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Using vector divisions in solving linear complementarity problem

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Abstract

The linear complementarity problem is to find vector z in IR^n satisfying $z^T(Mz+q)=0$, $Mz+q \geqslant 0$, $z \geqslant 0$, where M as a matrix and q as a vector, are given data; this problem becomes in present the subject of much important research because it arises in many areas and it includes important fields, we cite for example the linear and nonlinear programming, the convex quadratic programming and the variational inequalities problems, ...

It is known that the linear complementarity problem is completely equivalent to solving nonlinear equation $F(x)=0$ with F is a function from IR^n into itself defined by $F(x)=(M+I)x+(M-I)|x|+q$. In this paper we propose a globally convergent hybrid algorithm for solving this equation; this method is based on an algorithm given by Shi [22], he uses vector divisions with the secant method; but for using this method we must have a function continuous with partial derivatives on an open set of IR^n; so we built a sequence of functions $\tilde{F}(p,x) \in C^\infty$ which converges uniformly to the function $F(x)$; and we show that finding the zero of the function F is completely equivalent to finding the zero of the sequence of the functions $\tilde{F}(p,x)$. We close our paper with some numerical simulation examples to illustrate our theoretical results.

Key words and phrases: Linear complementarity problem, Vector division, Global convergence, *Newton's method, secant method* .

1. Introduction:

The complementarity problem noted (CP) is a classical problem of optimization theory of finding $(z, w) \in \mathrm{IR}^n \times \mathrm{IR}^n$ such that:

$$\begin{cases} < z, w >= 0 \\ w - f(z) = 0 \\ z, w \geqslant 0 \end{cases} \tag{1}$$

where f, a continuous operator from IR^n into itself, is given data.

The constraint $< z, w >= 0$ is called the complementarity condition since for any i, $1 \leqslant i \leqslant n, z_i = 0$ if $w_i > 0$, and vice versa; it may be the case that $z_i = w_i = 0$.

In the case that the function f is a nonlinear continuous operator from IR^n into itself, so the problem is called a **N**on**L**inear **C**omplementarity **P**roblem associated with the function f and noted $(NLCP)$.

In the case that the function f is affine, i.e

$$f(z) = q + Mz,$$

where q is an element of IR^n and M is a real square matrix of order n, so the problem is called a **L**inear **C**omplementarity **P**roblem associated with the matrix M and the vector q and noted (LCP).

Solving the (LCP) in general, however, appears to be difficult. One simple reason is that there is no known good characterization of the nonexistence of a solution to the system for any given function f.

The linear complementarity problem plays an important role in several fields (game theory, operational research ...); moreover, Cottle et Dantzig[1] et Lemke[12] have proved that all problems of linear programming (LP), convex quadratic programming (CQP), and also the problems of Nash equilibrium of a game bi-matrix can be written as a linear complementarity problem.

The question is precisely under what conditions on the matrix M and the vector q this problem admits one and only one solution, if this is the case, how can we express this solution as a function of the matrix and vector mentioned above. This question has not been completely resolved yet.

However, many results already exist, for instance Lemke[12] who gave sufficient conditions on the matrix M and the vector q under which the number of solutions of $LCP(M, q)$ is finite. Samelson [20], Ingeton[10], Murty[15], Watson[25], Kelly [11] and Cottle [2] have by contrast shown that the matrix M is a $P-matrix$ if and only if the linear complementarity problem associated with a matrix M and a vector q has a unique solution for all $q \in \mathrm{IR}^n$ (We should remind that a matrix M is called a $P-matrix$ if all principal minors are strictly positive (see [7]), and we should note

that any symmetric and positive definite matrix is a $P-matrix$, but not vice-versa).

2. Preliminaries:

In this section, we summarize some notations which will be used in this paper.

In particular, IR^n denotes the space of real $n-$dimensional vectors,

$\mathrm{IR}^n_+ := \{x \in \mathrm{IR}^n : x_i \geqslant 0, i = 1..n\}$ is the nonnegative orthant and its interior is $\mathrm{IR}^n_{++} := \{x \in \mathrm{IR}^n : x_i > 0, i = 1..n\}$.

With $x \in \mathrm{IR}^n$ we define $|x| = (|x_1|, .., |x_n|)^T \in \mathrm{IR}^n$.

We denote by I the identity matrix.

Let x, $y \in \mathrm{IR}^n$, $x^T y$ or $< x, y >$ is the inner product of the x and y; $||x||$ is the Euclidean norm.

For $x \in \mathrm{IR}^n$ and k a nonnegative integer, $x^{(k)}$ refers to the vector obtained after k iterations; for $1 \leqslant i \leqslant n$, x_i refers to the ith element of x, and $x_i^{(k)}$ refers to the ith element of the vector obtained after k iterations.

Let x, $y \in \mathrm{IR}^n$, the expression $x \leqslant y$ (respectively $x < y$) meaning that $x_i \leqslant y_i$ (respectively $x_i < y_i$) for each $i = 1..n$.

For $x \in \mathrm{IR}^n$ we denote by $e^x = (e^{x_1}, .., e^{x_n})^T \in \mathrm{IR}^n$ and for $x \in \mathrm{IR}^n_{++}$ we denote by $\ln(x) = (\ln(x_1), .., \ln(x_n))^T$.

The transpose of a vector is denoted by super script T, such as the transpose of the vector x is given by x^T.

Remember that the spectrum $\sigma(A)$ of the matrix A is the set of its eigenvalues and its spectral radius ρ is given by: $\rho(A) := \sup\{|\lambda|$ such that $\lambda \in \sigma(A)\}$.

3. Equivalent reformulation of LCP:

It is known in [17] that the linear complementarity problem $LCP(M, q)$ is completely equivalent to solving nonlinear equation

$$F(x) = 0$$

with F is a function from IR^n into itself defined by

$$F(x) := (M + I)x + (M - I)|x| + q;$$

More precisely (see [17]), on the one hand, if x^* is a zero of the function F, then

$$\begin{cases} z^* := |x^*| + x^* \\ w^* := |x^*| - x^* \end{cases} \tag{2}$$

define a solution of $LCP(M, q)$.

On the other hand, if (z^*, w^*) is a solution of $LCP(M, q)$, then

$$x^* := \frac{z^* - w^*}{2}$$

is a zero of the function F.

We mention that this equation is solved by the fixed point algorithm (see [21]), this algorithm is defined by:

$$\begin{cases} x^{(0)} \in \mathrm{IR}^n \text{ arbitrary,} \\ x^{(k+1)} = (I+M)^{-1}(I-M)|x^{(k)}| - (I+M)^{-1}q \end{cases} \tag{3}$$

For the case that M is symetric and positive definite, it was shown in [24] (see also Section 9.2 in [17]) that

$$D := (I+M)^{-1}(I-M)$$

it holds

$$||D||_2 = \sqrt{\rho(D^T D)} = \sqrt{\rho(D^2)} = \rho(D) < 1,$$

where $\rho(.)$ denotes the spectral radius of a matrix; hence (3) converges by the contraction-mapping theorem (see Theorem 5.1.3 in [18]) and

$$x^* = \lim_{k \to +\infty} x^{(k)}$$

is the unique solution of the $F(x) = 0$.

Therefore,

$$w^* := |x^*| - x^*$$

and

$$z^* := |x^*| + x^*$$

define the unique solution of the $LCP(M, q)$.

We mention also that the convergence of algorithm (3) is only linear; in this paper, we consider the use of vector divisions with the secant method (see [22]) in instead of the algorithm (3) decribed above, this algorithm has global convergence (see [22]); but for using this algorithm we must have a function F a continuous with partial derivatives on a set of IR^n; the next section can answer this problematic.

4. The main result

We consider again the function $F : \mathrm{IR}^n \to \mathrm{IR}^n$ defined by

$$F(x) := (M+I)x + (M-I)|x| + q$$

and let's consider the sequence of functions $\tilde{F} : \mathrm{IN}^* \times \mathrm{IR}^n \to \mathrm{IR}^n$defined by

$$\tilde{F}(p, x) := (M+I)x + \frac{1}{p}(M-I)\ln(e^0 + e^{px} + e^{-px}) + q$$

Proposition 1 *: $\tilde{F}(p,x)$ converges uniformly to $F(x)$ when $p \to +\infty$.*

Proof. : To show that we can start by

$$\begin{aligned}\frac{1}{p}\ln(e^0+e^{px}+e^{-px})-|x| &= \frac{1}{p}[\ln(e^0+e^{px}+e^{-px})+\ln(e^{-p|x|})]\\ &= \frac{1}{p}\ln[e^{-p|x|}*(1+e^{px}+e^{-px})]\\ &= \frac{1}{p}\ln(e^{-p|x|}+e^{p(x-|x|)}+e^{-p(x+|x|)})\end{aligned}$$

then we have

$$0 \leqslant \frac{1}{p}\ln(e^0+e^{px}+e^{-px})-|x| \leqslant \frac{1}{p}\ln(3)$$

so

$$\frac{1}{p}\ln(e^0+e^{px}+e^{-px})$$

is uniform convergence to $|x|$ when $p \to +\infty$;

Moreover, the operator $(M-I)$ is linear then we have

$\frac{1}{p}(M-I)\ln(e^0+e^{px}+e^{-px})$ converges uniformly to $(M-I)|x|$ as $p \to +\infty$

and from the expression of the sequence of the functions $\tilde{F}$ and the function F we have

$$\tilde{F}(p,x) \text{ converges uniformly to } F(x) \text{ when } p \to +\infty.$$

■

Theorem 2 *: Let $x^*(p)$ be a solution of the equation $\tilde{F}(p,x)=0$, then $x^*(p)$ is an approximation solution of $F(x)=0$ for p is large enough.*

Proof. :To show that, we use proposition(1) wich we can interpret as $\forall \epsilon > 0$, $\exists\ p^* > 0$ such that for all $p > p^*$ we have

$$\begin{aligned}||F(x^*(p))|| &= ||F(x^*(p))-\tilde{F}(p,x^*(p))||\\ &\leqslant \epsilon\end{aligned}$$

then we have for any $\epsilon > 0$, $x^*(p)$ is the approximation solution of

$$F(x)=0.$$

■

Remark 3 *: The uniqueness of the root of the function F results from the uniqueness of the solution of linear complementarity problem $LCP(M,q)$, in fact, supposing that x_1^* and x_2^*, two distinct roots of the function F,exist, then*

$$z_1^* := |x_1^*| + x_1^*.$$
$$z_2^* := |x_2^*| + x_2^*.$$

Since $z_1^* = z_2^*$ (uniqueness of the solution of $LCP(M,q)$) then

$$|x_1^*| + x_1^* = |x_2^*| + x_2^*. \tag{4}$$

Simularly, we use the same method for w_1^* and w_2^* we have

$$|x_1^*| - x_1^* = |x_2^*| - x_2^*. \tag{5}$$

so (4)−(5) means that $x_1^* = x_2^*$.

Now, we give the following algorithm for solving $\tilde{F}(p,x) = 0$ (see [22]):

Algorithm:

Step 0: Determine ϵ, p, k^*, ρ, σ such that k^* is a positive integer, $0 < \rho < 1/2$, and $\rho < \sigma < 1$;

Step 1: Select two points $x^{(0)}$ and $x^{(1)} \in \mathrm{IR}^n$;

Step 2: for $k = 1, 2, ...$ until termination, do the following:

1- Compute the steepset descent direction

$$d^{(k)} := -J(p, x^{(k)})^T \tilde{F}(p, x^{(k)}),$$

where

$$\tilde{F}(p,x) := (M+I)x + \frac{1}{p}(M-I)\ln(e^0 + e^{px} + e^{-px}) + q;$$

$$J(p,x) := (M+I) + (M-I)E(p,x)$$

and

$$E_{ij}(p,x) := \delta_{ij}\frac{e^{px_i} - e^{-px_i}}{1 + e^{px_i} + e^{-px_i}};$$

we racall that $\delta_{ii} = 1$ and $\delta_{ij} = 0$ if $i \neq j$.

2- if k equals a multiple of k^*, then insert a steepset descent direction step, that is, let $s^{(k)} := d^{(k)}$ and go to step 2.7;

3- Compute:

$$\begin{cases} u^{(k)} := \xi_1 \tilde{F}(p, x^{(k)}) \\ v^{(k)} := \xi_2 (x^{(k)} - x^{(k-1)}) \end{cases}$$

with

$$\xi_1 = -\frac{||x^{(k)} - x^{(k-1)}||^2}{< x^{(k)} - x^{(k-1)}, \tilde{F}(p, x^{(k)}) - \tilde{F}(p, x^{(k-1)}) >}$$

and

$$\xi_2 = -\frac{< \tilde{F}(p, x^{(k)}) - \tilde{F}(p, x^{(k-1)}), \tilde{F}(p, x^{(k)}) >}{||\tilde{F}(p, x^{(k)}) - \tilde{F}(p, x^{(k-1)})||^2}.$$

4- if $(u^{(k)} - v^{(k)})^T d^{(k)} \neq 0$, then choose

$$\alpha^{(k)} > \frac{- < v^{(k)}, d^{(k)} >}{< u^{(k)} - v^{(k)}, d^{(k)} >}$$

such that $\alpha^{(k)}$ maximizes the value of

$$Cos[s^{(k)}, d^{(k)}] = \frac{< s^{(k)}, d^{(k)} >}{||s^{(k)}||.||d^{(k)}||};$$

Set

$$s^{(k)} := \alpha^{(k)} u^{(k)} + (1 - \alpha^{(k)}) v^{(k)}$$

and go to step 2.7;

5- if $(u^{(k)} - v^{(k)})^T d^{(k)} = 0$ and $< v^{(k)}, d^{(k)} >> 0$, then set

$$s^{(k)} := \frac{u^{(k)} + v^{(k)}}{2}$$

and go to step 2.7;

6- if $(u^{(k)} - v^{(k)})^T d^{(k)} = 0$ and $< v^{(k)}, d^{(k)} > \leqslant 0$, then set $s^{(k)} := d^{(k)}$ and go to step 2.7;

7- take a line search along the direction $s^{(k)}$ to determine the step length γ suth that

$$f(p, x^{(k)} + \gamma s^{(k)}) \leqslant f(p, x^{(k)}) - \gamma\rho < d^{(k)}, s^{(k)} >$$

and

$$< \bigtriangledown f(p, x^{(k)} + \gamma s^{(k)}), s^{(k)} > \geqslant -\sigma < d^{(k)}, s^{(k)} >;$$

where

$$f(p, x) = \frac{1}{2}||\tilde{F}(p, x)||^2$$

8- set $x^{(k+1)} := x^{(k)} + \gamma s^{(k)}$ and go to next iteration.

5. Numerical examples

In this part, we consider some examples to test our method. The results of Fixed Point and using vector divisions methods for these examples are presented here for comparison purposes. The results and expected solutions for each example have been presented on the Tables 1 and 2.

Example 4 *: Consider the following linear complementarity problem:*
Find vector z in IR^n satisfying $z^T(Mz+q)=0$, $Mz+q \geqslant 0$, $z \geqslant 0$,

$$\text{where } M = \begin{bmatrix} 4 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 4 \end{bmatrix} \text{ and } q = \begin{bmatrix} -4 \\ 3 \\ -4 \\ 2 \end{bmatrix}.$$

The exact solutions is $x^* = (1,0,1,0)^T$.

We apply the fixed point and using vector divisions methods to solve this example with the initial approximation $x^{(0)} = (1.1, 0.1, 1.2, 0.2)^T$.

The solution of this problem with six significant digits is presented in Table1.

	Iteration	z_1	z_2	z_3	z_4
Fixed point method	*k=01*	*0,0000000*	*0,0000000*	*0,0000000*	*0,0000000*
	k=05	*0,7883251*	*0,0000000*	*1,3448593*	*0,0000000*
	k=10	*1,0197946*	*0,0000000*	*0,9884737*	*0,0000000*
	k=15	*0,9985643*	*0,0000000*	*1,0012907*	*0,0000000*
	k=20	*1,0001030*	*0,0000000*	*0,9999377*	*0,0000000*
	k=25	*0,9999918*	*0,0000000*	*1,0000058*	*0,0000000*
	k=30	*1,0000005*	*0,0000000*	*0,9999997*	*0,0000000*
	k=33	*0,9999999*	*0,0000000*	*1,0000001*	*0,0000000*
	k=34	*1,0000001*	*0,0000000*	*1,0000000*	*0,0000000*
	k=35	*1,0000000*	*0,0000000*	*1,0000000*	*0,0000000*
Using vector divisions method	*k=01*	*0,0000000*	*0,0000000*	*0,0000000*	*0,0000000*
	k=02	*4,0000000*	*0,0000000*	*4,0000000*	*0,0000000*
	k=03	*1,0000000*	*0,0000000*	*1,0000000*	*0,0000000*

Table 1: The results of different methods for example1.

Example 5 *: Let's solve the following linear complementarity problem*
Find vector z in IR^n satisfying $z^T(Mz+q)=0$, $Mz+q \geqslant 0$, $z \geqslant 0$,

$$\text{where } M = \begin{bmatrix} 8 & -1 & 0 & -5 \\ 1 & 5 & -1 & 0 \\ 2 & -1 & 6 & -1 \\ 6 & 0 & -1 & 7 \end{bmatrix} \text{ and } q = \begin{bmatrix} 1 \\ -2 \\ -3 \\ 4 \end{bmatrix}.$$

The exact solutions is $x^* = (0, \frac{15}{29}, \frac{17}{29}, 0)^T$.

We apply the fixed point and using vector divisions methods to solve this example with the initial approximation $x^{(0)} = (-1,-2,-3,-4)^T$.

The solution of this problem with six significant digits is presented in Ta-

ble2.

	Iteration	z_1	z_2	z_3	z_4
Fixed point method	*k=01*	*0,0000000*	*0,0000000*	*0,0000000*	*0,0000000*
	k=05	*0,1115059*	*0,6300237*	*0,3146258*	*0,0000000*
	k=10	*0,0000000*	*0,4906849*	*0,6581657*	*0,0000000*
	k=15	*0,0000000*	*0,5309163*	*0,5863067*	*0,0000000*
	k=20	*0,0000000*	*0,5192743*	*0,5936312*	*0,0000000*
	k=25	*0,0000000*	*0,5188339*	*0,5869623*	*0,0000000*
	k=30	*0,0000000*	*0,5173795*	*0,5862876*	*0,0000000*
	k=35	*0,0000000*	*0,5171712*	*0,5859293*	*0,0000000*
	k=40	*0,0000000*	*0,5171411*	*0,5860432*	*0,0000000*
	k=45	*0,0000000*	*0,5171911*	*0,5861363*	*0,0000000*
	k=50	*0,0000000*	*0,5172239*	*0,5861902*	*0,0000000*
	k=55	*0,0000000*	*0,5172388*	*0,5862079*	*0,0000000*
	k=60	*0,0000000*	*0,5172428*	*0,5862107*	*0,0000000*
	k=65	*0,0000000*	*0,5172428*	*0,5860930*	*0,0000000*
	k=70	*0,0000000*	*0,5172421*	*0,5862079*	*0,0000000*
	k=75	*0,0000000*	*0,5172416*	*0,5862071*	*0,0000000*
	k=79	*0,0000000*	*0,5172424*	*0,5862069*	*0,0000000*
Using vector divisions method	*k=01*	*0,0000000*	*2,0000000*	*3,0000000*	*0,0000000*
	k=02	*0,0000000*	*0,0000000*	*0,4367816*	*0,0000000*
	k=03	*0,0000000*	*2,6264367*	*0,5000000*	*0,0000000*
	k=04	*0,0000000*	*0,6983749*	*0,6163958*	*0,0000000*
	k=05	*0,0000000*	*0,5172414*	*0,5862069*	*0,0000000*
	k=06	*0,0000000*	*0,5172414*	*0,5862069*	*0,0000000*

Table 2: The results of different methods for example2.

Conclusion:

In this paper we have used that the linear complementarity problem is completely equivalent to solving nonlinear equation $F(x) = 0$ with F is a function from IR^n into itself defined by $F(x) = (M + I)x + (M - I)|x| + q$; for solving this equation we have used the Shi's method, this method uses vector divisions with the secant method; based on that, we have proposed a globally convergent hybrid algorithm for solving this equation; to do so, we had to build a sequence of functions $\tilde{F}(p, x) \in C^\infty$ which converges uniformly to the function $F(x)$; and we have shown that finding the zero of the function F is completely equivalent to finding the zero of the sequence of the functions $\tilde{F}$.

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