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# The canonical injection of the Hardy-Orlicz space $H^\Psi$ into the Bergman-Orlicz space $\mathfrak{B}^\Psi$

*Pascal Lefèvre, Daniel Li,  
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**Abstract.** *We study the canonical injection from the Hardy-Orlicz space  $H^\Psi$  into the Bergman-Orlicz space  $\mathfrak{B}^\Psi$ .*

**Mathematics Subject Classification.** Primary: 46E30 – Secondary: 30D55; 30H05; 32A35; 32A36; 42B30

**Key-words.** absolutely summing operator – Bergman-Orlicz space – compactness – Dunford-Pettis operator – Hardy-Orlicz space – weak compactness

## 1 Introduction and notation

### 1.1 Introduction

There are two natural Orlicz spaces of analytic functions on the unit disk  $\mathbb{D}$  of the complex plane: the Hardy-Orlicz space  $H^\Psi$  and the Bergman-Orlicz space  $\mathfrak{B}^\Psi$ . It is well-known that in the classical case  $\Psi(x) = x^p$ ,  $H^p \subseteq \mathfrak{B}^p$  and the canonical injection  $J_p$  from  $H^p$  to  $\mathfrak{B}^p$  is bounded, and even compact. In fact, for any Orlicz function  $\Psi$ , one has  $H^\Psi \subseteq \mathfrak{B}^\Psi$  and the canonical injection  $J_\Psi: H^\Psi \rightarrow \mathfrak{B}^\Psi$  is bounded, but we shall see in this paper that its compactness requires that  $\Psi$  does not grow too fast. We actually characterize in Section 2 the compactness:  $J_\Psi$  is compact if and only if  $\lim_{x \rightarrow +\infty} \Psi(Ax)/[\Psi(x)]^2 = 0$  for every  $A > 1$ , and the weak compactness:  $J_\Psi$  is weakly compact if and only if  $\limsup_{x \rightarrow +\infty} \Psi(Ax)/[\Psi(x)]^2 < +\infty$  for every  $A > 1$ . We show that, if these two properties are “often” equivalent (this happens for example if  $\Psi(x)/x$  is non-decreasing for  $x$  large enough), it is not always the case. We actually show a stronger result in Section 4: there is an Orlicz function  $\Psi$  such that  $J_\Psi$  is weakly compact and is Dunford-Pettis, but such that  $J_\Psi$  is not compact.

### 1.2 Notation

An Orlicz function is a non-decreasing convex function  $\Psi: [0, +\infty[ \rightarrow [0, +\infty[$  such that  $\Psi(0) = 0$  and  $\Psi(\infty) = \infty$ . One says that the Orlicz function  $\Psi$  has

property  $\Delta_2$  ( $\Psi \in \Delta_2$ ) if  $\Psi(2x) \leq C\Psi(x)$  for some constant  $C > 0$  and  $x$  large enough. It is equivalent to say that, for every  $\beta > 1$ ,  $\Psi(\beta x) \leq C_\beta\Psi(x)$ . It is known that if  $\Psi \in \Delta_2$ , then  $\Psi(x) = O(x^p)$  for some  $1 \leq p < +\infty$ . One says (see [6], [7]) that  $\Psi$  satisfies the condition  $\Delta^0$  if, for some  $\beta > 1$ , one has  $\lim_{x \rightarrow \infty} \Psi(\beta x)/\Psi(x) = +\infty$ . If  $\Psi \in \Delta^0$ , then  $\Psi(x)/x^p \xrightarrow{x \rightarrow \infty} +\infty$  for every  $1 \leq p < \infty$ . Indeed, let  $1 \leq p < \infty$ . For every  $\beta > 1$  one can find  $x_0 > 0$  such that  $\Psi(\beta x)/\Psi(x) \geq \beta^p$  for  $x \geq x_0$ ; then  $\Psi(\beta^n x_0) \geq \beta^{np}\Psi(x_0)$  for every  $n \geq 1$ . That implies that  $\Psi(x) \geq C_p x^p$  for every  $x > 0$  large enough. Since  $p \geq 1$  is arbitrary, we get  $x^p = o[\Psi(x)]$ .

We say that  $\Psi \in \nabla_0(1)$  if, for every  $A > 1$ ,  $\Psi(Ax)/\Psi(x)$  is non-decreasing for  $x$  large enough. This is equivalent to say (see [7], Proposition 4.7) that  $\log \Psi(e^x)$  is convex. When  $\Psi \in \nabla_0(1)$ , one has either  $\Psi \in \Delta_2$ , or  $\Psi \in \Delta^0$ .

If  $(S, \mathcal{S}, \mu)$  is a finite measure space, one defines the Orlicz space  $L^\Psi(\mu)$  as the set of all (classes of) measurable functions  $f: S \rightarrow \mathbb{C}$  for which there is a  $C > 0$  such that  $\int_S \Psi(|f|/C) d\mu$  is finite. The norm  $\|f\|_\Psi$  is the infimum of all  $C > 0$  for which the above integral is  $\leq 1$ . The Morse-Transue space  $M^\Psi(\mu)$  is the subspace of  $f \in L^\Psi(\mu)$  for which  $\int_S \Psi(|f|/C) d\mu$  is finite for all  $C > 0$ ; it is the closure of  $L^\infty(\mu)$  in  $L^\Psi(\mu)$ . One has  $M^\Psi(\mu) = L^\Psi(\mu)$  if and only if  $\Psi \in \Delta_2$ .

If  $\Psi(x)/x \xrightarrow{x \rightarrow +\infty} +\infty$ , the conjugate function  $\Phi$  of  $\Psi$  is defined by  $\Phi(y) = \sup_{x>0} (xy - \Psi(x))$ . It is an Orlicz function and  $[M^\Psi(\mu)]^* = L^\Phi(\mu)$ , isomorphically.

We may note that if  $\Psi(x)/x$  does not converges to infinity, we must have  $\Psi(x) \leq ax$  for some  $a \geq 1$  and  $x$  large enough. Then  $L^\Psi(\mu) = L^1(\mu)$  isomorphically and then  $\Phi(y) = +\infty$  for  $y > a$  (giving  $L^\Phi(\mu) = L^\infty(\mu)$  isomorphically).

We denote by  $\mathbb{D}$  the open unit disk of  $\mathbb{C}$  and by  $\mathbb{T} = \partial\mathbb{D}$  the unit circle. The normalized area-measure on  $\mathbb{D}$  is denoted by  $\mathcal{A}$  and the normalized Lebesgue measure on  $\mathbb{T}$  is denoted by  $m$ .

The Hardy-Orlicz space  $H^\Psi$  is defined as  $\{f \in H^1; f^* \in L^\Psi(m)\}$ , where  $f^*$  is the boundary values function of  $f$ , and  $HM^\Psi = H^\Psi \cap M^\Psi(m)$  is the closure of  $H^\infty$  in  $H^\Psi$ . The Bergman-Orlicz space  $\mathfrak{B}^\Psi$  is the subspace of analytic  $f \in L^\Psi(\mathcal{A})$ , and  $\mathfrak{B}M^\Psi = \mathfrak{B}^\Psi \cap M^\Psi(\mathcal{A})$  is the closure of  $H^\infty$  in  $\mathfrak{B}^\Psi$ . Since, for  $f \in H^\Psi$ ,  $\|f\|_{H^\Psi} = \sup_{0 < r < 1} \|f_r\|_{H^\Psi}$  (see [7], Proposition 3.1), where  $f_r(z) = f(rz)$ , one has:

$$\int_0^{2\pi} \Psi\left(\frac{|f(re^{it})|}{\|f\|_{H^\Psi}}\right) \frac{dt}{2\pi} \leq \int_0^{2\pi} \Psi\left(\frac{|f(re^{it})|}{\|f_r\|_{H^\Psi}}\right) \frac{dt}{2\pi} \leq 1;$$

hence:

$$\int_{\mathbb{D}} \Psi\left(\frac{|f(re^{it})|}{\|f\|_{H^\Psi}}\right) d\mathcal{A} = \int_0^1 \left[ \int_0^{2\pi} \Psi\left(\frac{|f(re^{it})|}{\|f\|_{H^\Psi}}\right) \frac{dt}{2\pi} \right] 2r dr \leq 1,$$

so  $f \in \mathfrak{B}^\Psi$  and  $\|f\|_{\mathfrak{B}^\Psi} \leq \|f\|_{H^\Psi}$ . It follows that  $H^\Psi \subseteq \mathfrak{B}^\Psi$  and the canonical injection  $J_\Psi: H^\Psi \rightarrow \mathfrak{B}^\Psi$  is bounded, and has norm 1. Let us point out that

the boundedness also follows from [7], Theorem 4.10, 2), since  $J_\Psi$  is a Carleson embedding  $J_\Psi: H^\Psi \rightarrow \mathfrak{B}^\Psi \subseteq L^\Psi(\mathcal{A})$ .

This injection is not onto, since there are functions  $f \in \mathfrak{B}^\Psi$  with no radial limit on a subset of  $\mathbb{T}$  of positive measure (the proof is the same as in  $\mathfrak{B}^p$ : see [4], § 3.2, Lemma 2, page 81). Note that  $J_\Psi$  is not an into-isomorphism: take  $f_n(z) = z^n$ , for every  $n \in \mathbb{N}$ ; it is easy to see that  $\{f_n\}_n$  tends to 0 in  $\mathfrak{B}^\Psi$ , but not in  $H^\Psi$ .

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## 2 Compactness and weak-compactness

In order to characterize the compactness and the weak-compactness of  $J_\Psi$ , we introduce the following quantity  $Q_A$ ,  $A > 1$ :

$$(2.1) \quad Q_A = \limsup_{x \rightarrow +\infty} \frac{\Psi(Ax)}{[\Psi(x)]^2},$$

which will turn out to be essential.

We are going to start with the compactness.

**Theorem 2.1** *The canonical injection  $J_\Psi: H^\Psi \rightarrow \mathfrak{B}^\Psi$  is compact if and only if*

$$(2.2) \quad \lim_{x \rightarrow +\infty} \frac{\Psi(Ax)}{[\Psi(x)]^2} = 0 \quad \text{for every } A > 1.$$

**Remarks.** 1) Condition (2.2) means that  $Q_A = 0$  for every  $A > 1$ . It is equivalent to say that:

$$(2.3) \quad \sup_{A > 1} Q_A < +\infty.$$

Indeed, assume that  $M := \sup_{A > 1} Q_A < +\infty$ . Let  $0 < \varepsilon \leq 1$  and  $A > 1$ ; we can find  $x_A = x_A(\varepsilon) > 0$  such that  $\Psi(Ax/\varepsilon)/[\Psi(x)]^2 \leq 2M$  for  $x \geq x_A$ . By convexity, one has  $\Psi(Ax) \leq \varepsilon \Psi(Ax/\varepsilon)$ , and hence  $\Psi(Ax)/[\Psi(x)]^2 \leq 2\varepsilon M$  for  $x \geq x_A$ . We get  $Q_A = 0$ .

2) It is clear that condition (2.2) is satisfied whenever  $\Psi \in \Delta_2$ , but  $\Psi(x) = e^{[\log(x+1)]^2} - 1$  satisfies (2.2) without being in  $\Delta_2$ . However, condition (2.2) implies that  $\Psi$  cannot grow too fast. More precisely, we must have

$$\Psi(x) = o(e^{x^\alpha}) \quad \text{for every } \alpha > 0.$$

Indeed, one has  $\Psi(At) \leq [\Psi(t)]^2$  for  $t \geq t_A$ , and, by iteration,  $\Psi(A^n t_A) \leq [\Psi(t_A)]^{2^n}$  for every  $n \geq 1$ . For every  $x > 0$  large enough, taking  $n \geq 1$  such that  $A^n t_A \leq x < A^{n+1} t_A$ , we get  $\Psi(x) \leq C_1 e^{C_2 x^\alpha}$ , with  $\alpha = \log 2 / \log A$ . Since  $A > 1$  is arbitrary,  $\alpha$  may be any positive number. The little-oh condition follows from the fact that the inequality is true for all  $\alpha > 0$ .

**Proof of Theorem 2.1.** By definition,  $\mathfrak{B}^\Psi$  is a subspace of  $L^\Psi(\mathbb{D}, \mathcal{A})$ ; hence we can see  $J_\Psi$  as a Carleson embedding  $J_\Psi: H^\Psi \rightarrow L^\Psi(\mathbb{D}, \mathcal{A})$ . If  $S(\xi, h) = \{z \in \mathbb{D}; |z - \xi| < h\}$ , the compactness of  $J_\Psi$  implies, by [7], Theorem 4.11, that, for every  $A > 1$ , every  $\varepsilon > 0$ , and  $h > 0$  small enough:

$$h^2 \leq 4 \mathcal{A}[S(\xi, h)] \leq \frac{4\varepsilon}{\Psi[A\Psi^{-1}(1/h)]},$$

that is, setting  $x = \Psi^{-1}(1/h)$ ,  $\Psi(Ax) \leq 4\varepsilon [\Psi(x)]^2$ , and (2.2) is satisfied.

Conversely, one has:

$$\sup_{0 < t \leq h} \sup_{|\xi|=1} \frac{\mathcal{A}[S(\xi, t)]}{t} \leq \sup_{0 < t \leq h} \frac{t^2}{t} = h,$$

which is  $o((1/h)/\Psi[A\Psi^{-1}(1/h)])$  for every  $A > 1$ , if (2.2) holds; hence, by [7], Theorem 4.11, again,  $J_\Psi$  is compact.  $\square$

We now turn ourself to the weak compactness.

**Theorem 2.2** *The following assertions are equivalent:*

- (a)  $J_\Psi: H^\Psi \rightarrow \mathfrak{B}^\Psi$  is weakly compact;
- (b)  $J_\Psi$  fixes no copy of  $c_0$ ;
- (c)  $J_\Psi$  fixes no copy of  $\ell_\infty$ ;
- (d)  $Q_A < +\infty$ , for every  $A > 1$ ;
- (e)  $H^\Psi \subseteq \mathfrak{B}M^\Psi$ ;
- (f)  $J_\Psi$  is strictly singular.

Recall that an operator  $T: X \rightarrow Y$  between two Banach spaces is said to be strictly singular if there is no infinite-dimensional subspace  $X_0$  of  $X$  on which  $T$  is an into-isomorphism.

The proof will be somewhat long, and before beginning it, we shall remark that if  $\Psi \in \Delta^0$ , then condition

$$(2.4) \quad Q_A < +\infty \quad \text{for every } A > 1$$

implies condition (2.2). Indeed, if  $\lim_{x \rightarrow +\infty} \frac{\Psi(\beta x)}{\Psi(x)} = +\infty$ , we get, for every  $A > 1$ :

$$\limsup_{x \rightarrow +\infty} \frac{\Psi(Ax)}{[\Psi(x)]^2} = \limsup_{x \rightarrow +\infty} \frac{\Psi(Ax)}{\Psi(\beta Ax)} \frac{\Psi(\beta Ax)}{[\Psi(x)]^2} \leq \limsup_{x \rightarrow +\infty} \frac{\Psi(Ax)}{\Psi(\beta Ax)} Q_{\beta A} = 0.$$

Now, if, for some  $A > 1$ ,  $\Psi(Ax)/\Psi(x)$  is non-decreasing for  $x$  large enough (in particular if  $\Psi \in \nabla_0(1)$ ), one has the dichotomy: either  $\Psi \in \Delta_2$ , and then  $J_\Psi$  is compact; or  $\Psi \in \Delta^0$  and hence the weak compactness of  $J_\Psi$  implies, by the two above theorems, its compactness. Hence:

**Proposition 2.3** *If, for some  $A > 1$ ,  $\Psi(Ax)/\Psi(x)$  is non-decreasing, for  $x$  large enough, then the weak compactness of  $J_\Psi$  is equivalent to its compactness.*

However, it is easy to construct an Orlicz function  $\Psi$  which satisfies condition (2.4), but not condition (2.2). We do not give an example here because we have a stronger result in Section 4.

In order to prove Theorem 2.2, we shall need several lemmas.

**Lemma 2.4** *Let  $\Psi$  be any Orlicz function. If we define  $\Psi_1(t) = [\Psi(t)]^2$ ,  $t \geq 0$ , then  $\Psi_1$  is an Orlicz function for which  $H^\Psi \subseteq \mathfrak{B}^{\Psi_1}$  and the canonical injection of  $H^\Psi$  into  $\mathfrak{B}^{\Psi_1}$  is continuous.*

**Proof.** It is enough to see that  $H^\Psi$  continuously embeds into  $L^{\Psi_1}(\mathcal{A})$ , and for this we can use Theorem 4.10 in [7]. Following the notation of that theorem for the measure  $\mu = \mathcal{A}$ , it is easy to see that, as  $h \rightarrow 0^+$ ,  $\rho_{\mathcal{A}}(h) \approx h^2$ , and  $K_{\mathcal{A}}(h) \approx h$ . Observe that, for  $t > 1$ , we have  $\Psi_1[\Psi^{-1}(t)] = t^2$ , and so, for  $h \in (0, 1)$ ,

$$\frac{1/h}{\Psi_1[\Psi^{-1}(1/h)]} = \frac{1/h}{1/h^2} = h \geq K_{\mathcal{A}}(h).$$

Using part 2) of Theorem 4.10 in [7], the lemma follows.  $\square$

**Lemma 2.5** *Let  $M > \delta > 0$  and  $\{f_n\}_n$  be a sequence in  $H^\Psi \cap \mathfrak{B}M^\Psi$  such that:*

- (a)  $\{f_n\}_n$  tends to 0 uniformly on compact subsets of  $\mathbb{D}$ ;
- (b)  $\|f_n\|_{\mathfrak{B}^\Psi} \geq \delta$ , for every  $n \geq 1$ ;
- (c)  $\|f_n\|_{H^\Psi} \leq M$ , for every  $n \geq 1$ .

*Then there exists a subsequence  $\{f_{n_k}\}_k$  such that  $\sum_k |f_{n_k}(z)| < +\infty$ , for every  $z \in \mathbb{D}$ , and for every  $\alpha = (\alpha_k)_k \in \ell_\infty$ , one has, writing  $T\alpha(z) = \sum_{k=1}^\infty \alpha_k f_{n_k}(z)$ :*

$$(2.5) \quad T\alpha \in \mathfrak{B}^\Psi \quad \text{and} \quad (\delta/2)\|\alpha\|_\infty \leq \|T\alpha\|_{\mathfrak{B}^\Psi} \leq 2M\|\alpha\|_\infty.$$

**Remark.** It is clear that, by (2.5), we are defining an operator  $T$  from  $\ell_\infty$  into  $\mathfrak{B}^\Psi$  which is an isomorphism between  $\ell_\infty$  and its image. In particular, the subsequence  $\{f_{n_k}\}_k$  is equivalent, in  $\mathfrak{B}^\Psi$ , to the canonical basis of  $c_0$ .

**Proof.** First we are going to construct, inductively, a subsequence  $\{f_{n_k}\}_k$  of  $\{f_n\}$ , and an increasing sequence  $\{r_k\}_k$  in  $(0, 1)$ , such that  $\lim_{k \rightarrow \infty} r_k = 1$  and, setting

$$D_k = \{z \in \mathbb{D}; |z| \leq r_k\}, \quad \text{for } k \geq 1,$$

and

$$C_1 = D_1, \quad C_k = D_k \setminus D_{k-1} = \{z \in \mathbb{D}; r_{k-1} < |z| \leq r_k\}, \quad k \geq 2,$$

we have:

$$(2.6) \quad |f_{n_k}(z)| \leq 2^{-k}, \quad \text{for every } z \in D_{k-1}, \text{ and every } k \geq 2;$$

and

$$(2.7) \quad \|f_{n_k} \mathbf{1}_{\mathbb{D} \setminus C_k}\|_{L^\Psi} < \delta 2^{-k-2}, \quad \text{for every } k \geq 1.$$

Start the construction by taking  $n_1 = 1$ . It is a known fact that, for every function  $f$  in the Morse-Transue space  $M^\Psi(\mathcal{A})$ , we have

$$(2.8) \quad \lim_{\mathcal{A}(A) \rightarrow 0} \|f \mathbf{1}_A\|_{L^\Psi} = 0.$$

Now, using (2.8), with  $f = f_{n_1}$  and considering sets  $A$  of the form  $A = \{z \in \mathbb{D}; r < |z| < 1\}$ , we get  $r_1 \in (0, 1)$  so that, for  $C_1 = D_1 = \{z \in \mathbb{D}; |z| \leq r_1\}$ , we have

$$\|f_1 \mathbf{1}_{\mathbb{D} \setminus C_1}\|_{L^\Psi} < \delta 2^{-3}.$$

By the uniform convergence of  $\{f_n\}_n$  to 0 on  $D_1$ , we can find  $n_2 > n_1$  such that

$$|f_{n_2}(z)| \leq 1/4, \quad \text{for every } z \in D_1, \quad \text{and} \quad \|f_{n_2} \mathbf{1}_{D_1}\|_{L^\Psi} < \delta 2^{-5}.$$

Using this last inequality and (2.8) again (for  $f = f_{n_2}$ ), we get  $r_2 \in (r_1, 1)$ ,  $r_2 > 1 - 1/2$ , such that, setting  $C_2 = \{z \in \mathbb{D}; r_1 < |z| \leq r_2\}$ , we have

$$\|f_{n_2} \mathbf{1}_{\mathbb{D} \setminus C_2}\|_{L^\Psi} < \delta 2^{-4}.$$

Now that we have (2.6) and (2.7) for  $k = 1$  and  $k = 2$ , it is clear how we must iterate the inductive construction. At the time of choosing  $r_k \in (r_{k-1}, 1)$ , we also impose the condition  $r_k > 1 - 1/k$  in order to get  $\lim_{k \rightarrow \infty} r_k = 1$ .

Once the construction is achieved, let us see why the subsequence  $\{f_{n_k}\}_k$  works. The condition (2.6) and the fact that  $\lim_{k \rightarrow \infty} r_k = 1$  imply that, for every compact set  $K$  in  $\mathbb{D}$  and  $z \in \mathbb{D}$ , there exists  $l_K \in \mathbb{N}$  such that:

$$|f_{n_k}(z)| \leq 2^{-k}, \quad \text{for every } z \in K, \quad \text{and every } k \geq l_K.$$

This yields two facts. First,  $\sum_k |f_{n_k}(z)| < +\infty$ , for every  $z \in \mathbb{D}$ , and secondly: for every bounded complex sequence  $\alpha = (\alpha_k)_k \in \ell_\infty$ , the series  $\sum_k \alpha_k f_{n_k}$  converges uniformly on compact subsets of  $\mathbb{D}$ , and its sum, the function  $T\alpha$ , is analytic on  $\mathbb{D}$ .

It remains to prove the estimates in (2.5) about the norm of  $T\alpha$  in  $L^\Psi(\mathcal{A})$ . By homogeneity, we may assume that  $\|\alpha\|_\infty = 1$ . Let us write  $g_k = f_{n_k} \mathbf{1}_{C_k}$  and  $h_k = f_{n_k} \mathbf{1}_{\mathbb{D} \setminus C_k}$ , for every  $k \geq 1$ ,

$$g = \sum_{k=1}^{\infty} \alpha_k g_k \quad \text{and} \quad h = \sum_{k=1}^{\infty} \alpha_k h_k.$$

We have  $T\alpha = g + h$ . By (2.7) and the fact that  $|\alpha_k| \leq 1$ , we have that  $h \in L^\Psi(\mathcal{A})$  and  $\|h\|_{L^\Psi} \leq \delta/4$ .

By the condition (c) in the statement and the definition of the norm in  $H^\Psi$  we have, for every  $n$  and every  $r \in (0, 1)$ :

$$(2.9) \quad \frac{1}{2\pi} \int_0^{2\pi} \Psi(|f_n(re^{it})|/M) dt \leq 1.$$

The function  $g_k$  is 0 outside of  $C_k$ , and the sequence  $\{C_k\}_k$  is a partition of  $\mathbb{D}$ . Therefore:

$$\begin{aligned} \int_{\mathbb{D}} \Psi(|g|/M) d\mathcal{A} &= \sum_{k=1}^{\infty} \int_{C_k} \Psi(|g|/M) d\mathcal{A} = \sum_{k=1}^{\infty} \int_{C_k} \Psi(|\alpha_k| |f_{n_k}|/M) d\mathcal{A} \\ &\leq \sum_{k=1}^{\infty} \int_{C_k} \Psi(|f_{n_k}|/M) d\mathcal{A}. \end{aligned}$$

Integrating in polar coordinates, setting  $r_0 = 0$ , and using (2.9), we get:

$$\begin{aligned} \int_{\mathbb{D}} \Psi(|g|/M) d\mathcal{A} &\leq \sum_{k=1}^{\infty} \int_{r_{k-1}}^{r_k} 2r \frac{1}{2\pi} \int_0^{2\pi} \Psi(|f_{n_k}(re^{it})|/M) dt dr \\ &\leq \sum_{k=1}^{\infty} \int_{r_{k-1}}^{r_k} 2r dr = 1, \end{aligned}$$

and therefore  $\|g\|_{L^\Psi} \leq M$ , and  $\|T\alpha\|_{L^\Psi} \leq \delta/4 + M \leq 2M$ .

On the other hand, for every  $k$ , we have:

$$\|g\|_{L^\Psi} \geq \|g \mathbf{1}_{C_k}\|_{L^\Psi} = |\alpha_k| \|f_{n_k} - h_k\|_{L^\Psi} \geq |\alpha_k| (\delta - \delta/2^{2+k}) \geq \frac{3\delta}{4} |\alpha_k|.$$

Taking the supremum on  $k$ , we get  $\|g\|_{L^\Psi} \geq (3\delta/4) \|\alpha\|_\infty = 3\delta/4$ . Consequently,

$$\|T\alpha\|_{L^\Psi} \geq \|g\|_{L^\Psi} - \|h\|_{L^\Psi} \geq (3\delta/4) - \delta/4 \geq \delta/2,$$

and Lemma 2.5 is fully proved.  $\square$

In the following lemma we isolate the proof of the implication (c)  $\implies$  (d) in the statement of Theorem 2.2.

**Lemma 2.6** *Assume that the Orlicz function  $\Psi$  is such that, for some  $A > 1$ ,*

$$(2.10) \quad \limsup_{x \rightarrow +\infty} \frac{\Psi(Ax)}{[\Psi(x)]^2} = +\infty$$

*Then the injection  $J_\Psi: H^\Psi \rightarrow \mathfrak{B}^\Psi$  fixes a copy of  $\ell_\infty$ .*

**Proof.** Let us take a sequence of positive numbers  $\{d_n\}_n$ , and a sequence  $\{\xi_n\}_n$  in  $\mathbb{T}$ , such that the disks  $\{D(\xi_n, d_n)\}_n$  are pairwise disjoint in  $\mathbb{D}$ . In particular, we should have  $\lim_{n \rightarrow \infty} d_n = 0$ .

The convexity of  $\Psi$  implies the existence of some  $c > 0$  such that  $\Psi(x) \geq cx$  for every  $x \geq 1$ . Given a sequence  $\{\beta_n\}_n$  in  $(4, +\infty)$  to be fixed later, we can find, thanks to (2.10), an increasing sequence  $\{x_n\}$  satisfying:

$$(2.11) \quad x_n > 1, \quad \Psi(x_n) > 1, \quad \Psi(Ax_n) > \beta_n [\Psi(x_n)]^2, \quad \text{for every } n \in \mathbb{N}.$$



Define  $y_n$  as the point in the interval  $(x_n, Ax_n)$  such that

$$(2.12) \quad [\Psi(y_n)]^2 = \Psi(Ax_n).$$

Put now  $h_n = 1/\Psi(y_n)$  and  $r_n = 1 - h_n$ . By (2.11) and (2.12), we have  $[\Psi(y_n)]^2 > \beta_n > 4$ , and therefore  $h_n \in (0, 1/2)$ . Define

$$u_n(z) = \left( \frac{h_n}{1 - r_n \overline{\xi_n z}} \right)^2, \quad \text{and} \quad f_n(z) = y_n u_n(z).$$

It is easy to see that  $\|u_n\|_\infty = 1$ , and that  $\|u_n\|_{H^1} \leq h_n$ .

The first condition imposed to  $\beta_n$  is  $\beta_n > 16/d_n^2$ . That gives  $[\Psi(y_n)]^2 > 16/d_n^2$  and  $h_n < d_n/4$ . Let us write  $D_n$  for the disk  $D(\xi_n, d_n)$ . Observe that, for  $z \in \overline{\mathbb{D}} \setminus D_n$ , we have

$$|1 - r_n \overline{\xi_n z}| = |1 - r_n + r_n \xi_n \overline{\xi_n} - r_n \overline{\xi_n z}| \geq r_n |\xi_n - z| - h_n \geq (1/2)d_n - h_n \geq d_n/4,$$

and therefore, since  $[\Psi(x_n)]^2 \geq \Psi(x_n) \geq cx_n$ ,

$$|f_n(z)| \leq y_n \left( \frac{4h_n}{d_n} \right)^2 = \frac{16y_n}{d_n^2 [\Psi(y_n)]^2} \leq \frac{16Ax_n}{d_n^2 \beta_n [\Psi(x_n)]^2} \leq \frac{16A}{c d_n^2 \beta_n}.$$

We also impose the condition  $\beta_n > 16An^2/cd_n^2$ , and so we have:

$$(2.13) \quad |f_n(z)| \leq \frac{1}{n^2}, \quad \text{for } z \in \overline{\mathbb{D}} \setminus D_n.$$

From (2.13) we deduce that  $\{f_n\}_n$  converges to 0 uniformly on compact subsets of  $\mathbb{D}$ . Moreover (2.13) yields that, for every bounded sequence  $\{\alpha_n\}_n$  of complex numbers, the series  $\sum_{n \geq 1} \alpha_n f_n$  is uniformly convergent on compact subsets of  $\mathbb{D}$ . Let us write  $f_n^*$  for the boundary value (on  $\mathbb{T} = \partial\mathbb{D}$ ) of the function  $f_n$ . We claim that :

$$(2.14) \quad S = \sum_{n=1}^{\infty} |f_n^*| \in L^\Psi(\mathbb{T}, m).$$

From this, it is not difficult to deduce that, for every bounded sequence  $\{\alpha_n\}_n$  of complex numbers, the function  $\sum_{n=1}^{\infty} \alpha_n f_n$  is in  $H^\Psi$  and, for  $M = \|S\|_{L^\Psi(\mathbb{T})}$ ,

$$(2.15) \quad \left\| \sum_{n=1}^{\infty} \alpha_n f_n \right\|_{H^\Psi} \leq M \|\{\alpha_n\}_n\|_\infty.$$

On the other hand, taking  $A_n = \{z \in \mathbb{D}; |z - \xi_n| \leq h_n\}$ , there exists a constant  $\gamma \in (0, 1)$  such that  $\mathcal{A}(A_n) \geq \gamma h_n^2$ , and, for every  $z \in A_n$ , we have:

$$|1 - r_n \overline{\xi_n z}| \leq |1 - r_n| + |r_n \xi_n \overline{\xi_n} - r_n \overline{\xi_n z}| = h_n + r_n |z - \xi_n| \leq 2h_n,$$

and consequently  $|u_n(z)| \geq 1/4$ . If  $\delta = \gamma/4A$ , we have, for every  $n$ ,

$$\begin{aligned} \int_{\mathbb{D}} \Psi\left(\frac{|f_n|}{\delta}\right) d\mathcal{A} &\geq \int_{A_n} \Psi\left(\frac{y_n}{4\delta}\right) d\mathcal{A} \geq \gamma h_n^2 \Psi\left(\frac{1}{\gamma} A y_n\right) \\ &\geq h_n^2 \Psi(A y_n) > h_n^2 \Psi(A x_n) = 1. \end{aligned}$$

Thus  $\|f_n\|_{\mathfrak{B}^\Psi} \geq \delta$ , for every  $n \in \mathbb{N}$ . We can apply Lemma 2.5. Using this lemma and (2.15), we get a subsequence  $\{f_{n_k}\}_k$  such that, for every  $\alpha = (\alpha_k)_k \in \ell_\infty$ , we have:

$$(\delta/2) \|\{\alpha_k\}_k\|_\infty \leq \left\| \sum_{k=1}^{\infty} \alpha_k f_{n_k} \right\|_{\mathfrak{B}^\Psi} \leq \left\| \sum_{k=1}^{\infty} \alpha_k f_{n_k} \right\|_{H^\Psi} \leq M \|\{\alpha_k\}_k\|_\infty.$$

This clearly says that  $J_\Psi$  fixes a copy of  $\ell_\infty$ .

It remains to prove (2.14). For obtaining this we impose the last condition to the sequence  $\{\beta_n\}_n$ . We shall need:

$$(2.16) \quad \sum_{n=1}^{\infty} 1/\sqrt{\beta_n} \leq 1.$$

Let us set  $g_n = |f_n^*| \mathbf{1}_{D_n}$ . Thanks to (2.13),  $S - \sum_{n=1}^{\infty} g_n$  is a bounded function. Thus we just need to prove that  $G = \sum_{n=1}^{\infty} g_n$  is in  $L^\Psi(\mathbb{T})$ . We have  $\|G\|_{L^\Psi(\mathbb{T})} \leq A$ . Indeed, recalling that the  $D_n$ 's are pairwise disjoint, and that each  $g_n$  is 0 out of  $D_n$ , we have:

$$\begin{aligned} \int_{\mathbb{T}} \Psi\left(\frac{G}{A}\right) dm &= \sum_{n=1}^{\infty} \int_{D_n \cap \mathbb{T}} \Psi\left(\frac{G}{A}\right) dm = \sum_{n=1}^{\infty} \int_{D_n \cap \mathbb{T}} \Psi\left(\frac{|f_n^*|}{A}\right) dm \\ &\leq \sum_{n=1}^{\infty} \int_{\mathbb{T}} \Psi\left(\frac{y_n |u_n^*|}{A}\right) dm \end{aligned}$$

and by the convexity of  $\Psi$ , and the fact that  $|u_n| \leq 1$ ,

$$\begin{aligned} &\leq \sum_{n=1}^{\infty} \int_{\mathbb{T}} |u_n^*| \Psi\left(\frac{y_n}{A}\right) dm = \sum_{n=1}^{\infty} \|u_n\|_{H^1} \Psi\left(\frac{y_n}{A}\right) \\ &\leq \sum_{n=1}^{\infty} \frac{\Psi(y_n/A)}{\Psi(y_n)} \leq \sum_{n=1}^{\infty} \frac{\Psi(x_n)}{\Psi(y_n)} = \sum_{n=1}^{\infty} \frac{\Psi(x_n)}{\sqrt{\Psi(Ax_n)}} \leq \sum_{n=1}^{\infty} \frac{1}{\sqrt{\beta_n}} \leq 1, \end{aligned}$$

by the required condition (2.16), and that ends the proof of Lemma 2.6.  $\square$

We are now in position to prove Theorem 2.2.

**Proof of Theorem 2.2.** We shall prove that:

$$(a) \implies (b) \implies (c) \implies (d) \implies (e) \implies (a),$$

and that (b)  $\iff$  (f).

The implications (a)  $\implies$  (b)  $\implies$  (c) and (f)  $\implies$  (b) are trivial, and we have seen in Lemma 2.6 that (c)  $\implies$  (d).

(d)  $\implies$  (e). By Lemma 2.4, there exists a constant  $C > 0$  such that, for every  $f$  in the unit ball of  $H^\Psi$ , we have:

$$(2.17) \quad \int_{\mathbb{D}} [\Psi(|f|/C)]^2 d\mathcal{A} \leq 1.$$

For every  $A > 0$ , there exist  $x_A$ , such that  $\Psi(Ax) \leq (Q_A + 1)[\Psi(x)]^2$ , for every  $x \geq x_A$ . Thus for every  $x \geq 0$  we have  $\Psi(Ax) \leq (Q_A + 1)[\Psi(x)]^2 + \Psi(Ax_A)$ . Then, by (2.17), we have

$$\int_{\mathbb{D}} \Psi(A|f|/C) d\mathcal{A} < +\infty, \quad \text{for every } A > 0.$$

Therefore  $f \in \mathfrak{B}M^\Psi$ , for every  $f$  in the unit ball of  $H^\Psi$ , and thus for every  $f$  in  $H^\Psi$ .

(e)  $\implies$  (a). Let  $\{f_n\}_n$  be in the unit ball of  $H^\Psi$ . We have to prove that  $\{f_n\}_n$  has a subsequence which converges in the weak topology of  $\mathfrak{B}^\Psi$ . By Montel's Theorem  $\{f_n\}_n$  has a subsequence converging uniformly on compact subsets of  $\mathbb{D}$ , to a function  $g$  which, by Fatou's lemma, also belongs to the unit ball of  $H^\Psi$ . If this subsequence converges to  $g$  in the norm of  $\mathfrak{B}^\Psi$  we are done. If not, after perhaps a new extraction of subsequence, there exist  $\delta > 0$  and a subsequence  $\{f_{n_k}\}_k$ , such that

$$\|f_{n_k} - g\|_{\mathfrak{B}^\Psi} \geq \delta, \quad \text{and} \quad \|f_{n_k} - g\|_{H^\Psi} \leq 2.$$

Since moreover  $\{f_{n_k} - g\}_k$  converges to 0 uniformly on compact subsets of  $\mathbb{D}$  and, by condition (e),  $f_{n_k} - g \in \mathfrak{B}M^\Psi$ , we may apply Lemma 2.5 and we get that  $\{f_{n_k} - g\}_k$  has a subsequence equivalent to the canonical basis of  $c_0$  in  $\mathfrak{B}^\Psi$ , and is therefore weakly null. This yields that  $\{f_n\}_n$  has a subsequence converging to  $g$  in the weak topology of  $\mathfrak{B}^\Psi$ .

(b)  $\implies$  (f). Suppose there exists an infinite-dimensional subspace  $X$  of  $H^\Psi$  on which the norms  $\|\cdot\|_{\mathfrak{B}^\Psi}$  and  $\|\cdot\|_{H^\Psi}$  are equivalent. We shall have finished if we prove that  $X$  contains a subspace isomorphic to  $c_0$  because then  $J_\Psi$  will fix a copy of  $c_0$ .

We can assume that  $X$  is contained in  $\mathfrak{B}M^\Psi$  because we already know that (b) implies (e).  $X$  being infinite-dimensional, there exists, for every  $n \in \mathbb{N}$ ,  $f_n \in X$ , such that  $\|f_n\|_{H^\Psi} = 1$ , and  $\widehat{f_n}(k) = 0$ , for  $k = 0, 1, \dots, n$ . By the equivalence of the norms in  $X$ , there exists  $\delta > 0$  such that  $\|f_n\|_{\mathfrak{B}^\Psi} \geq \delta$ , for every  $n$ . The unit ball of  $H^\Psi$  is compact in the topology of  $\mathcal{H}(\mathbb{D})$ . Since

$$\lim_{n \rightarrow \infty} \widehat{f_n}(k) = 0, \quad \text{for every } k \geq 0,$$

the only possible limit of a subsequence of  $\{f_n\}_n$  is the function 0. So  $\{f_n\}_n$  converges to 0 uniformly on compact subsets of  $\mathbb{D}$ . As  $f_n \in X \subseteq \mathfrak{B}M^\Psi$ , for

every  $n$ , we can apply Lemma 2.5, and we get that  $\{f_n\}_n$  has a subsequence generating an space  $Y$  isomorphic to  $c_0$  in  $\mathfrak{B}^\Psi$ . This space  $Y$  is contained in  $X$ , where the norms are equivalent, so  $Y$  is also isomorphic to  $c_0$  for the norm of  $H^\Psi$ .  $\square$

### 3 Other properties

#### 3.1 Dunford-Pettis

Recall that an operator  $T: X \rightarrow Y$  between two Banach spaces  $X$  and  $Y$  is said to be *Dunford-Pettis* if  $\{Tx_n\}_n$  converges in norm whenever  $\{x_n\}_n$  converges weakly. Every compact operator is Dunford-Pettis. The next proposition shows that, in “most” of the cases, these two properties are equivalent for  $J_\Psi$ .

**Proposition 3.1** *If the conjugate function of  $\Psi$  satisfies condition  $\Delta_2$ , then  $J_\Psi: H^\Psi \rightarrow \mathfrak{B}^\Psi$  is Dunford-Pettis if and only if it is compact.*

We shall see in Section 4 that without condition  $\Delta_2$  for the conjugate function,  $J_\Psi$  may be Dunford-Pettis without being compact.

**Proof.** Remark first that speaking of the conjugate function of  $\Psi$  implicitly assume that  $\Psi(x)/x$  tends to  $+\infty$  as  $x$  goes to  $+\infty$ .

Assume that  $J_\Psi$  is not compact. By Theorem 2.1, there are some  $A > 1$  and a sequence  $\{x_j\}_j$  going to  $+\infty$  such that  $\Psi(Ax_j) \geq [\Psi(x_j)]^2$ . Setting  $r_j = 1 - 1/\Psi(x_j)$ , this is equivalent to say that  $A\Psi^{-1}(1/(1-r_j)) \geq \Psi^{-1}(1/(1-r_j)^2)$ . Define:

$$f_j(z) = x_j \left( \frac{1-r_j}{1-r_j z} \right)^2.$$

One has  $f_j \in HM^\Psi$  and  $\|f_j\|_{H^\Psi} \leq 1$  (see [7], Corollary 3.10). Since  $\{f_j\}_j$  converges to 0 uniformly on compact subsets of  $\mathbb{D}$ ,  $\{f_j\}_j$  converges to 0 in the weak-star topology of  $H^\Psi$  ([7], Proposition 3.7). But, since the conjugate function of  $\Psi$  satisfies condition  $\Delta_2$ ,  $H^\Psi$  is the bidual of  $HM^\Psi$  ([7], Corollary 3.3); hence  $\{f_j\}_j$  converges weakly to 0 in  $HM^\Psi$ .

On the other hand, if  $S_j = D(1, 1-r_j) \cap \mathbb{D}$ , one has  $|1-r_j z| \leq 2(1-r_j)$  for  $z \in S_j$ ; hence, writing  $K = \|f_j\|_{\mathfrak{B}^\Psi}$ , one has:

$$1 = \int_{\mathbb{D}} \Psi(|f_j|/K) d\mathcal{A} \geq \int_{S_j} \Psi(|f_j|/K) d\mathcal{A} \geq \mathcal{A}(S_j) \Psi(x_j/4K).$$

Since  $\mathcal{A}(S_j) \geq \alpha(1-r_j)^2$ , with  $0 < \alpha < 1$ , we get (since  $\Psi(\alpha x_j/4K) \leq \alpha \Psi(x_j/4K)$ , by convexity):

$$\|f_j\|_{\mathfrak{B}^\Psi} \geq (\alpha/4) \frac{x_j}{\Psi^{-1}(1/(1-r_j)^2)} = (\alpha/4) \frac{\Psi^{-1}(1/(1-r_j))}{\Psi^{-1}(1/(1-r_j)^2)} \geq \frac{\alpha}{4A}.$$

Therefore  $J_\Psi$  is not Dunford-Pettis.  $\square$

On the other hand, one has:

**Proposition 3.2** *If  $J_\Psi$  is Dunford-Pettis, then  $J_\Psi$  is weakly compact.*

**Proof.** By Theorem 2.2, if  $J_\Psi$  is not weakly compact, there is a subspace  $X_0$  of  $H^\Psi$  isomorphic to  $c_0$  on which  $J_\Psi$  is an into-isomorphism; hence  $J_\Psi$  cannot be Dunford-Pettis.  $\square$

We shall see in the next section that  $J_\Psi$  may be weakly compact without being Dunford-Pettis.

### 3.2 Absolutely summing

Every  $p$ -summing operator is weakly compact and Dunford-Pettis; so it may be expected that  $J_\Psi$  is  $p$ -summing for some  $p < \infty$ . The next results show that this is never the case as soon as  $\Psi$  grows faster than all the power functions.

Recall that an operator  $T: X \rightarrow Y$  between two Banach spaces  $X$  and  $Y$  is called  $(p, q)$ -summing if there is a constant  $C > 0$  such that

$$\left( \sum_{k=1}^n \|Tx_k\|^p \right)^{1/p} \leq C \sup_{\|x^*\|_{X^*} \leq 1} \left( \sum_{k=1}^n |x^*(x_k)|^q \right)^{1/q},$$

for every finite sequence  $(x_1, \dots, x_n)$  in  $X$ . If  $q = p$ , it is said  $p$ -summing. Every  $p$ -summing operator is  $(p, q)$ -summing for  $q \leq p$ .

**Theorem 3.3** *If  $J_\Psi: H^\Psi \rightarrow \mathfrak{B}^\Psi$  is  $p$ -summing, then, for every  $q > p$ ,  $\Psi(x) = O(x^q)$  for  $x$  large enough. Moreover, if  $p < 2$ , then  $J_\Psi$  is compact.*

In order to prove this, we need two lemmas.

**Lemma 3.4** *If the canonical injection  $I_\Psi: A \rightarrow \mathfrak{B}^\Psi$  is  $(p, 1)$ -summing, where  $A = A(\mathbb{D})$  is the disk algebra, then  $\Psi(x) = O(x^{2p})$  for  $x$  large enough.*

*In particular,  $J_r: H^r \rightarrow \mathfrak{B}^r$  is  $(p, 1)$ -summing for no  $p < r/2$ , and, if  $\Psi \in \Delta^0$ , then  $J_\Psi$  is  $(p, 1)$ -summing for no  $p < \infty$ .*

Recall that the disk algebra is the space of continuous functions on  $\overline{\mathbb{D}}$  which are analytic in  $\mathbb{D}$ .

We refer to [9] for a detailed study of  $r$ -summing Carleson embeddings  $H^r \rightarrow L^r(\mu)$ . In particular, it follows from these results that  $J_r: H^r \rightarrow \mathfrak{B}^r$  is 1-summing for  $1 \leq r < 2$ . On the other hand, it is easy to see that  $J_2: H^2 \rightarrow \mathfrak{B}^2$  is not Hilbert-Schmidt (*i.e.* not 2-summing): for the canonical orthonormal basis  $\{z^n\}_n$  and  $\{\sqrt{n+1}z^n\}_n$  of  $H^2$  and  $\mathfrak{B}^2$ ,  $J_2$  is the diagonal operator of multiplication by  $\{1/\sqrt{n+1}\}_n$ . It also follows from [9] that, for  $r \geq 2$ ,  $J_r$  is  $p$ -summing for no finite  $p$ .

**Proof.** Assume that we do not have  $\Psi(x) = O(x^{2p})$  for  $x$  large enough. Then  $\limsup_{x \rightarrow +\infty} \Psi(x)/x^{2p} = +\infty$ . Given any  $K > 0$ , take  $y > 0$  such that  $\Psi(y)/y^{2p} \geq K$  and such that  $h = 1/\sqrt{\Psi(y)} \leq 1/2$ . Let  $N$  be the integer part of  $(1/h) + 1$ . Writing  $\xi_j = e^{2\pi i j/N}$ , we set:

$$u_j(z) = \frac{h^2}{[1 - (1-h)\overline{\xi_j}z]^2}.$$

We have  $u_j \in A(\mathbb{D})$ . By [7], Lemma 5.6, one has, since  $h \geq 1/N$ :

$$\sum_{j=0}^{N-1} |u_j(e^{it})| \leq N h^2 \frac{1 - (1-h)^{2N}}{[1 - (1-h)^2][1 - (1-h)^N]^2} \leq \frac{e^2}{(1-e)^2} := C.$$

Hence:

$$\sup_{\|x^*\|_{A^*} \leq 1} \sum_{j=0}^{N-1} |x^*(u_j)| \leq C.$$

On the other hand, it is easy to see that  $|u_j(z)| \geq 1/9$  when  $|z - (1-h)\xi_j| < h$ ; hence, if  $S_j = \{z \in \mathbb{D}; |z - (1-h)\xi_j| < h\}$ , one has, since  $\mathcal{A}(S_j) = h^2$ :

$$1 = \int_{\mathbb{D}} \Psi\left(\frac{|u_j(z)|}{\|u_j\|_{\mathfrak{B}^\Psi}}\right) d\mathcal{A}(z) \geq \int_{S_j} \Psi\left(\frac{1/9}{\|u_j\|_{\mathfrak{B}^\Psi}}\right) d\mathcal{A} \geq h^2 \Psi\left(\frac{1/9}{\|u_j\|_{\mathfrak{B}^\Psi}}\right),$$

so  $\|u_j\|_{\mathfrak{B}^\Psi} \geq 1/9\Psi^{-1}(1/h^2)$ . Since  $y = \Psi^{-1}(1/h^2)$ , one gets:

$$\sum_{j=0}^{N-1} \|u_j\|_{\mathfrak{B}^\Psi}^p \geq (1/9)^p \frac{N}{y^p} \geq (1/9)^p \left[\frac{\Psi(y)}{y^{2p}}\right]^{1/2} \geq \frac{K^{1/2}}{9^p}.$$

This yields that the  $(p, 1)$ -summing norm of  $I_\Psi$  should be greater than  $K^{1/2p}/9C$ , and, as  $K$  is arbitrary, that  $I_\Psi$  is not  $(p, 1)$ -summing.  $\square$

**Remark.** When  $I_\Psi: A \hookrightarrow \mathfrak{B}^\Psi$  is  $p$ -summing, we have this shorter argument. By Pietsch's factorization theorem, this  $I_\Psi$  factors through  $H^p$ . It follows from [7], Theorem 4.10, that  $\alpha h^2 \leq \rho_{\mathcal{A}}(h) \leq 1/\Psi^{-1}(A/h^{1/p})$ , for some constants  $0 < \alpha < 1$  and  $A > 0$ , and  $h$  small enough. That means that  $\Psi(x) \leq Cx^{2p}$  for  $x$  large enough.

**Lemma 3.5** *If the canonical injection  $I_\Psi: A \rightarrow \mathfrak{B}^\Psi$  is 1-summing, then  $J_\Psi$  is compact.*

**Proof.** The canonical injection  $J_1: H^1 \rightarrow \mathfrak{B}^1$  (as well as  $J_\Psi$  whenever  $\Psi \in \Delta_2$ ) is compact. Hence we may assume that  $H^\Psi$  is not  $H^1$  and hence that  $\Psi(x)/x$  tends to  $+\infty$  as  $x$  tends to  $+\infty$ .

Assume that  $J_\Psi$  is not compact. Then, as in the proof of Proposition 3.1, there are some  $A > 1$  and a sequence  $\{x_k\}_k$  going to  $+\infty$  such that  $\Psi(Ax_k) \geq [\Psi(x_k)]^2$ . Setting  $h_k = 1/\Psi(x_k)$ , we define, as in the proof of Proposition 3.4:

$$u_{k,j}(z) = \frac{h_k^2}{[1 - (1-h_k)\overline{\xi_{k,j}}z]^2},$$

where  $\xi_{k,j} = e^{2\pi ij/N_k}$ , with  $N_k$  the integer part of  $(1/h_k) + 1$ . One has  $u_{k,j} \in A$  and (see the proofs of the two quoted propositions):

$$\sum_{j=0}^{N_k-1} |u_{k,j}(e^{it})| \leq C \quad \text{and} \quad \|u_{k,j}\|_{\mathfrak{B}^\Psi} \geq \frac{\delta\alpha}{A} \frac{1}{\Psi^{-1}(1/h_k)}.$$

It follows that:

$$\sum_{j=0}^{N_k-1} \|u_{k,j}\|_{\mathfrak{B}^\Psi} \geq \frac{\delta\alpha}{A} \frac{N_k}{\Psi^{-1}(1/h_k)} \geq \frac{\delta\alpha}{A} \frac{1/h_k}{\Psi^{-1}(1/h_k)} = \frac{\delta\alpha}{A} \frac{\Psi(x_k)}{x_k} \xrightarrow{k \rightarrow \infty} +\infty.$$

Hence  $I_\Psi$  is not 1-summing.  $\square$

**Proof of Theorem 3.3.** Since  $J_\Psi: H^\Psi \rightarrow \mathfrak{B}^\Psi$  is  $p$ -summing and the canonical injection  $I_\Psi: A \rightarrow \mathfrak{B}^\Psi$  factors as  $I_\Psi: A \rightarrow H^\Psi \rightarrow \mathfrak{B}^\Psi$ , this injection is  $p$ -summing. By Lemma 3.4,  $\Psi(x) = O(x^{2p})$  for  $x$  large enough. Hence we have the factorization  $A \rightarrow H^{2p} \rightarrow H^\Psi \rightarrow \mathfrak{B}^\Psi$ . Since the first injection is  $2p$ -summing and the last one is  $p$ -summing, the composition is  $\max(1, p_1)$ -summing, with  $\frac{1}{p_1} = \frac{1}{2p} + \frac{1}{p}$  (see [2], Theorem 2.22), *i. e.*  $p_1 = \frac{2}{3}p$ . If  $p_1 > 1$ , we can use again Lemma 3.4 with  $p_1$  instead of  $2p$ ; we get that  $\Psi(x) = O(x^{2p_1})$ , for  $x$  large enough, and that the factorization  $I_\Psi: A \rightarrow H^{2p_1} \rightarrow H^\Psi \rightarrow \mathfrak{B}^\Psi$  is  $\max(1, p_2)$ -summing, with  $\frac{1}{p_2} = \frac{1}{2p_1} + \frac{1}{p}$ . Going on the same way, we get a decreasing sequence  $\{p_n\}_n$  such that the canonical injection  $A \rightarrow \mathfrak{B}^\Psi$  is  $\max(1, p_n)$ -summing and  $\frac{1}{p_{n+1}} = \frac{1}{2p_n} + \frac{1}{p}$ . Writing  $p_n = \alpha_n p$ , we get  $\alpha_{n+1} = \frac{2\alpha_n}{2\alpha_n+1}$ ; hence  $p_n \xrightarrow{n \rightarrow \infty} p/2$ . In particular,  $\Psi(x) = O(x^q)$  for every  $q > p$ .

If  $p < 2$ , one has  $\max(1, p_n) = 1$  for  $n$  large enough, and Lemma 3.4 implies that  $J_\Psi$  is compact.  $\square$

**Remark 1.** It is not clear whether  $J_\Psi$   $p$ -summing, with  $p \geq 2$ , implies that  $J_\Psi$  is compact. However, when  $r \geq 2$ ,  $J_r: H^r \rightarrow \mathfrak{B}^r$  is  $p$ -summing for no  $p < \infty$  (see [9]).

**Remark 2.** An operator  $T: X \rightarrow Y$  between two Banach spaces is said to be *finitely strictly singular* (or *superstrictly singular*) if for every  $\varepsilon > 0$ , there is an integer  $N_\varepsilon \geq 1$  such that, for every subspace  $X_0$  of  $X$  of dimension  $\geq N_\varepsilon$ , there is an  $x \in X_0$  such that  $\|Tx\| \leq \varepsilon \|x\|$ . Every finitely strictly singular operator is strictly singular. It is not difficult to see that every compact operator is finitely strictly singular and it is shown in [10] (see also [5], Corollary 2.3) that every  $p$ -summing operator is finitely strictly singular. We do not know when  $J_\Psi$  is finitely strictly singular.

### 3.3 Order boundedness

Recall that an operator  $T: X \rightarrow Y$  from a Banach space  $X$  into a Banach lattice  $Y$  is said to be *order bounded* if there is  $y \in Y_+$  such that  $|Tx| \leq y$  for every  $x$  in the unit ball of  $X$ . Since the Bergman-Orlicz space  $\mathfrak{B}^\Psi$  is a subspace of the Banach lattice  $L^\Psi(\mathbb{D}, \mathcal{A})$ , we may study the order boundedness of  $J_\Psi$ . Actually, we are going to see that the natural space for the order boundedness of  $J_\Psi$  is not  $L^\Psi(\mathbb{D}, \mathcal{A})$ , but the *weak Orlicz space*  $L^{\Psi, \infty}(\mathbb{D}, \mathcal{A})$ , the definition of which we are recalling below (see [7], Definition 3.16).

**Definition 3.6** Let  $(S, \mathcal{S}, \mu)$  be a measure space; the weak- $L^\Psi$  space  $L^{\Psi, \infty}$  is the set of the (classes of) measurable functions  $f: S \rightarrow \mathbb{C}$  such that, for some constant  $c > 0$ , one has, for every  $t > 0$ :

$$\mu(|f| > t) \leq \frac{1}{\Psi(ct)}.$$

One has  $L^\Psi \subseteq L^{\Psi, \infty}$  and ([7], Proposition 3.18) the equality  $L^\Psi = L^{\Psi, \infty}$  implies that  $\Psi \in \Delta^0$ . On the other hand, this equality holds when  $\Psi$  grows sufficiently; for example, if  $\Psi$  satisfies the condition  $\Delta^1$ :  $x\Psi(x) \leq \Psi(\alpha x)$ , for some constant  $\alpha > 1$  and  $x$  large enough.

**Proposition 3.7**  $J_\Psi: H^\Psi \rightarrow \mathfrak{B}^\Psi$  is always order bounded into  $L^{\Psi, \infty}(\mathbb{D}, \mathcal{A})$ .

**Proof.** Since (see [7], Lemma 3.11):

$$(3.1) \quad \frac{1}{4}\Psi^{-1}\left(\frac{1}{1-|z|}\right) \leq \sup_{\|f\|_{H^\Psi} \leq 1} |f(z)| \leq 4\Psi^{-1}\left(\frac{1}{1-|z|}\right),$$

one has, denoting by  $S(z)$  the supremum in (3.1), for  $t$  large enough:

$$\mathcal{A}(|S| > t) \leq \mathcal{A}(\{z \in \mathbb{D}; |z| > 1 - 1/\Psi(t/4)\}) \leq \frac{2}{\Psi(t/4)} \leq \frac{1}{\Psi(t/8)},$$

and the result follows.  $\square$

Since we also have, for  $t$  large enough:

$$\mathcal{A}(|S| > t) \geq \mathcal{A}(\{z \in \mathbb{D}; |z| > 1 - 1/\Psi(4t)\}) \geq \frac{1}{\Psi(4t)},$$

we get:

**Corollary 3.8**  $J_\Psi$  is order bounded into  $L^\Psi(\mathbb{D}, \mathcal{A})$  if and only if  $L^\Psi = L^{\Psi, \infty}$ . This is the case if  $\Psi \in \Delta^1$ .

**Remark.** Contrary to the compactness, or the weak compactness, which requires that  $\Psi$  does not grow too fast, the order boundedness of  $J_\Psi$  into  $L^\Psi(\mathbb{D}, \mathcal{A})$  holds when  $\Psi$  grows fast enough. Nevertheless, for  $\Psi(x) = \exp[(\log(x+1))^2] - 1$ ,  $J_\Psi$  is compact and order bounded into  $L^\Psi(\mathbb{D}, \mathcal{A})$ .

When  $J_\Psi$  is weakly compact,  $J_\Psi$  maps  $H^\Psi$  into  $\mathfrak{B}M^\Psi$  (Theorem 2.2); hence, we may ask whether  $J_\Psi$  may be order bounded into  $M^\Psi(\mathbb{D}, \mathcal{A})$ ; however, we have:

**Proposition 3.9**  $J_\Psi$  is never order bounded into  $M^\Psi(\mathbb{D}, \mathcal{A})$ .

**Proof.** If it were the case, we should have  $S \in M^\Psi(\mathbb{D}, \mathcal{A})$ , and hence

$$\int_{\mathbb{D}} \Psi \left[ 4 \times \frac{1}{4} \Psi^{-1} \left( \frac{1}{1-|z|} \right) \right] d\mathcal{A}(z) < +\infty,$$

which is false.  $\square$



## 4 An example

**Theorem 4.1** *There exists an Orlicz function  $\Psi$  such that  $J_\Psi$  is weakly compact and Dunford-Pettis, but which is not compact.*

Note that such an Orlicz function is very irregular:  $\Psi \notin \Delta_2$ ,  $\Psi \notin \Delta^0$ , so, for every  $A > 1$ ,  $\Psi(Ax)/\Psi(x)$  is not non-decreasing for  $x$  large enough, and the conjugate function of  $\Psi$  does not satisfies condition  $\Delta_2$ .

The following lemma is undoubtedly well-known, but we have found no reference, so we shall give a proof. Recall that a sublattice  $X$  of  $L^0(\mu)$  is solid if  $|f| \leq |g|$  and  $g \in X$  implies  $f \in X$  and  $\|f\| \leq \|g\|$ .

**Lemma 4.2** *Let  $(S, \mathcal{S}, \mu)$  be a measure space, and let  $X$  be a solid Banach sublattice of  $L^0(\mu)$ , the space of all measurable functions. Then, for every weakly null sequence  $\{f_n\}_n$  in  $X$  and every sequence  $\{A_n\}_n$  of disjoint measurable sets, the sequence  $\{f_n \mathbb{1}_{A_n}\}_n$  converges weakly to 0 in  $X$ .*

**Proof.** If the conclusion does not hold, there are a continuous linear functional  $\sigma: X \rightarrow \mathbb{C}$  and some  $\delta > 0$  such that, up to taking a subsequence,  $|\sigma(f_n \mathbb{1}_{A_n})| \geq \delta$ . Set, for every measurable set  $A \in \mathcal{S}$ :

$$\mu_n(A) = \sigma(f_n \mathbb{1}_A).$$

Then  $\mu_n$  is a finitely additive measure with bounded variation. By Rosenthal's lemma (see [3], Lemma I.4.1, page 18, or [1], Chapter VII, page 82), there is an increasing sequence of integers  $\{n_k\}_k$  such that:

$$\left| \mu_{n_k} \left( \bigcup_{l \neq k} A_{n_l} \right) \right| \leq |\mu_{n_k}| \left( \bigcup_{l \neq k} A_{n_l} \right) \leq \delta/2.$$

Now, if  $A = \bigcup_{l \geq 1} A_{n_l}$ ,  $\{f_{n_k} \mathbb{1}_A\}_k$  is weakly null, but:

$$|\sigma(f_{n_k} \mathbb{1}_A)| \geq |\sigma(f_{n_k} \mathbb{1}_{A_{n_k}})| - |\mu_{n_k}| \left( \bigcup_{l \neq k} A_{n_l} \right) \geq \delta - \frac{\delta}{2} = \frac{\delta}{2},$$

so we get a contradiction.  $\square$

**Proof of Theorem 4.1.** We begin by defining a sequence  $\{x_n\}_n$  of positive numbers in the following way: set  $x_1 = 4$  and, for every  $n \geq 1$ ,  $x_{n+1} > 2x_n$  is the abscissa of the second intersection point of the parabola  $y = x^2$  with the straight line containing  $(x_n, x_n^2)$  and  $(2x_n, x_n^4)$ ; we have  $x_{n+1} = x_n^3 - 2x_n$  (for example,  $x_2 = 56$ ). Define  $\Psi: [0, +\infty) \rightarrow [0, +\infty)$  by  $\Psi(x) = 4x$  for  $0 \leq x \leq 4$ , and, for  $n \geq 1$ :

$$(4.1) \quad \Psi(x_n) = x_n^2, \quad \Psi(2x_n) = x_n^4, \quad \Psi \text{ affine between } x_n \text{ and } x_{n+1}.$$

Then  $\Psi$  is an Orlicz function and

$$(4.2) \quad x^2 \leq \Psi(x) \leq x^4 \quad \text{for } x \geq 4.$$

For this Orlicz function  $\Psi$ ,  $J_\Psi$  is not compact, since  $\Psi(2x)/[\Psi(x)]^2$  does not tend to 0. However,  $J_\Psi$  is weakly compact, because one has the factorization  $H^\Psi \hookrightarrow H^2 \hookrightarrow \mathfrak{B}^4 \hookrightarrow \mathfrak{B}^\Psi$  (by (4.2) and Lemma 2.4).

Assume that  $J_\Psi$  is not Dunford-Pettis: there exists a weakly null sequence  $\{f_n\}_n$  in the unit ball of  $H^\Psi$  which does not converge for the norm in  $\mathfrak{B}^\Psi$ . Then  $\{f_n\}_n$  converges uniformly to 0 on the compact subsets of  $\mathbb{D}$  (since it is weakly null) and we may assume that  $\|f_n\|_{\mathfrak{B}^\Psi} \geq \delta$  for some  $\delta > 0$ . We may also assume that  $\|f_n\|_\infty \xrightarrow{n \rightarrow \infty} +\infty$  because if  $\{f_n\}_n$  were uniformly bounded, we should have  $\|f_n\|_{\mathfrak{B}^\Psi} \xrightarrow{n \rightarrow \infty} 0$ , by dominated convergence.

We are going to show that there exist a subsequence  $\{f_{n_k}\}_k$  and pairwise disjoint measurable sets  $A_k \subseteq \mathbb{T}$  such that the sequence  $\{f_{n_k} \mathbf{1}_{A_k}\}_k \subseteq L^\Psi(\mathbb{T}, m)$  is equivalent to the canonical basis of  $\ell_1$ , whence a contradiction with Lemma 4.2.

It is worth to note from now that the Poisson integral  $\mathcal{P}$  maps boundedly  $L^2(\mathbb{T})$  into  $L^4(\mathbb{D})$ . Indeed,  $L^2(\mathbb{T}) = H^2 \oplus \overline{H_0^2}$  and the canonical injection is bounded from  $H^2$  into  $\mathfrak{B}^4$ , by Lemma 2.4.

We have seen in the proof of Lemma 2.5 that there exist a subsequence  $\{f_{n_k}\}_k$  and disjoint measurable annuli  $C_1 = \{z \in \mathbb{D}; |z| \leq r_1\}$  and  $C_k = \{z \in \mathbb{D}; r_{k-1} < |z| \leq r_k\}$ ,  $k \geq 2$ , with  $0 < r_1 < r_2 < \dots < r_n < \dots < 1$ , such that  $\|f_{n_k} \mathbf{1}_{C_k}\|_{L^\Psi(\mathbb{D})} \geq \delta/2$ . The assumptions of that lemma are satisfied here:  $\|f_n\|_{H^\Psi} \leq 1$ ,  $\|f_n\|_{\mathfrak{B}^\Psi} \geq \delta$ ,  $\{f_n\}_n$  converges uniformly to 0 on the compact subsets of  $\mathbb{D}$ , and  $f_n \in \mathfrak{B}M^\Psi$  because  $H^\Psi \subseteq \mathfrak{B}M^\Psi$ , since  $J_\Psi$  is weakly compact. Then:

**Fact 1.** *There exist two sequences  $\{\alpha_k\}_k$  and  $\{\beta_k\}_k$ , with  $\beta_n > \alpha_n \xrightarrow{n \rightarrow \infty} +\infty$  such that, if  $g_k = f_{n_k}^* \mathbf{1}_{\{\alpha_k \leq |f_{n_k}^*| \leq \beta_k\}}$ , then:*

$$\|\mathcal{P}(g_k)\|_{L^\Psi(\mathbb{D})} \geq \delta/3,$$

where  $f_{n_k}^*$  is the boundary value of  $f_{n_k}$  on  $\mathbb{T}$ .

**Proof.** 1) Let  $\alpha_k = \frac{\delta}{12} \Psi^{-1}(1/\mathcal{A}(C_k))$  and  $v_k = \mathcal{P}(f_{n_k}^* \mathbf{1}_{\{|f_{n_k}^*| < \alpha_k\}}) \mathbf{1}_{C_k}$ . One has:

$$\int_{\mathbb{D}} \Psi(|v_k|/(\delta/12)) d\mathcal{A} = \int_{C_k} \Psi(|v_k|/(\delta/12)) d\mathcal{A} \leq \Psi(\alpha_k/(\delta/12)) \mathcal{A}(C_k) = 1,$$

so  $\|v_k\|_{L^\Psi(\mathbb{D})} \leq \delta/12$ . Since  $\mathcal{P}(f_{n_k}^*) = f_{n_k}$ , we have  $\|\mathcal{P}(f_{n_k}^*) \mathbf{1}_{C_k}\|_{L^\Psi(\mathbb{D})} = \|f_{n_k} \mathbf{1}_{C_k}\|_{L^\Psi(\mathbb{D})} \geq \delta/2$ , and we get:

$$\|\mathcal{P}(f_{n_k}^* \mathbf{1}_{\{|f_{n_k}^*| \geq \alpha_k\}}) \mathbf{1}_{C_k}\|_{L^\Psi(\mathbb{D})} \geq \|f_{n_k} \mathbf{1}_{C_k}\|_{L^\Psi(\mathbb{D})} - \|v_k\|_{L^\Psi(\mathbb{D})} \geq \frac{\delta}{2} - \frac{\delta}{12} = \frac{5\delta}{12}.$$

2) Let  $w_k = f_{n_k}^* \mathbf{1}_{\{|f_{n_k}^*| \geq \alpha_k\}}$ . Since  $\mathcal{P}(w_k \mathbf{1}_{\{|w_k| > \beta\}})$  tends to 0 uniformly on  $C_k$  when  $\beta$  goes to infinity, Lebesgue's dominated convergence theorem gives:

$$\|\mathcal{P}(w_k \mathbf{1}_{\{|w_k| > \beta\}}) \mathbf{1}_{C_k}\|_{L^\Psi(\mathbb{D})} \leq \|\mathcal{P}(w_k \mathbf{1}_{\{|w_k| > \beta\}}) \mathbf{1}_{C_k}\|_{L^4(\mathbb{D})} \xrightarrow{\beta \rightarrow +\infty} 0,$$

so there is some  $\beta_k > \alpha_k$  such that  $\|\mathcal{P}(w_k \mathbf{1}_{\{|w_k| > \beta\}}) \mathbf{1}_{C_k}\|_{L^\Psi(\mathbb{D})} \leq \delta/12$ .

We then have, with  $g_k = f_{n_k}^* \mathbf{1}_{\{\alpha_k \leq |f_{n_k}^*| \leq \beta_k\}}$ :

$$\|\mathcal{P}(g_k)\|_{L^\Psi(\mathbb{D})} \geq \|\mathcal{P}(g_k) \mathbf{1}_{C_k}\|_{L^\Psi(\mathbb{D})} \geq \frac{5\delta}{12} - \frac{\delta}{12} = \frac{\delta}{3},$$

and that ends the proof of Fact 1.  $\square$

**Fact 2.** *There are a further subsequence, denoted yet by  $\{f_{n_k}\}_k$ , and pairwise disjoint measurable subsets  $E_k \subseteq \{\alpha_k \leq |f_{n_k}^*| \leq \beta_k\}$ , such that, if  $h_k = f_{n_k}^* \mathbf{1}_{E_k}$ , then:*

$$\|\mathcal{P}(h_k)\|_{L^\Psi(\mathbb{D})} \geq \delta/4.$$

**Proof.** First, since  $g_k \in L^\infty(\mathbb{T}) \subseteq M^\Psi(\mathbb{T})$ , there exists  $\varepsilon_k > 0$  such that  $m(A) \leq \varepsilon_k$  implies  $\|g_k \mathbf{1}_A\|_{L^\Psi(\mathbb{T})} \leq \delta/(12 \|\mathcal{P}\|)$  (where  $\|\mathcal{P}\|$  stands for the norm of  $\mathcal{P}: L^2(\mathbb{T}) \rightarrow L^4(\mathbb{D})$ ). Now,  $\mathcal{P}: L^\Psi(\mathbb{T}) \rightarrow L^\Psi(\mathbb{D})$  is bounded and its norm is  $\leq \|\mathcal{P}\|$ , thanks to the factorization  $L^\Psi(\mathbb{T}) \hookrightarrow L^2(\mathbb{T}) \hookrightarrow L^4(\mathbb{D}) \hookrightarrow L^\Psi(\mathbb{D})$ . Hence  $\|\mathcal{P}(g_k \mathbf{1}_A)\|_{L^\Psi(\mathbb{D})} \leq \delta/12$  for  $m(A) \leq \varepsilon_k$ .

Let  $B_k = \{\alpha_k \leq |f_{n_k}^*| \leq \beta_k\}$ . Up to taking a subsequence, we may assume that  $\sum_{l>k} m(B_l) \leq \varepsilon_k$ . Let

$$E_k = B_k \setminus \bigcup_{l>k} B_l.$$

The sets  $E_k$ ,  $k \geq 1$ , are pairwise disjoint, and

$$\|\mathcal{P}(g_k \mathbf{1}_{E_k})\|_{L^\Psi(\mathbb{D})} \geq \|\mathcal{P}(g_k \mathbf{1}_{B_k})\|_{L^\Psi(\mathbb{D})} - \|\mathcal{P}(g_k \mathbf{1}_{\bigcup_{l>k} B_l})\|_{L^\Psi(\mathbb{D})} \geq \frac{\delta}{3} - \frac{\delta}{12} = \frac{\delta}{4},$$

so we get the Fact 2 with  $h_k = g_k \mathbf{1}_{E_k} = f_{n_k}^* \mathbf{1}_{E_k}$ .  $\square$

Set

$$F_k = \{z \in E_k; \Psi(|f_{n_k}^*(z)|) \leq M |f_{n_k}^*(z)|^2\}.$$

For  $z \in E_k \setminus F_k$ , one has:

$$\int_{E_k \setminus F_k} |f_{n_k}^*|^2 dm \leq \frac{1}{M} \int_{\mathbb{T}} \Psi(|f_{n_k}^*|) dm \leq \frac{1}{M},$$

so  $\|f_{n_k}^* \mathbf{1}_{E_k \setminus F_k}\|_{L^2(\mathbb{T})} \leq 1/\sqrt{M}$  and:

$$\begin{aligned} \|\mathcal{P}(f_{n_k}^* \mathbf{1}_{E_k \setminus F_k})\|_{L^\Psi(\mathbb{D})} &\leq \|\mathcal{P}(f_{n_k}^* \mathbf{1}_{E_k \setminus F_k})\|_{L^4(\mathbb{D})} \\ &\leq \|\mathcal{P}\| \|f_{n_k}^* \mathbf{1}_{E_k \setminus F_k}\|_{L^2(\mathbb{T})} \leq \frac{\|\mathcal{P}\|}{\sqrt{M}} \leq \frac{\delta}{8}, \end{aligned}$$

for  $M$  large enough. It follows that, for  $M$  large enough,  $\|\mathcal{P}(f_{n_k}^* \mathbf{1}_{F_k})\|_{L^\Psi(\mathbb{D})} \geq \delta/8$  and

$$(4.3) \quad \|f_{n_k}^* \mathbf{1}_{F_k}\|_{L^\Psi(\mathbb{D})} \geq \delta/(8 \|\mathcal{P}\|).$$

Now, we may assume that, for some  $\alpha > 0$ ,

$$\int_{\mathbb{T}} |f_{n_k}^*|^2 \mathbb{1}_{F_k} dm \geq \alpha,$$

because, if not, there would be a subsequence  $\{f_{n_{k_j}}^* \mathbb{1}_{F_{k_j}}\}_j$  converging to 0 in  $L^2(\mathbb{T})$ ; but then  $\{\mathcal{P}(f_{n_{k_j}} \mathbb{1}_{F_{k_j}})\}_j$  would converge to 0 in  $\mathfrak{B}^4$ , and hence in  $\mathfrak{B}^\Psi$ , contrary to (4.3). It follows, using (4.2), that:

$$(4.4) \quad \int_{F_k} \Psi(|f_{n_k}^*|) dm \geq \alpha.$$

The following lemma is now the key of the proof.

**Lemma 4.3** *Let  $\delta_n = 2x_{n-1}/x_n = 2/(x_{n-1}^2 - 2)$ . If  $\Psi(x) \leq Mx^2$  and  $x \geq x_n$ , then, for  $n$  large enough ( $n \geq N$ ), one has  $\Psi(\varepsilon x) \geq C_M \varepsilon \Psi(x)$  for  $\delta_n \leq \varepsilon \leq 1$ .*

**Proof.** We may assume that  $x_n \leq x < x_{n+1}$ , because if  $x_k \leq x < x_{k+1}$  with  $k \geq n$ , then  $\varepsilon \geq \delta_n$  implies  $\varepsilon \geq \delta_k$ .

Now, remark that:

$$(4.5) \quad \frac{\Psi(y)}{\Psi(x)} \leq 4 \frac{y}{x}, \quad \text{for } 2x_n \leq x \leq y \leq x_{n+1}.$$

Indeed, on the one hand,  $\frac{\Psi(y) - \Psi(x_n)}{\Psi(x) - \Psi(x_n)} = \frac{y - x_n}{x - x_n} \leq \frac{y}{x/2} = 2 \frac{y}{x}$ ; and, on the other hand,  $\Psi(y) - \Psi(x_n) \geq \Psi(y) - \Psi(y/2) \geq \Psi(y) - \frac{1}{2} \Psi(y) = \frac{1}{2} \Psi(y)$ , so  $\frac{\Psi(y)}{\Psi(x)} \leq \frac{\Psi(y)}{\Psi(x) - \Psi(x_n)} \leq 2 \frac{\Psi(y) - \Psi(x_n)}{\Psi(x) - \Psi(x_n)} \leq 4 \frac{y}{x}$ .

We shall separate three cases:

1)  $\varepsilon x \leq x_n \leq x \leq 2x_n$ . Then  $\varepsilon x \geq \varepsilon x_n$  and hence  $\Psi(\varepsilon x) \geq \Psi(\varepsilon x_n)$ . But  $2x_{n-1} \leq \varepsilon x_n \leq x_n$ , since  $\varepsilon \geq \delta_n$ ; hence (4.5) implies that  $\Psi(\varepsilon x) \geq (\varepsilon/4) \Psi(x_n) = (\varepsilon/4) x_n^2$ . On the other hand, one has, by hypothesis,  $\Psi(x) \leq Mx^2 \leq M(2x_n)^2$ , so we get  $\Psi(\varepsilon x) \geq (\varepsilon/16M) \Psi(x)$ .

2)  $x_n \leq \varepsilon x \leq x \leq 2x_n$ . Then, since  $1 \leq 1/\varepsilon$ :

$$\frac{\Psi(x)}{\Psi(\varepsilon x)} \leq \frac{Mx^2}{\Psi(x_n)} \leq \frac{M(2x_n)^2}{x_n^2} = 4M \leq \frac{4M}{\varepsilon}.$$

3) For  $x \geq 2x_n$ , remark that the conditions  $\Psi(x) \leq Mx^2$  and  $x \geq 2x_n$  imply that  $x \geq x_n^2/\sqrt{M}$ . Indeed, if  $x \geq 2x_n$ , then  $\Psi(x) \geq \Psi(2x_n) = x_n^4$ , and the condition  $\Psi(x) \leq Mx^2$  implies  $x_n^4 \leq Mx^2$ , i.e.  $x \geq x_n^2/\sqrt{M}$ .

In this case, one has  $\varepsilon x \geq \varepsilon x_n^2/\sqrt{M} \geq \delta_n x_n^2/\sqrt{M} = 2(x_{n-1}/x_n)x_n^2/\sqrt{M} = 2x_{n-1}x_n/\sqrt{M} \geq 2x_n$ , if  $x_{n-1} \geq \sqrt{M}$ . Hence (4.5) gives, for  $2x_n \leq x < x_{n+1}$  (since then  $2x_n \leq \varepsilon x \leq x < x_{n+1}$ ):

$$\frac{\Psi(x)}{\Psi(\varepsilon x)} \leq 4 \frac{x}{\varepsilon x} = \frac{4}{\varepsilon}.$$

That ends the proof of Lemma 4.3.  $\square$

Extract now a further subsequence of  $\{f_{n_k}\}$ , yet denoted by  $\{f_{n_k}\}$ , in order that (see Fact 1)  $\alpha_k \geq x_{N+k}$ . Lemma 4.3 holds, with  $x = \Psi(|f_{n_k}^*(z)|)$ ,  $z \in F_k$ , for every  $k \geq 1$ ; one has (since, by definition,  $\Psi(|f_{n_k}|) \leq M|f_{n_k}|^2$  on  $F_k$ ):

$$\int_{F_k} \Psi(\varepsilon |f_{n_k}^*|) dm \geq \varepsilon C/\alpha := c\varepsilon, \quad \text{for } \delta_{N+k} \leq \varepsilon \leq 1.$$

The proof of Theorem 4.1 reaches now its end: put  $u_k = f_{n_k}^* \mathbb{1}_{F_k}$ , and take an arbitrary sequence of complex numbers such that  $\sum_{k \geq 1} |\lambda_k| = 1$ . Let  $\delta_0 = \sum_{k \geq N} \delta_k$ . One has  $\delta_0 < 1$ , because we may assume that  $N$  had been taken large enough. One gets:

$$\begin{aligned} \int_{\mathbb{T}} \Psi\left(\left|\sum_{k \geq 1} \lambda_k u_k\right|\right) dm &= \sum_{k \geq 1} \int_{F_k} \Psi(|\lambda_k f_{n_k}|) dm \\ &\geq \sum_{|\lambda_k| \geq \delta_{N+k}} c|\lambda_k| + \sum_{|\lambda_k| < \delta_{N+k}} \int_{F_k} \Psi(|\lambda_k f_{n_k}|) dm \\ &\geq \sum_{|\lambda_k| \geq \delta_{N+k}} c|\lambda_k| = c\left(1 - \sum_{|\lambda_k| < \delta_{N+k}} |\lambda_k|\right) \\ &\geq c\left(1 - \sum_{k \geq N} \delta_k\right) = c(1 - \delta_0) := c_0. \end{aligned}$$

Since  $c_0 < 1$ , this implies, by convexity, that

$$\left\| \sum_{k \geq 1} \lambda_k u_k \right\|_{L^\Psi(\mathbb{T})} \geq c_0.$$

Hence  $\{u_k\}_k$  is equivalent to the canonical basis of  $\ell_1$ , and that achieves the proof of Theorem 4.1.  $\square$

**Remarks.** 1) It follows from Theorem 3.3 that, for this  $\Psi$ ,  $J_\Psi$  is not  $p$ -summing for  $p < 4$ . By modifying the definition of  $\Psi$  (taking  $\Psi(x_n) = x_n^{r/2}$  and  $\Psi(2x_n) = x_n^r$ ), we get, for every  $4 \leq r < \infty$ , an Orlicz function  $\Psi$  such that  $J_\Psi$  is Dunford-Pettis and weakly compact, without being  $p$ -summing for  $p < r$ , and without being compact. We do not know whether it is possible to have  $J_\Psi$   $p$ -summing for no finite  $p$ .

2) Let us point out that the fact that  $J_\Psi$  is Dunford-Pettis does not trivially follow from its weak compactness:  $H^\Psi$  does not have the Dunford-Pettis property. In fact, if it were the case, the weakly compact injection  $H^\Psi \hookrightarrow H^2$  would be Dunford-Pettis, and hence also  $H^4 \hookrightarrow H^2$  (since  $H^4 \hookrightarrow H^\Psi \hookrightarrow H^2$ ). But it is not the case: the sequence  $\{z^n\}_n$  converges weakly to 0 in  $H^4$ , whereas it does not converge in norm to 0 in  $H^2$ .

**Proposition 4.4** *There is an Orlicz function  $\Psi$  for which  $J_\Psi$  is weakly compact, but not Dunford-Pettis.*

**Proof.** Let us call  $\Psi_0$  the Orlicz function constructed in Theorem 4.1, and let  $\Psi(x) = \Psi_0(x^2)$ . Then, with  $\beta = 2$ ,  $\Psi(\beta x) = \Psi_0(4x^2) \geq 4\Psi_0(x^2) = (2\beta)\Psi(x)$ ; that means that the conjugate function of  $\Psi$  satisfies  $\Delta_2$ .

$J_\Psi$  is weakly compact (since  $J_\Psi$  factors as  $H^\Psi \hookrightarrow H^4 \hookrightarrow \mathfrak{B}^8 \hookrightarrow \mathfrak{B}^\Psi$ ), but is not compact, since  $[\Psi(\sqrt{x_n})]^2 = \Psi(\sqrt{2}\sqrt{x_n})$ . Since the conjugate function satisfies  $\Delta_2$ ,  $J_\Psi$  is not Dunford-Pettis, by Proposition 3.1.  $\square$

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