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An introduction to Mahler's method for transcendence and algebraic independence.

Federico Pellarin*

Abstract. Here we propose a survey on Mahler's theory for transcendence and algebraic independence focusing on certain applications to the arithmetic of periods of Anderson t -motives.

Mathematics Subject Classification (2000). Primary 11J85; Secondary 11G09.

Keywords. Mahler's method, transcendence, algebraic independence, periods of Drinfeld modules and t -motives.

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1. Introduction

In his mathematical production, Kurt Mahler (1903-1988) introduced entire new subjects. One of them, perhaps chronologically the first one, aimed three fundamental papers [27, 28, 29] and was, later in 1977, baptised "Mahler's method" by Loxton and van der Poorten. By this locution we mean a general method to prove transcendence and algebraic independence of values of a certain class of functions by means of the following classical scheme of demonstration whose terminology will be explained in the present text (¹):

(AP) - Construction of *auxiliary polynomials*,

(UP) - Obtaining an *upper bound*, by means of *analytic estimates*,

(NV) - Proving the *non-vanishing*, by means of *zeros estimates*,

*The presentation of the paper essentially follows, in an expanded form, the instructional talk the author gave in the conference " t -motives: Hodge structures, transcendence and other motivic aspects", held in Banff, Alberta, (September 27 - October 2, 2009). The author is thankful to the organisers of this excellent conference for giving the opportunity to present these topics, and thankful to the Banff Centre for the exceptional environment of working it provided. The author also wishes to express his gratitude to B. Adamczewski and P. Philippon for discussions and hints that helped to improve the presentation of this text.

¹We borrowed this presentation from Masser's article [31, p. 5], whose point of view influenced our point of view.

(LB) - Obtaining a *lower bound*, by means of *arithmetic estimates*.

For example, with Mahler's method, and with the help of the basic theory of *heights*, it is possible to show the transcendence of values at algebraic complex numbers of transcendental analytic solutions $f(x) \in \mathcal{L}[[x]]$ (with \mathcal{L} a number field embedded in \mathbb{C}) of functional equations such as

$$f(x^d) = R(x, f(x)), \quad d > 1, \quad R \in \mathcal{L}(X, Y), \quad (1)$$

with d integer, see [27].

In this text, we will also be interested in analogues of these functions over complete, algebraically closed fields other than \mathbb{C} and for this purpose it will be advantageous to choose right away an appropriate terminology. Indeed, in the typical situation we will analyse, there will be a *base field* \mathcal{K} , together with a distinguished absolute value that will be denoted by $|\cdot|$. Over \mathcal{K} there will be other absolute values as well, and a *product formula* will hold. We will consider the completion of \mathcal{K} with respect to $|\cdot|$, its algebraic closure that will be embedded in its completion \mathbb{K} with respect to an extension of $|\cdot|$. The algebraic closure of \mathcal{K}^{alg} , embedded in \mathbb{K} , will also be endowed with a *absolute logarithmic height* that will be used to prove transcendence results. Here, an element of \mathbb{K} is *transcendental* if it does not belong to \mathcal{K}^{alg} .

If \mathcal{L} is a finite extension of \mathcal{K} in \mathbb{K} and $f \in \mathcal{L}[[x]]$ is a formal series solution of (1), we will say that f is a *Mahler's function over \mathcal{K}* . If f converges at $\alpha \in \mathcal{K}^{\text{alg}} \setminus \{0\}$ (for the distinguished absolute value), we will say that $f(\alpha) \in \mathbb{K}$ is a *Mahler's value* and α is a *base point* for this value. In spite of the generality of this terminology, in this text we will restrict our attention to the base fields $\mathbb{Q}, K = \mathbb{F}_q(\theta)$ and $C(t)$ where C is the completion of an algebraic closure of the completion of K for the unique extension of the absolute value defined by $|a| = q^{\deg_\theta a}$, with $a \in K$.

The interest of the method introduced by Mahler in [27] is that it can be generalised, as it was remarked by Mahler himself in [28], to explicitly produce finitely generated subfields of \mathbb{C} of arbitrarily large transcendence degree. This partly explains, after that the theory was long-neglected for about forty years, a regain of interest in it, starting from the late seventies, especially due to the intensive work of Loxton and van der Poorten, Masser, Nishioka as well as other authors we do not mention here but that are quoted, for example, in Nishioka's book [37].

In some sense, Mahler's functions and Siegel's E -functions share similar properties; large transcendence degree subfields of \mathbb{C} can also be explicitly constructed by the so-called Siegel-Shidlovski theorem on values of Siegel E -functions at algebraic numbers (see Lang's account on the theory in [26]). However, this method makes fundamental use of the fact that E -functions are entire, with finite analytic growth order. This strong assumption is not at all required when it is possible to apply Mahler's method, where the functions involved have natural boundaries for analytic continuation; this is certainly an advantage that this theory has. Unfortunately, no complex "classical constant" (period, special value of exponential function at algebraic numbers. . .) seems to occur as a complex Mahler's value, as far as we can see.

More recently, a variety of results by Becker, Denis [9, 17, 18, 19, 20] and other authors changed the aspect of the theory, especially that of Mahler's functions over fields of positive characteristic. It was a fundamental discovery of Denis, that every period of Carlitz's exponential function is a Mahler's value, hence providing a new proof of its transcendency. This motivates our choice of terminology; we hope the reader will not find it too heavy. At least, it will be useful to compare the theory over \mathbb{Q} and that over K .

The aim of this paper is to provide an overview of the theory from its beginning (transcendence), through its recent development in algebraic independence and its important excursions in positive characteristic, where it is in "competition" with more recent, and completely different techniques inspired by the theory of Anderson's t -motives (see, for example, the work of An-

derson, Brownawell, Papanikolas, Chieh-Yu Chang, Jing Yu, other authors [6, 14, 39] and the related bibliographies).

Here is what the paper contains. In Section 2, we give an account of transcendence theory of Mahler's values with \mathbb{Q} as a base field; this is part of the classical theory, essentially contained in one of the first results by Mahler. In Section 3, we will outline the transcendence theory with, as a base field, a function field of positive characteristic (topic which is closer to the themes of the conference). Here, the main two features are some applications to the arithmetic of periods of Anderson's t -motives and some generalisations of results of the literature (cf. Theorem 3.3). In Section 4, we first make an overview of known results of algebraic independence over \mathbb{Q} of Mahler's values, then we describe more recent results in positive characteristic (with the base field $K = \mathbb{F}_q(\theta)$) and finally, we mention some quantitative aspects. The main features of this section are elementary proofs of two results: one by Papanikolas [39], describing algebraic dependence relations between certain special values of Carlitz's logarithms, and another one, by Chieh-Yu Chang and Jing Yu, describing all the algebraic dependence relations of values of Carlitz-Goss zeta function at positive integers.

This paper does not contain a complete survey on Mahler's method. For example, Mahler's method was also successful in handling several variable functions. To keep the size of this survey reasonable, we made the arguable decision of not describing this part of the theory, concentrating on the theory in one variable, which seemed closer to the other themes of the conference.

2. Transcendence theory over the base field \mathbb{Q}

2.1. An example to begin with. The example that follows gives an idea of the method. We consider the formal series:

$$f_{\text{TM}}(x) = \prod_{n=0}^{\infty} (1 - x^{2^n}) = \sum_{n=0}^{\infty} (-1)^{a_n} x^n \in \mathbb{Z}[[x]]$$

$(a_n)_{n \geq 0}$ being the Thue-Morse sequence (a_n is the reduction modulo 2 of the sum of the digits of the binary expansion of n and, needless to say, the subscript TM in f_{TM} stands for "Thue-Morse"). The formal series f_{TM} converges in the open unit ball $B(0, 1)$ to an analytic function and satisfies the functional equation

$$f_{\text{TM}}(x^2) = \frac{f_{\text{TM}}(x)}{1-x}, \quad (2)$$

(in (1), $d = 2$ and $R = \frac{Y}{1-X}$ so that f_{TM} is a Mahler's function. In all the following, we fix an embedding in \mathbb{C} of the algebraic closure \mathbb{Q}^{alg} of \mathbb{Q} . We want to prove:

Theorem 2.1. *For all $\alpha \in \mathbb{Q}^{\text{alg}}$ with $0 < |\alpha| < 1$, $f_{\text{TM}}(\alpha)$ is transcendental.*

This is a very particular case of a result of Mahler [27] reproduced as Theorem 2.2 in the present paper. The proof, contained in 2.1.3 uses properties of *Weil's logarithmic absolute height* reviewed in 2.1.2. It will also use the property that f_{TM} is transcendental over $\mathbb{C}(x)$, proved below in 2.1.1.

2.1.1. Transcendence of f_{TM} . The transcendence of f_{TM} over $\mathbb{C}(x)$ can be checked in several ways. A first way to proceed appeals to Pólya-Carlson Theorem (1921), (statement and proof can be found on p. 265 of [48]). It says that a given formal series $\phi \in \mathbb{Z}[[x]]$ converging with

radius of convergence 1, either has $\{z \in \mathbb{C}, |z| = 1\}$ as natural boundary for holomorphy, or can be extended to a rational function of the form $\frac{P(x)}{(1-x^m)^n}$, with $P \in \mathbb{Z}[x]$. To show that f_{TM} is transcendental, it suffices to prove that f_{TM} is not of the form above, which is evident from the functional equation (2), which implies that f_{TM} has bounded integral coefficients. Indeed, if rational, f_{TM} should have ultimately periodic sequence of the coefficients. However, it is well known (and easy to prove, see [47, Chapter 5, Proposition 5.1.2]) that this is not the case for the Thue-Morse sequence.

Another way to check the transcendence of f_{TM} is that suggested in Nishioka's paper [35]. Assuming that f_{TM} is algebraic, the field $F = \mathbb{C}(x, f_{\text{TM}}(x))$ is an algebraic extension of $\mathbb{C}(x)$ of degree, say n , and we want to prove that this degree is 1. It is possible to contradict this property observing that the extension F of $\mathbb{C}(x^d)$ ramifies at the places 0 and ∞ only and applying Riemann-Hurwitz formula. Hence, f_{TM} is rational and we know already from the lines above how to exclude this case.

2.1.2. A short account on heights.. Here we closely follow Lang [26, Chapter 3] and Waldschmidt [54, Chapter 3]. Let L be a number field. The *absolute logarithmic height* $h(\alpha_0 : \dots : \alpha_n)$ of a projective point $(\alpha_0 : \dots : \alpha_n) \in \mathbb{P}_n(L)$ is the following weighted average of logarithms of absolute values:

$$h(\alpha_0 : \dots : \alpha_n) = \frac{1}{[L : \mathbb{Q}]} \sum_{v \in M_L} d_v \log \max\{|\alpha_0|_v, \dots, |\alpha_n|_v\},$$

where v runs over a complete set M_L of non-equivalent places of L , where $d_v = [L_v : \mathbb{Q}_p]$ with $v|_{\mathbb{Q}} = p$ the *local degree* at the place v (one then writes that $v|p$) (L_v, \mathbb{Q}_p are completions of L, \mathbb{Q} at the respective places so that if $v|\infty$, $L_v = \mathbb{R}$ or $L_v = \mathbb{C}$ according to whether the place v is real or complex), and where $|\cdot|_v$ denotes, for all v , an element of v chosen in such a way that the following *product formula* holds:

$$\prod_{v \in M_L} |\alpha|_v^{d_v} = 1, \quad \alpha \in L^\times, \quad (3)$$

where we notice that only finitely many factors of this product are distinct from 1. A common way to normalise the $|\cdot|_v$'s is to set $|x|_v = x$ if $x \in \mathbb{Q}$, $x > 0$, and $v|\infty$, and $|p|_v = 1/p$ if $v|p$.

This formula implies that h does not depend on the choice of the number field L , so that we have a well defined function

$$h : \mathbb{P}_n(\mathbb{Q}^{\text{alg.}}) \rightarrow \mathbb{R}_{\geq 0}.$$

If $n = 1$ we also write $h(\alpha) = h(1 : \alpha)$. For example, we have $h(p/q) = h((1 : p/q)) = \log \max\{|p|, |q|\}$ if p, q are relatively prime and $q \neq 0$. With the convention $h(0) := 0$, this defines a function

$$h : \mathbb{Q}^{\text{alg.}} \rightarrow \mathbb{R}_{\geq 0} \quad (4)$$

satisfying, for $\alpha, \beta \in \mathbb{Q}^{\text{alg.}\times}$:

$$\begin{aligned} h(\alpha + \beta) &\leq h(\alpha) + h(\beta) + \log 2, \\ h(\alpha\beta) &\leq h(\alpha) + h(\beta), \\ h(\alpha^n) &= |n|h(\alpha), \quad n \in \mathbb{Z}. \end{aligned}$$

More generally, if $P \in \mathbb{Z}[X_1, \dots, X_n]$ and if $\alpha_1, \dots, \alpha_n$ are in $\mathbb{Q}^{\text{alg.}}$,

$$h(P(\alpha_1, \dots, \alpha_n)) \leq \log L(P) + \sum_{i=1}^n (\deg_{X_i} P) h(\alpha_i), \quad (5)$$

where $L(P)$ denotes the *length* of P , that is, the sum of the absolute values of the coefficients of P . Proofs of these properties are easy collecting metric information at every place. More details can be found in [54, Chapter 3].

Liouville's inequality, a sort of “fundamental theorem of transcendence”, reads as follows. Let L be a number field, v an archimedean place of L , n an integer. For $i = 1, \dots, n$, let α_i be an element of L . Further, let P be a polynomial in n variables X_1, \dots, X_n , with coefficients in \mathbb{Z} , which does not vanish at the point $(\alpha_1, \dots, \alpha_n)$. Assume that P is of degree at most N_i with respect to the variable X_i . Then,

$$\log |P(\alpha_1, \dots, \alpha_n)|_v \geq -([L : \mathbb{Q}] - 1) \log L(P) - [L : \mathbb{Q}] \sum_{i=1}^n N_i h(\alpha_i).$$

The proof of this inequality is again a simple application of product formula (3): [54, Section 3.5]. It implies that for $\beta \in \mathbb{Q}^{\text{alg}}$, $\beta \neq 0$,

$$\log |\beta| \geq -[L : \mathbb{Q}] h(\beta). \quad (6)$$

This inequality suffices for most of the arithmetic purposes of this paper (again, see [54] for the details of these basic tools).

2.1.3. Proof of Theorem 2.1. *Step (AP).* For all $N \geq 0$, we choose a polynomial $P_N \in \mathbb{Q}[X, Y] \setminus \{0\}$ of degree $\leq N$ in both X and Y , such that the *order of vanishing* $\nu(N)$ at $x = 0$ of the formal series

$$F_N(x) := P_N(x, f_{\text{TM}}(x)) = c_{\nu(N)} x^{\nu(N)} + \dots \quad (c_{\nu(N)} \neq 0).$$

(not identically zero because f_{TM} is transcendental by 2.1.1), is $\geq N^2$.

The existence of P_N follows from the existence of a non-trivial solution of a homogeneous linear system with N^2 linear equations defined over \mathbb{Q} in $(N+1)^2$ indeterminates. We will not need to control the size of the coefficients of P_N and this is quite unusual in transcendence theory.

Step (NV). Let α be an algebraic number such that $0 < |\alpha| < 1$ and let us suppose by contradiction that $f_{\text{TM}}(\alpha)$ is also algebraic, so that there exists a number field L containing at once α and $f_{\text{TM}}(\alpha)$. Then, by the functional equation (2), for all $n \geq 0$,

$$F_N(\alpha^{2^{n+1}}) = P_N(\alpha^{2^{n+1}}, f_{\text{TM}}(\alpha^{2^{n+1}})) = P_N\left(\alpha^{2^{n+1}}, \frac{f_{\text{TM}}(\alpha)}{(1-\alpha) \cdots (1-\alpha^{2^n})}\right) \in L.$$

We know that $F_N(\alpha^{2^{n+1}}) \neq 0$ for all n big enough depending on N and α ; indeed, f_N is not identically zero and analytic at 0.

Step (UB). Writing the expansion of f_N at 0

$$f_N(x) = \sum_{m \geq \nu(N)} c_m x^m = x^{\nu(N)} \left(c_{\nu(N)} + \sum_{i \geq 1} c_{\nu(N)+i} x^i \right)$$

with the leading coefficient $c_{\nu(N)}$ which is a non-zero rational integer (whose size we do not control), we see that for all $\epsilon > 0$, if n is big enough depending on N, α and ϵ :

$$\log |F_N(\alpha^{2^{n+1}})| \leq \log |c_{\nu(N)}| + 2^{n+1} \nu(N) \log |\alpha| + \epsilon.$$

Step (LB). At once, by (5) and (6),

$$\begin{aligned}
\log |F_N(\alpha^{2^{n+1}})| &\geq \\
&\geq -[L : \mathbb{Q}](L(P_N) + Nh(\alpha^{2^{n+1}}) + Nh(f(\alpha)/(1-\alpha) \cdots (1-\alpha^{2^n}))) \\
&\geq -[L : \mathbb{Q}](L(P_N) + N2^{n+1}h(\alpha) + Nh(f_{\text{TM}}(\alpha)) + \sum_{i=0}^n h(1-\alpha^{2^i})) \\
&\geq -[L : \mathbb{Q}](L(P_N) + N2^{n+2}h(\alpha) + Nh(f_{\text{TM}}(\alpha)) + (n+1)\log 2).
\end{aligned}$$

The four steps allow to conclude: for all n big enough,

$$\begin{aligned}
2^{-n-1} \log |c_{\nu(N)}| + \nu(N) \log |\alpha| + 2^{-n-1}\epsilon &\geq \\
&\geq -[L : \mathbb{Q}](L(P_N)2^{-n-1} + 2Nh(\alpha) + N2^{-n-1}h(f_{\text{TM}}(\alpha)) + (n+1)2^{-n-1}\log 2).
\end{aligned}$$

Letting n tend to infinity and using that $\nu(N) \geq N^2$ (recall that $\log |\alpha|$ is negative), we find the inequality

$$N \log |\alpha| \geq -2[L : \mathbb{Q}]h(\alpha).$$

But the choice of the ‘‘auxiliary’’ polynomial P_N can be done for *every* $N > 0$. With

$$N > \frac{2[L : \mathbb{Q}]h(\alpha)}{|\log |\alpha||}, \quad (7)$$

we encounter a contradiction.

2.1.4. A more general result. For $R = N/D \in \mathbb{C}(X, Y)$ with N, D relatively prime polynomials in $\mathbb{C}(X)[Y]$, we write $h_Y(R) := \max\{\deg_Y N, \deg_Y D\}$. With the arguments above, the reader can be easily prove the following theorem originally due to Mahler [27].

Theorem 2.2 (Mahler). *Let $L \subset \mathbb{C}$ be a number field, R be an element of $L(X, Y)$, $d > 1$ an integer such that $h_Y(R) < d$. Let $f \in L[[x]]$ be a transcendental formal series such that, in $L((x))$,*

$$f(x^d) = R(x, f(x)).$$

Let us suppose that f converges for $x \in \mathbb{C}$ with $|x| < 1$. Let α be an element of L such that $0 < |\alpha| < 1$.

Then, for all n big enough, $f(\alpha^{d^n})$ is transcendental over \mathbb{Q} .

Obviously, for all n big enough, $L(f(\alpha^{d^n}))^{\text{alg.}} = L(f(\alpha^{d^{n+1}}))^{\text{alg.}}$. It can happen, under the hypotheses of Theorem 2.2, that $f(\alpha)$ is well defined and algebraic for certain $\alpha \in \mathbb{Q}^{\text{alg.}} \setminus \{0\}$. For example, the formal series

$$f(x) = \prod_{i=0}^{\infty} (1 - 2x^{2^i}) \in \mathbb{Z}[[x]],$$

converging for $x \in \mathbb{C}$ such that $|x| < 1$ and satisfying the functional equation

$$f(x^2) = \frac{f(x)}{1 - 2x}, \quad (8)$$

vanishes at every α such that $\alpha^{2^i} = 1/2$, $i \geq 0$. In particular, f being non-constant and having infinitely many zeroes, it is transcendental. By Theorem 2.2, $f(1/4) = \lim_{x \rightarrow 1/2} \frac{f(x)}{1-2x}$ is transcendental.

Remark 2.3. Nishioka strengthened Theorem 2.2 allowing the rational function R satisfying only the relaxed condition $h_Y(R) < d^2$ (see [37, Theorem 1.5.1] for an even stronger result). The proof, more involved than the proof of Theorem 2.2, follows most of the principles of it, with the following notable difference. To achieve the proof, a more careful choice of the polynomials P_N is needed. In step (AP) it is again needed to choose a sequence of polynomials $(P_N)_N$ with $P_N \in \mathbb{Q}[X, Y]$ of degree $\leq N$ in X and Y , such that the function $F_N(x) = P_N(x, f(x))$ vanishes at $x = 0$ with order of vanishing $\geq c_1 N^2$ for a constant c_1 depending on α and f . Since for $h_Y(R) \geq d$ the size of the coefficients of P_N influences the conclusion, the use of Siegel's Lemma is now needed to accomplish this choice [53, Section 1.3]. To make good use of these refinements we need an improvement of the step (NV), since an explicit upper bound like $c_2 N \log N$ for the integer k such that $F(x^{d^s}) = 0$ for $s = 0, \dots, k$ is required ⁽²⁾.

2.2. Some further discussions.. In this subsection we discuss about some variants of Mahler's method and applications to modular functions (in 2.2.1). We end with 2.2.2, where we quote a criterion of transcendence by Corvaia and Zannier quite different from Mahler's method, since it can be obtained as a corollary of Schmidt's subspace theorem.

2.2.1. The "stephanese" Theorem. We refer to [50] for a precise description of the tools concerning elliptic curves and modular forms and functions, involved in this subsection.

Let

$$J(q) = \frac{1}{q} + 744 + \sum_{i \geq 1} c_i q^i \in (1/q)\mathbb{Z}[[q]]$$

be the q -expansion of the classical *hauptmodul* for $\mathbf{SL}_2(\mathbb{Z})$, converging for $q \in \mathbb{C}$ such that $0 < |q| < 1$. The following theorem was proved in 1996; see [8]:

Theorem 2.4 (Barré-Sirieix, Diaz, Gramain and Philibert). *For q complex such that $0 < |q| < 1$, one at least of the two complex numbers $q, J(q)$ is transcendental.*

This *stephanese* theorem ⁽³⁾ furnished a positive answer to Mahler's conjecture on values of the modular j -invariant (see [30]). Although we will not say much more about, we mention that a similar conjecture was independently formulated by Manin, for p -adic values of J at algebraic α 's, as the series J also converges in all punctured p -adic unit disks, for every prime p . Manin's conjecture is proved in [8] as well. Manin's conjecture is relevant for its connections with the values of p -adic L -functions and its consequences on p -adic variants of Birch and Swinnerton Dyer conjecture.

Mahler's conjecture was motivated by the fact that the function J satisfies the autonomous non-linear *modular equation* $\Phi_2(J(q), J(q^2)) = 0$, where

$$\begin{aligned} \Phi_2(X, Y) = & X^3 + Y^3 - X^2 Y^2 + 1488XY(X + Y) - 162000(X^2 + Y^2) + \\ & 40773375XY + 8748000000(X + Y) - 15746400000000. \end{aligned}$$

Mahler hoped to apply some suitable generalisation of Theorem 2.2. It is still unclear, at the time being, if this intuition is correct; we remark that Theorem 2.2 does not apply here.

The proof of Theorem 2.4 relies on a *variant* of Mahler's method that we discuss now. We first recall from [50] that there exists a collection of *modular equations*

$$\Phi_n(J(q), J(q^n)) = 0, \quad n > 0,$$

²This is not difficult to obtain; see 2.2.1 below for a similar, but more difficult estimate.

³Sometimes, this result is called *stephanese theorem* from the name of the city of Saint-Etienne, where the authors of this result currently live.

with explicitly calculable polynomials $\Phi_n \in \mathbb{Z}[X, Y]$ for all n . The stephanese team make use of the full collection of polynomials $(\Phi_n)_{n>0}$ so let us briefly explain how these functional equations occur.

For q complex such that $0 < |q| < 1$, $J(q)$ is the *modular invariant* of an elliptic curve analytically isomorphic to the complex torus $\mathbb{C}^\times/q^\mathbb{Z}$; if $z \in \mathbb{C}$ is such that $\Im(z) > 0$ and $e^{2\pi iz} = q$, then there also is a torus analytic isomorphism $\mathbb{C}^\times/q^\mathbb{Z} \cong \mathbb{C}/(\mathbb{Z} + z\mathbb{Z})$. Since the lattice $\mathbb{Z} + nz\mathbb{Z}$ can be embedded in the lattice $\mathbb{Z} + z\mathbb{Z}$, the natural map $\mathbb{C}^\times/q^\mathbb{Z} \rightarrow \mathbb{C}^\times/q^{n\mathbb{Z}}$ amounts to a cyclic *isogeny* of the corresponding elliptic curves which, being projective smooth curves, can be endowed with *Weierstrass models* $y^2 = 4x^3 - g_2x - g_3$ connected by algebraic relations independent on the choice of z . At the level of the modular invariants, these algebraic relations for n varying are precisely the modular equations, necessarily autonomous, defined over \mathbb{Z} as a simple Galois argument shows.

Assuming that for a given q with $0 < |q| < 1$, $J(q)$ is algebraic, means that there exists an *elliptic curve* E analytically isomorphic to the torus $\mathbb{C}^\times/q^\mathbb{Z}$, which is definable over a number field (it has Weierstrass model $y^2 = 4x^3 - g_2x - g_3$ with $g_2, g_3 \in \mathbb{Q}^{\text{alg}}$). The discussion above, with the fact that the modular polynomials Φ_n are defined over \mathbb{Z} , implies that $J(q^n)$ is algebraic as well.

Arithmetic estimates involved in the (LB) step of the proof of Theorem 2.4 require a precise control, for $J(q)$ algebraic, of the height of $J(q^n)$ and the degree d_n of $\mathbb{Q}(J(q^n), J(q))$ over $\mathbb{Q}(J(q))$. the degree d_n can be easily computed counting lines in \mathbb{F}_p^2 for p prime dividing n ; it thus divides the number $\psi(n) = \prod_{p|n} (1 + 1/p)$ and is bounded from above by $c_3 n^{1+\epsilon}$, for all $\epsilon > 0$. As for the height $h_n = h(J(q^n))$, we said that the modular polynomial Φ_n is related to a family of cyclic isogenies of degree n connecting two families of elliptic curves. We then have, associated to the algebraic modular invariants $J(q), J(q^n)$, two isogenous elliptic curves defined over a number field, and the isogeny has degree n . Faltings theorem asserting that the modular heights of two isogenous elliptic curves may differ of at most the half of the logarithm of a minimal degree of isogeny gives the bound $c_3(h(J(q)) + (1/2)\log n)$ for the logarithmic height $h(J(q^n))$ (this implies the delicate estimates the authors do in [8]).

With these information in mind, the proof of Theorem 2.4 proceeds as follows. As in remark 2.3, we use standard estimates of the growth of the absolute values of the (integral) coefficients of the q -expansions of the normalised Eisenstein series E_4, E_6 of weights 4, 6 to apply Siegel's Lemma and construct a sequence of auxiliary polynomials (AP) $(P_N)_{N \geq 1}$ in $\mathbb{Z}[X, Y] \setminus \{0\}$ with $\deg_X P_N, \deg_Y P_N \leq N$, such that $F_N(x) := P_N(x, xJ(x))$ vanishes with order $\geq N^2/2$ at $x = 0$.

The (UB) estimate is then exactly as in the Proof of Theorem 2.2. All the authors of [8] need to achieve their proof is the (NV) step; and it is here that a new idea occurs. They use that the coefficients of J are *rational integers* to deduce a sharp estimate of the biggest integer n such that $F_N(x)$ vanishes at q^m for all $m = 1, \dots, n - 1$. This idea, very simple and appealing to Schwarz lemma, does not seem to occur elsewhere in Mahler's theory; it was later generalised by Nesterenko in the proof of his famous theorem in [33, 34], implying the algebraic independence of the three numbers $\pi, e^\pi, \Gamma(1/4)$ and the stephanese theorem ⁽⁴⁾. We will come back to the latter result in Section 4.

2.2.2. Effects of Schmidt's Subspace Theorem. We mention the following result in [16] whose authors Corvaja and Zannier deduce from Schmidt's Subspace Theorem.

Theorem 2.5 (Corvaja and Zannier). *Let us consider a formal series $f \in \mathbb{Q}^{\text{alg}}((x)) \setminus \mathbb{Q}^{\text{alg}}[x, x^{-1}]$ and assume that f converges for x such that $0 < |x| < 1$. Let $L \subset \mathbb{C}$ be a number field and S*

⁴We take the opportunity to notice that a proof of an analog of the stephanese theorem for the so-called "Drinfeld modular invariant" by Ably, Recher and Denis is contained in [1].

a finite set of places of L containing the archimedean ones. Let $\mathcal{A} \subset \mathbb{N}$ be an infinite subset. Assume that:

- (1) $\alpha \in L$, $0 < |\alpha| < 1$
- (2) $f(\alpha^n) \in L$ is an S -integer for all $n \in \mathcal{A}$.

Then,

$$\liminf_{n \in \mathcal{A}} \frac{h(f(\alpha^n))}{n} = \infty$$

This theorem has as an immediate application with $\mathcal{A} = \{d, d^2, d^3, \dots\}$, $d > 1$ being an integer. If $f \in \mathbb{Q}^{\text{alg}}[[x]]$ is not a polynomial, converges for $|x| < 1$ and is such that $f(x^d) = R(x, f(x))$ with $R \in \mathbb{Q}^{\text{alg}}(X, Y)$ with $h_Y(R) < d$, then, $f(\alpha^{d^n})$ is transcendental for α algebraic with $0 < |\alpha| < 1$ and for all n big enough. This implies a result (at least apparently) stronger than Theorem 2.2; indeed, the hypothesis that the coefficients of the series f all lie in a given number field is dropped.

3. Transcendence theory in positive characteristic

The reduction modulo 2 in $\mathbb{F}_2[[x]]$ of the formal series $f_{\text{TM}}(x) \in \mathbb{Z}[[x]]$ is an algebraic formal series. In this section we will see that several interesting transcendental series in positive characteristic are analogues of the series satisfying the functional equation (8).

Let $q = p^e$ be an integer power of a prime number p with $e > 0$, let \mathbb{F}_q be the field with q elements. Let us write $A = \mathbb{F}_q[\theta]$ and $K = \mathbb{F}_q(\theta)$, with θ an indeterminate over \mathbb{F}_q , and define an absolute value $|\cdot|$ on K by $|a| = q^{\deg_\theta a}$, a being in K , so that $|\theta| = q$. Let $K_\infty := \mathbb{F}_q((1/\theta))$ be the completion of K for this absolute value, let K_∞^{alg} be an algebraic closure of K_∞ , let C be the completion of K_∞^{alg} for the unique extension of $|\cdot|$ to K_∞^{alg} , and let K^{alg} be the algebraic closure of K embedded in C . There is a unique degree map $\deg_\theta : C^\times \rightarrow \mathbb{Q}$ which extends the map $\deg_\theta : K^\times \rightarrow \mathbb{Z}$.

Let us consider the power series

$$f_{\text{De}}(x) = \prod_{n=1}^{\infty} (1 - \theta x^{q^n}),$$

which converges for all $x \in C$ such that $|x| < 1$ and satisfies, just as in (8), the functional equation:

$$f_{\text{De}}(x^q) = \frac{f_{\text{De}}(x)}{1 - \theta x^q} \quad (9)$$

(the subscript De stands for Denis, who first used this series for transcendence purposes).

For $q = 2$, we notice that $f_{\text{De}}(x) = \sum_{n=0}^{\infty} \theta^{b_n} x^{2^n}$, where

$$(b_n)_{n \geq 0} = 0, 1, 1, 2, 1, 2, 2, 3, 1, 2, 2, 3, \dots$$

is the sequence with b_n equal to the sum of the digits of the binary expansion of n (and whose reduction modulo 2 precisely is Thue-Morse sequence of Section 2.1). It is very easy to show that f_{De} is transcendental, because it is plain that f_{De} has infinitely many zeros $\theta^{-1/q}, \theta^{-1/q^2}, \dots$ (just as the function occurring at the end of 2.1.4). We shall prove:

Theorem 3.1. *For all $\alpha \in K^{\text{alg}}$ with $0 < |\alpha| < 1$, $f_{\text{De}}(\alpha)$ is transcendental.*

3.1. Proof of Theorem 3.1. The proof of Theorem 3.1 follows the essential lines of Section 2.1, once the necessary tools are introduced.

3.1.1. Transcendence of functions. Not all the arguments of 2.1.1 work well to show the transcendence of formal series such as f_{De} ; in particular, the so-called Riemann-Hurwitz-Hasse formula does not give much information for functional equations such as $f_{\text{De}}(x^d) = af_{\text{De}}(x) + b$ with the characteristic that divides d . Since in general it is hard to detect zeros of Mahler's functions, we report another way to check the transcendency of f_{De} , somewhat making use of "automatic methods", which can also be generalised as it does not depend on the location of the zeroes. To simplify the presentation, we assume, in the following discussion, that $q = 2$ but at the same time, we relax certain conditions so that, in all this subsection, we denote by ϑ an element of C and by f_ϑ the formal series

$$f_\vartheta(x) = \prod_{n=0}^{\infty} (1 - \vartheta x^{q^n}) = \sum_{n=0}^{\infty} \vartheta^{b_n} x^n \in F[[x]] \subset C[[x]]$$

with F the perfect field $\cup_{n \geq 0} \mathbb{F}_2(\vartheta^{1/2^n})$, converging for $x \in C$ with $|x| < 1$. In particular, we have

$$f_\vartheta(x) = f_{\text{De}}(x^{1/2}) \in \mathbb{F}_2[\vartheta][[x]].$$

We shall prove:

Theorem 3.2. *The formal series f_ϑ is algebraic over $F(x)$ if and only if ϑ belongs to $\overline{\mathbb{F}_q}$, embedded in C .*

Proof. If $\vartheta \in \overline{\mathbb{F}_q}$, it is easy to show that f_ϑ is algebraic, so let us assume by contradiction that f_ϑ is algebraic, with ϑ that belongs to $C \setminus \overline{\mathbb{F}_q}$.

We have the functional equation:

$$(1 - \vartheta x) f_\vartheta(x^2) = f_\vartheta(x). \quad (10)$$

We introduce the operators

$$f = \sum_i c_i x^i \in F((x)) \mapsto f^{(k)} = \sum_i c_i^{2^k} x^i \in F((x)),$$

well defined for all $k \in \mathbb{Z}$. Since $f(x^2) = f^{(-1)}(x)^2$ for any $f \in F[[x]]$, we deduce from (10) the collection of functional equations

$$f_\vartheta^{(-1-k)}(x)^2 (1 - \vartheta^{1/2^k} x) = f_\vartheta^{(-k)}(x), \quad k \geq 0. \quad (11)$$

For any $f \in F[[x]]$ there exist two series $f_0, f_1 \in F[[x]]$, uniquely determined, with the property that

$$f = f_0^2 + x f_1^2.$$

We define $E_i(f) := f_i$ ($i = 0, 1$). It is plain that, for $f, g \in F[[x]]$,

$$\begin{aligned} E_i(f + g) &= E_i(f) + E_i(g), \quad (i = 0, 1), \\ E_0(fg) &= E_0(f)E_0(g) + x E_1(f)E_1(g), \\ E_1(fg) &= E_0(f)E_1(g) + E_1(f)E_0(g), \\ E_0(f^2) &= f, \\ E_1(f^2) &= 0. \end{aligned}$$

Therefore,

$$E_i(f^2g) = fE_i(g), \quad i = 0, 1.$$

By (11) we get

$$\begin{aligned} E_0(f_\vartheta^{(-k)}) &= f_\vartheta^{(-1-k)} E_0(1 - \vartheta^{1/2^k} x) = f_\vartheta^{(-1-k)}, \\ E_1(f_\vartheta^{(-k)}) &= E_1(1 - \vartheta^{1/2^k} x) f_\vartheta^{(-1-k)} = -\vartheta^{1/2^k} f_\vartheta^{(-1-k)}, \end{aligned}$$

and we see that if V is a F -subvector space of $F[[x]]$ containing f_ϑ and stable under the action of the operators E_0, E_1 , then V contains the F -subvector space generated by $f_\vartheta, f_\vartheta^{(-1)}, f_\vartheta^{(-2)}, \dots$

By a criterion for algebraicity of Sharif and Woodcock [49, Theorem 5.3] there is a subvector space V as above, with finite dimension, containing f_ϑ . The formal series $f_\vartheta^{(-k)}$ are F -linearly dependent and there exists $s > 0$ such that $f_\vartheta, f_\vartheta^{(1)}, \dots, f_\vartheta^{(s-1)}$ are F -linearly dependent.

Going back to the explicit x -expansion of f_ϑ , the latter condition is equivalent to the existence of $c_0, \dots, c_{s-1} \in F$, not all zero, such that for all $n \geq 0$:

$$\sum_{i=0}^{s-1} c_i \vartheta^{2^i b_n} = 0.$$

The sequence $b : \mathbb{N} \cup \{0\} \rightarrow \mathbb{N} \cup \{0\}$ is known to be surjective, so that the Moore determinant

$$\det((\vartheta^{2^i j})_{0 \leq i, j \leq s-1})$$

vanishes. But this means that $1, \vartheta, \vartheta^2, \dots, \vartheta^{s-1}$ are \mathbb{F}_2 -linearly dependent (Goss, [23, Lemma 1.3.3]), or in other words, that ϑ is algebraic over \mathbb{F}_2 ; a contradiction which completes the proof that f_ϑ and in particular f_{D_e} are transcendental over $F(x)$ (and the fact that the image of b has infinitely many elements suffices to achieve the proof).

3.1.2. Heights under a more general point of view. A good framework to generalise logarithmic heights to other base fields is that described by Lang in [26, Chapter 3] and by Artin and Whaples [7, Axioms 1, 2]. Let \mathcal{K} be any field together with a proper set of non-equivalent places $M_{\mathcal{K}}$. Let us choose, for every place $v \in M_{\mathcal{K}}$ an absolute value $|\cdot|_v \in v$ and assume that for all $x \in \mathcal{K}^\times$, the following product formula holds (cf. [26] p. 23):

$$\prod_{v \in M_{\mathcal{K}}} |x|_v = 1, \quad x \in \mathcal{K}^\times, \quad (12)$$

with the additional property that if α is in \mathcal{K}^\times , then $|\alpha|_v = 1$ for all but finitely many $v \in M_{\mathcal{K}}$. Let us suppose that $M_{\mathcal{K}}$ contains at least one absolute value associated to either a discrete, or an archimedean valuation of \mathcal{K} . It is well known that under these circumstances [7], \mathcal{K} is either a number field, or a function field of one variable over a field of constants.

Given a finite extension \mathcal{L} of \mathcal{K} , there is a proper set $M_{\mathcal{L}}$ of absolute values on \mathcal{L} , extending those of $M_{\mathcal{K}}$, again satisfying the product formula

$$\prod_{v \in M_{\mathcal{L}}} |\alpha|_v^{d_v} = 1, \quad (13)$$

where, if v is the place of \mathcal{K} such that $w|_{\mathcal{K}} = v$ (one then writes $w|v$), we have defined $d_w = [L_w, K_v]$, so that $\sum_{w|v} d_w = [\mathcal{L} : \mathcal{K}]$.

An analogue of the absolute logarithmic height h is available, by the following definition (see [26, Chapter 3]). Let $(\alpha_0 : \cdots : \alpha_n)$ be a projective point defined over \mathcal{L} . Then we define:

$$h(\alpha_0 : \cdots : \alpha_n) = \frac{1}{[\mathcal{L} : \mathcal{K}]} \sum_{w \in M_{\mathcal{L}}} d_w \log \max\{|\alpha_0|_w, \dots, |\alpha_n|_w\}.$$

Again, we have a certain collection of properties making this function useful in almost every proof of transcendence over function fields.

First of all, product formula (13) implies that $h(\alpha_0 : \cdots : \alpha_n)$ does not depend on the choice of the field \mathcal{L} and defines a map

$$h : \mathbb{P}_n(\mathcal{K}^{\text{alg.}}) \rightarrow \mathbb{R}_{\geq 0}.$$

We write $h(\alpha) := h(1 : \alpha)$. If the absolute values of $M_{\mathcal{K}}$ are all ultrametric, it is easy to prove, with the same indications as in 2.1.2, that for $\alpha, \beta \in \mathcal{K}^{\text{alg.}\times}$:

$$\begin{aligned} h(\alpha + \beta), h(\alpha\beta) &\leq h(\alpha) + h(\beta), \\ h(\alpha^n) &= |n|h(\alpha), \quad n \in \mathbb{Z}. \end{aligned}$$

More generally, if P is a polynomial in $\mathcal{L}[X_1, \dots, X_n]$ and if $(\alpha_1, \dots, \alpha_n)$ is a point of \mathcal{L}^n , we write $h(P)$ for the height of the projective point whose coordinates are 1 and its coefficients. We have:

$$h(P(\alpha_1, \dots, \alpha_n)) \leq h(P) + \sum_{i=1}^n (\deg_{X_i} P) h(\alpha_i). \quad (14)$$

Product formula (13) also provides a Liouville's type inequality. Let $[\mathcal{L} : \mathcal{K}]_{\text{sep.}}$ be the separable degree of \mathcal{L} over \mathcal{K} . Let us choose a distinguished absolute value $|\cdot|$ of \mathcal{L} and $\beta \in \mathcal{L}^{\times}$. We have:

$$\log |\beta| \geq -[\mathcal{L} : \mathcal{K}]_{\text{sep.}} h(\beta). \quad (15)$$

The reason of the presence of the separable degree in (15) is the following. If $\alpha \in \mathcal{L}^{\times}$ is separable over \mathcal{K} then $\log |\alpha| \geq -[\mathcal{K}(\alpha) : \mathcal{K}] h(\alpha) = -[\mathcal{K}(\alpha) : \mathcal{K}]_{\text{sep.}} h(\alpha)$. Let β be any element of \mathcal{L}^{\times} . There exists $s \geq 0$ minimal with $\alpha = \beta^{p^s}$ separable and we get $p^s \log |\beta| \geq -[\mathcal{K}(\alpha) : \mathcal{K}] h(\alpha) = -p^s [\mathcal{K}(\beta) : \mathcal{K}]_{\text{sep.}} h(\beta)$.

3.1.3. Transcendence of the values of f_{De} . We now follow Denis and we take $\mathcal{K} = K$, $M_{\mathcal{K}}$ the set of all the places of K and we choose in each of these places an absolute value normalised so that product formula (12) holds, with the distinguished absolute value $|\cdot|$ chosen so that $|\alpha| = q^{\deg_{\theta} \alpha}$ for $\alpha \in K^{\times}$.

As we already did in 2.1.3, we choose for all $N \geq 0$, a polynomial $P_N \in K[X, Y]$, non-zero, of degree $\leq N$ in both X, Y , such that the order of vanishing $\nu(N) < \infty$ of the function $F_N(x) := P_N(x, f_{\text{De}}(x))$ at $x = 0$ satisfies $\nu(N) \geq N^2$. We know that this is possible by simple linear algebra arguments as we did before.

Let $\alpha \in K^{\text{alg.}}$ be such that $0 < |\alpha| < 1$; as in 2.1.3, the sequence $(P_N)_N$ need not to depend on it but the choice of N we will do does.

By the identity principle of analytic functions on \mathcal{C} , if ϵ is a positive real number, for n big enough depending on N and α, l, ϵ , we have $F_N(\alpha^{q^{n+1}}) \neq 0$ and

$$\log |F_N(\alpha^{q^{n+1}})| \leq \nu(N) q^{n+1} \log |\alpha| + \log |c_{\nu(N)}| + \epsilon,$$

where $c_{\nu(N)}$ is a non-zero element of K depending on N (it is the leading coefficient of the formal series F_N).

Let us assume by contradiction that

$$f_{\text{De}}(\alpha) \in K^{\text{alg.}},$$

let L be a finite extension of K containing α and $f_{\text{De}}(\alpha)$.

By the variant of Liouville's inequality (15) and from the basic facts on the height h explained above

$$\begin{aligned} \log |F_N(\alpha^{q^{n+1}})| &\geq \\ &\geq -[L : K]_{\text{sep.}} \left(\deg_{\theta} P_N + Nh(\alpha^{q^{n+1}}) + Nh \left(\frac{f_{\text{De}}(\alpha)}{(1 - \theta\alpha^q) \cdots (1 - \theta\alpha^{q^{n+1}})} \right) \right). \end{aligned}$$

Dividing by Nq^{n+1} and using that $\nu(N) \geq N^2$ we get, for all n big enough,

$$\begin{aligned} N \log |\alpha| + \log |c_{\nu(N)}| + \epsilon &\geq \\ &\geq -[L : K]_{\text{sep.}} (\deg_{\theta} P_N N^{-1} q^{-n-1} + (1 + (q - q^{-n-1})/(q - 1))h(\alpha) + \\ &\quad q^{-n-1}h(f_{\text{De}}(\alpha)) + (n + 1)N^{-1}q^{-n-1}h(\theta)). \end{aligned}$$

Letting n tend to infinity, we obtain the inequality:

$$N \log |\alpha| \geq -[L : K]_{\text{sep.}} \left(1 + \frac{q}{q - 1} \right) h(\alpha)$$

for all $N > 0$. Just as in the proof of Theorem 2.1, if N is big enough, this is contradictory with the assumptions showing that $f_{\text{De}}(\alpha)$ is transcendental.

3.1.4. A first application to periods. The transcendence of values of f_{De} at algebraic series has interesting applications, especially when one looks at what happens with the base point $\alpha = \theta^{-1}$. Indeed, let

$$\tilde{\pi} = \theta(-\theta)^{1/(q-1)} \prod_{i=1}^{\infty} (1 - \theta^{1-q^i})^{-1} \quad (16)$$

be a fundamental period of Carlitz's module (it is defined up to multiplication by an element of \mathbb{F}_q^{\times}). Then,

$$\tilde{\pi} = \theta(-\theta)^{1/(q-1)} f_{\text{De}}(\theta^{-1})^{-1},$$

so that it is transcendental over K .

If $\alpha = \theta^{-1}$, $h(\alpha) = \log q$ so that to show that $\tilde{\pi}$ is not in K , it suffices to choose $N \geq 4$ if $q = 2$ and $N \geq 3$ if $q \neq 2$ in the proof above. Let us look, for $q \neq 2$ given, at a polynomial (depending on q) $P \in A[X, Y] \setminus \{0\}$ with relatively prime coefficients in X of degree ≤ 3 in X and in Y , such that $P(u, f_{\text{De}}(u))$ vanishes at $u = 0$ with the biggest possible order $\nu > 9$ (which also depends on q).

It is possible to prove that for all $q \geq 4$,

$$P = X^3(Y - 1)^3 \in \mathbb{F}_q[X, Y].$$

This means that to show that $f_{\text{De}}(\theta^{-1}) \notin K$, it suffices to work with the polynomial $Q = Y - 1$ ⁽⁵⁾. Indeed, $f_{\text{De}}(u) - 1 = -\theta u^q + \cdots$. Therefore, for n big enough, if by contradiction $f_{\text{De}}(\theta^{-1}) \in K$, then $\log |f_{\text{De}}(\alpha^{q^{n+1}}) - 1| \geq -(1 + q/(q - 1))h(\alpha^{q^{n+1}})$ which is contradictory even taking $q = 3$, but not for $q = 2$, case that we skip.

⁵Other reasons, related to the theory of Carlitz module, allow to show directly that $\tilde{\pi} \notin \mathbb{F}_q((\theta^{-1}))$ for $q > 2$

If $q = 3$, we get a completely different kind of polynomial P of degree ≤ 3 in each indeterminate:

$$P = 2 + \theta^2 + \theta X^3 + 2\theta^3 X^3 + 2\theta^2 Y + Y^3 + 2\theta X^3 Y^3.$$

It turns out that

$$P(u, f_{\text{De}}(u)) = (\theta^6 - \theta^4)u^\nu + \dots$$

with $\nu = 36$ so that the order of vanishing is three times as big as the quantity expected from the computations with $q \geq 4$: $12 = 3 \cdot 3 + 3$. How big can ν be? It turns out that this question is important, notably in the search for quantitative measures of transcendence and algebraic independence; we will discuss about this problem in Section 4.1.

3.2. A second transcendence result. With essentially the same arguments of Section 3, it is possible to deal with a more general situation and prove the Theorem below. We first explain the data we will work with.

Let F be a field and t an indeterminate. We denote by $F\langle\langle t \rangle\rangle$ the field of *Hahn generalised series*. This is the set of formal series

$$\sum_{i \in S} c_i t^i, \quad c_i \in F,$$

with S a *well ordered subset* of \mathbb{Q} ⁽⁶⁾, endowed with the standard addition and Cauchy's multiplication from which it is plain that every non-zero formal series is invertible.

We have a field $\mathbb{F}_q\langle\langle t \rangle\rangle$ -automorphism

$$\tau : C\langle\langle t \rangle\rangle \rightarrow C\langle\langle t \rangle\rangle$$

defined by

$$\alpha = \sum_{i \in S} c_i t^i \mapsto \tau\alpha = \sum_{i \in S} c_i^q t^i.$$

Assume that, with the notations of 3.1.2, $\mathcal{K} = C(t)$, with t an independent indeterminate. Let $|\cdot|$ be an absolute value associated to the t -adic valuation and $\widehat{\mathcal{K}}$ the completion of \mathcal{K} for this absolute value. Let \mathbb{K} be the completion of an algebraic closure of $\widehat{\mathcal{K}}$ for the extension of $|\cdot|$, so that we have an embedding of \mathcal{K}^{alg} in \mathbb{K} . We have an embedding $\iota : \mathbb{K} \rightarrow C\langle\langle t \rangle\rangle$ (see Kedlaya, [24, Theorem 1]); there exists a rational number $c > 1$ such that if α is in \mathbb{K} and $\iota(\alpha) = \sum_{i \in S} c_i t^i$, then $|\alpha| = c^{-i_0}$, where $i_0 = \min(S)$. We identify \mathbb{K} with its image by ι . It can be proved that $\tau\mathbb{K} \subset \mathbb{K}$, $\tau\mathcal{K}^{\text{alg}} \subset \mathcal{K}^{\text{alg}}$ and $\tau\mathcal{K} \subset \mathcal{K}$.

The definition of τ implies immediately that, for all $\alpha \in \mathbb{K}$,

$$|\tau\alpha| = |\alpha| \tag{17}$$

We choose $M_{\mathcal{K}}$ a complete set of non-equivalent absolute values of \mathcal{K} such that the product formula (12) holds. On $\mathbb{P}_n(\mathcal{K}^{\text{alg}})$, we have the absolute logarithmic height whose main properties have been described in 3.1.2.

There is a useful expression for the height $h(\alpha)$ of a non-zero element α in \mathcal{K}^{alg} of degree D . If $P = a_0 X^d + a_1 X^{d-1} + \dots + a_{d-1} X + a_d$ is a polynomial in $C[t][X]$ with relatively prime coefficients such that $P(\alpha) = 0$, we have:

$$h(\alpha) = \frac{1}{D} \left(\log |a_0| + \sum_{\sigma: \mathcal{K}^{\text{alg}} \rightarrow \mathbb{K}} \log \max\{1, |\sigma(\alpha)|\} \right), \tag{18}$$

⁶By definition, every nonempty subset of S has a least element for the order \leq .

where the sum runs over all the \mathcal{K} -embeddings of \mathcal{K}^{alg} in \mathbb{K} . The proof of this formula follows the same ideas as that of [54, Lemma 3.10].

Let α be in \mathcal{K}^{alg} . From (17) and (18), it follows that:

$$h(\tau\alpha) = h(\alpha). \quad (19)$$

Let us also consider, over the ring of formal series $\mathbb{K}[[x]]$, the $\mathbb{F}_q\langle\langle t \rangle\rangle$ -linear extension of τ defined in the following way:

$$f := \sum_i c_i x^i \mapsto \tau f := \sum_i (\tau c_i) x^{qi}.$$

We can now state the main result of this section.

Theorem 3.3. *Let $f \in \mathcal{K}[[x]]$ be converging for $x \in \mathbb{K}$, $|x| < 1$, let $\alpha \in \mathcal{K}$ be such that $0 < |\alpha| < 1$. Assume that:*

- (1) f is transcendental over $\mathcal{K}(x)$,
- (2) $\tau f = af + b$, where a, b are elements of $\mathcal{K}(x)$.

Then, for all n big enough, $(\tau^n f)(\alpha)$ is transcendental over \mathcal{K} .

Proof. We begin with a preliminary discussion about heights. Let $r = r_0 x^n + \dots + r_n$ be a polynomial in $\mathcal{K}[x]$. We have, for all $j \geq 0$, $\tau^j r = (\tau^j r_0) x^{q^j n} + \dots + (\tau^j r_n)$. Therefore, if α is an element of \mathcal{K}^{alg} , we deduce from (14), (19) and from elementary height estimates:

$$\begin{aligned} h((\tau^j r)(\alpha)) &\leq h(1 : \tau^j r_0 : \dots : \tau^j r_n) + q^j n h(\alpha), \\ &\leq \sum_{i=0}^n h(\tau^j r_i) + q^j n h(\alpha) \\ &\leq \sum_{i=0}^n h(r_i) + q^j n h(\alpha), \end{aligned}$$

where we have applied (19). Therefore, if a is a rational function in $\mathcal{K}(x)$ such that $(\tau^j a)(\alpha)$ is well defined, we have

$$h((\tau^j a)(\alpha)) \leq c_1 + q^j c_2, \quad (20)$$

where c_1, c_2 are two constants depending on a, α only.

The condition on f implies that, for all $k \geq 0$,

$$\tau^k f = f \prod_{i=0}^{k-1} (\tau^i a) + \sum_{i=0}^{k-1} (\tau^i b) \prod_{j=i+1}^{k-1} (\tau^j a) \quad (21)$$

(where empty sums are equal to zero and empty products are equal to one). Hence, the field

$$\mathcal{L} = \mathcal{K}(\alpha, f(\alpha), (\tau f)(\alpha), (\tau^2 f)(\alpha), \dots)$$

is equal to $\mathcal{K}(\alpha, (\tau^n f)(\alpha))$ for all n big enough.

The transcendence of f implies that $a \neq 0$. If α is a zero or a pole of $\tau^k a$ and a pole of $\tau^k b$ for all k , then it is a simple exercise left to the reader to prove that $|\alpha| = 1$, case that we have excluded.

Let us suppose by contradiction that the conclusion of the theorem is false. Then, α is not a pole or a zero of $\tau^n a, \tau^n b, (\tau^n f)(\alpha)$ is algebraic over \mathcal{K} for all n big enough, and \mathcal{L} is a finite extension of \mathcal{K} .

An estimate for the height of this series can be obtained as follows.

A joint application of (19), (20) and (21) yields:

$$\begin{aligned} h((\tau^k f)(\alpha)) &\leq h(f(\alpha)) + \sum_{i=0}^{k-1} h((\tau^i a)(\alpha)) + \sum_{i=0}^{k-1} h((\tau^i b)(\alpha)) \\ &\leq c_3 + c_4 k + c_5 q^k. \end{aligned}$$

Therefore, if P is a polynomial in $\mathcal{K}[X, Y]$ of degree $\leq N$ in X and Y , writing F_k for the formal series $\tau^k P(x, f(x)) = P^{\tau^k}(x^{q^k}, (\tau^k f)(x))$ (P^{τ^k} is the polynomial obtained from P , replacing the coefficients by their images under τ^k), we get:

$$\begin{aligned} h(F_k(\alpha)) &\leq h(P^{\tau^k}) + q^k (\deg_X P) h(\alpha) + (\deg_Y P) h((\tau^k f)(\alpha)) \\ &\leq c_7(P) + c_6 N q^k, \end{aligned} \tag{22}$$

where c_7 is a constant depending on P .

Let N be a positive integer. There exists $P_N \in \mathcal{K}[X, Y]$ with partial degrees in X, Y not bigger than N , with the additional property that $F_N(x) := P_N(x, f(x)) = c_{\nu(N)} x^{\nu(N)} + \dots$, with $\nu(N) \geq N^2$ and $c_{\nu(N)} \neq 0$.

Let us write $F(x) = \sum_{i \geq 0} c_i x^i$. In ultrametric analysis, Newton polygons suffice to locate the absolute values of the zeroes of Taylor series. The Newton polygons of the series $\sum_{i \geq 0} (\tau^k c_i) x^i \in \mathcal{K}[[x]]$ for $k \geq 0$ are all equal by (17). By [23, Propositions 2.9, 2.11], we have $\sum_{i \geq 0} (\tau^k c_i) \alpha^{q^k i} \neq 0$ for k big enough. Now, since for $k \geq 0$,

$$(\tau^k F_N)(x) = (\tau^k c_{\nu(N)}) x^{\nu(N)q^k} + \dots,$$

we find, when the logarithm is well defined and by (19), that

$$-\infty < \log |(\tau^k F_N)(\alpha)| \leq \nu(N) q^k \log |\alpha| + \log |c_{\nu(N)}| + \epsilon. \tag{23}$$

On the other side, by (22),

$$h((\tau^k F_N)(\alpha)) \leq c_8(N) + c_9 N q^k, \tag{24}$$

where c_8 is a constant depending on f, α, N and c_9 is a constant depending on f, α .

A good choice of N (big) and inequality (15) with k big enough depending on N give a contradiction (⁷).

3.2.1. Applications of Theorem 3.3. We look at solutions $f \in \mathcal{K}[[x]]$ of

$$\tau f = af + b, \quad a, b \in \mathcal{K}(x). \tag{25}$$

Theorem 3.3 allows to give some information about the arithmetic properties of their values.

First application. Assume that in (25), $a, b \in \mathbb{F}_q(t)$. Then, since $\mathbb{F}_q(t)$ is contained in the field of constants of τ , solutions of this difference equation are related to the variant of Mahler's method of Section 3.

⁷It is likely that Nishioka's proof of Theorem 2.2 can be adapted to strengthen Theorem 3.3, but we did not enter into the details of this verification.

If $a = (1 - t^{-1}x)^{-1}$, $b = 0$, the equation above has the solution

$$f_{\text{De2}}(x) = \prod_{n=1}^{\infty} (1 - t^{-1}x^{q^n}),$$

which converges for $x \in \mathbb{K}$, $|x| < 1$.

If $x = t$, Theorem 3.3 yields the transcendence of $f_{\text{De2}}(t) = \prod_{n=1}^{\infty} (1 - t^{q^n-1}) \in \mathbb{F}_q[[t]]$ over $\mathbb{F}_q(t)$ and we get (again) the transcendence of $\tilde{\pi}$ over K (we also notice the result of the paper [2], which allows some other applications). More generally, all the examples of functions in [40, Section 3.1] have a connection with this example.

Second application. Theorem 3.3 also has some application which does not seem to immediately follow from results such as Theorem 2.2. Consider equation (25) with $b = 0$ and $a = (1 + \vartheta x)^{-1}$, where $\vartheta \in \mathbb{F}_q(t, \theta)$ is non-zero. We have the following solution of (25) in $\mathcal{K}[[x]]$:

$$\phi(x) = \prod_{n=0}^{\infty} (1 + (\tau^n \vartheta)x^{q^n}).$$

It is easy to show that $\phi = \sum_{j \geq 0} c_j x^j$ is a formal series of $\mathcal{K}[[x]]$ converging for $x \in \mathbb{K}$, $|x| < 1$.

The coefficients c_j can be computed in the following way. We have $c_j = 0$ if the q -ary expansion of j has its set of digits not contained in $\{0, 1\}$. Otherwise, if $j = j_0 + j_1 q + \dots + j_n q^n$ with $j_0, \dots, j_n \in \{0, 1\}$, we have, writing ϑ_i for $\tau^i \vartheta$, $c_j = \vartheta_0^{j_0} \vartheta_1^{j_1} \dots \vartheta_n^{j_n}$. Therefore, if $\psi(x) = \sum_{k=0}^{\infty} \vartheta_0 \vartheta_1 \dots \vartheta_k x^{1+q+\dots+q^k}$, we have $\phi(x) - 1 = \sum_{j=0}^{\infty} \tau^j \psi(x)$.

For $\vartheta = -t^{-1}(1 + t/\theta)^{-1}$, the series $\phi(x)$ vanishes at every $x_n = t^{1/q^n}(1 + t^{1/q^n}/\theta)$. The x_n 's are elements of \mathbb{K} which are distinct with absolute value < 1 (we recall that we are using the t -adic valuation). Having thus infinitely many zeros in the domain of convergence and not being identically zero, ϕ is transcendental.

The series ϕ converges at $x = t$. Theorem 3.3 implies that the formal series $\phi(t) \in C((t))$ is transcendental over \mathcal{K} . We notice that the arguments of 3.1.1 can be probably extended to investigate the transcendence of the series ϕ associated to, say, $\vartheta = -(1 + t/\theta)^{-1}$, case in which we do not have infinitely many zeros. The reason is that, over $\mathbb{K}((x))$, the \mathbb{F}_q -linear Frobenius map $F : a \mapsto a^q$ (for all a) splits as

$$F = \tau\chi = \chi\tau$$

where τ is Anderson's $\mathbb{F}_q((t))$ -linear map and χ is Mahler's $C((x))$ -linear map, and most of the arguments of 3.1.1 can be generalised to this setting.

3.2.2. Further examples of difference equations. The arguments of the previous subsection deal with formal series in $\mathcal{K}[[x]] = C(t)[[x]]$. We have another important ring of formal series, also embedded in $\mathbb{K}[[x]]$, which is $C(x)[[t]]$. Although the arithmetic of values of these series seems to be not deducible from Theorem 3.3, we discuss here about some examples because solutions $f \in C(x)[[t]]$ of τ -difference equations such as

$$\tau f = af + b, \quad a, b \in C(x) \tag{26}$$

are often related to Anderson-Brownawell-Papanikolas linear independence criterion in [6] (see the corresponding contribution in this volume and the refinement [15]).

With ζ_θ a fixed $(q-1)$ -th root of $-\theta$, the transcendental formal series

$$\Omega(t) := \zeta_\theta^{-q} \prod_{i=1}^{\infty} \left(1 - \frac{t}{\theta^{q^i}}\right) = \sum_{i=0}^{\infty} d_i t^i \in K^{\text{alg.}}[[t]] \tag{27}$$

is convergent for all $t \in C$, such that $\Omega(\theta) \in \mathbb{F}_q^\times \tilde{\pi}^{-1}$ and satisfies the functional equation

$$\Omega(t^q) = (t^q - \theta^q)\Omega(t)^q \quad (28)$$

(see [6]). By a direct inspection it turns out that there is no finite extension of K containing all the coefficients d_i of the t -expansion of Ω ⁽⁸⁾. Hence, there is no available variant of Mahler's method which seems to apply to prove the transcendence of Ω at algebraic elements (and a suitable variant of Theorem 2.5 is not yet available).

The map $\tau : C((t)) \rightarrow C((t))$ acts on in the following way:

$$c = \sum_i c_i t^i \mapsto \tau c := \sum_i c_i^q t^i.$$

Let $s(t)$ be the formal series $\tau^{-1}\Omega^{-1} \in C[[t]]$ (where τ^{-1} is the reciprocal map of τ). After (28) this function is solution of the τ -difference equation:

$$(\tau s)(t) = (t - \theta)s(t), \quad (29)$$

hence it is a solution of (26) with $a = t - \theta$ and $b = 0$. Transcendence of values of this kind of function does not seem to follow from Theorem 3.3, but can be obtained with [6, Theorem 1.3.2].

More generally, let Λ be an A -lattice of C of rank r and let

$$e_\Lambda(z) = z \prod_{\lambda \in \Lambda \setminus \{0\}} \left(1 - \frac{z}{\lambda}\right) \quad (30)$$

be its exponential function, in Weierstrass product form. The function e_Λ is an entire, surjective, \mathbb{F}_q -linear function. Let $\phi_\Lambda : A \rightarrow \mathbf{End}_{\mathbb{F}_q\text{-lin.}}(\mathbb{G}_a(C)) = C[\tau]$ ⁽⁹⁾ be the associated Drinfeld module. We have, for $a \in A$, $\phi_\Lambda(a)e_\Lambda(z) = e_\Lambda(az)$.

Let us extend ϕ_Λ over \mathcal{K} by means of the endomorphism τ ($\tau t = t$). After having chosen an element $\omega \in \Lambda \setminus \{0\}$, define the function

$$s_{\Lambda, \omega}(t) := \sum_{n=0}^{\infty} e_\Lambda\left(\frac{\omega}{\theta^{n+1}}\right) t^n \in C[[t]], \quad (31)$$

convergent for $|t| < q$. We have, for all $a \in A$,

$$\phi_\Lambda(a)s_{\Lambda, \omega} = \bar{a}s_{\Lambda, \omega}, \quad (32)$$

where, if $a = a(\theta) \in \mathbb{F}_q[\theta]$, we have defined $\bar{a} := a(t) \in \mathbb{F}_q[t]$. This means that $s_{\Lambda, \omega}$, as a formal series of $C[[t]]$, is an eigenfunction for all the $\mathbb{F}_q((t))$ -linear operators $\phi_\Lambda(a)$, with eigenvalue \bar{a} , for all $a \in A$.

If $\Lambda = \tilde{\pi}A$, ϕ_Λ is Carlitz's module, and equation (32) implies the τ -difference equation (29).

4. Algebraic independence

In [28], Mahler proved his first result of algebraic independence obtained modifying and generalising the methods of his paper [27]. The result obtained involved m formal series in several variables but we describe its consequences on the one variable theory only. Let L be a number field embedded in \mathbb{C} and let $d > 1$ be an integer.

⁸They generate an infinite tower of Artin-Schreier extensions.

⁹Polynomial expressions in τ with the product satisfying $\tau c = c^q \tau$, for $c \in C$.

Theorem 4.1 (Mahler). *Given m formal series $f_1, \dots, f_m \in L[[x]]$, satisfying functional equations*

$$f_i(x^d) = a_i f_i(x) + b_i(x), \quad 1 \leq i \leq m$$

with $a_i \in L$, $b_i \in L(x)$ for all i , converging in the open unit disk. If α is algebraic such that $0 < |\alpha| < 1$, then the transcendence degree over $L(x)$ of the field $L(x, f_1(x), \dots, f_m(x))$ is equal to the transcendence degree over \mathbb{Q} of the field $\mathbb{Q}(f_1(\alpha), \dots, f_m(\alpha))$.

4.1. Criteria of algebraic independence and applications.. Mahler's result remained nearly unobserved for several years. It came back to surface notably thanks to the work of Loxton and van der Poorten in the seventies, and then by Nishioka and several other authors. At the beginning, these authors developed criteria for algebraic independence tailored for application to algebraic independence of Mahler's values. Later, criteria for algebraic independence evolved in very general results, especially in the hands of Philippon. Here follows a particular case of a criterion of algebraic independence by Philippon. The statement that follows merges the results [44, Theorem 2] and [43, Theorem 2.11] and uses the data $\mathbb{K}, \mathcal{L}, |\cdot|, \mathcal{K}, \mathcal{A}, \dots$ where \mathbb{K} is a complete algebraically closed field in two cases. We examine only the cases in which \mathbb{K} is either \mathbb{C} or C , but it is likely that the principles of the criterion extend to several other fields, like that of Section 3.2.

In the case $\mathbb{K} = \mathbb{C}$, \mathcal{L} is a number field embedded in \mathbb{C} , $|\cdot|$ is the usual absolute value, h is the absolute logarithmic height of projective points defined over \mathbb{Q}^{alg} , \mathcal{K} denotes \mathbb{Q} and \mathcal{A} denotes \mathbb{Z} .

In the case $\mathbb{K} = C$, \mathcal{L} is a finite extension of $K = \mathbb{F}_q(\theta)$ embedded in C , $|\cdot|$ is an absolute value associated to the θ^{-1} -adic valuation, h is the absolute logarithmic height of projective points defined over K^{alg} , \mathcal{A} denotes the ring $A = \mathbb{F}_q[\theta]$ and we write $\mathcal{K} = K$.

In both cases, if P is a polynomial with coefficients in \mathcal{L} , we associate to it a projective point whose coordinates are 1 and its coefficients, and we write $h(P)$ for the logarithmic height of this point (which depends on P up to permutation of the coefficients).

Theorem 4.2 (Philippon). *Let $(\alpha_1, \dots, \alpha_m)$ be an element of \mathbb{K}^m and k an integer with $1 \leq k \leq m$. Let us suppose that there exist three increasing functions $\mathbb{Z}_{\geq 1} \rightarrow \mathbb{R}_{\geq 1}$*

$$\begin{aligned} \delta & \quad (\text{degree}) \\ \sigma & \quad (\text{height}) \\ \lambda & \quad (\text{magnitude}) \end{aligned}$$

and five positive real numbers $c_2, c_3, c_\delta, c_\lambda, c_\sigma$ satisfying the following properties.

- (1) $\lim_{n \rightarrow \infty} \delta(n) = \infty$,
- (2) $\lim_{n \rightarrow \infty} \frac{s(n+1)}{s(n)} = c_s$ for $s = \delta, \sigma, \lambda$,
- (3) $\sigma(n) \geq \delta(n)$, for all n big enough,
- (4) The sequence $n \mapsto \frac{\lambda(n)}{\delta(n)^{k+1} \sigma(n)}$ is ultimately increasing,
- (5) For all n big enough,

$$\lambda(n)^{k+1} > \sigma(n) \delta(n)^{k-1} (\lambda(n)^k + \delta(n)^k).$$

Let us suppose that there exists a sequence of polynomials $(Q_n)_{n \geq 0}$ with

$$Q_n \in \mathcal{L}[X_1, \dots, X_m],$$

with $\deg_{X_i} Q_n \leq \delta(n)$ for all i and n , with $h(Q_n) \leq \sigma(n)$ for all n , with coefficients integral over \mathcal{A} , such that, for all n big enough,

$$-c_2\lambda(n) < \log |Q_n(\alpha_1, \dots, \alpha_m)| < -c_3\lambda(n).$$

Then, the transcendence degree over \mathcal{L} of the field $\mathcal{L}(\alpha_1, \dots, \alpha_m)$ is $\geq k$.

Theorem 4.2 can be applied to prove the following result.

Theorem 4.3. *Let us assume that we are again in one of the cases above; $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = C$, let \mathcal{L} be as above. Let f_1, \dots, f_m be formal series of $\mathcal{L}[[x]]$ and $d > 1$ and integer, satisfying the following properties:*

- (1) f_1, \dots, f_m converge for $|x| < 1$,
- (2) f_1, \dots, f_m are algebraically independent over $\mathbb{K}(x)$,
- (3) For all $i = 1, \dots, m$, there exist $a_i, b_i \in \mathcal{L}(x)$ such that

$$f_i(x^d) = a_i(x)f_i(x) + b_i(x), \quad i = 1, \dots, m.$$

Let $\alpha \in \mathcal{L}$ be such that $0 < |\alpha| < 1$. Then, for all n big enough, $f_1(\alpha^{d^n}), \dots, f_m(\alpha^{d^n})$ are algebraically independent over \mathcal{L} .

This result is, for $\mathbb{K} = \mathbb{C}$, a corollary of Kubota's result [25, Theorem p. 10]. For $\mathbb{K} = C$, it is due to Denis [18, Theorem 2]. See also [9, 17].

Sketch of proof of Theorem 4.3 in the case $\mathbb{K} = C$. To simplify the exposition, we assume that $\mathcal{L} = K$. Let $N > 0$ be an integer; there exists at least one non-zero polynomial $P_N \in K[x, X_1, \dots, X_n]$ (that we choose) of degree $\leq N$ in each indeterminate, such that the order $\nu(N)$ of vanishing at $x = 0$ of the function

$$F_N(x) = P_N(x, f_1(x), \dots, f_m(x))$$

(not identically zero because of the hypothesis of algebraic independence of the functions f_i over $C(x)$), satisfies $\nu(N) \geq N^{m+1}$.

The choice of the parameter N will be made later. If $c(x) \in A[x]$ is a non-zero polynomial such that $ca_i, cb_i \in A[x]$ for $i = 1, \dots, m$, then we define, inductively, $R_0 = P_N$ and

$$R_n = c(x)^N R_{n-1}(x^d, a_1 X_1 + b_1, \dots, a_m X_m + b_m) \in A[x, X_1, \dots, X_m].$$

Elementary inductive computations lead to the following estimates, holding for n big enough depending on N, f_1, \dots, f_m , where c_4, c_5, c_6 are integer constants depending on f_1, \dots, f_m only:

$$\begin{aligned} \deg_{X_i}(R_n) &\leq N, \quad (i = 1, \dots, m), \\ \deg_x(R_n) &\leq c_4 d^n N, \\ \deg_\theta(R_n) &\leq c_5 d^n N, \\ h(R_n) &\leq c_7 + c_6 d^n N, \end{aligned}$$

where we wrote $c_7(N) = h(R_0)$; it is a real number depending on f_1, \dots, f_m and N .

Since

$$R_n(x, f_1(x), \dots, f_n(x)) = \left(\prod_{i=0}^{n-1} c(x^{d^i})^N \right) R_0(x^{d^n}, f_1(x^{d^n}), \dots, f_m(x^{d^n})),$$

one verifies the existence of two constants $c_2 > c_3 > 0$ such that

$$-c_2 d^n \nu(N) \leq \deg_{\theta}(R_n(\alpha, f_1(\alpha), \dots, f_m(\alpha))) \leq -c_3 d^n \nu(N)$$

for all n big enough, depending on N, f_1, \dots, f_m and α .

Let us define:

$$Q_n(X_1, \dots, X_n) = D^{c_1 d^n N} R_n(\alpha, X_1, \dots, X_m) \in A[\alpha][X_1, \dots, X_m],$$

where $D \in A \setminus \{0\}$ is such that $D\alpha$ is integral over K . The estimate above implies at once that, for n big enough:

$$\begin{aligned} \deg_{X_i} Q_n &\leq N, \\ h(Q_n) &\leq c_8(N) + c_9 d^n N, \end{aligned}$$

where $c_8(N)$ is a constant depending on f_1, \dots, f_m, N , and α (it can be computed with an explicit dependence on $c_7(N)$) and c_9 is a constant depending on f_1, \dots, f_m , and α but not on N .

Finally, Theorem 4.2 applies with the choices:

$$\begin{aligned} \alpha_i &= f_i(\alpha), \quad i = 1, \dots, m \\ k &= m \\ \lambda(n) &= d^n \nu(N) \\ \delta(n) &= N \\ \sigma(n) &= c_9 d^n \nu(N), \end{aligned}$$

provided that we choose N large enough depending on the constants c_1, \dots introduced so far. Then, one chooses n big enough (depending on the good choice of N). \square

4.1.1. An example with \mathbb{Q} as a base field.. Theorem 4.3 furnishes algebraic independence of Mahler's values if we are able to check algebraic independence of Mahler's functions but it does not say anything on the latter problem; this is not an easy task in general. With the following example, we would like to sensitise the reader to this problem which, the more we get involved in the subtleties of Mahler's method, the more it takes a preponderant place.

In the case $\mathbb{K} = \mathbb{C}$, we consider the formal series in $\mathbb{Z}[[x]]$:

$$L_0 = \prod_{i=0}^{\infty} (1 - x^{2^i})^{-1}, \quad L_r = \sum_{i=0}^{\infty} \frac{x^{2^i r}}{\prod_{j=0}^{i-1} (1 - x^{2^j})}, \quad r \geq 1$$

converging, on the open unit disk $|x| < 1$, to functions satisfying :

$$L_0(x^2) = (1 - x)L_0(x), \quad L_r(x^2) = (1 - x)(L_r(x) - x^r), \quad r \geq 1.$$

By a result of Kubota [25, Theorem 2], (see also Töpfer, [51, Lemma 6]), if L_1, \dots are algebraically dependent, then they also are \mathbb{C} -linearly dependent modulo $\mathbb{C}(x)$ in the following sense. There exist $c_1, \dots, c_m \in \mathbb{C}$ not all zero, such that:

$$\sum_{i=1}^m c_i L_i(x) = f(x)$$

with $f(x) \in \mathbb{C}(x)$. It is easy to see that the reductions modulo 2 of the series L_1, \dots, L_m generate a $(2, \dots, 2)$ -extension of $\mathbb{F}_2(x)$ of maximal degree 2^m . With this property in mind, it is rather easy to prove that L_1, \dots, L_m are \mathbb{C} -linearly independent modulo $\mathbb{C}(x)$. Therefore, by Theorem 4.3, for all $\alpha \in \mathbb{Q}^{\text{alg}}$ with $0 < |\alpha| < 1$, the numbers $L_1(\alpha), \dots, L_m(\alpha)$ are algebraically independent. The reduction modulo 2 of the series L_0 is a rational series and at the time being, we do not know if the functions L_0, L_1, \dots are algebraically independent ⁽¹⁰⁾.

4.2. Measuring algebraic independence. Beyond transcendence and algebraic independence, the next step in the study of the arithmetic of Mahler's numbers is that of quantitative results such as measures of algebraic independence. Very often, such estimates are not mere technical refinements of well known results but deep information on the diophantine behaviour of classical constants; everyone knows the important impact that Baker's theory on quantitative minorations of linear forms in logarithms on algebraic groups had in arithmetic geometry.

Rather sharp estimates are known for complex Mahler's values. We quote here a result of Nishioka [36, 37] and [38, Chapter 12] (it has been generalised by Philippon: [45, Theorem 6]).

Theorem 4.4 (Nishioka). *Let us assume that, in the notations previously introduced, $\mathbb{K} = \mathbb{C}$. Let L be a number field embedded in \mathbb{C} . Let f_1, \dots, f_m be formal series of $L[[x]]$, let us write $\underline{f} \in \mathbf{Mat}_{n \times 1}(L[[x]])$ for the column matrix whose entries are the f_i 's. Let $\mathcal{A} \in \mathbf{Mat}_{n \times n}(L(x))$, $\underline{b} \in \mathbf{Mat}_{n \times 1}(L(x))$ be matrices. Let us assume that:*

- (1) f_1, \dots, f_m are algebraically independent over $\mathbb{C}(x)$,
- (2) For all i , the formal series $f_i(x)$ converges for x complex such that $|x| < 1$,
- (3) $\underline{f}(x^d) = \mathcal{A}(x) \cdot \underline{f}(x) + \underline{b}(x)$.

Let $\alpha \in L$ be such that $0 < |\alpha| < 1$, not a zero or a pole of \mathcal{A} and not a pole of \underline{b} . Then, there exists a constant $c_1 > 0$ effectively computable depending on α, \underline{f} , with the following property.

For any $H, N \geq 1$ integers and any non-zero polynomial $P \in \mathbb{Z}[X_1, \dots, X_m]$ whose partial degrees in every indeterminate do not exceed N and whose coefficients are not greater than H in absolute value, the number $P(f_1(\alpha), \dots, f_m(\alpha))$ is non-zero and the inequality below holds:

$$\log |P(f_1(\alpha), \dots, f_m(\alpha))| \geq -c_1 N^m (\log H + N^{m+2}). \quad (33)$$

We sketch how Theorem 4.4 implies the algebraic independence of $f_1(\alpha), \dots, f_m(\alpha)$ and $g(\alpha)$ with $f_1, \dots, f_m, g \in \mathbb{Q}[[x]]$ algebraically independent over $\mathbb{Q}[[x]]$ satisfying linear functional equations as in Theorem 4.3 and g satisfying

$$g(x^d) = a(x)g(x) + b(x),$$

with $a, b \in \mathbb{Q}(x)$, $\mathcal{A}, \underline{b}$ with rational coefficients, and α not a pole of all these rational functions. Of course, this is a simple corollary of Theorem 4.4. However, we believe that the proof is instructive; it follows closely Philippon's ideas in [45]. The result is reached because the estimates of Theorem 4.4 are precise enough. In particular, the separation of the quantities H and N in (33) is crucial.

¹⁰P. Bundschuh pointed out that L_0, L_2, L_4, \dots are algebraically independent. To obtain this property, he studies the behaviour of these functions near the the unit circle.

4.2.1. Algebraic independence from measures of algebraic independence.. For the purpose indicated at the end of the last subsection, we assume that $\alpha \in \mathbb{Q}^\times$. This hypothesis is not strictly necessary and is assumed only to simplify the exposition of the proof; by the way, the reader will remark that several other hypotheses we assume are avoidable.

Step (AP). For all $N \geq 1$, we choose a non-zero polynomial $P_N \in \mathbb{Z}[x, X_1, \dots, X_m, Y]$ of partial degrees $\leq N$ in each indeterminate, such that, writing

$$F_N(x) := P_N(x, f_1(x), \dots, f_m(x), g(x)) = c_{\nu(N)} x^{\nu(N)} + \dots \in \mathbb{Q}[[x]], \quad c_{\nu(N)} \neq 0,$$

we have $\nu(N) \geq N^{m+2}$ (we have already justified why such a kind of polynomial exists).

Just as in the proof of Theorem 4.3 we construct, for each $N \geq 1$, a sequence of polynomials $(P_{N,k})_{k \geq 0}$ in $\mathbb{Z}[x, X_1, \dots, X_m, Y]$ recursively in the following way:

$$\begin{aligned} P_{N,0} &:= P_N, \\ P_{N,k} &:= c(x)^N P_{N,k-1}(x^d, a_1(x)X_1 + b_1(x), \dots, a_m(x)X_m + b_m(x), a(x)Y + b(x)), \end{aligned}$$

where $c(x) \in \mathbb{Z}[x] \setminus \{0\}$ is chosen so that ca_i, cb_j, ca, cb belong to $\mathbb{Z}[x]$. The following estimates are easily obtained:

$$\begin{aligned} \deg_Z P_{N,k} &\leq N, \quad \text{for } Z = X_1, \dots, X_m, Y, \\ \deg_x P_{N,k} &\leq c_2 d^k N, \\ h(P_{N,k}) &\leq c_3(N) + c_4 d^k N, \end{aligned}$$

where c_2, c_4 are positive real numbers effectively computable depending on α, \underline{f} and g , and $c_3(N) > 0$ depends on these data as well as on N (it depends on the choice of the polynomials P_N).

Let us assume by contradiction that $g(\alpha)$ is algebraic over the field

$$\mathcal{F} := \mathbb{Q}(f_1(\alpha), \dots, f_m(\alpha)),$$

of transcendence degree m over \mathbb{Q} . We observe that, after the identity principle of analytic functions we have, for k big enough depending on α, \underline{f}, g and N :

$$P_{N,k}(\alpha, f_1(\alpha), \dots, f_m(\alpha), g(\alpha)) \in \mathcal{F}^\times. \quad (34)$$

Let $\tilde{Q} \in \mathcal{F}[X] \setminus \{0\}$ be the minimal polynomial of $g(\alpha)$, algebraic over \mathcal{F} . We can write $\tilde{Q} = a_0 + a_1 X + \dots + a_{r-1} X^{r-1} + X^r$ with the a_i 's in \mathcal{F} . Multiplying by a common denominator, we obtain a non-zero polynomial $Q \in \mathbb{Z}[X_1, \dots, X_m, Y]$ such that $Q(f_1(\alpha), \dots, f_m(\alpha), g(\alpha)) = 0$, with the property that the polynomial $Q^* = Q(f_1(\alpha), \dots, f_m(\alpha), Y) \in \mathcal{F}[Y]$ is irreducible.

Step (NV). Let us denote by Δ_k the resultant $\mathbf{Res}_Y(P_{N,k}, Q) \in \mathbb{Z}[x, X_1, \dots, X_m]$. If $\delta_k := \Delta_k(\alpha, f_1(\alpha), \dots, f_m(\alpha)) \in \mathcal{F}$ vanishes for a certain k , then Q^* and

$$P_{N,k}^* := P_{N,k}(\alpha, f_1(\alpha), \dots, f_m(\alpha), Y) \in \mathcal{F}[Y]$$

have a common zero. Since Q^* is irreducible, we have that Q^* divides $P_{N,k}^*$ in $\mathcal{F}[Y]$ and

$$P_{N,k}(\alpha, f_1(\alpha), \dots, f_m(\alpha), g(\alpha)) = 0;$$

this cannot happen for k big enough by the identity principle of analytic functions (34) so that we can assume that for k big enough, $\delta_k \neq 0$, ensuring that Δ_k is not identically zero; the estimates

of the height and the degree of Δ_k quoted below are simple exercises and we do not give details of their proofs:

$$\begin{aligned} \deg_Z \Delta_k &\leq c_5 N, \quad Z = X_1, \dots, X_m, \\ \deg_x \Delta_k &\leq c_6 d^k N, \\ h(\Delta_k) &\leq c_7(N) + c_8 d^k N, \end{aligned}$$

where c_5, c_6, c_8 are positive numbers effectively computable depending on α, \underline{f} and g , while the constant $c_7(N)$ depends on these data and on N .

Let D be a non-zero positive integer such that $D\alpha \in \mathbb{Z}$. Then, writing

$$\Delta_k^* := D^{Nd^k} \Delta_k(\alpha, X_1, \dots, X_m),$$

we have $\Delta_k^* \in \mathbb{Z}[X_1, \dots, X_m] \setminus \{0\}$ and

$$\begin{aligned} \deg_{X_i} \Delta_k^* &\leq c_5 N, \\ h(\Delta_k^*) &\leq c_7(N) + c_8 d^k N. \end{aligned}$$

Step (LB). By Nishioka's Theorem 4.4, we have the inequality (for k big enough):

$$\log |\Delta_k^*(f_1(\alpha), \dots, f_m(\alpha))| \geq c_1 N^m (d^k N + c_9(N)), \quad (35)$$

where $c_9(N)$ is a constant depending on N .

To finish our proof, we need to find an upper bound contradictory with (35); it will be obtained by analytic estimates as usual.

Step (UB). Looking at the proof of Lemma 5.3.1 of [53], and using in particular inequality (1.2.7) of loc. cit., we verify the existence of constants c_{12}, c_{15} depending on α, \underline{f}, g , $c_{14}(\alpha, \epsilon)$ depending on α and ϵ , and $c_{13}(N)$ depending on α, \underline{f}, g and N , such that:

$$\begin{aligned} \log |\Delta_k^*(f_1(\alpha), \dots, f_m(\alpha))| &\leq \log(N + c_{10}) + c_{11} h(P_{N,k}^*) + (N + 1)h(Q) + \\ &\quad \log |c_{\nu_N}| + \nu(N) d^k \log |\alpha| + \epsilon \\ &\leq c_{12}(\log N + d^k N) + c_{13}(N) - c_{14}(\alpha, \epsilon) d^k \nu(N) \\ &\leq c_{15} d^k N + c_{13}(N) - c_{14}(\alpha, \epsilon) d^k N^{m+2}. \end{aligned} \quad (36)$$

Finally, it is easy to choose N big enough, depending on c_1, c_{15}, c_{14} but *not* on c_9, c_{13} so that, for k big enough, the estimates (35) and (36) are not compatible: this is due to the particular shape of (33), with the linear dependence in $\log H$.

4.2.2. Further remarks, comparisons with Nesterenko's Theorem.. In the sketch of proof of the previous subsection, the reader probably observed a kind of induction structure; a measure of algebraic independence for m numbers delivers algebraic independence for $m+1$ numbers. The question is then natural: *is it possible to obtain a measure of algebraic independence for $m+1$ numbers allowing continue the process and consider $m+2$ numbers?*

In fact *yes*, there always is an inductive structure of proof, but *no*, it is not just a measure for m numbers which alone implies a measure for $m+1$ numbers. Things are more difficult than they look at first sight and the inductive process one has to follow concerns other parameters as well. For instance, the reader can verify that it is unclear how to generalise the arguments of 4.2.1 and work directly with a polynomial Q which has a very small value at $\omega = (f_1(\alpha), \dots, f_m(\alpha), g(\alpha))$.

Algebraic independence theory usually appeals to *transfer techniques*, as an alternative to direct estimates at ω . A detour on a theorem of Nesterenko might be useful to understand what

is going on so our discussion now temporarily leaves Mahler's values, that will be reconsidered in a little while.

Precise multiplicity estimates in differential rings generated by Eisenstein's series obtained by Nesterenko, the criteria for algebraic independence by Philippon already mentioned in this paper and a trick of the stephanese team (cf. 2.2.1) allowed Nesterenko, in 1996, to prove the following theorem (see [33, 34]):

Theorem 4.5 (Nesterenko). *Let E_2, E_4, E_6 the classical Eisenstein's series of weights 2, 4, 6 respectively, normalised so that $\lim_{\Im(z) \rightarrow \infty} E_{2i}(z) = 1$ (for $\Re(z)$ bounded), let z be a complex number of strictly positive imaginary part. Then, three of the four complex numbers $e^{2\pi iz}, E_2, E_4, E_6$ are algebraically independent.*

Although Eisenstein's series are not directly related to Mahler's functions there is a hidden link and the ideas introduced to prove Theorem 4.5 influenced Nishioka in her proof of Theorem 33 as well as other results by Philippon that we will mention below. This is why we cannot keep silent on this aspect.

First of all, we recall that in [45], Philippon showed how to deduce the algebraic independence of $\pi, e^\pi, \Gamma(1/4)$ (¹¹), a well known corollary of Nesterenko's Theorem 4.5, from a measure of algebraic independence of $\pi, \Gamma(1/4)$ by Philibert in [42]. This implication was possible because Philibert's result was sharp enough. It is however virtually impossible to deduce Theorem 4.5 or the quantitative result in [33] which can also be deduced from corollary of [45, Theorem 3] (¹²) just by using Philibert's result.

In the proof of Theorem 33, Nishioka proves (just as Nesterenko does in [33]) a more general measure of the smallness of the values that a polynomial with rational integer coefficients assumes at $(1, f_1(\alpha), \dots, f_m(\alpha), g(\alpha))$, restricting the choice of that polynomial in a given unmixed homogeneous ideal I . The proof of such a kind of result (cf. [38, Lemma 2.3]) involves induction on the dimension of I .

Assuming the existence of an ideal I with minimal dimension with "very small" values at ω , it is possible, looking at its reduced primary decomposition, to concentrate our attention to $I = \mathfrak{p}$ prime. The "closest point principle" of [46, p. 89] allows to show the existence, in the projective variety V associated to \mathfrak{p} , of a point β which is at a very short distance from ω (see also [34, Proposition 1.5]). This shows that in this problem, to measure a polynomial or an ideal at ω it is more advantageous to do it at β ; indeed, all the polynomials of \mathfrak{p} vanish at β .

At this point, it remains to construct an unmixed homogeneous ideal J of dimension $\dim \mathfrak{p} - 1$, contradicting our assumptions. To do so, it is necessary to proceed as we did in 4.2.1 to construct $P_{N,k}$ etc., with the important difference that now, all the estimate depend very much on the choice, that must then use Siegel's Lemma. Another important tool that has to be used is a *multiplicity estimate*, that belongs to step (NV), proved by Nishioka [37, Theorem 4.3], that we reproduce here.

Theorem 4.6 (Nishioka). *Let f_1, \dots, f_m be satisfying the hypotheses of Theorem 4.4, so that for all $P \in \mathbb{C}[x, X_1, \dots, X_m] \setminus \{0\}$, the function $F(x) := P(x, f_1(x), \dots, f_m(x))$ has the expansion*

$$F(x) = c_\nu x^\nu + \dots,$$

¹¹Philippon's result is in fact more general than the algebraic independence of these three numbers, but less general than Nesterenko's theorem 4.5, although it historically followed it. Our arguments in 4.2.1 are strongly influenced by it. In [45], Philippon proposes alternative, simpler proofs for Nesterenko's theorem.

¹²A result asserting that, for a polynomial $P \in \mathbb{Z}[X_0, \dots, X_4] \setminus \{0\}$ whose partial degrees in every indeterminate do not exceed $N > 0$ and whose coefficients are not greater than $H > 0$ in absolute value, then $\log |P(e^\pi, \pi, \Gamma(1/4))| \geq -c_1(\epsilon)(N + \log H)^{4+\epsilon}$, where c_1 is an absolute constant depending on ϵ only. The dependence in ϵ is completely explicit.

with $c_\nu \neq 0$. There exists a constant $c_1 > 0$, depending on f_1, \dots, f_m only, with the following property. If P is as above and $N_1 := \max\{1, \deg_x P\}$ and $N_2 := \max\{1, \deg_{X_1} P, \dots, \deg_{X_m} P\}$, then

$$\nu \leq c_1 N_1 N_2^m.$$

This result is very similar to Nesterenko's multiplicity estimate [34, Chapter 10, Theorem 1.1] and again, its proof essentially follows Nesterenko's ideas.

The ideal J previously mentioned is defined as the ideal generated by \mathfrak{p} and a polynomial obtained from $P_{N,k}$ by homogenisation, substitution $x = \alpha$, and a good choice of N, k taking into account the magnitude of the coefficients of the series f_i, g . Indeed, one proves that such a polynomial cannot belong to \mathfrak{p} . The closest point principle is necessary in this kind of proof.

The arguments of the above discussion can be modified to obtain the analog of Theorem 4.5 for values of Mahler's functions at general complex numbers, obtained by Philippon (cf. [45, Theorem 4]). Here, L is again a number field embedded in \mathbb{C} and $d > 1$ is an integer.

Theorem 4.7 (Philippon). *Under the same hypotheses and notations of Theorem 4.4, if α is a complex number with $0 < |\alpha| < 1$, then, for n big enough, the complex numbers*

$$\alpha, f_1(\alpha^{d^n}), \dots, f_m(\alpha^{d^n}) \in \mathbb{C}$$

generate a subfield of \mathbb{C} of transcendence degree $\geq m$.

This result is a corollary of a more general quantitative result [45, Theorem 6] which follows from Philippon's algebraic independence criterion Philippon's criterion for measures of algebraic independence (loc. cit. p. 5).

4.2.3. Commentaries on the case of positive characteristic. Similar, although simpler arguments are in fact commonly used to obtain measures of transcendence. Several authors deduce them from measures of linear algebraic approximation; see for example Amou, Galochkin and Miller [3, 21, 32] ⁽¹³⁾. These results often imply that Mahler's values are Mahler's *S-numbers*.

In positive characteristic, it is well known that separability difficulties occur preventing to deduce good measures of transcendence from measures of linear algebraic approximation ⁽¹⁴⁾. In [20], Denis proves the following result, where $d > 1$ is an integer.

Theorem 4.8 (Denis). *Let us consider a finite extension \mathcal{L} of $K = \mathbb{F}_q(\theta)$, $f \in \mathcal{L}[[x]]$, $\alpha \in \mathcal{L}$ such that $0 < |\alpha| < 1$. Let us assume that f is transcendental over $C(x)$, convergent for $x \in \mathcal{C}$ such that $|x| < 1$, and satisfying the linear functional equation*

$$f(x^q) = a(x)f(x) + b(x),$$

with $a, b \in \mathcal{L}(x)$.

For all n big enough, we have the following property. Let $\beta = \alpha^{d^n}$. Then, there exists an effectively computable constant $c_1 > 0$ depending on β, f only, such that, given any non-constant polynomial $P \in A[X]$,

$$\log |P(f(\beta))| \geq -\deg_X(P)^4 (\deg_X(P) + \deg_\theta(P)). \quad (37)$$

¹³Some results hold for series which satisfy functional equations which are not necessarily linear.

¹⁴As first remarked by Lang, for complex numbers, there is equivalence between measures of transcendence and measures of linear algebraic approximation in the sense that, from a measure of linear algebraic approximation one can get a measure of transcendence and then again, a measure of linear algebraic approximation which is essentially that of the beginning, with a controllable degradation of the constants.

This result yields completely explicit measures of transcendence of $\tilde{\pi}$, of Carlitz's logarithms of elements of K , and of certain Carlitz-Goss's zeta values (see Section 4.3 for definitions).

To prove Theorem 4.8, Denis uses the following multiplicity estimate.

Theorem 4.9. *Let \mathcal{K} be any (commutative) field and $f \in \mathcal{K}((x))$ be transcendental satisfying the functional equation $f(x^d) = R(x, f(x))$ with $R \in \mathcal{K}(X, Y)$ and $h_Y(R) < d$. Then, if P is a polynomial in $\mathcal{K}[X, Y] \setminus \{0\}$ such that $\deg_X P \leq N$ and $\deg_Y P \leq M$, we have*

$$\text{ord}_{x=0} P(x, f(x)) \leq N(2Md + Nh_X(Q)).$$

It would be interesting to generalise such a multiplicity estimate for several algebraically independent formal series and obtain a variant of Töpfer's [52, Theorem 1] (see [11, 12, 41] to check the difficulty involved in the research of an analogue of Nesterenko's multiplicity estimate for Drinfeld quasi-modular forms). This could be helpful to obtain analogues of Theorem 4.4 for Mahler's values in fields of positive characteristic.

4.3. Algebraic independence of Carlitz's logarithms. For the rest of this chapter, we will give some application of Mahler's method and of Anderson-Brownawell-Papanikolas method to algebraic independence of Carlitz's logarithms of algebraic elements of C and of some special values of Carlitz-Goss zeta function at rational integers. Results of this part are not original since they are all contained in the papers [39] by Papanikolas and [14] by Chieh-Yu Chang and Jing Yu. But the methods we use here are slightly different and self-contained.

Both proofs of the main results in [39, 14] make use of a general statement [39, Theorem 5.2.2] which can be considered as a variant of Grothendieck period conjecture for a certain generalisation of Anderson's t -motives, also due Papanikolas. To apply this result, the computation of motivic Galois groups associated to certain t -motives is required.

Particular cases of these results are also contained in Denis work [19], where he applies Mahler's method and without appealing to Galois theory. Hence, we follow the ideas of the example in 4.1.1 and the main worry here is to develop analogous proofs in the Drinfeldian framework. In 4.1.1 the explicit computation of the transcendence degree of the field generated by L_0, L_1, \dots was pointed out as a problem. But we have already remarked there, that if L_0, L_1, \dots are algebraic dependent, then they also are \mathbb{C} -linearly dependent modulo $\mathbb{C}(x)$. This property, consequence of a result by Kubota, is easy to obtain because the matrix of the linear difference system of equations satisfied by the L_i 's has the matrix of its associated homogeneous system which is *diagonal*.

For $\Lambda = \tilde{\pi}A$ with $\tilde{\pi}$ as in (16), the exponential function $e_{\text{Car}} := e_\Lambda$ (30) can be explicitly written as follows:

$$e_{\text{Car}}(z) = \sum_{i \geq 0} \frac{z^{q^i}}{[i][i-1]^q \dots [1]^{q^{i-1}}},$$

where $[i] := \theta^{q^i} - \theta$ ($i \geq 1$). This series converges uniformly on every open ball with center in 0 to an \mathbb{F}_q -linear surjective function $e_{\text{Car}} : C \rightarrow C$. The formal series \log_{Car} , reciprocal of e_{Car} in 0, converges for $|z| < q^{q/(q-1)} = |\tilde{\pi}|$. Its series expansion can be computed explicitly:

$$\log_{\text{Car}}(z) = \sum_{i \geq 0} \frac{(-1)^i z^{q^i}}{[i][i-1] \dots [1]}.$$

The first Theorem we shall prove in a simpler way is the following (cf. [39, Theorem 1.2.6]):

Theorem 4.10 (Papanikolas). *Let $\ell_1, \dots, \ell_m \in C$ be such that $e_{\text{Car}}(\ell_i) \in K^{\text{alg}}$ ($i = 1, \dots, m$). If ℓ_1, \dots, ℓ_m are linearly independent over K , then they also are algebraically independent over K .*

4.3.1. Carlitz-Goss polylogarithms and zeta functions.. Let us write $A_+ = \{a \in A, a \text{ monic}\}$. In [22], Goss introduced a function ζ , defined over $C \times \mathbb{Z}_p$ with values in C , such that for $n \geq 1$ integer,

$$\zeta(\theta^n, n) = \sum_{a \in A_+} \frac{1}{a^n} \in K_\infty.$$

In the following, we will write $\zeta(n)$ for $\zeta(\theta^n, n)$. For $n \in \mathbb{N}$, let us also write $\Gamma(n) := \prod_{i=0}^s D_i^{n_i} \in K$, $n_0 + n_1q + \dots + n_sq^s$ being the expansion of $n - 1$ in base q and D_i being the polynomial $[i][i-1]^q \dots [1]^{q^{i-1}}$. It can be proved that $z/e_{\text{Car}}(z) = \sum_{n=0}^{\infty} B_n \frac{z^n}{\Gamma(n+1)}$ for certain $B_n \in K$. The so-called Bernoulli-Carlitz relations can be obtained by a computation involving the logarithmic derivative of $e_{\text{Car}}(z)$: for all $m \geq 1$,

$$\frac{\zeta(m(q-1))}{\tilde{\pi}^{m(q-1)}} = \frac{B_m}{\Gamma(m(q-1)+1)} \in K. \quad (38)$$

In particular, one sees that

$$\tilde{\pi}^{q-1} = (\theta^q - \theta)\zeta(q-1) \in K_\infty.$$

We also have the obvious relations:

$$\zeta(mp^k) = \zeta(m)^{p^k}, \quad m, k \geq 1. \quad (39)$$

The second theorem we are going to prove directly is:

Theorem 4.11 (Chang, Yu). *The algebraic dependence relations over K between the numbers*

$$\zeta(1), \zeta(2), \dots$$

are generated by Bernoulli-Carlitz's relations (38) and the relations (39).

4.3.2. Two propositions.. In this subsection we develop an analogue of [51, Lemma 6], for the same purpose we needed it in 4.1.1.

We consider here a perfect field U of characteristic $p > 0$ containing \mathbb{F}_q and a \mathbb{F}_q -automorphism $\tau : U \rightarrow U$. Let U_0 be the subfield of constants of τ , namely, the subset of U whose elements s are such that $\tau s = s$.

For example, we can consider $U = \bigcup_{n \geq 0} C(x^{1/p^n})$ with τ defined as the identity over C , with $\tau x = x^q$. Another choice is to consider $U = \bigcup_{n \geq 0} C(t^{1/p^n})$, with $\tau t = c^{1/q}$ for all $c \in C$ and $\tau t = t$. In the first example we have $U_0 = C$ while in the second, $U_0 = \bigcup_{n \geq 0} \mathbb{F}_q((t^{1/p^n}))$.

More generally, after 3.2, we can take either

$$U = \bigcup_{n \geq 0} \mathbb{K}(x^{1/p^n})$$

or the field

$$\bigcup_{n, m \geq 0} C(x^{1/p^n}, t^{1/p^m})$$

(which is contained in the previous field) with the corresponding automorphism τ (these settings will essentially include the two examples above). In the first case, we have $U_0 = \mathbb{F}_q\langle\langle t \rangle\rangle$, and in the second case, we have $U_0 = \bigcup_{n \geq 0} \mathbb{F}_q((t^{1/p^n}))$.

Let us also consider the ring $\mathcal{R} = U[X_1, \dots, X_N]$ and write, for a polynomial $P = \sum_{\underline{\lambda}} c_{\underline{\lambda}} X^{\underline{\lambda}} \in \mathcal{R}$, P^τ as the polynomial $\sum_{\underline{\lambda}} (\tau c_{\underline{\lambda}}) X^{\underline{\lambda}}$. Let D_1, \dots, D_N be elements of U^\times , B_1, \dots, B_N be elements of U and, for a polynomial $P \in \mathcal{R}$, let us write

$$\tilde{P} = P^\tau(D_1 X_1 + B_1, \dots, D_N X_N + B_N).$$

We prove the following two Propositions, which provide together the analogue in positive characteristic of Kubota [25, Theorem 2].

Proposition 4.12. *Let $P \in \mathcal{R}$ be a non-constant polynomial such that $\tilde{P}/P \in \mathcal{R}$. Then, there exists a polynomial $G \in \mathcal{R}$ of the form $G = \sum_i c_i X_i + B^p$ such that $\tilde{G}/G \in \mathcal{R}$, where $c_1, \dots, c_N \in U$ are not all vanishing and $B \in \mathcal{R}$. If W is the subfield generated by \mathbb{F}_q and the coefficients of P , then there exists $M \geq 1$ such that for each coefficient c of G , $c^{p^M} \in W$.*

Proof. If $P \in \mathcal{R}$ is such that $\tilde{P} = QP$ for $Q \in \mathcal{R}$ one sees, comparing the degrees of \tilde{P} and P , that $Q \in U$ and if P is non-zero, $Q \neq 0$. The subset of \mathcal{R} of these polynomials is a semigroup \mathcal{S} containing U . If $P \in \mathcal{S}$ satisfies $\tilde{P} = QP$, then $F := \partial P / \partial X_i$ belongs to \mathcal{S} since $\tilde{F} = D_i^{-1} Q F$. Similarly, if $P = F^p \in \mathcal{S}$ with $F \in \mathcal{S}$ then $F \in \mathcal{S}$ as one sees easily that in this case, $\tilde{F} = Q^{1/p} F$.

By hypothesis, \mathcal{S} contains a non-constant polynomial P . We now show that the polynomial $G \in \mathcal{S}$ as in the Proposition can be constructed by iterated applications of partial derivatives $\partial_1 = \partial / \partial X_1, \dots, \partial_N = \partial / \partial X_N$ and p -root extractions starting from P .

Let P be as in the hypotheses. We can assume that P is not a p -th power. We can write:

$$P = \sum_{\underline{\lambda}=(\lambda_1, \dots, \lambda_N) \in \{0, \dots, p-1\}^N} c_{\underline{\lambda}} X^{\underline{\lambda}}, \quad c_{\underline{\lambda}} \in \mathcal{R}^p.$$

Let $M := \max\{\lambda_1 + \dots + \lambda_N, c_{\underline{\lambda}} \neq 0\}$. We can write $P = P_1 + P_2$ with

$$P_1 := \sum_{\lambda_1 + \dots + \lambda_N = M} c_{\underline{\lambda}} X^{\underline{\lambda}}.$$

There exists $(\beta_1, \dots, \beta_N) \in \{0, \dots, p-1\}^N$ with $\beta_1 + \dots + \beta_N = M-1$ and

$$P' := \partial_1^{\beta_1} \dots \partial_N^{\beta_N} P = \sum_{i=1}^N c'_i X_i + c'_0 \in \mathcal{S} \setminus \{0\}, \quad c'_0, c'_1, \dots, c'_N \in \mathcal{R}^p,$$

where

$$\partial_1^{\beta_1} \dots \partial_N^{\beta_N} P_1 = \sum_{i=1}^N c'_i X_i, \quad \partial_1^{\beta_1} \dots \partial_N^{\beta_N} P_2 = c'_0.$$

If (case 1) the polynomials c'_1, \dots, c'_N are all in U , then we are done. Otherwise, (case 2), there exists i such that c'_i is non-constant (its degree in X_j is then $\geq p$ for some j). Now, $c'_i = \partial_i P'$ belongs to $(\mathcal{R}^p \cap \mathcal{S}) \setminus \{0\}$ and there exists $s > 0$ with $c'_i = P''^{p^s}$ with $P'' \in \mathcal{S}$ which is not a p -th power. We have constructed an element P'' of \mathcal{S} which is not a p -th power, whose degrees in X_j are all strictly smaller than those of P for all j (if the polynomial depends on X_j).

We can repeat this process with P'' at the place of P and so on. Since at each stage we get a polynomial P'' with partial degrees in the X_j strictly smaller than those of P for all j (if P'' depends on X_j), we eventually terminate with a polynomial P which has all the partial degrees $< p$ in the indeterminates on which it depends, for which the case 1 holds.

As for the statement on the field W , we remark that we have applied to P an algorithm which constructs G from P applying finitely many partial derivatives and p -th roots extractions

successively, the only operations bringing out of the field W being p -root extractions. Hence, the existence of the integer M is guaranteed. \square

We recall that U_0 is the subfield of U whose elements are the $s \in U$ such that $\tau s = s$. Let V be a subgroup of U^\times such that $V \setminus V^p \neq \emptyset$.

Proposition 4.13. *Under the hypotheses of Proposition 4.12, let us assume that for all $D \in V \setminus \{1\}$, the only solution $s \in U$ of $\tau s = Ds$ is zero and that $D_1, \dots, D_N \in V \setminus V^p$. Then, the polynomial $G \in \mathcal{R}$ given by this Proposition is of the form $G = \sum_i c_i X_i + c_0$ with $c_1, \dots, c_N \in U_0$ and $c_0 \in U$. Moreover, if $c_i, c_j \neq 0$ for $1 \leq i < j \leq N$, then $D_i = D_j$. Let \mathcal{I} be the non-empty subset of $\{1, \dots, N\}$ whose elements i are such that $c_i \neq 0$, let $D_i = D$ for all $i \in \mathcal{I}$. Then,*

$$c_0 = \frac{\tau(c_0)}{D} + \frac{1}{D} \sum_{i \in \mathcal{I}} c_i B_i.$$

Proof. Proposition 4.12 gives us a polynomial G with $\tilde{G}/G \in \mathcal{R}$, of the form $\sum_i c_i X_i + B^p$ with $c_i \in U$ not all vanishing and $B \in \mathcal{R}$. Let $sX^{p\Delta}$ be a monomial of maximal degree in B^p . Since $\tilde{G} = QG$ with $Q \in U^\times$, we have $\tau s = (D_1^{\lambda_1} \cdots D_N^{\lambda_N})^{-p} Qs$. Moreover, $\tau(c_i) = D_i^{-1} Qc_i$ for all i . Hence, if i is such that $c_i \neq 0$, $r := s/c_i$ satisfies $\tau r = D_i(D_1^{\lambda_1} \cdots D_N^{\lambda_N})^{-p} r$. Now, $D_i(D_1^{\lambda_1} \cdots D_N^{\lambda_N})^{-p} \neq 1$ (because $D_i \in V \setminus V^p$) and $r = 0$, that is $s = 0$. This shows that $B \in U$. Let us suppose that $1 \leq i, j \leq N$ are such that $i \neq j$ and $c_i, c_j \neq 0$. Let us write $r = c_i/c_j$; we have $\tau r = D_j/D_i r$, from which we deduce $r \in U_0$ in case $D_j/D_i = 1$ and $r = 0$ otherwise. The Proposition is proved dividing $\sum_i c_i X_i + B^p$ by c_j with $j \neq 0$ and by considering the relation $\tilde{P} = QP$, once observed that $Q = D$. \square

We proceed, in the next two subsections, to prove Theorems 4.10 and 4.11. We will prove the first theorem applying Propositions 4.12 and 4.13 to the field $U = \bigcup_{n \geq 0} C(t^{1/p^n})$ and then by using the criterion [6, Theorem 1.3.2] and we will prove the second theorem applying these propositions to the field $U = \bigcup_{n \geq 0} K^{\text{alg.}}(x^{1/p^n})$ and then by using Theorem 4.3.

4.3.3. Direct proof of Theorem 4.10.. For $\beta \in K^{\text{alg.}}$ such that $|\beta| < q^{q/(q-1)}$, we will use the formal series in $K^{\text{alg.}}((t))$

$$L_\beta(t) = \beta + \sum_{i=1}^{\infty} \frac{(-1)^i \beta^{q^i}}{(\theta^q - t) \cdots (\theta^{q^i} - t)},$$

defining holomorphic functions for $|t| < q^q$ with $L_\beta(\theta) = \log_{\text{Car}} \beta$ ⁽¹⁵⁾.

We denote by W one of the following fields: $K^{\text{alg.}}, K_\infty^{\text{alg.}}, C$. For $f = \sum_i c_i t^i \in W((t))$ and $n \in \mathbb{Z}$ we write $f^{(n)} := \sum_i c_i^{q^n} t^i \in W((t))$, so that $f^{(-1)} = \sum_i c_i^{1/q} t^i$. We have the functional equation $L_\beta^{(-1)}(t) = \beta^{1/q} + \frac{L_\beta(t)}{t-\theta}$. The function L_β allows meromorphic continuation to the whole C , with simple poles at the points $\theta^q, \theta^{q^2}, \dots, \theta^{q^n}, \dots$ of residue

$$(\log_{\text{Car}} \beta)^q, \frac{(\log_{\text{Car}} \beta)^{q^2}}{D_1^q}, \dots, \frac{(\log_{\text{Car}} \beta)^{q^n}}{D_{n-1}^q}, \dots \quad (40)$$

¹⁵Papanikolas uses these series in [39]. It is also possible to work with the series $\sum_{i=0}^{\infty} e_{\text{Car}}((\log_{\text{Car}} \beta)/\theta^{i+1})t^i$.

Let β_1, \dots, β_m be algebraic numbers with $|\beta| < q^{q/(q-1)}$, let us write $L_i = L_{\beta_i}$ for $i = 1, \dots, m$. Let us also consider the infinite product Ω in (27), converging everywhere to an entire holomorphic function with zeros at $\theta^q, \theta^{q^2}, \dots$, and write $L_0 = -\Omega^{-1}$, which satisfies the functional equation

$$L_0^{(-1)}(t) = \frac{L_0(t)}{t - \theta},$$

with $L_0(\theta) = \tilde{\pi}$, meromorphic with simple poles at the points $\theta^q, \theta^{q^2}, \dots, \theta^{q^n}, \dots$, with residues

$$\tilde{\pi}^q, \frac{\tilde{\pi}^{q^2}}{D_1^q}, \dots, \frac{\tilde{\pi}^{q^n}}{D_{n-1}^q}, \dots \quad (41)$$

We now prove the following Proposition.

Proposition 4.14. *If the functions L_0, L_1, \dots, L_m are algebraically dependent over $K^{\text{alg}}(t)$, then $\tilde{\pi}, \log_{\text{Car}} \beta_1, \dots, \log_{\text{Car}} \beta_m$ are linearly dependent over K .*

Proof. The functions L_i are transcendental, since they have infinitely many poles. Without loss of generality, we may assume that $m \geq 1$ is minimal so that for all $0 \leq n \leq m$ the functions obtained from the family (L_0, L_1, \dots, L_m) discarding L_n are algebraically independent over $K^{\text{alg}}(t)$.

We now apply Propositions 4.12 and 4.13. We take $U := \bigcup_{n \geq 0} C(t^{1/p^n})$, which is perfect, and $\tau : U \rightarrow U$ the q -th root map on C (inverse of the Frobenius map), such that $\tau(t) = t$; this is an \mathbb{F}_q -automorphism. Moreover, we take $N = m + 1$, $D_1 = \dots = D_N = (t - \theta)^{-1}$,

$$(B_1, \dots, B_N) = (0, \beta_1^{1/q}, \dots, \beta_m^{1/q}),$$

and $V = (t - \theta)^{\mathbb{Z}}$.

Let $\mathcal{T} \subset C[[t]]$ be the subring of formal series converging for all $t \in C$ with $|t| \leq 1$, let \mathcal{L} be its fraction field. Let $f \in \mathcal{L}$ be non-zero. A variant of Weierstrass preparation theorem (see [4, Lemma 2.9.1]) yields a unique factorisation:

$$f = \lambda \left(\prod_{|a|_\infty \leq 1} (t - a)^{\text{ord}_a(f)} \right) \left(1 + \sum_{i=1}^{\infty} b_i t^i \right), \quad (42)$$

where $0 \neq \lambda \in C$, $\sup_i |b_i| < 1$, and $|b_i| \rightarrow 0$, the product being over a finite index set. Taking into account (42), it is a little exercise to show that $U_0 = \bigcup_{i \geq 0} \mathbb{F}_q(t^{1/p^i})$ and that for $D \in V \setminus \{1\}$, the solutions in U of $f^{(-1)} = Df$ are identically zero (for this last statement, use the transcendence over U of Ω).

Let $P \in \mathcal{R}$ be an irreducible polynomial such that $P(L_0, L_1, \dots, L_m) = 0$; we clearly have $\tilde{P} = QP$ with $Q \in U$ and Propositions 4.12 and 4.13 apply to give $c_1(t), \dots, c_m(t) \in U_0$ not all zero and $c(t) \in U$ such that

$$c(t) = (t - \theta)c^{(-1)}(t) + (t - \theta) \sum_{i=1}^m c_i(t) \beta_i^{1/q}. \quad (43)$$

We get, for all $k \geq 0$:

$$\begin{aligned} c(t) &= - \sum_{i=1}^m c_i(t) \left(\beta_i + \sum_{h=1}^k \frac{(-1)^h \beta_i^{q^h}}{(\theta^q - t)(\theta^{q^2} - t) \dots (\theta^{q^h} - t)} \right) \\ &\quad + \frac{c^{(k+1)}(t)}{(\theta^q - t)(\theta^{q^2} - t) \dots (\theta^{q^{k+1}} - t)}. \end{aligned} \quad (44)$$

We endow \mathcal{L} with a norm $\|\cdot\|$ in the following way: if $f \in \mathcal{L}^\times$ factorises as in (42), then $\|f\| := |\lambda|$. Let g be a positive integer. Then $\|\cdot\|$ extends in a unique way to the subfield $\mathcal{L}_g := \{f : f^{p^g} \in \mathcal{L}\}$. If $(f_i)_{i \in \mathbb{N}}$ is a uniformly convergent sequence in \mathcal{L}_g (on a certain closed ball centered at 0) such that $\|f_i\| \rightarrow 0$, then $f_i \rightarrow 0$ uniformly.

We observe that there exists $g \geq 0$ such that $c(t), c_1(t), \dots, c_m(t) \in \mathcal{L}_g$. Hence $c_1(t), \dots, c_m(t) \in \mathbb{F}_q(t^{1/p^g})$ and $\|c_i\| = 1$ if $c_i \neq 0$. This implies that

$$\left\| \sum_{i=1}^m c_i \beta_i^{1/q} \right\| \leq \max_i \{|\beta_i^{1/q}|\} < q^{1/(q-1)}.$$

By (43), $\|c\| \leq q^{q/(q-1)}$. Indeed, two cases occur. The first case when $\|c^{(-1)}\| \leq \max_i \{|\beta_i^{1/q}|\}$; here we have $\|c\| < q^{q/(q-1)}$ because $\|c^{(-1)}\| = \|c\|^{1/q}$ by (42) and $\max_i \{|\beta_i|\} < q^{q/(q-1)}$ by hypothesis. The second case occurs when the inequality $\|c^{(-1)}\| > \max\{|\beta_1^{1/q}|, \dots, |\beta_m^{1/q}|\}$ holds. In this case, $\max\{\|c^{(-1)}(t - \theta)\|, \|(t - \theta) \sum_{i=1}^m c_i(t) \beta_i^{1/q}\|\} = \|c^{(-1)}(t - \theta)\|$ which yields $\|c\| = q^{q/(q-1)}$ by (43).

Going back to (44) we see that the sequence of functions

$$E_h(t) = \frac{c^{(h+1)}(t)}{(\theta^q - t)(\theta^{q^2} - t) \dots (\theta^{q^{h+1}} - t)}$$

converges uniformly in every closed ball included in $\{t : |t| < q^q\}$, as the series defining the functions L_i ($i = 1, \dots, m$) do. We want to compute the limit of this sequence: we have two cases.

First case. If $\|c\| < q^{q/(q-1)}$, then, there exists $\epsilon > 0$ such that $\|c\| = q^{(q-\epsilon)/(q-1)}$. Then, for all $h \geq 0$, $\|c^{(h+1)}\| = \|c\|^{q^{h+1}} = q^{(q^{h+2} - \epsilon q^{h+1})/(q-1)}$. On the other side:

$$\begin{aligned} \|(\theta^q - t)(\theta^{q^2} - t) \dots (\theta^{q^{h+1}} - t)\| &= |\theta|^{q + \dots + q^{h+1}} \\ &= q^{q(q^{h+1} - 1)/(q-1)}. \end{aligned}$$

Hence,

$$\|E_h\| = q^{\frac{q^{h+2} - \epsilon q^{h+1}}{q-1} - \frac{q^{h+2} - q}{q-1}} = q^{\frac{q - \epsilon q^{h+1}}{q-1}} \rightarrow 0,$$

which implies $E_h \rightarrow 0$ (uniformly on every ball as above).

This means that $\sum_{i=1}^m c_i(t)L_i(t) + c(t) = 0$. Let g be minimal such that there exists a non-trivial linear relation as above, with $c_1, \dots, c_m \in U_0 \cap \mathcal{L}_g$; we claim that $g = 0$. Indeed, if $g > 0$, $c_1, \dots, c_m \notin \mathbb{F}_q$ and there exists a non-trivial relation $\sum_{i=1}^m d_i(t)L_i(t)^{p^g} + d(t) = 0$ with $d_1, \dots, d_m \in \mathbb{F}_q[t]$ not all zero, $d(t) \in C(t)$ and $\max_i \{\deg_t d_i\}$ minimal, non-zero. But letting the operator d/dt act on this relation we get a non-trivial relation with strictly lower degree because $dF^p/dt = 0$, leading to a contradiction.

Hence, $g = 0$ and $c_1, \dots, c_m \in \mathbb{F}_q(t)$. This also implies that $c \in C$; multiplying by a common denominator, we get a non-trivial relation $\sum_{i=1}^m c_i(t)L_i(t) + c(t) = 0$ with $c_1, \dots, c_m \in \mathbb{F}_q[t]$ and $c \in C(t)$. The function c being algebraic, it has finitely many poles. This means that

$$\sum_{i=1}^m c_i(t)L_i(t)$$

has finitely many poles but for all i , L_i has poles at $\theta^q, \theta^{q^2}, \dots$ with residues as in (40), which implies that $\sum_{i=1}^m c_i(t)L_i(t)$ has poles in $\theta^q, \theta^{q^2}, \dots$. Since the functions c_i belong to $\mathbb{F}_q[t]$,

they vanish only at points of absolute value 1, and the residues of the poles are multiples of $\sum_{i=0}^m c_i(\theta)^{q^k} (\log_{\text{Car}} \beta_i)^{q^k}$ ($k \geq 1$) by non-zero factors in A . They all must vanish: this happens if and only if

$$\sum_{i=1}^m c_i(\theta) \log_{\text{Car}} \beta_i = 0,$$

where we also observe that $c_i(\theta) \in K$; the Proposition follows in this case.

Second case. Here we know that the sequence E_h converges, but not to 0 and we must compute its limit. Let ν be in C with $|\nu| = 1$. Then, there exists $\mu \in \mathbb{F}_q^{\text{alg.}\times}$, unique such that $|\nu - \mu| < 1$. Hence, if $\lambda \in C$ is such that $|\lambda| = q^{q/(q-1)}$, there exists $\mu \in \mathbb{F}_q^{\text{alg.}\times}$ unique with

$$|\lambda - \mu(-\theta)^{q/(q-1)}| < q^{q/(q-1)}. \quad (45)$$

We have:

$$c(t) = \lambda \prod_{|a| \leq 1} (t^{1/p^g} - a)^{\text{ord}_a c} \left(1 + \sum_{i \geq 1} b_i t^{i/p^g} \right),$$

with $\lambda \in C^\times$, the product being finite and $|b_i| < 1$ for all i so that $\|c\| = |\lambda|$.

Let $\mu \in \mathbb{F}_q^{\text{alg.}\times}$ be such that (45) holds, and write:

$$\begin{aligned} c_1(t) &= (\lambda - \mu(-\theta)^{q/(q-1)}) \prod_{|a| \leq 1} (t^{1/p^g} - a)^{\text{ord}_a c} \left(1 + \sum_{i \geq 1} b_i t^{i/p^g} \right), \\ c_2(t) &= \mu(-\theta)^{q/(q-1)} \prod_{|a| \leq 1} (t^{1/p^g} - a)^{\text{ord}_a c} \left(1 + \sum_{i \geq 1} b_i t^{i/p^g} \right), \end{aligned} \quad (46)$$

so that $c(t) = c_1(t) + c_2(t)$, $\|c_1\| < q^{q/(q-1)}$ and $\|c_2\| = q^{q/(q-1)}$. For all h , we also write:

$$E_{1,h}(t) = \frac{c_1^{(h+1)}(t)}{(\theta^q - t)(\theta^{q^2} - t) \cdots (\theta^{q^{h+1}} - t)}, \quad E_{2,h}(t) = \frac{c_2^{(h+1)}(t)}{(\theta^q - t)(\theta^{q^2} - t) \cdots (\theta^{q^{h+1}} - t)}.$$

Following the first case, we easily check that $E_{1,h}(t) \rightarrow 0$ on every closed ball of center 0 included in $\{t : |t| < q^q\}$. It remains to compute the limit of $E_{2,h}(t)$.

We look at the asymptotic behaviour of the images of the factors in (46) under the operators $f \mapsto f^{(n)}$, $n \rightarrow \infty$. The sequence of functions $(1 + \sum_{i \geq 1} b_i t^{i/p^g})^{(n)}$ converges to 1 for $n \rightarrow \infty$ uniformly on every closed ball as above. Let \mathcal{E} be the finite set of the a 's involved in the finite product (46), take $a \in \mathcal{E}$. If $|a| < 1$, then $a^{(n)} \rightarrow 0$ and $(t^{1/p^g} - a)^{(n)} \rightarrow t^{1/p^g}$. If $|a| = 1$, there exists $\mu_a \in \mathbb{F}_q^{\text{alg.}\times}$ such that $|a - \mu_a| < 1$ and we can find $n_a > 0$ integer such that $\lim_{s \rightarrow \infty} a^{(sn_a)} = \mu_a$, whence $\lim_{s \rightarrow \infty} (t^{1/p^g} - a)^{(sn_a)} = t^{1/p^g} - \mu_a$.

Let us also denote by $\tilde{n} > 0$ the smallest positive integer such that $\mu^{\tilde{n}} = \mu$. Let N be the lowest common multiple of \tilde{n} and the n_a 's with a varying in \mathcal{E} . Then the sequence of functions:

$$\left(\left(\prod_{|a| \leq 1} (t^{1/p^g} - a)^{\text{ord}_a c} \right) \left(1 + \sum_{i \geq 1} b_i t^{i/p^g} \right) \right)^{(Ns)}, \quad s \in \mathbb{N}$$

converges to a non-zero element $Z \in \mathbb{F}_q^{\text{alg.}}(t^{1/p^g})$.

For $n \in \mathbb{N}$, let us write:

$$V_n(t) := \mu^{q^n} \frac{(-\theta)^{q^{n+1}/(q-1)}}{(\theta^q - t)(\theta^{q^2} - t) \cdots (\theta^{q^{n+1}} - t)}.$$

We have:

$$\begin{aligned} (-\theta)^{q/(q-1)} \prod_{i=1}^{n+1} \left(1 - \frac{t}{\theta^{q^i}}\right)^{-1} &= (-1)^{q/(q-1)} \theta^{q/(q-1)} \theta^{(q+\cdots+q^{n+1})} \prod_{i=1}^{n+1} (\theta^{q^i} - t)^{-1} \\ &= (-1)^{q/(q-1)} \theta^{q^{n+2}/(q-1)} \prod_{i=1}^{n+1} (\theta^{q^i} - t)^{-1}. \end{aligned}$$

Hence, $\lim_{n \rightarrow \infty} \theta^{q/(q-1)} / ((\theta^q - t)(\theta^{q^2} - t) \cdots (\theta^{q^{n+1}} - t))^{-1} = \Omega(t)^{-1}$ from which we deduce that $\lim_{s \rightarrow \infty} E_{2,sN}(t) = c_0(t)L_0(t)$ with $c_0 \in \mathbb{F}_q^{\text{alg.}\times}(t^{1/p^g})$. We have proved that for some $c_1, \dots, c_m \in \mathbb{F}_q(t^{1/p^g})$, $c_0 \in \mathbb{F}_q^{\times}(t^{1/p^g})$ and $c \in K^{\text{alg.}}(t^{1/p^g})$, $\sum_{i=0}^m c_i L_i + c = 0$. Applying the same tool used in the first case we can further prove that in fact, $g = 0$. If c_0 is not defined over \mathbb{F}_q , then applying the operator $f \mapsto f^{(-1)}$ we get another non-trivial relation $c'_0 + \sum_{i=1}^m c_i L_i = c'$ with $c' \in K^{\text{alg.}}(t)$ and $c'_0 \in \mathbb{F}_q^{\times}(t)$ not equal to c_0 ; subtracting it from the former relation yields $L_0 \in K^{\text{alg.}}(t)$ which is impossible since Ω is transcendental over $C(t)$. Hence c_0 belongs to $\mathbb{F}_q(t)$ too. Multiplying by a common denominator in $\mathbb{F}_q[t]$ and applying arguments of the first case again (by using the explicit computation of the residues of the poles of L_0 at $\theta^q, \theta^{q^2}, \dots$), we find a non-trivial relation $c_0(\theta)\tilde{\pi} + \sum_{i=1}^m c_i(\theta) \log_{\text{Car}} \beta_i = 0$. \square

Proof of Theorem 4.10. If $\ell \in C$ is such that $e_{\text{Car}}(\ell) \in K^{\text{alg.}}$, then there exist $a, b \in A$, $\beta \in K^{\text{alg.}}$ with $|\beta| < q^{q/(q-1)}$ such that $\ell = a \log_{\text{Car}} \beta + b\tilde{\pi}$. This well known property (also used in [39], see Lemma 7.4.1), together with Theorem 3.1.1 of [6], implies Theorem 4.10. \square

4.3.4. Direct proof of Theorem 4.11.. Let $s \geq 1$ be an integer and let Li_s denote the s -th Carlitz's *polylogarithm* by:

$$\text{Li}_s(z) = \sum_{k=0}^{\infty} \frac{(-1)^{ks} z^{q^k}}{([k][k-1] \cdots [1])^s},$$

so that $\text{Li}_1(z) = \log_{\text{Car}}(z)$ (the series $\text{Li}_s(z)$ converges for $|z| < q^{sq/(q-1)}$).

For $\beta \in K^{\text{alg.}} \cap K_{\infty}$ such that $|\beta| < q^{sq/(q-1)}$ (a discussion about this hypothesis follows in 4.3.5), we will use as in [19] the series

$$F_{s,\beta}(x) = \tilde{\beta}(x) + \sum_{i=1}^{\infty} \frac{(-1)^{is} \tilde{\beta}(x)^{q^i}}{(x^q - \theta)^s \cdots (x^{q^i} - \theta)^s},$$

where $\tilde{\beta}(x)$ is the formal series in $\mathbb{F}_q((1/x))$ obtained from the formal series of $\beta \in \mathbb{F}_q((1/\theta))$ by replacing θ with x , an independent indeterminate.

Let us assume that $x \in C$, with $|x| > 1$. We have, for i big enough,

$$\left| \frac{\tilde{\beta}(x)^{q^i}}{(x^q - \theta)^s \cdots (x^{q^i} - \theta)^s} \right| = |x|^{q^i \deg_{\theta} \beta - q \frac{q^i - 1}{q-1}},$$

so that the series $F_{s,\beta}(x)$ converges for $|x| > 1$ provided that $|\beta| < q^{\frac{q}{q-1}}$ and x is not of the form θ^{1/q^i} .

We have the functional equations:

$$F_{s,\beta}(x^q) = (x^q - \theta)^s (F_{s,\beta}(x) - \tilde{\beta}(x)),$$

moreover,

$$F_{s,\beta}(\theta) = \beta_j + \sum_{i=1}^{\infty} \frac{(-1)^i \beta^{q^i}}{(\theta^q - \theta)^s \cdots (\theta^{q^i} - \theta)^s} = \text{Li}_s(\beta).$$

Therefore, these series define holomorphic functions for $|x| > q^{1/q}$ and allow meromorphic continuations to the open set $\{x \in C, |x| > 1\}$, with poles at the points θ^{1/q^i} . We have “deformed” certain Carlitz's logarithms and got in this way Mahler's functions (except that the open unit disk is replaced with the complementary of the closed unit disk, but changing x to x^{-1} allows us to work in the neighbourhood of the origin).

Let \mathcal{J} be a finite non-empty subset of $\{1, 2, \dots\}$ such that if $n \in \mathcal{J}$, p does not divide n . Let us consider, for all $s \in \mathcal{J}$, an integer $l_s \geq 1$ and elements $\beta_{s,1}, \dots, \beta_{s,l_s} \in K^{\text{alg}} \cap K_{\infty}$ with $|\beta_{s,i}| < q^{qs/(q-1)}$ ($i = 1, \dots, l_s$). We remark that if s is divisible by $q-1$ then, for all $r > q^{1/q}$ the product:

$$(-x)^{sq/(q-1)} \prod_{i=1}^{\infty} \left(1 - \frac{\theta}{x^{q^i}}\right)^{-s}$$

converges uniformly in the region $\{x \in C, |x| \geq r\}$ to a holomorphic function $F_{s,0}(x)$, which is the $(q-1)$ -th power of a formal series in $K((1/(-x)^{1/(q-1)}))$, hence in $K((1/x))$ (compare with the function of 3). Moreover, $F_{s,0}(\theta) = \tilde{\pi}^s$.

Proposition 4.15. *If the functions $(F_{s,\beta_{s,1}}, \dots, F_{s,\beta_{s,l_s}})_{s \in \mathcal{J}}$ are algebraically dependent over $K^{\text{alg}}(x)$, there exists $s \in \mathcal{J}$ and a non-trivial relation*

$$\sum_{i=1}^{l_s} c_i F_{s,\beta_{s,i}}(x) = f(x) \in K^{\text{alg}}(x)$$

with $c_1, \dots, c_{l_s} \in K^{\text{alg}}$ if $q-1$ does not divide s , or a non-trivial relation:

$$\sum_{i=1}^{l_s} c_i F_{s,\beta_{s,i}}(x) + \lambda F_{s,0}(x) = f(x) \in K^{\text{alg}}(x)$$

with $c_1, \dots, c_{l_s}, \lambda \in K^{\text{alg}}$ if $q-1$ divides s . In both cases, non-trivial relations can be found with $c_1, \dots, c_{l_s}, \lambda \in A$.

Proof. Without loss of generality, we may assume that \mathcal{J} is minimal so that for all $n \in \mathcal{J}$ and $i \in \{1, \dots, l_n\}$ the functions obtained from the family $(F_{s,\beta_{s,1}}, \dots, F_{s,\beta_{s,l_s}})_{s \in \mathcal{J}}$ discarding $F_{n,\beta_{n,1}}$ are algebraically independent over $K^{\text{alg}}(x)$.

We want to apply Propositions 4.12 and 4.13. We take $U := \bigcup_{n \geq 0} K^{\text{alg}}(x^{1/p^n})$, and $\tau : U \rightarrow U$ the identity map on K^{alg} extended to U so that $\tau(x) = x^q$. We also take:

$$\begin{aligned} (X_1, \dots, X_N) &= (Y_{s,1}, \dots, Y_{s,l_s})_{s \in \mathcal{J}}, \\ (D_1, \dots, D_N) &= \underbrace{((x^q - \theta)^s, \dots, (x^q - \theta)^s)}_{l_s \text{ times}}_{s \in \mathcal{J}}, \\ (B_1, \dots, B_N) &= (\widetilde{\beta_{s,1}}(x), \dots, \widetilde{\beta_{s,l_s}}(x))_{s \in \mathcal{J}}. \end{aligned}$$

We take $V = (x^q - \theta)^{\mathbb{Z}}$. We have $U_0 = K^{\text{alg}}$ and for $D \in V \setminus \{1\}$, the solutions of $f(x^q) = Df(x)$ are identically zero as one sees easily writing down a formal power series for a solution $f \in U$.

Let $P \in \mathcal{R}$ be an irreducible polynomial such that $P((F_{s,\beta_{s,1}}, \dots, F_{s,\beta_{s,l_s}})_{s \in \mathcal{J}}) = 0$; we clearly have $\tilde{P} = QP$ with $Q \in U$ and Propositions 4.12 and 4.13 apply. They give $s \in \mathcal{J}$, $c_1, \dots, c_{l_s} \in K^{\text{alg}}$ not all zero and $c_0 \in U$ such that

$$c_0(x) = \frac{c_0(x^q)}{(x^q - \theta)^s} - \frac{1}{(x^q - \theta)^s} \sum_{i=1}^{l_s} c_i \widetilde{\beta_{s,i}}(x). \quad (47)$$

We now inspect this relation in more detail. To ease the notations, we write $l_s = m$ and $F_i(x) := F_{s,\beta_{s,i}}$ ($i = 1, \dots, m$). Since $\tilde{\beta}(x)^q = \tilde{\beta}(x^q)$ for all $\beta \in K$, from (47) we get, for all $k \geq 0$:

$$\begin{aligned} c_0(x) &= - \sum_{i=1}^m c_i \left(\tilde{\beta}_i(x) + \sum_{h=1}^k \frac{(-1)^{hs} \tilde{\beta}_i(x)^{q^h}}{((x^q - \theta)(x^{q^2} - \theta) \dots (x^{q^{h+1}} - \theta))^s} \right) \\ &\quad + \frac{c_0(x^{q^{k+1}})}{((x^q - \theta)(x^{q^2} - \theta) \dots (x^{q^{k+1}} - \theta))^s}. \end{aligned} \quad (48)$$

By Proposition 4.12, there exists $M > 0$ such that $c_0(x)^{q^M} \in K^{\text{alg}}(x)$, which implies that $c_0(x^{q^M}) \in K^{\text{alg}}(x)$. By equation (48) we see that $c_0(x) \in K^{\text{alg}}(x)$.

We write $c_0(x) = \sum_{i \geq i_0} d_i x^{-i}$ with $d_i \in K^{\text{alg}}$. The sequence $(|d_i|)_i$ is bounded; let κ be an upper bound. If $x \in C$ is such that $|x| \geq r > q^{1/q}$ with r independent on x , then $|c_0(x)| = \sup_i |d_i| |x|^{-i} \leq \kappa \sup_i |x|^{-i} \leq \kappa |x|^{\deg_x c_0}$. Moreover, for $|x| > r$ as above, $|x|^{q^s} > |\theta| = q$ for all $s \geq 1$ so that $|x^{q^s} - \theta| = \max\{|x|^{q^s}, |\theta|\} = |x|^{q^s}$. Hence we get:

$$|(x^q - \theta)(x^{q^2} - \theta) \dots (x^{q^{k+1}} - \theta)| = |x|^{q+q^2+\dots+q^{k+1}} = |x|^{\frac{q(q^{k+1}-1)}{q-1}}.$$

Let us write:

$$R_k(x) := \frac{c_0(x^{q^{k+1}})}{((x^q - \theta)(x^{q^2} - \theta) \dots (x^{q^{k+1}} - \theta))^s}.$$

We have, for $|x| \geq r > q^{1/q}$ and for all k :

$$|R_k(x)| \leq \kappa |x|^{q^{k+1} \deg_x c_0 - \frac{sq(q^{k+1}-1)}{q-1}}. \quad (49)$$

Since $|\beta_i| < q^{sq/(q-1)}$, $\deg_x \tilde{\beta}_i(x) < sq/(q-1)$ for all i . In (47) we have two cases: one if $\deg_x(c_0(x^q)/(\theta - x^q)^s) \leq \max_i \{\deg_x \tilde{\beta}_i\}$, one if $\deg_x(c_0(x^q)/(\theta - x^q)^s) > \max_i \{\deg_x \tilde{\beta}_i\}$. In the first case we easily see that $\deg_x c_0 < sq/(q-1)$ (notice that $\deg_x c_0(x^q) = q \deg_x c_0$). In the second case, $\deg_x c_0 = q \deg_x c_0 - sq$ which implies $\deg_x c_0 = sq/(q-1)$.

First case. Here, there exists $\epsilon > 0$ such that $\deg_x c_0 = (sq - \epsilon)/(q-1)$. We easily check (assuming that $|x| \geq r > q^{1/q}$):

$$\begin{aligned} |R_k(x)| &\leq \kappa |x|^{\frac{sq-\epsilon}{q-1} q^{k+1} - \frac{sq(q^{k+1}-1)}{q-1}} \\ &\leq \kappa |x|^{\frac{sq-\epsilon q^{k+1}}{q-1}} \end{aligned}$$

and the sequence of functions $(R_k(x))_k$ converges uniformly to zero in the domain $\{x, |x| \geq r\}$ for all $r > q^{1/q}$. Letting k tend to infinity in (48), we find $\sum_i c_i F_i(x) + c_0(x) = 0$; that is what we expected.

Second case. In this case, the sequence $|R_k(x)|$ is bounded but does not tend to 0. Notice that this case does not occur if $q-1$ does not divide s , because $c_0 \in K^{\text{alg.}}(x)$ and its degree is a rational integer. Hence we suppose that $q-1$ divides s .

Let us write:

$$c_0(x) = \lambda x^{sq/(q-1)} + \sum_{i>sq/(q-1)} d_i x^i,$$

with $\lambda \in K^{\text{alg.}\times}$. We have

$$\lim_{k \rightarrow \infty} \frac{\sum_{i>sq/(q-1)} d_i x^{q^k i}}{((x^q - \theta)(x^{q^2} - \theta) \cdots (x^{q^{k+1}} - \theta))^s} = 0$$

(uniformly on $|x| > r > q^{1/q}$), as one verifies following the first case.

For all $k \geq 0$ we have:

$$\begin{aligned} (-x)^{sq/(q-1)} \prod_{i=1}^{k+1} \left(1 - \frac{\theta}{x^{q^i}}\right)^{-s} &= (-1)^{sq/(q-1)} x^{sq/(q-1)} x^{s(q+\cdots+q^{k+1})} \prod_{i=1}^{k+1} (x^{q^i} - \theta)^{-s} \\ &= (-1)^{sq/(q-1)} x^{sq^{k+2}/(q-1)} \prod_{i=1}^{k+1} (x^{q^i} - \theta)^{-s}. \end{aligned}$$

Hence we have $\lim_{k \rightarrow \infty} \lambda x^{sq/(q-1)} / ((x^q - \theta)(x^{q^2} - \theta) \cdots (x^{q^{k+1}} - \theta))^s = \lambda F_{s,0}(x)$ and $\sum_i c_i F_i(x) + \lambda F_{s,0}(x) + c_0(x) = 0$.

We now prove the last statement of the Proposition: this follows from an idea of Denis. The proof is the same in both cases and we work with the first only. There exists $a \geq 0$ minimal such that the p^a -th powers of c_1, \dots, c_s are defined over the separable closure K^{sep} of K . The trace $K^{\text{sep}} \rightarrow K$ can be extended to formal series $K^{\text{sep}}((1/x)) \rightarrow K((1/x))$; its image does not vanish. We easily get, multiplying by a denominator in A , a non-trivial relation

$$\sum_i b_i F_i(x)^{q^a} + b_0(x) = 0$$

with $b_i \in A$ and $b_0(x) \in K^{\text{alg.}}(x)$. If the coefficients b_i are all in \mathbb{F}_q , this relation is the p^a -th power of a linear relation as we are looking for. If every relation has at least one of the coefficients b_i not belonging to \mathbb{F}_q , the one with $\max_i \{\deg_\theta b_i\}$ and a minimal has in fact $a = 0$ (otherwise, we apply the operator $d/d\theta$ to find one with smaller degree, because $dg^{p^a}/d\theta = 0$ if $a > 0$). \square

The following proposition reproduces Denis' criterion of algebraic independence in [17, 18]. It follows immediately from Theorem 4.3.

Proposition 4.16. *Let $L \subset K^{\text{alg.}}$ be a finite extension of K . We consider f_1, \dots, f_m holomorphic functions in a domain $|x| > r \geq 1$ with Taylor's expansions in $L((1/x))$. Let us assume that there exist elements $a_i, b_i \in L(x)$ ($i = 1, \dots, m$) such that*

$$f_i(x) = a_i(x) f_i(x^q) + b_i(x), \quad 1 \leq i \leq m.$$

Let α be in L , $|\alpha| > r$, such that for all n , α^{q^n} is not a zero nor a pole of any of the functions a_i, b_j .

If the series f_1, \dots, f_m are algebraically independent over $K^{\text{alg.}}(x)$, then the values

$$f_1(\alpha), \dots, f_m(\alpha)$$

are algebraically independent over K .

The next step is the following Proposition.

Proposition 4.17. *If the numbers $(Li_s(\beta_{s,1}), \dots, Li_s(\beta_{s,l_s}))_{s \in \mathcal{J}}$ are algebraically dependent over K^{alg} , there exists $s \in \mathcal{J}$ and a non-trivial linear relation*

$$\sum_{i=1}^{l_s} c_i Li_s(\beta_{s,i}) = 0$$

with $c_1, \dots, c_{l_s} \in A$. If $q-1$ does not divide s , or a non-trivial relation:

$$\sum_{i=1}^{l_s} c_i Li_s(\beta_{s,i}) + \lambda \tilde{\pi}^s = 0$$

with $c_1, \dots, c_{l_s}, \lambda \in A$ if $q-1$ divides s .

Proof. By Proposition 4.16, the functions $F_{s,i}$ ($s \in \mathcal{J}, 1 \leq i \leq l_s$) are algebraically dependent over $K^{\text{alg}}(t)$. Proposition 4.15 applies and gives $s \in \mathcal{J}$ as well as a non-trivial linear dependence relation. If $q-1$ does not divide s , by Proposition 4.15 there exists a non-trivial relation

$$\sum_{i=1}^{l_s} c_i F_{s,\beta_{s,i}}(x) = f(x) \in K^{\text{alg}}(x)$$

with $c_1, \dots, c_{l_s} \in A$. We substitute $x = \theta$ in this relation:

$$\sum_{i=1}^{l_s} c_i Li_s(\beta_{s,i}) = f \in K^{\text{alg}}.$$

After [5] pp.172-176, for all $x \in C$ such that $|x| < q^{qs/(q-1)}$, there exist

$$v_1(x), \dots, v_{s-1}(x) \in C$$

such that

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ x \end{pmatrix} = \exp_s \begin{pmatrix} v_1(x) \\ \vdots \\ v_{s-1}(x) \\ Li_s(x) \end{pmatrix},$$

\exp_s being the exponential function associated to the s -th tensor power of Carlitz's module. Moreover:

$$\exp_s \begin{pmatrix} c_j v_1(\beta_{s,j}) \\ \vdots \\ c_j v_{s-1}(\beta_{s,j}) \\ c_j Li_s(\beta_{s,j}) \end{pmatrix} = \phi_{\text{Car}}^{\otimes s}(c_j) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \beta_{s,j} \end{pmatrix} \in (K^{\text{alg}})^s, \quad j = 1, \dots, n_s,$$

where $\phi_{\text{Car}}^{\otimes s}(c_j)$ denotes the action of the s -th tensor power of Carlitz's module. By \mathbb{F}_q -linearity, there exist numbers $w_1, \dots, w_{s-1} \in C$ such that

$$\exp_s \begin{pmatrix} w_1 \\ \vdots \\ w_{s-1} \\ c \end{pmatrix} \in (K^{\text{alg}})^s.$$

Yu's sub- t -module Theorem (in [55]) implies the following analogue of Hermite-Lindemann's Theorem. Let $G = (\mathbb{G}_a^s, \phi)$ be a regular t -module with exponential function e_ϕ , with $\phi(g) = a_0(g)\tau^0 + \dots$, for all $g \in A$. Let $u \in C^s$ be such that $e_\phi(u) \in \mathbb{G}_a^s(K^{\text{alg.}})$. Let V the smallest vector subspace of C^s containing u , defined over $K^{\text{alg.}}$, stable by multiplication by $a_0(g)$ for all $g \in A$. Then the \mathbb{F}_q -subspace $e_\phi(V)$ of C^s equals $H(C)$ with H sub- t -module of G .

This result with G the s -th twist of Carlitz's module and $e_\phi = \exp_s$ implies the vanishing of c and the K -linear dependence of the numbers

$$\text{Li}_s(\beta_{s,1}), \dots, \text{Li}_s(\beta_{s,l_s}).$$

If $q - 1$ divides s then by Proposition 4.15 there exists a non-trivial relation

$$\sum_{i=1}^{l_s} c_i F_{s,\beta_{s,i}}(x) + \lambda F_{s,0}(x) = f(x) \in K^{\text{alg.}}(x)$$

with $c_1, \dots, c_{l_s}, \lambda \in K$. We substitute $x = \theta$ in this relation:

$$\sum_{i=1}^{l_s} c_i \text{Li}_s(\beta_{s,i}) + \lambda \tilde{\pi}^s = f \in K^{\text{alg.}}$$

The Proposition follows easily remarking that, after [5] again, there exist

$$v_1, \dots, v_{s-1} \in C$$

such that

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} = \exp_s \begin{pmatrix} v_1 \\ \vdots \\ v_{s-1} \\ \tilde{\pi}^s \end{pmatrix}.$$

□

Proof of Theorem 4.11. To deduce Theorem 4.11 from Proposition 4.17 we quote Theorem 3.8.3 p. 187 of Anderson-Thakur in [5] and proceed as in [14]. For all $i \leq nq/(q-1)$ there exists $h_{n,i} \in A$ such that if we set

$$P_n := \sum_i \phi_{\text{Car}}^{\otimes n}(h_{n,i}) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \theta^i \end{pmatrix},$$

then the last coordinate P_n is equal to $\Gamma(n)\zeta(n)$ (where $\Gamma(n)$ denotes Carlitz's arithmetic Gamma function). Moreover, there exists $a \in A \setminus \{0\}$ with $\phi_{\text{Car}}^{\otimes n}(a)P_n = 0$ if and only if $q-1$ divides n . This implies that

$$\Gamma(n)\zeta(n) = \sum_{i=0}^{[nq/(q-1)]} h_{n,i} \text{Li}_n(\theta^i).$$

The numbers $h_{n,i}$ are explicitly determined in [5]. In particular, one has

$$\zeta(s) = \text{Li}_s(1), \quad s = 1, \dots, q-1.$$

We apply Proposition 4.16 and Proposition 4.17 with $\mathcal{J} = \mathcal{J}^\# \cup \{q-1\}$, $\mathcal{J}^\#$ being the set of all the integers $n \geq 1$ with $p, q-1$ not dividing n , $l_{q-1} = 1$, $\beta_{q-1,1} = 1$, and for $s \in \mathcal{J}^\#$,

$$(\beta_1, \dots, \beta_{l_s}) = (\theta^{i_0}, \dots, \theta^{i_{m_s}}),$$

where the exponents $0 \leq i_0 < \dots < i_{m_s} \leq sq/(q-1)$ are chosen so that

$$\zeta(s) \in K\text{Li}_s(1) + \dots + K\text{Li}_s(\theta^{\lfloor sq/(q-1) \rfloor}) = K\text{Li}_s(\theta^{i_0}) \oplus \dots \oplus K\text{Li}_s(\theta^{i_{m_s}}).$$

□

Remark 4.18. With $\beta \in K$ as above, we can identify, replacing θ with t^{-1} , the formal series $F_\beta \in C((x))$ with a formal series $F_\beta^* \in \mathbb{F}_q[t][[x]] \subset \mathcal{K}[[x]]$ (as in 3.2), over which the operator τ defined there acts. Carlitz's module $\phi_{\text{Car}} : \theta \mapsto \theta\tau^0 + \tau^1$ acts on $\mathbb{F}_q[t][[x]]$ and it is easy to compute the image of F_β^* under this action. from this we get:

$$F_{\phi_{\text{Car}}(\theta)\beta}(x) = \theta F_\beta(x) + (x - \theta)\beta(x),$$

which implies that, for all $a \in A$, $F_{\phi_{\text{Car}}(a)\beta}(x) \in aF_\beta(x) + \mathbb{F}_q(\theta, x)$. In some sense, the functions F_β are ‘‘eigenfunctions’’ of the Carlitz module (a similar property holds for the functions L_β and the functions (31), which also have Mahler's functions as counterparts).

4.3.5. Final remarks. The fact that we could obtain Theorems 4.10 and 4.11 in a direct way should not induce a false optimism about Mahler's approach to algebraic independence; the matrices of the linear τ -difference equation systems involved are diagonal and we benefitted of this very special situation. In the general case, it seems more difficult to compute the transcendence degree of the field generated by solutions $\underline{f} = {}^t(f_1, \dots, f_m) \in \mathcal{K}((x))$ of a system like:

$$\underline{f}(x^d) = \mathcal{A}(x) \cdot \underline{f}(x) + \underline{b}(x)$$

(see for example, the difficulties encountered in [37, Section 5.2]). One of the reasons is that tannakian approach to this kind of equation is, the time being, not yet explored.

This point of view should be considered since it has been very successful in the context of t -motives as in Papanikolas work [39], which is fully compatible with Galois' approach. We could expect, once the tannakian theory of Mahler's functions is developed enough, to reach more general results by computing dimensions of motivic Galois groups (noticing the advantageous fact that the field of constants is here algebraically closed).

However, there is an important question we shall deal with: *is any ‘‘period’’ of a trivially analytic t -motive (in the sense of Papanikolas in [39]) a Mahler's value?* For example, in 4.3.4, we made strong restrictions on the $\beta \in K^{\text{alg}}$. so that finally, Denis Theorem in [19] is weaker than Papanikolas Theorem 4.10. *Is it possible to avoid these restrictions in some way?*

We presently do not have a completely satisfactory answer to this question, but there seem to be some elements in favour of a positive answer. We will explain this in the next few lines.

The method in 4.3.4 of deforming Carlitz logarithms $\log_{\text{Car}}(\beta)$ into Mahler's functions requires that the β 's in K^{alg} correspond to analytic functions at infinity. If $\beta = \sum_i c_i \theta^{-i}$ lies in $K_\infty = \mathbb{F}_q((1/\theta))$, the series $\tilde{\beta}(x) := \sum_i c_i x^{-i}$ converges for $x \in C$ such that $|x| > 1$ and $\beta = \tilde{\beta}(\theta)$.

This construction still works in the perfect closure of the maximal tamely ramified extension F of $\mathbb{F}_q((x^{-1}))$ but cannot be followed easily for general $\beta \in K^{\text{alg}} \setminus K_\infty$. Artin-Schreier's polynomial $X^p - X - \theta$ does not split over F . Hence, if $\xi \in K^{\text{alg}}$ is a root of this polynomial (it has absolute value $|\xi| = q^{1/p}$), the construction fails with the presence of divergent series.

Let us consider $\ell_1 = \log_{\text{Car}}(\xi)$ and $\ell_2 = \log_{\text{Car}}(\theta) = \log_{\text{Car}}(\xi^p) - \ell_1$. It is easy to show that ℓ_1, ℓ_2 are K -linearly independent. By Theorem 4.10, ℓ_1, ℓ_2 are algebraically independent over K . The discussion above shows that it is virtually impossible to apply Proposition 4.16 with the base point $\alpha = \theta$.

We now show that it is possible to modify the arguments of 4.3.4 and apply Proposition 4.16 with the base point $\alpha = \xi$.

To do so, let us consider, for $\beta \in K$, the formal series

$$\tilde{F}_\beta(x) = \tilde{\beta}(x) + \sum_{n=1}^{\infty} (-1)^n \frac{\tilde{\beta}(x^{q^n})}{\prod_{j=1}^n (x^{q^{j+1}} - x^{q^j} - \theta)}.$$

If $|\beta| < q^{\frac{q^2}{q-1}}$ and if $|x| > 1$, $x \notin \{\xi^{1/q^j} + \lambda, j \geq 1, \lambda \in \mathbb{F}_q\}$, these series converge. In particular, under the condition on $|\beta|$ above, they all converge at $x = \xi$ since they define holomorphic functions on the domain $\{x \in C, |x| > q^{1/pq}\}$, which contains ξ . More precisely, the value at $x = \xi$ is:

$$\tilde{F}_\beta(\xi) = \log_{\text{Car}}(\beta(\xi)).$$

We have the functional equations

$$\tilde{F}_\beta(x^q) = (\theta - x^{q^2} + x^q)(\tilde{F}_\beta(x) - \beta(x))$$

which tells us that the functions \tilde{F}_β define meromorphic functions in the open set $\{x \in C, |x| > 1\}$. With all these observations, it is a simple exercise to apply Proposition 4.16 with $\alpha = \xi$ and show the algebraic independence of ℓ_1, ℓ_2 .

The reader can extend these computations and show the algebraic independence of other logarithms of elements of K^{alg} . However, the choice of the base point α has to be made cleverly, and there is no general recipe yet. Here, the occurrence of Artin-Schreier extensions is particularly meaningful since it is commonly observed that every finite normal extension of $\mathbb{F}_q((1/\theta))$ is contained in a finite tower of Artin-Schreier extensions of $\mathbb{F}_q((1/\theta^{1/n}))$ for some n [24, Lemma 3].

Remark 4.19. Just as in Remark 4.18, the action of Carlitz's module yields the following formula

$$\tilde{F}_{(\theta^q - \theta)\beta + \beta^q}(x) = \theta \tilde{F}_\beta(x) + (x^q - x - \theta) \tilde{\beta}(x).$$

Since there are natural ring isomorphisms $K[X]/(X^p - X - \theta) \cong \mathbb{F}_q[\xi] \cong A$, it could be interesting to see if there is some \mathbb{F}_q -algebra homomorphism $A \rightarrow C[[\tau]]$ of which the functions \tilde{F}_β are "eigenfunctions".

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