{m-2} \int_{\mathbb{R}^d} |f|^2 B_\sigma^{2-m} dx \leq \frac{2}{m} \mathcal{F}_\sigma[u] \leq h^{2-m} \int_{\mathbb{R}^d} |f|^2 B_\sigma^{2-m} dx$$

where $f := (w-1)B_\sigma^{m-1}$, $h_1(t) := \inf_{\mathbb{R}^d} w(t, \cdot)$, $h_2(t) := \sup_{\mathbb{R}^d} w(t, \cdot)$ and $h := \max\{h_2, 1/h_1\}$. The fact that h is bounded for any $t > 0$ is easy to prove by the Maximum Principle if $h(0)$ is finite. See for instance [5] for more details. Even if $h(0)$ is infinite, h is anyway bounded for any $t > 0$, large enough, when $m \in (m_c, 1)$; see [8, Theorem 1.2]. By [5, Corollary 1], we also know that $\lim_{t \rightarrow \infty} h(t) = 1$.

According to [5, Lemma 7] (also see [6]), the generalized Fisher information satisfies the bounds

$$(13) \quad \int_{\mathbb{R}^d} |\nabla f|^2 B_\sigma dx \leq [1 + X(h)] \mathcal{I}_\sigma[u] + Y(h) \int_{\mathbb{R}^d} |f|^2 B_\sigma^{2-m} dx$$

where

$$\begin{aligned} h_1^{-1} h_2^{2(2-m)} \leq h^{5-2m} =: 1 + X(h) , \\ d(1-m) \left[\left(\frac{h_2}{h_1} \right)^{2(2-m)} - 1 \right] \leq d(1-m) [h^{4(2-m)} - 1] =: Y(h) . \end{aligned}$$

Notice that $X(1) = Y(1) = 0$.

Third step: orthogonality conditions. To obtain decay rates of $t \mapsto \mathcal{F}_{\sigma(t)}[u(t, \cdot)]$ is now reduced to establish a relation between $\int_{\mathbb{R}^d} |f|^2 B_\sigma^{2-m} dx$ and $\int_{\mathbb{R}^d} |\nabla f|^2 B_\sigma dx$. This is the purpose of the next section, but before let us make a few additional observations on the properties of $f = B_\sigma^{m-2} u - B_\sigma^{m-1}$.

Lemma 3. *Let u be a solution of (7) and $f = B_\sigma^{m-1}(u/B_\sigma - 1)$ where $\sigma = \sigma(t)$ is defined by (5). With these notations, the function f has the following properties, for any $t > 0$:*

- (i) Mass conservation: $\int_{\mathbb{R}^d} f(t, x) B_\sigma^{2-m} dx = 0$ if $m > m_c$.
- (ii) Position of the center of mass: $\int_{\mathbb{R}^d} x f(t, x) B_\sigma^{2-m} dx = 0$ if $m > (d-1)/(d+1)$.
- (iii) Conservation of the second moment: $\int_{\mathbb{R}^d} |x|^2 f(t, x) B_\sigma^{2-m} dx = 0$ if $m > \tilde{m}_1$.

Notice that Property (ii) has already been used in [6, Theorem 7] to obtain improved rates of convergence for $m \in (m_1, 1)$. Property (iii) is new and arises from the fact that the change of variables (6) is chosen in Lemma 2 by imposing a condition on the moment and not according to the self-similar variables corresponding to the Barenblatt solutions. Restrictions on m are such that mass, first and second moments are well defined for B_σ . As in [6], such conditions could be relaxed by considering moment conditions on f only, or relative moment conditions; we will however not pursue in this direction, as such an approach does not improve on the rates of convergence.

Proof. It is straightforward to rewrite the conservation of mass along evolution into Property (i).

Equation (1) being independent of y , it is clear that $\int_{\mathbb{R}^d} y v(\tau, y) dy = \int_{\mathbb{R}^d} y v_0(y) dy$ for any $\tau > 0$. As a consequence, the center of mass of u is located at $x = 0$: $\int_{\mathbb{R}^d} x u(t, x) dx = 0$, and so we get Property (ii).

Finally, Property (iii) is a direct consequence of (9). \square

3. IMPROVED HARDY-POINCARÉ INEQUALITIES

When $M = M^*$ and $\sigma = 1$, with the notations of Section 1, the quantities $\int_{\mathbb{R}^d} |f|^2 B_\sigma^{2-m} dx$ and $\int_{\mathbb{R}^d} |\nabla f|^2 B_\sigma dx$ involve various powers of $(1 + |x|^2)$. On \mathbb{R}^d , we shall therefore consider the measure $d\mu_\alpha := \mu_\alpha dx$, where the weight μ_α is defined by $\mu_\alpha(x) := (1 + |x|^2)^\alpha$, with $\alpha = 1/(m-1) < 0$, and study the operator

$$\mathcal{L}_{\alpha,d} := -\mu_{1-\alpha} \operatorname{div} [\mu_\alpha \nabla \cdot]$$

on the weighted space $L^2(d\mu_{\alpha-1})$. The operator $\mathcal{L}_{\alpha,d}$ is such that

$$\int_{\mathbb{R}^d} f(\mathcal{L}_{\alpha,d} f) d\mu_{\alpha-1} = \int_{\mathbb{R}^d} |\nabla f|^2 d\mu_\alpha.$$

Notice that in the range $m \in (m_c, 1)$, that is for $\alpha \in (-\infty, -d/2)$, $d\mu_\alpha$ is a bounded positive measure. If additionally $m > \tilde{m}_1$, that is for $\alpha < -(d+2)/2$, then $d\mu_{\alpha-1}$ is a bounded measure and $\int_{\mathbb{R}^d} |x|^2 d\mu_\alpha$ is also finite.

Let $\alpha_* := -(d-2)/2$. Based on [17, 18, 4, 6], we have the following result.

Proposition 4. *The bottom of the continuous spectrum of the operator $\mathcal{L}_{\alpha,d}$ on $L^2(d\mu_{\alpha-1})$ is $\lambda_{\alpha,d}^{\operatorname{cont}} := (\alpha - \alpha_*)^2$. Moreover, $\mathcal{L}_{\alpha,d}$ has some discrete*

spectrum only for $\alpha < \alpha_*$. For $d \geq 2$, the discrete spectrum is made of the eigenvalues

$$\lambda_{\ell k} = -2\alpha(\ell + 2k) - 4k\left(k + \ell + \frac{d}{2} - 1\right)$$

with $\ell, k = 0, 1, \dots$ provided $(\ell, k) \neq (0, 0)$ and $\ell + 2k - 1 < -(d + 2\alpha)/2$. If $d = 1$, the discrete spectrum is made of the eigenvalues $\lambda_k = k(1 - 2\alpha - k)$ with $k \in \mathbb{N} \cap [1, 1/2 - \alpha]$.

Let $\alpha_* := -(d - 2)/2$. The following result has been established in [5].

Corollary 5 (Sharp Hardy-Poincaré inequalities). *Let $d \geq 2$. For any $\alpha \in (-\infty, \alpha_*) \cup (\alpha_*, 0)$, there is a positive constant $\Lambda_{\alpha, d}$ such that*

$$(14) \quad \Lambda_{\alpha, d} \int_{\mathbb{R}^d} |f|^2 d\mu_{\alpha-1} \leq \int_{\mathbb{R}^d} |\nabla f|^2 d\mu_{\alpha} \quad \forall f \in L^2(d\mu_{\alpha-1})$$

under the additional condition $\int_{\mathbb{R}^d} f d\mu_{\alpha-1} = 0$ if $\alpha < \alpha_*$. Moreover, for $d \geq 3$, the sharp constant $\Lambda_{\alpha, d}$ is given by

$$\Lambda_{\alpha, d} = \begin{cases} \frac{1}{4}(d - 2 + 2\alpha)^2 & \text{if } \alpha \in \left[-\frac{d+2}{2}, \alpha_*\right) \cup (\alpha_*, 0), \\ -4\alpha - 2d & \text{if } \alpha \in \left[-d, -\frac{d+2}{2}\right), \\ -2\alpha & \text{if } \alpha \in (-\infty, -d). \end{cases}$$

For $d = 2$, Inequality (14) holds for all $\alpha < 0$, with the corresponding values of the best constant $\Lambda_{\alpha, 2} = \alpha^2$ for $\alpha \in [-2, 0)$ and $\Lambda_{\alpha, 2} = -2\alpha$ for $\alpha \in (-\infty, -2)$. For $d = 1$, (14) holds, but the values of $\Lambda_{\alpha, 1}$ are given by $\Lambda_{\alpha, 1} = -2\alpha$ if $\alpha < -1/2$ and $\Lambda_{\alpha, 1} = (\alpha - 1/2)^2$ if $\alpha \in [-1/2, 0)$.

The constant $\Lambda_{\alpha, d}$ is determined by the spectral gap and corresponds either to the lowest positive eigenvalue, $\lambda_{1,0}$ or $\lambda_{0,1}$, or to the bottom of the continuous spectrum, $\lambda_{\alpha, d}^{\text{cont}} := \frac{1}{4}(d + 2\alpha - 2)^2$ (see Fig. 1).

With additional orthogonality conditions, one improves on the spectral gap in the range for which discrete spectrum exists. A first result in this direction has been achieved in [6] for solutions with center of mass at the origin. Here we give a refined version of it, by going to the next order, that is, by considering functions with zero moments up to order two.

Corollary 6 (Improved Hardy-Poincaré inequalities). *Under the assumptions of Corollary 5, if $\alpha < -(d + 2)/2$ and $f \in L^2(d\mu_{\alpha-1})$ is such that*

$$\int_{\mathbb{R}^d} f d\mu_{\alpha-1} = 0, \quad \int_{\mathbb{R}^d} x f d\mu_{\alpha-1} = 0 \quad \text{and} \quad \int_{\mathbb{R}^d} |x|^2 f d\mu_{\alpha-1} = 0,$$

then (14) holds for any $d \geq 3$ with

$$\Lambda_{\alpha, d} = \begin{cases} \frac{1}{4}(d - 2 + 2\alpha)^2 & \text{if } \alpha \in \left[-\frac{d+6}{2}, -\frac{d+2}{2}\right), \\ -8\alpha - 4(d + 2) & \text{if } \alpha \in \left[-(d + 2), -\frac{d+6}{2}\right], \\ -4\alpha & \text{if } \alpha \in (-\infty, -(d + 2)]. \end{cases}$$

For $d = 2$, $\Lambda_{\alpha, 2} = \alpha^2$ for $\alpha \in [-4, -2)$ and $\Lambda_{\alpha, 2} = -4\alpha$ for $\alpha \in (-\infty, -4)$.

The constant $\Lambda_{\alpha,d}$ is now determined either by the lowest of the two eigenvalues, $\lambda_{1,1}$ or $\lambda_{2,0}$, or by the bottom of the continuous spectrum, $\lambda_{\alpha,d}^{\text{cont}}$ (see Fig. 1), since the components corresponding to the eigenspaces associated to $\lambda_{1,0}$ and $\lambda_{0,1}$ have been removed.

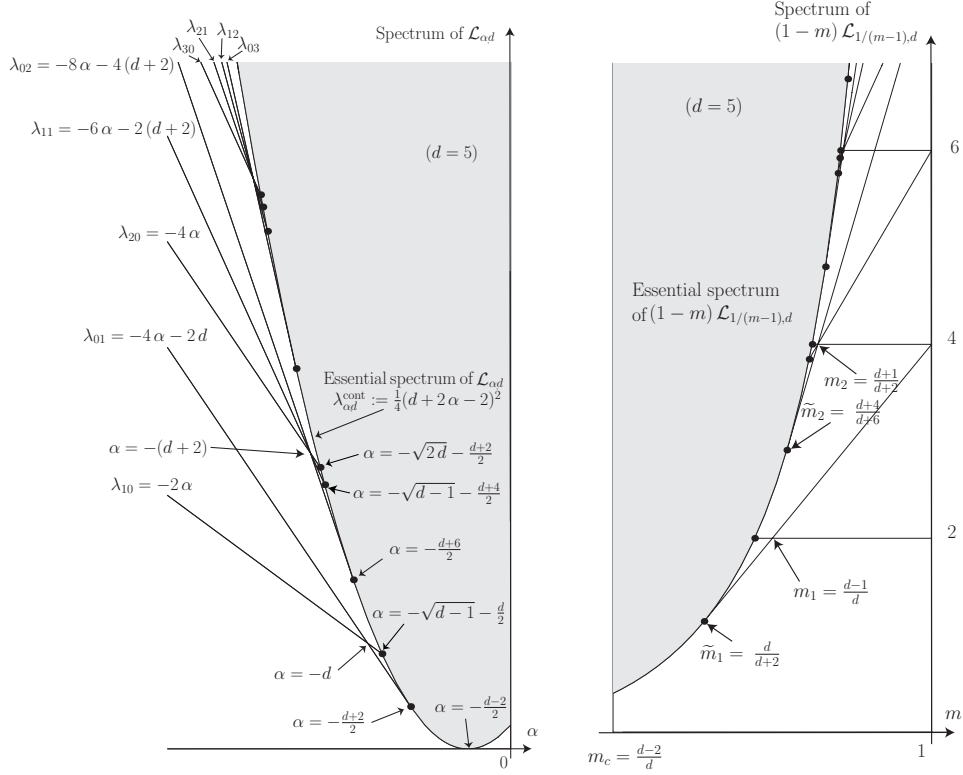


FIGURE 1. Spectrum of $\mathcal{L}_{\alpha,d}$ as a function of α (left), and spectrum of $(1-m)\mathcal{L}_{1/(m-1),d}$ as a function of m (right), for $d=5$.

A crucial observation is that we can scale Inequality (14).

Corollary 7. *Let $M > 0$ and $d \geq 2$. With B_σ defined by (2), let f be a function in $L^2(B_\sigma^{2-m} dx)$ such that*

$$\int_{\mathbb{R}^d} (1, x, |x|^2) f B_\sigma^{2-m} dx = (0, 0, 0) \quad \text{and} \quad \nabla f \in L^2(B_\sigma dx).$$

Then the inequality

$$\Lambda_{\alpha,d} \int_{\mathbb{R}^d} |f|^2 B_\sigma^{2-m} dx \leq \sigma^{\frac{d}{2}(m-m_c)} \int_{\mathbb{R}^d} |\nabla f|^2 B_\sigma dx$$

holds with $\Lambda_{\alpha,d}$ as in Corollary 6.

Proof. The proof relies on a simple change of variables. From Corollary 6, we know that

$$\Lambda_{\alpha,d} \int_{\mathbb{R}^d} |g|^2 (1 + |x|^2)^{\alpha-1} dx \leq \int_{\mathbb{R}^d} |\nabla g|^2 (1 + |x|^2)^\alpha dx$$

for any $g \in L^2(d\mu_{\alpha-1})$ satisfying the conditions of Corollary 6. Then Corollary 7 holds for f such that $f(x) = g(x/\sqrt{C_M\sigma})$ for any $x \in \mathbb{R}^d$, which concludes the proof. \square

4. ESTIMATES ON THE SECOND MOMENT

Up to now, we have not determined the behavior of $R(\tau)$ as $\tau \rightarrow \infty$, nor the fact that σ has a finite, positive limit as $t \rightarrow \infty$. These properties can be deduced for instance from [5]. For the convenience of the reader, let us give some details. As in [5], consider the standard change of variables

$$(\tau, y) \mapsto \left(t = \frac{1}{2} \log R_0(\tau), x = \frac{y}{R_0(\tau)} \right)$$

based on the self-similar behavior of the Barenblatt solution $B(\tau, y)$, where

$$R_0(\tau) := (1 + 2d(m - m_c)\tau)^{\frac{1}{d(m-m_c)}} .$$

With the above change of variables, if v is a solution of (1), then the function $U(t, x) := v(\tau, y)$ solves

$$\frac{\partial U}{\partial t} + \nabla \cdot [U (\nabla U^{m-1} - 2x)] = 0 \quad t > 0, \quad x \in \mathbb{R}^d,$$

which is nothing else than (7), except that here σ is replaced by 1. It has been established in [5] that

$$\lim_{t \rightarrow \infty} e^{2ct} \mathcal{F}_1[U(t, \cdot)] = 0$$

for some positive constant c . By Hölder's inequality, we know that

$$\begin{aligned} \int_{\mathbb{R}^d} U^m dx &= \int_{\mathbb{R}^d} (U B_1^{m-1})^m B_1^{m(1-m)} dx \\ &\leq \left(\int_{\mathbb{R}^d} U B_1^{m-1} dx \right)^m \left(\int_{\mathbb{R}^d} B_1^m dx \right)^{1-m}, \end{aligned}$$

thus proving that

$$\begin{aligned} \mathcal{F}_1[U] &\geq \int_{\mathbb{R}^d} B_1^m dx + \frac{m}{1-m} \int_{\mathbb{R}^d} U B_1^{m-1} dx \\ &\quad - \frac{1}{1-m} \left(\int_{\mathbb{R}^d} U B_1^{m-1} dx \right)^m \left(\int_{\mathbb{R}^d} B_1^m dx \right)^{1-m}. \end{aligned}$$

With

$$k(t) := \frac{\int_{\mathbb{R}^d} U B_1^{m-1} dx}{\int_{\mathbb{R}^d} B_1^m dx} = \frac{\int_{\mathbb{R}^d} |x|^2 U dx + M C_M}{K_M + M C_M},$$

this can be rewritten as

$$\mathcal{F}_1[U] \geq \int_{\mathbb{R}^d} B_1^m dx \frac{k^m - 1 - m(k-1)}{m-1} \sim |k(t) - 1|^2 \quad \text{as } t \rightarrow \infty,$$

hence showing

$$\lim_{t \rightarrow \infty} e^{ct} |k(t) - 1| = 0.$$

As a consequence, we observe that

$$\left| \int_{\mathbb{R}^d} |x|^2 U(t, x) dx - K_M \right| = O(e^{-ct}) \quad \text{as } t \rightarrow \infty,$$

Undoing the change of variables, we find that

$$\left| \frac{1}{R_0(\tau)^2} \int_{\mathbb{R}^d} |y|^2 v(\tau, y) dy - K_M \right| = O(R_0(\tau)^{-c/2}) \quad \text{as } \tau \rightarrow \infty,$$

which, by definition of σ , gives

$$\sigma = \frac{1}{R(\tau)^2} (1 + 2d(m - m_c) \tau)^{\frac{2}{d(m-m_c)}} \left(1 + O(R_0(\tau)^{-c/2}) \right)$$

as $\tau \rightarrow \infty$, where $\tau \mapsto R(\tau)$ is given by (4). Using (5), this means that

$$\frac{1}{R(\tau)} \frac{dR}{d\tau} = 2(R_0(\tau)^2 \sigma)^{-\frac{d}{2}(m-m_c)} = \frac{1}{d(m-m_c)} \frac{1}{\tau} (1 + O(\tau^{-\varepsilon}))$$

with $\varepsilon = \min\{1, c/2\} > 0$. With these estimates, we can prove the following result.

Lemma 8. *With the notations of Sections 1 and 2,*

$$R(\tau) \sim \tau^{\frac{1}{d(m-m_c)}} \quad \text{as } \tau \rightarrow \infty$$

and, as a function of t , $t \mapsto \sigma(t)$ is positive, decreasing, with

$$\lim_{t \rightarrow \infty} \sigma(t) =: \sigma_\infty > 0.$$

More precisely, we know that for some $C_\infty > 0$,

$$R(\tau) = C_\infty \tau^{\frac{1}{d(m-m_c)}} (1 + o(1)) \quad \text{as } \tau \rightarrow \infty$$

and $\sigma_\infty = (2d(m - m_c))^{\frac{2}{d(m-m_c)}} / C_\infty^2$. However, the value of σ_∞ in terms of v_0 is not known.

Proof. The asymptotic behaviors of R and σ are direct consequences of the above computations. We only have to prove the monotonicity of σ . According to Lemma 2, we know that

$$\sigma(t) = \frac{1}{K_M} \int_{\mathbb{R}^d} |x|^2 u(t, x) dx.$$

Using (7) and integrating by parts, we compute

$$\begin{aligned} \frac{d\sigma}{dt} &= \frac{1}{K_M} \frac{d}{dt} \int_{\mathbb{R}^d} |x|^2 u(t, x) dx \\ &= \frac{2d(1-m)}{m K_M} \sigma^{\frac{d}{2}(m-m_c)} \int_{\mathbb{R}^d} u^m dx - 4\sigma . \end{aligned}$$

On the other hand, by Lemma 2, we have

$$\int_{\mathbb{R}^d} B_{\sigma(t)}^{m-1} u(t, x) dx = \int_{\mathbb{R}^d} B_{\sigma(t)}^m dx ,$$

so that

$$\begin{aligned} \mathcal{F}_{\sigma(t)}[u(t, \cdot)] &= \int_{\mathbb{R}^d} \frac{u^m - B_{\sigma(t)}^m}{m-1} dx \\ &= \frac{2m}{d(1-m)^2} K_M \sigma^{\frac{d}{2}(1-m)} - \frac{1}{1-m} \int_{\mathbb{R}^d} u^m dx . \end{aligned}$$

Hence we have proved that

$$\frac{d\sigma}{dt} = -2d \frac{(1-m)^2}{m K_M} \sigma^{\frac{d}{2}(m-m_c)} \mathcal{F}_{\sigma(t)}[u(t, \cdot)] \leq 0 ,$$

which completes the proof. \square

Remark 2. We may notice that the second moment converges to a well defined value:

$$\lim_{t \rightarrow \infty} \int_{\mathbb{R}^d} |x|^2 u(t, x) dx = K_M \sigma_{\infty} .$$

This is not the one which is usually found by considering the time-dependent rescaling corresponding to the self-similar Barenblatt solutions.

5. PROOF OF THEOREM 1

We are now ready to resume with the relative entropy estimates and conclude the proof of Theorem 1. Using (12), (13) and Corollary 7, we find as in [6] that

$$(15) \quad \mathcal{F}_{\sigma}[u] \leq \frac{h^{2-m} [1 + X(h)]}{2 [\Lambda_{\alpha, d} - \sigma Y(h)]} m \sigma^{\frac{d}{2}(m-m_c)} \mathcal{I}_{\sigma}[u]$$

as soon as $0 < h < h_* := \min\{h > 0 : \Lambda_{\alpha, d} - \sup_{t \in \mathbb{R}^+} \sigma(t) Y(h) \geq 0\}$. Two differences with [6] arise: $\Lambda_{\alpha, d}$ has been improved in Corollary 7, to the price of a factor σ , which however plays no role because it also appears in the computation of $\frac{d}{dt} \mathcal{F}_{\sigma(t)}[u(t, \cdot)]$. The m factor is present because \mathcal{F}_{σ} has not been normalized as in [6], and also because the equation for v is not the same.

As in [6], uniform relative estimates hold, according to [7, Inequality (5.33)]: for some positive constant C , we have

$$0 \leq h(t) - 1 \leq C \mathcal{F}_{\sigma(t)}[u(t, \cdot)]^{\frac{1-m}{d+2-(d+1)m}} .$$

Summarizing, we end up with a system of nonlinear differential inequalities, with h as above and, at least for some $t_* > 0$ large enough,

$$\frac{d}{dt} \mathcal{F}_{\sigma(t)}[u(t, \cdot)] \leq -2 \frac{\Lambda_{\alpha, d} - \sigma(t) Y(h)}{[1 + X(h)] h^{2-m}} (1 - m) \mathcal{F}_{\sigma(t)}[u(t, \cdot)]$$

for any $t > t_*$. Gronwall type estimates then show that

$$\limsup_{t \rightarrow \infty} e^{2(1-m)\Lambda_{1/(m-1), d} t} \mathcal{F}_{\sigma(t)}[u(t, \cdot)] < \infty .$$

Notice as in [5] that for some constant $c > 0$, $\lim_{t \rightarrow \infty} e^{ct}(h(t) - 1) = 0$, so that the fact that the quotient $\mathcal{I}_{\sigma}[u]/\mathcal{F}_{\sigma}[u]$ in (15) is not estimated exactly by $(2\Lambda_{\alpha, d})/(m\sigma)$ plays no role for the rate of convergence. This completes the proof of Theorem 1 with $\gamma(m) = (1 - m)\Lambda_{1/(m-1), d}$ (see Fig. 2). \square

Remark 3. Exactly as in [6, Corollary 3], explicit estimates of the constants can be obtained. If v_0/B_1 is uniformly bounded from above and from below by two positive constants, an estimate of $\limsup_{\tau \rightarrow \infty} R(\tau)^{\gamma(m)} \mathcal{F}_{\sigma(t)}[u]$ in Theorem 1 can be given in terms of v_0 by computing a Gronwall estimate.

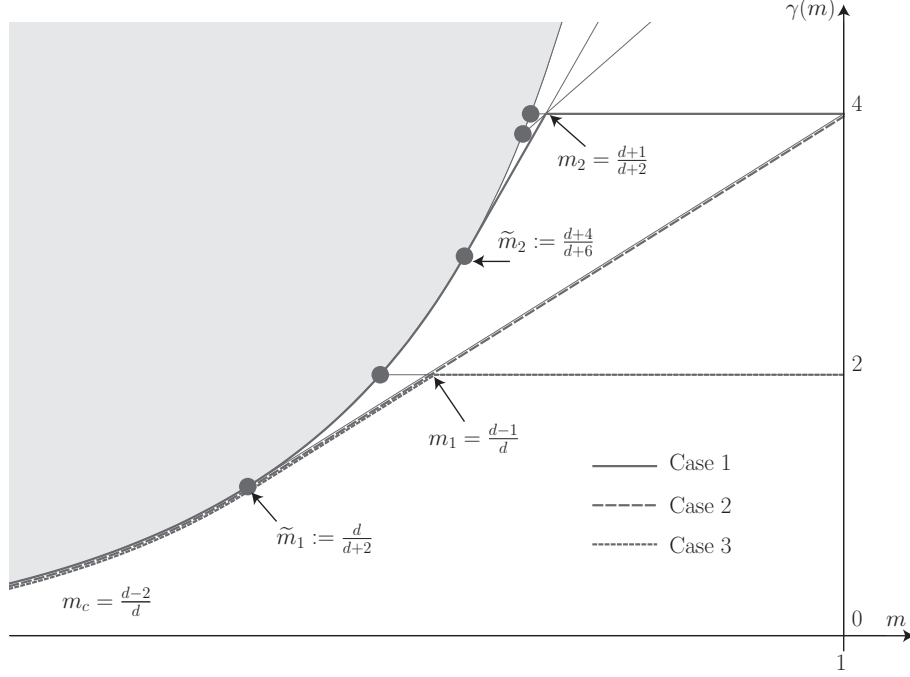


FIGURE 2. For $d = 5$, the value of $m \mapsto \gamma(m)$ is given by the curve of Case 1 when no assumptions are made on the initial data. The curve of Case 2 corresponds to the case where the center of mass is chosen at the origin, as in [6], while the curve of Case 3 corresponds to the exponent found in Theorem 1.

6. CONCLUDING REMARKS

The case of the heat equation, *i.e.* $m = 1$, is not covered in our approach. However, we may pass to the limit as $m \rightarrow 1_-$ in Corollary 7, for the special choice $1/\sigma = 2(1-m)$. Both weights B_σ^{2-m} and B_σ converge to the Gaussian weight, so that the conditions of Corollary 7 become

$$\int_{\mathbb{R}^d} (1, x, |x|^2) f e^{-|x|^2/2} dx = (0, 0, 0) \quad \text{and} \quad \nabla f \in L^2(e^{-|x|^2/2} dx).$$

By requiring the orthogonality with respect to all Hermite polynomials up to order two, we achieve the improved Poincaré inequality

$$2 \int_{\mathbb{R}^d} |f|^2 e^{-|x|^2/2} dx \leq \int_{\mathbb{R}^d} |\nabla f|^2 e^{-|x|^2/2} dx.$$

Compared with the results of [6], nothing is gained, as

$$\lim_{m \rightarrow 1_-} (1-m) \lambda_{01} = \lim_{m \rightarrow 1_-} (1-m) \lambda_{20} = 4,$$

where $\lambda_{\ell k}$ are defined in Proposition 4 and $\alpha = 1/(m-1)$. See [3] for more details on improved convergence rates of relative entropies in case $m = 1$.

For completeness, let us extend our results to the case $d = 1$, which is very simple. Eigenvalues of $\mathcal{L}_{\alpha,1}$ are ordered uniformly with respect to α , according to Proposition 4. Let $\alpha < -1/2$ and consider $f \in L^2(d\mu_{\alpha-1})$ such that

$$\int_{\mathbb{R}} f d\mu_{\alpha-1} = 0, \quad \int_{\mathbb{R}} x f d\mu_{\alpha-1} = 0 \quad \text{and} \quad \int_{\mathbb{R}} |x|^2 f d\mu_{\alpha-1} = 0.$$

Then (14) holds with

$$\Lambda_{\alpha,d} = \begin{cases} (\alpha - \frac{1}{2})^2 & \text{if } \alpha \in [-\frac{5}{2}, -\frac{1}{2}), \\ -6(\alpha + 1) & \text{if } \alpha \in (-\infty, -\frac{5}{2}]. \end{cases}$$

Theorem 9. *Assume that $d = 1$, $m \in (1/3, 1)$. Let v be a solution of (1) with initial datum $v_0 \in L^1_+(\mathbb{R})$ such that v_0^m and $|y|^2 v_0$ are integrable. With the above notations, we have $\limsup_{\tau \rightarrow \infty} R(\tau)^{\gamma(m)} \mathcal{E}(\tau) < \infty$, where $R(\tau) \sim \tau^{\frac{1}{m+1}}$ as $\tau \rightarrow \infty$ and*

$$\gamma(m) = \begin{cases} \frac{(3-m)^2}{4(1-m)} & \text{if } m \in (\frac{1}{3}, \frac{3}{5}] , \\ 6m & \text{if } m \in [\frac{3}{5}, 1) , \end{cases}$$

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REFERENCES

- [1] A. ARNOLD, J. A. CARRILLO, L. DESVILLETES, J. DOLBEAULT, A. JÜNGEL, C. LEDERMAN, P. A. MARKOWICH, G. TOSCANI, AND C. VILLANI, *Entropies and equilibria of many-particle systems: an essay on recent research*, Monatsh. Math., 142 (2004), pp. 35–43.
- [2] G. I. BARENBLATT, *On some unsteady motions of a liquid and gas in a porous medium*, Akad. Nauk SSSR. Prikl. Mat. Meh., 16 (1952), pp. 67–78.
- [3] J.-P. BARTIER, A. BLANCHET, J. DOLBEAULT, AND M. ESCOBEDO, *Improved intermediate asymptotics for the heat equation*, Applied Mathematics Letters, 24 (2011), pp. 76 – 81.
- [4] A. BLANCHET, M. BONFORTE, J. DOLBEAULT, G. GRILLO, AND J.-L. VÁZQUEZ, *Hardy-Poincaré inequalities and applications to nonlinear diffusions*, Comptes Rendus Mathématique, 344 (2007), pp. 431–436.
- [5] ———, *Asymptotics of the fast diffusion equation via entropy estimates*, Archive for Rational Mechanics and Analysis, 191 (2009), pp. 347–385.
- [6] M. BONFORTE, J. DOLBEAULT, G. GRILLO, AND J. L. VÁZQUEZ, *Sharp rates of decay of solutions to the nonlinear fast diffusion equation via functional inequalities*, Proceedings of the National Academy of Sciences, 107 (2010), pp. 16459–16464.
- [7] M. BONFORTE, G. GRILLO, AND J. VÁZQUEZ, *Special fast diffusion with slow asymptotics: Entropy method and flow on a riemannian manifold*, Archive for Rational Mechanics and Analysis, 196 (2010), pp. 631–680.
- [8] M. BONFORTE AND J.-L. VÁZQUEZ, *Global positivity estimates and Harnack inequalities for the fast diffusion equation*, Journal of Functional Analysis, 240 (2006), pp. 399–428.
- [9] M. J. CÁCERES AND G. TOSCANI, *Kinetic approach to long time behavior of linearized fast diffusion equations*, J. Stat. Phys., 128 (2007), pp. 883–925.
- [10] J. A. CARRILLO, M. DI FRANCESCO, AND G. TOSCANI, *Strict contractivity of the 2-Wasserstein distance for the porous medium equation by mass-centering*, Proc. Amer. Math. Soc., 135 (2007), pp. 353–363.
- [11] J. A. CARRILLO, A. JÜNGEL, P. A. MARKOWICH, G. TOSCANI, AND A. UNTERREITER, *Entropy dissipation methods for degenerate parabolic problems and generalized Sobolev inequalities*, Monatsh. Math., 133 (2001), pp. 1–82.
- [12] J. A. CARRILLO, C. LEDERMAN, P. A. MARKOWICH, AND G. TOSCANI, *Poincaré inequalities for linearizations of very fast diffusion equations*, Nonlinearity, 15 (2002), pp. 565–580.
- [13] J. A. CARRILLO AND G. TOSCANI, *Asymptotic L^1 -decay of solutions of the porous medium equation to self-similarity*, Indiana Univ. Math. J., 49 (2000), pp. 113–142.
- [14] D. CORDERO-ERAUSQUIN, B. NAZARET, AND C. VILLANI, *A mass-transportation approach to sharp Sobolev and Gagliardo-Nirenberg inequalities*, Adv. Math., 182 (2004), pp. 307–332.
- [15] P. DASKALOPOULOS AND N. SESUM, *On the extinction profile of solutions to fast diffusion*, J. Reine Angew. Math., 622 (2008), pp. 95–119.
- [16] M. DEL PINO AND J. DOLBEAULT, *Best constants for Gagliardo-Nirenberg inequalities and applications to nonlinear diffusions*, J. Math. Pures Appl. (9), 81 (2002), pp. 847–875.
- [17] J. DENZLER AND R. J. MCCANN, *Phase transitions and symmetry breaking in singular diffusion*, Proc. Natl. Acad. Sci. USA, 100 (2003), pp. 6922–6925.
- [18] ———, *Fast diffusion to self-similarity: complete spectrum, long-time asymptotics, and numerology*, Arch. Ration. Mech. Anal., 175 (2005), pp. 301–342.

- [19] A. FRIEDMAN AND S. KAMIN, *The asymptotic behavior of gas in an n -dimensional porous medium*, Trans. Amer. Math. Soc., 262 (1980), pp. 551–563.
- [20] C. LEDERMAN AND P. A. MARKOWICH, *On fast-diffusion equations with infinite equilibrium entropy and finite equilibrium mass*, Comm. Partial Differential Equations, 28 (2003), pp. 301–332.
- [21] R. J. MCCANN AND D. SLEPČEV, *Second-order asymptotics for the fast-diffusion equation*, Int. Math. Res. Not. Art. ID 24947, (2006), p. 22.
- [22] W. I. NEWMAN, *A Lyapunov functional for the evolution of solutions to the porous medium equation to self-similarity. I*, J. Math. Phys., 25 (1984), pp. 3120–3123.
- [23] F. OTTO, *The geometry of dissipative evolution equations: the porous medium equation*, Comm. Partial Differential Equations, 26 (2001), pp. 101–174.
- [24] J. RALSTON, *A Lyapunov functional for the evolution of solutions to the porous medium equation to self-similarity. II*, J. Math. Phys., 25 (1984), pp. 3124–3127.
- [25] G. TOSCANI, *A central limit theorem for solutions of the porous medium equation*, J. Evol. Equ., 5 (2005), pp. 185–203.
- [26] J.-L. VÁZQUEZ, *Asymptotic behaviour for the porous medium equation posed in the whole space*, J. Evol. Equ., 3 (2003), pp. 67–118.

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