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SEMICLASSICAL AND SPECTRAL ANALYSIS OF OCEANIC WAVES

CHRISTOPHE CHEVERRY, ISABELLE GALLAGHER, THIERRY PAUL,
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ABSTRACT. In this work we prove that the shallow water flow, subject to strong wind forcing and linearized around an adequate stationary profile, develops for large times closed trajectories due to the propagation of Rossby waves, while Poincaré waves are shown to disperse. The methods used in this paper involve semi-classical analysis and dynamical systems for the study of Rossby waves, while some refined spectral analysis is required for the study of Poincaré waves, due to the large time scale involved which is of diffractive type.

1. INTRODUCTION

The problem we consider in the present paper is motivated by large-scale oceanography: the main physical phenomenon leading this study is indeed the existence of persistent oceanic eddies, which are coherent structures of vortex type, spreading over dozens of kilometers and propagating slowly over periods from one year to one decade. These structures have been observed long past by physicists, among which [15, 16, 18, 24, 26, 27] who gave heuristic arguments (reproduced below) to explain their formation due both to wind forcing and to convection by a macroscopic zonal current.

Giving a (much less precise) mathematical counterpart of those arguments, even at a linear level, requires careful multiscale analysis and rather sophisticated tools of semiclassical and microlocal analysis. In this paper we simplify the model by considering particular macroscopic currents, which are stationary solutions of the forced equations. This allows to produce trapped Rossby waves, by solving the dynamics associated with an appropriate integrable Hamiltonian system. We prove also that the other waves produced by the dynamics, namely Poincaré waves, disperse on the same time scales (which turn out to be of diffractive type).

1.1. Physical observations.

Let us first give some details on the physical context. Simple observations show that **large-scale ocean dynamics** can be decomposed as the sum of

- the solid-body rotation together with the Earth,
- convection by macroscopic currents (such as the Gulf Stream in the North Atlantic, the Kuroshio in the North Pacific, Equatorial or Circumpolar currents),
- motion on smaller geographical zones, due for instance to the fluctuations of the wind and more generally to the coupling with the atmosphere.

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While the spatial extent of macroscopic currents is of the order of a hundred to a thousand kilometers, those fluctuations are typically on dozens of kilometers. We therefore expect eddies to be particular forms of those fluctuations. The point is to understand why they are quasi-stationary, or in other words why they do not disperse as other waves.

At this stage we have to describe briefly the **different kinds of waves** that can be found in the ocean as linear responses to exterior forcing. They are usually classified into two families, depending on their typical period and on their dynamical structure.

- **Poincaré waves**, the period of which is of the order of a day, are fast dispersive waves. They are essentially due to the Coriolis force, that is to the rotation of the Earth.
- **Rossby waves** propagate much slower, since the departure from geostrophy (that is equilibrium between pressure and Coriolis force) is very small. They are actually related to the variations of the Coriolis parameter with latitude. In particular, they propagate only eastwards.

The exact dispersion relation of all these waves can be computed explicitly [4, 10, 12, 13, 24] in simplified cases (no convection, linear approximation of the Coriolis parameter). This shows for instance that, contrary to the case of Poincaré waves, the frequency of Rossby waves increases as wave-number decreases.

The **heuristic argument leading to the existence of quasi-stationary coherent structures** is then as follows (as suggested by Physicists):

- the wind forcing produces waves, in particular Rossby waves which would propagate, in the absence of convection, with a speed comparable to the bulk velocity of the fluid $\bar{v} \sim 10 \text{ ms}^{-1}$;
- the convection by zonal flow may then stop the propagation, creating ventilation zones which are not influenced by external dynamics, in particular by continental recirculation.

We are then led to studying wave propagation under the coupled effects of the pressure, the Coriolis force and zonal convection, that is to studying a system of linear PDEs with non constant coefficients.

1.2. The model.

The system we will consider is actually a toy model insofar as many physical phenomena are neglected. Our aim here is only to get a qualitative mechanism to explain the trapping of Rossby waves.

More precisely, we consider the ocean as an **incompressible, inviscid fluid with free surface** submitted to gravitation and wind forcing, and further make the following classical assumptions :

(H1) the density of the fluid is homogeneous $\rho = \rho_0 = \text{constant}$

(H2) the pressure law is given by the hydrostatic approximation $p = \rho_0 g z$

(H3) the motion is essentially horizontal and does not depend on the vertical coordinate, leading to the so-called **shallow water approximation**.

For the sake of simplicity, we will not discuss the effects of the interaction with the boundaries, describing neither the vertical boundary layers, known as Ekman layers, nor the lateral boundary layers, known as Munk and Stommel layers. We indeed consider a purely horizontal model, and assume an infinite domain for the longitude (omitting the stopping conditions on the continents) as well as for the latitude (this may be heuristically justified using the exponential decay of the equatorial waves to neglect the boundary).

The evolution of the water height h and velocity v is then governed by the **Saint-Venant equations with Coriolis force**

$$(1.1) \quad \begin{aligned} \partial_t(\rho_0 h) + \nabla \cdot (\rho_0 h v) &= 0 \\ \partial_t(\rho_0 h v) + \nabla \cdot (\rho_0 h v \otimes v) + \omega(\rho_0 h v)^\perp + \rho_0 g h \nabla h &= \rho_0 h \tau \end{aligned}$$

where ω denotes the vertical component of the Earth rotation vector Ω , $v^\perp := (-v_2, v_1)$, g is the gravity and τ is the - stationary - forcing responsible for the macroscopic flow. It depends in particular on time averages of the wind forcing, temperature gradients and topography. The equations are written in cartesian coordinates (x_1, x_2) , where x_1 corresponds to the longitude, and x_2 to the latitude (both will be chosen in \mathbf{R}). The vertical component of the Earth rotation is therefore $\Omega \sin(x_2/R)$, where R is the radius of the Earth, but it is classical in the physical literature to consider the linearization of ω (known as the betaplane approximation) $\omega(x_2) = \Omega x_2/R$; most of our results will actually hold for more general functions ω , but in some situations we shall particularize the betaplane case in order to improve on the results.

In order to analyze the **influence of the macroscopic convection** on the trapping of Rossby waves, we will consider small fluctuations around the stationary solution

$$h = \bar{h}, \quad \nabla \cdot (\bar{v} \otimes \bar{v}) + \omega \bar{v}^\perp = \tau, \quad \operatorname{div} \bar{v} = 0,$$

where \bar{h} is a constant. Physical observations show that the nonlinear convection term is essentially negligible compared to the Coriolis term, so that the previous equation is nothing else than the Sverdrup relation (see [27]).

For technical reasons we shall restrict our attention in this paper to the case of a shear flow, in the sense that

$$\bar{v}(x) = (\bar{v}_1(x_2), 0),$$

where \bar{v}_1 is a smooth, compactly supported function. We shall further assume for simplicity that the zeros of \bar{v}_1 , in the interior of its support, are **of order one**.

1.3. Orders of magnitude and scaling.

Exhibiting trapped waves relies on some asymptotic study, which requires to compare the orders of magnitude of the different physical parameters.

- Let us first introduce the observation length, time and velocity scales l_0 (of the size of the radius of the Earth R), t_0 and v_0 , and the nondimensional variables

$$\tilde{x} = x/l_0, \quad \tilde{t} = t/t_0, \quad \text{and } u = (v - \bar{v})/v_0.$$

We further define the typical height variation δh and the corresponding dimensionless variable

$$\eta = \frac{h - \bar{h}}{\delta h}.$$

Finally, we denote by v_c the typical value of the velocity of the macroscopic current

$$\bar{u} = (\bar{u}_1, 0) = \bar{v}/v_c.$$

With these notations, the Saint-Venant equations with Coriolis force can be rewritten, with obvious notation,

$$(1.2) \quad \begin{aligned} \tilde{\partial}_t \eta + \frac{t_0 v_0}{l_0} \frac{\bar{h}}{\delta h} \tilde{\nabla} \cdot u + \frac{t_0 v_c}{l_0} \tilde{\nabla} \cdot (\eta \bar{u}) + \frac{t_0 v_0}{l_0} \tilde{\nabla} \cdot (\eta u) &= 0 \\ \tilde{\partial}_t u + t_0 |\Omega| \frac{\omega}{|\Omega|} u^\perp + \frac{t_0 g \delta h}{l_0 v_0} \tilde{\nabla} \eta + \frac{t_0 v_c}{l_0} (\bar{u} \cdot \tilde{\nabla} u + u \cdot \tilde{\nabla} \bar{u}) + \frac{t_0 v_0}{l_0} (u \cdot \tilde{\nabla} u) &= 0. \end{aligned}$$

- The length scale l_0 and the convection velocity v_c are fixed by the macroscopic flow: typical values for the Gulf Stream are

$$l_0 \sim 10^4 \text{ km} \quad \text{and} \quad v_c \sim 10 \text{ ms}^{-1}.$$

As we are interested in structures persisting during many months, a relevant choice for the observation time scale is

$$t_0 = 10^6 \text{ s} (\sim 0,38 \text{ months}).$$

The associated **Rosby number** is then

$$\text{Ro} := \frac{1}{t_0 |\Omega|} = 0.01$$

recalling that $|\Omega| = 7.3 \times 10^{-5} \text{ s}^{-1}$. Although the local normal component of the Earth's rotation would double this value at a latitude of 30° , it is still clear that such currents meet the criterion of large-scale motion.

The variations of water height which can be observed are typically of the order

$$\delta h \sim 1 \text{ m} \text{ to be compared to } \bar{h} \sim 10^3 \text{ m}.$$

The influence of gravity (through hydrostatic pressure) is then measured by the **Froude number**

$$\text{Fr}^2 := \frac{v_0 l_0}{t_0 g \delta h} \sim 0.1$$

considering namely fluctuations of order

$$v_0 \sim 0.1 \text{ ms}^{-1}.$$

Defining $\varepsilon := \text{Fr}^2$ and dropping the tildas (note that as often in Physics, ε is not really a very small number), we therefore end up with the following scaled system

$$(1.3) \quad \begin{aligned} \partial_t \eta + \frac{1}{\varepsilon} \nabla \cdot u + \bar{u} \cdot \nabla \eta + \varepsilon^2 \nabla \cdot (\eta u) &= 0, \\ \partial_t u + \frac{1}{\varepsilon^2} b u^\perp + \frac{1}{\varepsilon} \nabla \eta + \bar{u} \cdot \nabla u + u \cdot \nabla \bar{u} + \varepsilon^2 u \cdot \nabla u &= 0, \end{aligned}$$

where $b := \omega/|\Omega|$.

• For the sake of simplicity, we shall compute the response to the wind forcing, assuming that the wind induces a **pulse at time** $t = 0$: since the wind undergoes oscillations on small spatial scales, the initial data is further assumed to depend both on x and x/ε . Typically

$$(1.4) \quad (\eta_\varepsilon, u_\varepsilon)|_{t=0} = (\eta_k(x), u_k(x)) \exp\left(i \frac{k \cdot x}{\varepsilon}\right),$$

for some $k \in \mathbb{Z}^2$. More generally we shall consider initial data which is microlocalized (in the sense of Appendix B) in some compact set of $T^*\mathbf{R}^2$.

1.4. Local well-posedness. The local existence of a solution to the scaled Saint-Venant Coriolis system (1.3) supplemented with initial data in the form (1.4) comes from the general theory of hyperbolic quasilinear symmetrizable systems.

• Defining the sound speed u_0 by

$$\eta = [(1 + \varepsilon^3 u_0/2)^2 - 1]/\varepsilon^3,$$

we indeed obtain that (1.3) is equivalent to

$$(1.5) \quad \varepsilon^2 \partial_t U + A(x_2, \varepsilon D)U + \varepsilon^3 Q(U) = 0, \quad U = (u_0, u_1, u_2)$$

where $A(x_2, \varepsilon D_x)$ is the linear propagator

$$(1.6) \quad A(x_2, \varepsilon D) := \begin{pmatrix} \varepsilon \bar{u} \cdot \varepsilon \nabla & \varepsilon \partial_1 & \varepsilon \partial_2 \\ \varepsilon \partial_1 & \varepsilon \bar{u} \cdot \varepsilon \nabla & -b(x_2) + \varepsilon^2 \bar{u}'_1 \\ \varepsilon \partial_2 & b(x_2) & \varepsilon \bar{u} \cdot \varepsilon \nabla \end{pmatrix},$$

and $Q(U) := S_1(U)\varepsilon \partial_1 U + S_2(U)\varepsilon \partial_2 U$ with

$$(1.7) \quad S_1(U) := \begin{pmatrix} u_1 & \frac{1}{2}u_0 & 0 \\ \frac{1}{2}u_0 & u_1 & 0 \\ 0 & 0 & u_1 \end{pmatrix} \text{ and } S_2(U) := \begin{pmatrix} u_2 & 0 & \frac{1}{2}u_0 \\ 0 & u_2 & 0 \\ \frac{1}{2}u_0 & 0 & u_2 \end{pmatrix}.$$

• Because of the specific form of the initial data, involving fast oscillations with respect to x , we introduce suitable functional spaces, namely semi-classical Sobolev spaces

$$H_\varepsilon^s = \{U \in L^2 / \|U\|_{H_\varepsilon^s} < +\infty\} \text{ with } \|U\|_{H_\varepsilon^s}^2 = \sum_{|k| \leq s} \|(\varepsilon \nabla)^k U\|_{L^2}^2.$$

We will also need in the following to define weighted semi-classical Sobolev spaces (in the spirit of [10]), adapted to the linear propagator as explained in Section 7:

$$(1.8) \quad W_\varepsilon^s := \left\{ f \in L^2(\mathbf{R}^2) / (1 - \varepsilon^2 \partial_1^2)^{\frac{s}{2}} (1 - \varepsilon^2 \partial_2^2 + b^2(x_2))^{\frac{s}{2}} f \in L^2(\mathbf{R}^2) \right\}.$$

A classical result based on the Sobolev embedding (see Section 7 for related results)

$$\|\varepsilon \nabla U\|_{L^\infty} \leq \frac{C}{\varepsilon} \|\nabla U\|_{H_\varepsilon^s} \text{ for any } s > 1,$$

implies that (1.3) has a unique local solution $U_\varepsilon \in L^\infty([0, T_\varepsilon], H_\varepsilon^{s+1})$. Note that the life span of U_ε depends a priori on ε . One of the goals of this article is to show that actually the life span does not depend on ε .

2. MAIN RESULTS AND STRATEGY OF THE PROOFS

Actually most of this paper is concerned with the analysis of the (global) solution to the linear equation

$$(2.1) \quad \varepsilon^2 \partial_t V + A(x_2, \varepsilon D)V = 0, \quad V = (v_0, v_1, v_2)$$

which is expected to dominate the dynamics since we consider small fluctuations. The description of the linear dynamics is provided in Theorem 1 below. The comparison between linear and nonlinear solutions is postponed to the final section of the paper (the statement is given in Theorem 2).

Additionally to the assumptions on \bar{u}_1 made above (see Section 1.2), we shall suppose throughout the paper that b is a smooth function with a symbol-like behaviour:

$$(2.2) \quad \forall \alpha \in \mathbf{N}, \exists C_\alpha, \forall y \in \mathbf{R}, \quad |b^{(\alpha)}(y)| \leq C_\alpha(1 + b^2(y)),$$

and we shall further assume that

$$\lim_{y \rightarrow \infty} b^2(y) = \infty,$$

and that b has a finite number of critical points (that is to say points where b' vanishes). This assumption will allow to obtain trapped Rossby waves.

We shall also suppose that the initial data is **microlocalized** (see Appendix B) in some compact set of $T^*\mathbf{R}^2$ (which we shall identify to \mathbf{R}^4 in the following), denoted \mathcal{C} and satisfying

$$(2.3) \quad \mathcal{C} \cap \{\xi_1 = 0\} = \emptyset.$$

This assumption, which is propagated by the linear flow, allows to diagonalize the system into Rossby and Poincaré modes.

Finally in order to avoid pathological trapped Rossby trajectories we shall also require that

$$(2.4) \quad \mathcal{C} \cap \Sigma = \emptyset,$$

where Σ is a codimension 1 subset of \mathbf{R}^4 defined in Proposition 4.6.

2.1. Statement of the main results. In this paragraph we shall state the two main theorems proved in this paper. The first result deals with the linear system (2.1).

Theorem 1 (The linear case). *There is a submanifold Λ of \mathbf{R}^4 , invariant under translations in the x_1 -direction, such that the following properties hold.*

Let $U_{\varepsilon,0}$ be ε -microlocalized in a compact set \mathcal{C} satisfying Assumptions (2.3)-(2.4). For any parameter $\varepsilon > 0$, denote by V_ε the associate solution to (2.1). Then for all $t \geq 0$ one can write $V_\varepsilon(t)$ as the sum of a ‘‘Rossby’’ vector field and a ‘‘Poincaré’’ vector field: $V_\varepsilon(t) = V_\varepsilon^R(t) + V_\varepsilon^P(t)$, satisfying the following properties:

- (1) *There is a compact set K of \mathbf{R}^2 such that*

$$\forall t \geq 0, \quad \|V_\varepsilon^R(t)\|_{L^2(K)} \neq O(\varepsilon^\infty)$$

if and only if the ε -frequency set of $V_\varepsilon^R(0)$ intersects Λ .

- (2) Suppose that b^2 has only one non degenerate critical value (meaning that $(b^2)'$ only vanishes at one point, where $(b^2)''$ does not vanish). Then for any compact set Ω in \mathbf{R}^2 , one has

$$\forall t > 0, \quad \|V_\varepsilon^P(t)\|_{L^2(\Omega)} = O(\varepsilon^\infty).$$

In particular supposing that b^2 has only one non degenerate critical value, then there is a compact set K of \mathbf{R}^2 such that

$$\forall t > 0, \quad \|V_\varepsilon(t)\|_{L^2(K)} \neq O(\varepsilon^\infty)$$

if and only if the ε -frequency set of $V_\varepsilon^R(0)$ intersects Λ .

Remark 2.1. (1) Actually Λ corresponds to the set of initial positions and frequencies in the phase space giving rise to trapped trajectories for the Rossby hamiltonian. This will be made more precise in Section 4, where we shall prove that under some additional (non restrictive) assumptions on \bar{u} , Λ is of codimension one. In particular it will be shown that some of those trapped trajectories actually exhibit a singular behaviour in large times, in the sense that they converge in physical space towards a point, while the ξ_2 frequency goes to infinity. This could be interpreted like the creation of some sort of oceanic eddies.

- (2) The result (2) is related to dispersive properties of the Poincaré hamiltonian on diffractive type times (of the type $O(1/\varepsilon^2)$), which requires some spectral analysis. The assumption on b^2 allows for a rather simple proof; more general conditions could be treated, but the Bohr Sommerfeld quantization would require to decompose the phase space into various zones according to the geometry of the level sets of the Hamiltonian, which is much more technical and beyond the scope of this article.

The final section of this paper is devoted to the proof of the following theorem, which states that the very weak coupling chosen in this paper implies that nonlinear dynamics are governed by the linear equation. We also consider more generally the following weakly nonlinear system (with the notation (1.6) and (1.7)):

$$(2.5) \quad \varepsilon^2 \partial_t U + A(x, \varepsilon D_x)U + \varepsilon^{3+\eta} S_1(U) \varepsilon \partial_1 U + \varepsilon^{3+\eta} S_2(U) \varepsilon \partial_2 U = 0, \quad \eta \geq 0.$$

The case $\eta = 0$ corresponds of course to the original system (1.5) presented in the introduction.

Theorem 2 (The nonlinear case). *Let $U_{\varepsilon,0}$ be any initial data bounded in W_ε^4 . Then the following results hold.*

- (1) *There exists some $T^* > 0$ such that the initial value problem (2.5) with $\eta = 0$ has a unique solution U_ε on $[0, T^*[$ for any $\varepsilon > 0$.*
- (2) *Assume that the solution V_ε to the linear equation (2.1) satisfies*

$$\|\varepsilon V_\varepsilon\|_{L^2([0, T^*]; L^\infty)} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

Then the solution U_ε to (2.5) with $\eta = 0$ satisfies

$$\|U_\varepsilon - V_\varepsilon\|_{L^2} \rightarrow 0 \text{ uniformly on } [0, T^*[\text{ as } \varepsilon \rightarrow 0.$$

- (3) *Let $\eta > 0$ and $T > 0$ be fixed. Then there is $\varepsilon_0 > 0$ such that for any $\varepsilon \leq \varepsilon_0$, the equation (2.5) has a unique solution U_ε on $[0, T]$. Moreover,*

$$\|U_\varepsilon - V_\varepsilon\|_{L^2} \rightarrow 0 \text{ uniformly on } [0, T] \text{ as } \varepsilon \rightarrow 0.$$

Remark 2.2. *Note that result (2) implies in particular that for positive times, the energy of U_ε on any fixed compact subset is carried only by Rossby waves. The refined L^∞ estimate on the linear solution required in result (2) should be proved, for instance for WKB initial data as in (1.4), using*

- *some generalized WKB expansion for the Rossby waves (on Fourier side near caustics), together with estimates on Airy functions (note that before the first caustic the WKB expansion gives immediately the required estimate);*
- *some stationary phase lemma to estimate the contribution of Poincaré waves.*

Depending on the degeneracy of stationary points, we should gain some small power of ε due to the oscillatory behaviour of the integrals. For the sake of simplicity, we will not consider such technical estimates here, all the less that we do not expect them to be enough to get an optimal result regarding the nonlinear problem (see Remark 7.1). That is the reason why we consider, in result (3), a weaker coupling still. That result implies in particular that the L^2 norm of U_ε on any fixed compact subset may remain bounded from below only if there are trapped Rossby waves, i.e. if the ε -frequency set of the initial data does intersect Λ (with the notation of Theorem 1).

2.2. Some related studies. This work follows a long tradition of mathematical studies of fast rotating fluids, following the works of [28] and [17]; we refer for instance to [4] and [13] for a number of references to such works. One of the features of those studies is that fast oscillatory (in time) waves can be identified by means of a Fourier transform (the penalization matrix linked to the fast rotation has indeed constant coefficients), and the envelope equations can be derived by studying the interactions between those waves. Actually the case of a discrete spectrum is very different from the case of a continuous spectrum, as in the latter case dispersive effects may be used to get rid of fast oscillations, while in the former it is necessary to take oscillations into account at all times. These works are actually rather closely related to general methods in geometric optics (see the works of J.-L. Joly, G. Métivier and J. Rauch, as in [22] for instance), using furthermore the fact that explicit computations may be carried out (via the Fourier transform).

The present study concerns the case when the penalization matrix does not have constant coefficients : this is due to the fact that the Coriolis force actually depends on the latitude. A first study in this type of situation may be found in [11], where a rather general penalization matrix was considered. Due to the generality of the situation, explicit computations were ruled out and no study of waves was carried out (only weak convergence results were obtained). In order to compute explicitly the modes created by the penalization matrix, various authors (see [8], [9], [10] as well as [12]) studied the betaplane approximation, in which the rotation vector depends linearly on the latitude. In that case explicit calculations may again be carried out (or some explicit commuting vector fields may be computed) which allow again to derive envelope equations. In this paper we choose again to work with a more general rotation vector, this choice being made possible by a semi-classical setting (see the next paragraph); in particular that setting enables us diagonalize the system approximately, therefore to compute waves; note that a related study is performed by two of the authors in [5], where no explicit diagonalization is performed (actually the initial data is strongly polarized so that only Rossby modes are present, including in the non linear setting) with a purely geometric optics approach.

Another feature of our study is that it is a multi-scale problem, in the sense that the oscillation frequency is much bigger than the variation of the coefficients of the system. This is dealt with by using semi-classical analysis (which, compared to the previous paragraph, allows to compute *almost commuting* vector fields although the penalization matrix no longer depends only linearly on the latitude). Such techniques are classical in geometrical optics, but in our case the additional difficulty is that the propagators are linked to different time scales: this is due to the fact that the system has eigenvalues at different scales (one is actually a subsymbol). In particular we are mostly interested in the role of the subsymbol in the dynamics, as this subsymbol is responsible for the trapping phenomenon we want to exhibit: this implies, by semi-classical analysis, the need to study the dynamical system induced by that subsymbol. On the other hand this also means that the dynamics linked to other eigenvalues must be analyzed on diffractive-type time scales, therefore much longer than that allowed by semi-classical analysis. We are able to show the dispersion of those waves by using spectral analysis and Bohr-Sommerfeld quantization.

2.3. Organization of the paper. The proof of Theorems 1 and 2 requires a number of steps which are described in this paragraph.

2.3.1. Reduction to scalar propagators. As mentioned at the beginning of the introduction, persistent structures are related to the propagation of Rossby waves. Our first task is therefore to transform the original linear system (2.1) into three scalar equations, among which one is polarized on Rossby waves (the two others are polarized on Poincaré waves). This is done in Section 3 by proving some necessary conditions for the existence of those propagators. One constructs Rossby propagators and Poincaré propagators by the following strategy.

- (1) Consider the system $A(x_2, \varepsilon D)U = i\tau U$. Extract from this system (by linear combinations and substitutions) a linear equation of the type $h(x_2, \varepsilon D_2; \xi_1, \varepsilon, \tau)u_k = 0$, where u_k is one component of U (where the Fourier transform in x_1 has first been taken, which is possible since the equation is translation invariant in x_1).
- (2) The symbolic equation corresponding to that PDE writes $h(x_2, \xi_2; \xi_1, \varepsilon, \tau) = 0$. It has three roots (with respect to τ) noted $\tau_{\pm}(x_2, \xi; \varepsilon)$ (Poincaré roots) and $\tau_R(x_2, \xi; \varepsilon)$ (Rossby root). We find

$$\tau_{\pm}(x_2, \xi; \varepsilon) = \tau_{\pm}(x_2, \xi) + O(\varepsilon) \quad \text{and} \quad \tau_R(x_2, \xi; \varepsilon) = \varepsilon \tilde{\tau}_R(x_2, \xi) + O(\varepsilon^2).$$
- (3) Those roots are not necessarily symbols. To guarantee these are indeed symbols one needs a microlocalization. Given a compact set and a truncation on that compact set χ , one can construct three operators T_j^χ (via a general theorem, stated and proved in an abstract way in Theorem 3 in Appendix A) whose principal symbols are precisely the Rossby and Poincaré symbols τ_{\pm} and τ_R .

2.3.2. Trapping of Rossby waves. Section 4 is devoted to the study of the Rossby propagator T_0 , and in particular to the proof of result (1) in Theorem 1. For the time scale considered, it is easy to see that the energy propagates according to the trajectories of the semiclassical Rossby hamiltonian $\tilde{\tau}_R$. The first step of the analysis therefore consists in studying the dynamical system giving rise to those trajectories. It turns out that the trajectories are always bounded in the x_2 direction. One is therefore reduced to studying the trajectories in the x_1 variable and in identifying the set Λ of initial data in the cotangent space giving rise to trapping in x_1 . One then checks that Λ is of codimension one under some additional assumptions

on \bar{u} , and the last step of the study consists in studying more precisely the trajectories in some specific situations, in particular in the case of the betaplane approximation.

2.3.3. Dispersion of Poincaré waves. The next step of our analysis of wave propagation consists in proving, in Section 5, that Poincaré waves propagate so fast that they exit from any (bounded) domain of observation on the time scale that we consider, which proves result (2) in Theorem 1. Note that, because of the very long time scaling, usual tools of semiclassical analysis cannot be applied for the Poincaré waves : we actually need deeper arguments such as the Bohr-Sommerfeld quantization to conclude.

2.3.4. A diagonalization result. Once the Rossby and Poincaré propagators have been well understood, it is possible to retrace the steps followed in Section 3 to prove that the necessary conditions on the scalar propagators are in fact sufficient. The difficulty is that the operators Π allowing to go from the original system to the scalar equations and back (computing an approximate left inverse Q of Π at the order $O(\varepsilon^\infty)$) are only continuous on microlocalized functions; moreover the scalar propagators T_j^X are themselves only defined on microlocalized functions. So that requires understanding the persistency of the microlocalization of the solutions to the scalar equations. That is in fact achieved in the two previous sections, where it is proved that if the initial data is conveniently microlocalized, then for any time $t \geq 0$ one can find a compact set K and one can construct T_j^X as in Section 3, so that the solution to the scalar equations with propagators T_j^X is microlocalized in K (actually for Poincaré modes the microlocalization is in the variables (x_2, ξ_1, ξ_2) only, which is enough for our purpose). This allows in Section 6 to conclude rather easily by computing explicitly the matrix principal symbols of Π and Q .

2.3.5. The analysis of the nonlinear equation. Section 7 is devoted to the proof that the solution of the nonlinear equation remains close to that of the linear equation. The precise result is stated in Theorem 2. The method of proof consists first in proving the wellposedness of the nonlinear equation on a uniform time interval, by using semi-classical weighted Sobolev type spaces, whose additional feature is to be well adapted to the penalization operator $A(x_2, \varepsilon D)$: one therefore constructs a matrix-valued pseudo-differential operator which approximately commutes with $A(x_2, \varepsilon D)$. The convergence of $U_\varepsilon - V_\varepsilon$ to zero relies on a standard L^2 energy estimate on $U_\varepsilon - V_\varepsilon$, and Gronwall's lemma.

2.3.6. Two appendixes. In Appendix A one can find the statement and the proof of a general theorem, used in Section 3, allowing to associate to a linear evolution PDE a number of operators describing the dynamics of the equation; those operators are constructed by writing down the symbolic equation associated to the PDE and in quantizing the roots of that polynomial (in the time derivative).

Appendix B finally collects a number of prerequisites on microlocal and semiclassical analysis, that are used throughout the paper.

3. REDUCTION TO SCALAR PROPAGATORS

3.1. **Introduction.** Let us first recall that the propagator

$$A(x, \varepsilon D) = \begin{pmatrix} \varepsilon \bar{u} \cdot \varepsilon \nabla & \varepsilon \partial_1 & \varepsilon \partial_2 \\ \varepsilon \partial_1 & \varepsilon \bar{u} \cdot \varepsilon \nabla & -b + \varepsilon^2 \bar{u}'_1 \\ \varepsilon \partial_2 & b & \varepsilon \bar{u} \cdot \varepsilon \nabla \end{pmatrix}$$

can, in the particular case when $\bar{u} \equiv 0$ and $b(x_2) = \beta x_2$, be diagonalized without any error term (in particular for any finite ε), using a Fourier basis $(\exp(\frac{i}{\varepsilon} x_1 \xi_1))_{\xi_1 \in \mathbf{R}}$ in x_1 and a Hermite basis $(\psi_n^\varepsilon(x_2))_{n \in \mathbf{N}}$ in x_2 . Precisely, the following statement is proved in [12].

Proposition 3.1 (Gallagher & Saint-Raymond,[12]). *For all $(\xi_1, n, j) \in \mathbf{R} \times \mathbf{N} \times \{-1, 0, 1\}$, denote by $\tau(\xi_1, n, j)$ the three roots (in increasing order in j) of*

$$(3.1) \quad \tau^3 - (\xi_1^2 + \beta\varepsilon(2n+1))\tau + \varepsilon\beta\xi_1 = 0.$$

Then there exists a complete family of $L^2(\mathbf{R} \times \mathbf{R}, \mathbf{R}^3)$ of pseudo-eigenvectors $(\Psi_{\xi_1, n, j}^\varepsilon)$ of the operator $A(x, \varepsilon D)$ (where $\bar{u} \equiv 0$ and $b(x_2) = \beta x_2$):

$$(3.2) \quad \forall (\xi_1, n, j) \in \mathbf{R} \times \mathbf{N} \times \{-1, 0, 1\}, \quad A(x, \varepsilon D) \Psi_{\xi_1, n, j}^\varepsilon = i\tau(\xi_1, n, j) \Psi_{\xi_1, n, j}^\varepsilon$$

where $\Psi_{\xi_1, n, j}^\varepsilon$ can be computed explicitly in terms of the n -th Hermite function $\psi_n^\varepsilon(x_2)$ and its derivatives.

In other words, the three scalar propagators (numbered by j) can be obtained from the symbolic equation (3.1) remarking that $\beta\varepsilon(2n+1)$ is the quantization of the harmonic oscillator $-\varepsilon^2 \partial_2^2 + \beta^2 x_2^2$. It is proved in [12] that as ξ_1 and n go to infinity

$$\tau(\xi_1, n, \pm) \sim \pm \sqrt{\xi_1^2 + \beta\varepsilon(2n+1)}, \quad \text{and} \quad \tau(\xi_1, n, 0) \sim \frac{\varepsilon\beta\xi_1}{\xi_1^2 + \beta\varepsilon(2n+1)}.$$

We are interested here in deriving a symbolic equation similar to (3.1) for a general zonal current $\bar{u} = (\bar{u}_1(x_2), 0)$ and Coriolis parameter $b = b(x_2)$. The difficulty comes from the fact that the propagation of waves is governed by a matrix of differential operators with non-constant coefficients, the diagonalization of which is not a standard computation. Of course, in the semiclassical limit $\varepsilon \rightarrow 0$, we expect to get a good approximation of the propagation at leading order by considering the matrix of principal symbols

$$(3.3) \quad A_0(x, \xi) := \begin{pmatrix} 0 & i\xi_1 & i\xi_2 \\ i\xi_1 & 0 & -b(x_2) \\ i\xi_2 & b(x_2) & 0 \end{pmatrix}$$

and by computing the scalar propagators associated to each eigenvalue

$$(3.4) \quad \tau_\pm(x_2, \xi_1, \xi_2) := \pm \sqrt{\xi_1^2 + \xi_2^2 + b^2(x_2)}$$

and 0. Note that the eigenvalue 0 corresponds to the Rossby modes, whereas the two $O(1)$ eigenvalues $\pm \sqrt{\xi_1^2 + \xi_2^2 + b^2(x_2)}$ are the Poincaré modes. Nevertheless, this approximation is relevant only for times of order $O(\frac{1}{\varepsilon})$, and we are interested here in much longer times, of order $O(\frac{1}{\varepsilon^2})$. This means in particular that we need to compute the next order of the expansion

of the eigenvalue 0. Once that is done, we need to quantify these symbol eigenvalues to deduce scalar propagators.

The structure of this section is the following: in Paragraphs 3.2 and 3.3 we show how to obtain the characteristic equation on the Rossby and Poincaré modes respectively (for the Rossby modes we also compute the $O(\varepsilon)$ correction of the eigenvalue 0), and we compute Rossby and Poincaré propagators. This is done through an abstract theorem, Theorem 3 stated and proved in Appendix A. We also show how to obtain in each case the “spectral projectors” on each mode (at least formally).

Note that in both Paragraphs 3.2 and 3.3, we obtain the characteristic equation and the projectors by a rather simple method, based on substitutions and linear combinations. Such a method can be compared to Gauss’ algorithm for the computation of characteristic polynomials in the case of complex matrices. This method can be applied successfully here because of the simple structure of the propagator, and especially because of the fact that the principal symbols have coefficients that depend only on x_2 .

3.2. The Rossby modes. As pointed out in the previous paragraph, finding precisely the Rossby modes (up to an $O(\varepsilon^2)$ error) requires more intricate calculations than merely diagonalizing the matrix of principal symbols $A_0(x, \xi)$ given in (3.3). So let $i\tau$ be an eigenvalue of the propagator, assumed to be of the form $\tau = \varepsilon\tilde{\tau}_R + O(\varepsilon^2)$. Explicit computations lead to the subsystem

$$\begin{pmatrix} \varepsilon\bar{u}_1\xi_1 - \tau & \xi_1 \\ \xi_1 & \varepsilon\bar{u}_1\xi_1 - \tau \end{pmatrix} \begin{pmatrix} \rho \\ u_1 \end{pmatrix} = \begin{pmatrix} i\varepsilon\partial_2 u_2 \\ -i(b - \varepsilon^2\bar{u}'_1)u_2 \end{pmatrix}$$

and defining

$$\alpha_\tau(x_2, \xi_1) := \varepsilon\bar{u}_1\xi_1 - \tau \quad \text{and} \quad p_R(x_2, \xi_1) = -\xi_1^2 + \alpha_\tau^2$$

to the scalar equation

$$(3.5) \quad (\varepsilon\partial_2 p_R^{-1}(i\alpha_\tau\varepsilon\partial_2 + i\xi_1(b - \varepsilon^2\bar{u}'_1)) - bp_R^{-1}(i\xi_1\varepsilon\partial_2 + i\alpha_\tau(b - \varepsilon^2\bar{u}'_1)) + i\alpha_\tau) u_2 = 0$$

where ξ_1 is the Fourier variable corresponding to x_1/ε . From now on we assume that ξ_1 is fixed, and is bounded away from zero (recalling Assumption (2.3)). Note that equation (3.5) makes sense because we are assuming here that $\tau = \varepsilon\tilde{\tau}_R + O(\varepsilon^2)$, so p_R^{-1} is well defined. That would not be the case for the Poincaré modes (where $\tau = \pm\sqrt{|\xi|^2 + b^2(x_2)} + O(\varepsilon)$) so we shall use another subsystem in the next paragraph to deal with the Poincaré operators.

In order to derive a symbolic equation associated with the differential equation (3.5), we shall proceed by transforming (3.5) into a differential equation which is the left quantization (in the sense recalled in Appendix B) of a symbol, polynomial in ε and τ . This leads to a differential equation of the type

$$(3.6) \quad \mathcal{P}_2(x_2; \varepsilon, \tau)(\varepsilon\partial_2)^2 u_2 + \mathcal{P}_1(x_2; \varepsilon, \tau)\varepsilon\partial_2 u_2 + \mathcal{P}_0(x_2; \varepsilon, \tau)u_2 = 0,$$

where each $\mathcal{P}_j(x_2; \varepsilon, \tau)$ is a smooth function in x_2 , and has polynomial dependence in ε and in τ (precisely of degree at most 5 in ε and τ). This generalizes (3.1); one can compute in particular, using the fact that $\varepsilon\partial_2\alpha_\tau = O(\varepsilon^2)$, that

$$\begin{aligned} \mathcal{P}_2(x_2; \varepsilon, \tau) &= ip_R\alpha_\tau + O(\varepsilon^2), & \mathcal{P}_1(x_2; \varepsilon, \tau) &= O(\varepsilon^2), & \text{and} \\ \mathcal{P}_0(x_2; \varepsilon, \tau) &= ip_R(\xi_1\varepsilon b' - (b^2 + \xi_1^2)\alpha_\tau) + O(\varepsilon^2). \end{aligned}$$

The differential operator appearing on the left-hand side of (3.6) is the left quantization of the following symbol:

$$(3.7) \quad h(x_2, \xi_2; \varepsilon, \tau) := -\mathcal{P}_2(x_2; \varepsilon, \tau)\xi_2^2 + i\mathcal{P}_1(x_2; \varepsilon, \tau)\xi_2 + \mathcal{P}_0(x_2; \varepsilon, \tau)$$

which belongs for each τ to $S_2(g_\tau)$ for some function g_τ of the type

$$g_\tau(x_2, \xi_2) = (1 + \tau^5) (1 + b^2(x_2) + \xi_2^2)$$

recalling that \bar{u}_1 and \bar{u}'_1 are bounded from above, as well as Assumption (2.2).

Now we recall that $h(x_2, \xi_2; \varepsilon, \tau)$ is a polynomial of degree 5 in τ hence has five roots, among which 2 are actually spurious: these are of the form $\pm\xi_1 + O(\varepsilon)$, and they appear because we have multiplied the equation by a polynomial in τ which cancels at the point $\pm\xi_1$ at first order in ε . An easy computation allows to obtain the two other $O(1)$ roots, which are precisely the Poincaré roots $\pm\sqrt{|\xi|^2 + b^2(x_2)} + O(\varepsilon)$, and an asymptotic expansion allows also easily to derive the Rossby $O(\varepsilon)$ root: one finds

$$(3.8) \quad \begin{aligned} \tau_R(x_2, \xi_1, \xi_2; \varepsilon) &:= \varepsilon\tilde{\tau}_R(x_2, \xi_1, \xi_2; \varepsilon) + O(\varepsilon^2), \quad \text{where} \\ \tilde{\tau}_R(x_2, \xi_1, \xi_2) &:= \frac{b'(x_2)\xi_1}{\xi_1^2 + \xi_2^2 + b^2(x_2)} + \xi_1\bar{u}_1(x_2). \end{aligned}$$

Now that the root τ_R has been computed, our next task is to prove that there is a pseudodifferential operator $T_R = \varepsilon\tilde{T}_R$ whose principal symbol is precisely $\varepsilon\tilde{\tau}_R$ such that the following result holds. Actually this result is a direct consequence of Theorem 3 stated and proved in Appendix A: with the notation of Theorem 3, one has here $\nu = 1$ and

$$\partial_\tau h_0(x, \xi, 0) = -i\xi_1^2(\xi_1^2 + \xi_2^2 + b^2(x_2)).$$

Note that in the following statement, the time variable s is defined as $s = t/\varepsilon^2$.

Proposition 3.2 (The Rossby propagator). *Let $\tilde{\tau}_R$ be the symbol defined in (3.8). Then for any compact set \mathcal{K} satisfying Assumptions (2.3, 2.4) there exists a formally self-adjoint pseudo-differential operator \tilde{T}_R of principal symbol $\tilde{\tau}_R$ such that if φ_R is microlocalized in \mathcal{K} and solves*

$$(3.9) \quad \partial_s \varphi_R = i\varepsilon\tilde{T}_R \varphi_R,$$

then

$$\mathcal{P}_2(x_2; \varepsilon, \partial_s)(\varepsilon\partial_2)^2 \varphi_R + \mathcal{P}_1(x_2; \varepsilon, \partial_s)(\varepsilon\partial_2) \varphi_R + \mathcal{P}_0(x_2; \varepsilon, \partial_s) \varphi_R = O(\varepsilon^\infty).$$

Definition 3.3 (The Rossby operator). *We shall call Π_R the Rossby operator defined by*

$$\Pi_R := \begin{pmatrix} P_R^{-1} (i(-i\varepsilon^2\bar{u}_1\partial_1 - \varepsilon\tilde{T}_R)\varepsilon\partial_2 + \varepsilon\partial_1(b - \varepsilon^2\bar{u}'_1)) \\ -P_R^{-1}(-\varepsilon^2\partial_{12} + i(-i\varepsilon^2\bar{u}_1\partial_1 - \varepsilon\tilde{T}_R)(b - \varepsilon^2\bar{u}'_1)) \\ \text{Id} \end{pmatrix}$$

where $P_R := \varepsilon^2\partial_1^2 - (i\varepsilon^2\bar{u}_1\partial_1 + \varepsilon\tilde{T}_R)^2$.

Remark 3.4. (1) Notice that Π_R is well defined since the principal symbol of P_R is bounded away from below (see Appendix B).

(2) Proposition 4.6 proved in Section 4 indicates that if $\varphi_{R|t=0}$ is microlocalized in a compact set \mathcal{K}_0 satisfying (2.4), then the solution to (3.9) is microlocalized for all times $t \geq 0$ in a compact set \mathcal{K}_t .

- (3) *The above computations allow to formally recover the original shallow-water equation, up to $O(\varepsilon^\infty)$. Indeed retracing the steps which enabled us above to derive equation (3.6) (noticing that \tilde{T}_R has constant coefficients in x_1) shows that if φ^0 is a smooth function conveniently microlocalized, and if φ solves*

$$\begin{cases} \varepsilon \partial_t \varphi = i \tilde{T}_R \varphi \\ \varphi|_{t=0} = \varphi^0, \end{cases}$$

then the vector field $U := \Pi_R \varphi$ satisfies (2.1) up to $O(\varepsilon^\infty)$. This property will be made rigorous in Section 6.

3.3. The Poincaré modes. In this paragraph we shall follow the method used above in the case of Rossby modes to infer Poincaré propagators T_\pm and operators Π_\pm . Actually one cannot use precisely the same method since the symbol $(\varepsilon \bar{u}_1 \xi_1 - \tau)^2 - \xi_1^2$ may vanish when $\tau = \tau_\pm + O(\varepsilon)$. So we shall instead consider the subsystem

$$\begin{pmatrix} \varepsilon \bar{u}_1 \xi_1 - \tau & -i\varepsilon \partial_2 \\ -i\varepsilon \partial_2 & \varepsilon \bar{u}_1 \xi_1 - \tau \end{pmatrix} \begin{pmatrix} \rho \\ u_2 \end{pmatrix} = \begin{pmatrix} -\xi_1 u_1 \\ i b u_1 \end{pmatrix}$$

and the scalar equation

$$(3.10) \quad \left(\xi_1 p_P^{-1} (-\alpha_\tau \xi_1 - \varepsilon \partial_2 b) + \alpha_\tau + (b - \varepsilon^2 \bar{u}'_1) p_P^{-1} (\varepsilon \partial_2 - \alpha_\tau b) \right) u_1 = 0,$$

where as before $\alpha_\tau(x_2, \xi_1) := \varepsilon \bar{u}_1 \xi_1 - \tau$, and

$$p_P(x_2, \xi_1) := \varepsilon^2 \partial_2^2 + \alpha_\tau^2.$$

We notice here that α_τ^{-1} is well defined when $\tau = \tau_\pm$ since τ_\pm is bounded away from zero by the assumption on ξ_1 , and the same goes for $\tau_\pm^2 - \xi_2^2$ so p_P^{-1} is also well defined. Then it remains to follow the steps of Paragraph 3.2 to obtain a new scalar PDE of the same type as (3.6), as well as a symbol equation of the type (3.7):

$$(3.11) \quad \tilde{h}(x_2, \xi_2; \varepsilon, \tau) := -\tilde{\mathcal{P}}_2(x_2; \varepsilon, \tau) \xi_2^2 + i \tilde{\mathcal{P}}_1(x_2; \varepsilon, \tau) \xi_2 + \tilde{\mathcal{P}}_0(x_2; \varepsilon, \tau)$$

of which of course τ_\pm are roots up to $O(\varepsilon)$, and $\varepsilon \tau_R$ is a root up to $O(\varepsilon^2)$. Then the application of Theorem 3 implies a similar result to Proposition 3.2, noticing that with the notation of Theorem 3, $\nu = 0$ and

$$\partial_\tau h_0(x, \xi, \tau_\pm) = -2(\xi_1^2 + \xi_2^2 + b^2(x_2)).$$

Proposition 3.5 (The Poincaré propagator). *Let τ_\pm be the symbol defined in (3.4). Consider a compact set $\mathcal{K}_P \subset \mathbf{R}^* \times T^*\mathbf{R}$. Then there exists formally self-adjoint pseudo-differential operators T_\pm of principal symbols τ_\pm such that if φ_\pm solves*

$$\partial_s \varphi_\pm = iT_\pm \varphi_\pm,$$

and φ_\pm is micolocalized in $\mathbf{R} \times \mathcal{K}_P$, then

$$\tilde{\mathcal{P}}_2(x_2; \varepsilon, \partial_s)(\varepsilon \partial_2)^2 \varphi_\pm + \tilde{\mathcal{P}}_1(x_2; \varepsilon, \partial_s)(\varepsilon \partial_2) \varphi_\pm + \tilde{\mathcal{P}}_0(x_2; \varepsilon, \partial_s) \varphi_\pm = O(\varepsilon^\infty).$$

Definition 3.6 (The Poincaré operator). *We shall call Π_\pm the Poincaré operator defined by*

$$\Pi_\pm := \begin{pmatrix} P_\pm^{-1}(i(-i\varepsilon^2 \bar{u}_1 \partial_1 - T_\pm)\varepsilon \partial_1 + \varepsilon \partial_2 b) \\ \text{Id} \\ P_\pm^{-1}(i(-i\varepsilon^2 \bar{u}_1 \partial_1 - T_\pm)b - \varepsilon \partial_1 \varepsilon \partial_2) \end{pmatrix}$$

where $P_\pm := (i\varepsilon^2 \bar{u}_1 \partial_1 + T_\pm)^2 + \varepsilon^2 \partial_2^2$.

Remark 3.7. *As in Remark 3.4, one sees formally that if φ^0 is a smooth function conveniently microlocalized, and if φ solves*

$$\begin{cases} \varepsilon^2 \partial_t \varphi = iT_{\pm} \varphi \\ \varphi|_{t=0} = \varphi^0, \end{cases}$$

then the vector field $U := \Pi_{\pm} \varphi$ satisfies (2.1) up to $O(\varepsilon^\infty)$. This property will be made rigorous in Section 6.

4. STUDY OF THE ROSSBY WAVES

In this section we shall prove (1) of Theorem 1.

This section is organized as follows: we first identify (in Paragraph 4.1) the dynamics of Rossby waves by writing down a system of ODEs describing the trajectories in phase space. Their projection onto the physical space describes the propagation of the energy. To obtain the result, the goal is therefore to prove that there is a set Λ of initial positions, which generate trajectories remaining in a compact set of the physical space.

The main part of this section therefore consists in the analysis of the system (4.2) describing the trajectories. In Paragraph 4.2 we analyze the trajectories in the reduced phase space (x_2, ξ_2) and prove in particular that the projection of all trajectories onto the x_2 -axis is bounded — the trapping phenomenon in physical space therefore only depends on the behaviour of the x_1 variable. Moreover we describe the qualitative features of the trajectories (which can be periodic or exhibit some asymptotic behaviour for large times).

Paragraph 4.3 is then devoted to the study of the projection of the trajectories onto the x_1 -axis and in proving that trapping is equivalent to the fact that a certain function of the initial values of (x_1, x_2, ξ_1, ξ_2) vanishes. The codimension of the set of such initial values can be computed under some additional assumptions on the velocity profile \bar{u}_1 and on the Coriolis function b . We recall that the useful assumptions on \bar{u}_1 and b were given in the introduction of this article. It should be noticed that due to the particular geometry of our problem (which is translation invariant in the x_1 direction) the codimension of the set (if it is not empty) will always be at most 3. We shall prove it is smaller (equal to 1) in some cases.

4.1. The dynamical system.

For the time scale considered here, the propagation of energy by Rossby waves is given by the transport equation (see Appendix B)

$$\partial_t f + \{\tilde{\tau}_R, f\} = 0$$

where we recall that the Poisson bracket is given by

$$\{\tilde{\tau}_R, f\} := \nabla_{\xi} \tilde{\tau}_R \cdot \nabla_x f - \nabla_x \tilde{\tau}_R \cdot \nabla_{\xi} f,$$

while $\tilde{\tau}_R$ is the principal symbol of the Rossby mode computed in (3.8):

$$(4.1) \quad \tilde{\tau}_R(\xi_1, x_2, \xi_2) = \frac{b'(x_2)\xi_1}{\xi_1^2 + \xi_2^2 + b^2(x_2)} + \bar{u}_1(x_2) \xi_1.$$

As $\tilde{\tau}_R$ is a smooth function of (x_2, ξ_1, ξ_2) , the energy is propagated along the bicharacteristics, i.e. along the integral curves of the following system of ODEs:

$$\begin{cases} \dot{x}^t = \nabla_{\xi} \tilde{\tau}_R(\xi_1^t, x_2^t, \xi_2^t), & x^0 = (x_1^0, x_2^0) \\ \dot{\xi}^t = -\nabla_x \tilde{\tau}_R(\xi_1^t, x_2^t, \xi_2^t), & \xi^0 = (\xi_1^0, \xi_2^0). \end{cases}$$

Since the condition (2.3) avoids the set $\{\xi_1 = 0\}$, we can suppose that $\xi_1^0 \neq 0$. Moreover since $\tilde{\tau}_R$ does not depend on the variable x_1 , we find $\xi_1^t \equiv \xi_1^0$. The ODE to be studied is therefore

$$(4.2) \quad \begin{cases} \dot{x}_1^t = \bar{u}_1(x_2^t) + \frac{b'(x_2^t)(-\xi_1^2 + \xi_2^2 + b^2(x_2^t))}{(\xi_1^2 + \xi_2^2 + b^2(x_2^t))^2} \\ \dot{x}_2^t = \frac{-2b'(x_2^t)\xi_1\xi_2^t}{(\xi_1^2 + \xi_2^2 + b^2(x_2^t))^2} \\ \dot{\xi}_2^t = -\bar{u}'_1(x_2^t)\xi_1 + \frac{2b(x_2^t)b'(x_2^t)^2\xi_1}{(\xi_1^2 + \xi_2^2 + b^2(x_2^t))^2} - \frac{b''(x_2^t)\xi_1}{\xi_1^2 + \xi_2^2 + b^2(x_2^t)}. \end{cases}$$

Due to the assumptions on \bar{u}_1 and on b , the map $(x, \xi) \mapsto (\nabla_{\xi} \tilde{\tau}_R, -\nabla_x \tilde{\tau}_R)(x, \xi)$ is bounded, so the integral curves are globally defined in time. The strategy to study their qualitative behaviours is to first (in Section 4.2) consider the motion in the reduced phase space $(x_2, \xi_2) \in \mathbf{R}^2$ and then (in Section 4.3) to study the motion in the x_1 direction.

4.2. Trajectories in the reduced (x_2, ξ_2) phase space. In this section we study the trajectories in the reduced (x_2, ξ_2) phase space. We shall denote $\xi_1 := \xi_1^0$.

4.2.1. Energy surfaces. Since the Hamiltonian $\tilde{\tau}_R$ and ξ_1 are conserved along any trajectory, trajectories are submanifolds of

$$\mathcal{E}_{\tau, \xi_1} := \{(x_2, \xi_2) \in \mathbf{R}^2; \tilde{\tau}_R(\xi_1, x_2, \xi_2) = \tau\}.$$

In the following we shall note for any energy τ and any $\xi_1 \in \mathbf{R}^*$

$$V_{\tau, \xi_1}(x_2) := \frac{b'(x_2)\xi_1}{\tau - \bar{u}_1(x_2)\xi_1} - \xi_1^2 - b^2(x_2),$$

so that if $\mathcal{D} := \{x_2/V_{\tau, \xi_1}(x_2) \geq 0\}$, then $\mathcal{E}_{\tau, \xi_1} = \{(x_2, \pm\sqrt{V_{\tau, \xi_1}(x_2)}), x_2 \in \mathcal{D}\}$. Note that $V_{\tau, \xi_1}(x_2)$ becomes singular if x_2^t reaches a point x_2 such that $\tau = \bar{u}_1(x_2)\xi_1$.

Proposition 4.1. *The projection of $\mathcal{E}_{\tau, \xi_1}$ on the x_2 -axis is bounded.*

Proof. We recall that on $\mathcal{E}_{\tau, \xi_1}$ we have

$$\frac{b'(x_2)\xi_1}{\xi_1^2 + \xi_2^2 + b^2(x_2)} + \bar{u}_1(x_2)\xi_1 = \tau.$$

Suppose the trajectory in x_2 is not bounded, then in particular it escapes the support of \bar{u}_1 . In the case when $\tau \neq 0$, letting $|x_2|$ go to infinity yields a contradiction due to the assumptions on b . In the case $\tau = 0$, for x_2 out of the support of \bar{u}_1 we have

$$\frac{b'(x_2)\xi_1}{\xi_1^2 + \xi_2^2 + b^2(x_2)} = 0$$

and the only possibility is for x_2 to be fixed on a zero point of b' (hence in particular does not go to infinity). \square

Proposition 4.1 shows that to prove that some trajectories are trapped in physical space, it suffices to study their behaviour in the x_1 direction. However before doing so, let us prepare that study by classifying the trajectories in the reduced phase space.

Up to some change of parametrization, namely expressing time t as a function of x_2

$$dt = \pm \frac{b'(x_2)\xi_1}{(\tau - \xi_1 \bar{u}_1(x_2))^2 \sqrt{V_{\tau, \xi_1}(x_2)}} dx_2,$$

which is justified locally, and will give the convenient global behaviour by suitable gluing, we are brought back to the study of the hamiltonian system $\xi_2^2 - V_{\tau, \xi_1}(x_2)$ describing the motion of a particle in the potential $-V_{\tau, \xi_1}$.

For smooth potentials V such that $V(x) \rightarrow -\infty$ as $|x| \rightarrow \infty$, the possible behaviours of such a system are well-known and the trajectories are usually classified as follows (see for instance [1, 2, 3, 19, 21])

- periodic orbits;
- fixed points;
- homoclinic and heteroclinic orbits connecting unstable fixed points.

Here the situation is more complex insofar as V_{τ, ξ_1} admits singularities. We shall classify in the following the trajectories according to their motion in the x_2 variable.

4.2.2. *Periodic trajectories.* These correspond to the case when there exists an interval $[x_{min}, x_{max}]$ of \mathbf{R} with $x_{min} \neq x_{max}$, containing x_2^0 such that

- V_{τ, ξ_1} has no singularity and does not vanish on $]x_{min}, x_{max}[$;
- $V_{\tau, \xi_1}(x_{min}) = V_{\tau, \xi_1}(x_{max}) = 0$;
- the points x_{min} and x_{max} are reached in finite time.

The extremal points x_{min} and x_{max} are then turning points, meaning that the motion is periodic.

The fact that x_{min} and x_{max} are reached in finite time is equivalent to $V'_{\tau, \xi_1}(x_{min}) \neq 0$ and $V'_{\tau, \xi_1}(x_{max}) \neq 0$. Indeed if $V'_{\tau, \xi_1}(x_{max}) = 0$ (resp. $V'_{\tau, \xi_1}(x_{min}) = 0$), then $(x_{max}, 0)$ (resp. $(x_{min}, 0)$) is a fixed point, which contradicts the uniqueness given by the Cauchy-Lipschitz theorem. And conversely, if $V'_{\tau, \xi_1}(x_{max}) \neq 0$ (resp. $V'_{\tau, \xi_1}(x_{min}) \neq 0$), an asymptotic expansion in the vicinity of x_{max} (resp. x_{min}) shows that the extremal point is reached in finite time.

Definition 4.2. We will denote by \mathcal{P} the subset of the phase space $T^*\mathbf{R}^2$ consisting of initial data corresponding to periodic motions along x_2 .

A rather simple continuity argument allows to prove that \mathcal{P} is an open subset of $\mathbf{R} \times \mathbf{R}^* \times \mathbf{R}^2$. Denote indeed by $(\tilde{x}_1^0, \tilde{\xi}_1, \tilde{x}_2^0, \tilde{\xi}_2^0)$ any point of \mathcal{P} , and by \tilde{x}_{min} and \tilde{x}_{max} the extremal points of the corresponding (periodic) trajectory along x_2 . As $V'_{\tilde{\tau}, \tilde{\xi}_1}(\tilde{x}_{max}) \neq 0$ and V_{τ, ξ_1} is a smooth function of τ and ξ_1 outside from the closed subset of singularity points, the implicit function theorem gives the existence of a neighborhood of $(\tilde{\xi}_1, \tilde{\tau})$ such that there exists a unique x_{max} which satisfies $V_{\tau, \xi_1}(x_{max}) = 0$. Furthermore x_{max} depends continuously on τ and ξ_1 , in particular

$$V'_{\tau, \xi_1}(x_{max}) \neq 0.$$

Using the same arguments to build a suitable x_{min} , we finally obtain that there exists a neighborhood of $(\tilde{x}_1^0, \tilde{\xi}_1, \tilde{x}_2^0, \tilde{\xi}_2^0)$ for which the motion along x_2 is a non degenerate periodic motion. Moreover, x_{min} , x_{max} and also ξ_{min} , ξ_{max} and the period T depend continuously on the initial data.

Fixed points correspond to the degenerate case when $x_{min} = x_{max} = x_2^0$, which implies that either $\xi_2^0 = 0$ or $b'(x_2^0) = 0$. The latter case is completely characterized by the condition $b'(x_2^0) = 0$, so let us focus on the case when $b'(x_2^0) \neq 0$. Fixed points correspond then to local extrema of the potential V_{τ, ξ_1} , they can be either stable or unstable depending on the sign of V''_{τ, ξ_1} .

Stable fixed points are obtained as a limit of periodic orbits when the period $T \rightarrow 0$ (bottom of a potential well), whereas unstable fixed points are obtained in the limit $T \rightarrow \infty$ as explained below.

Stopping trajectories belong to the same energy surfaces as unstable fixed points and reach some unstable fixed point in infinite time : they correspond to the case when there exists an interval $[x_{min}, x_{max}]$ of \mathbf{R} containing x_2^0 such that

- V_{τ, ξ_1} has no singularity and does not vanish on $]x_{min}, x_{max}[$;
- x_{min} and x_{max} are either zeros or singularities of V_{τ, ξ_1} ;
- as $t \rightarrow \infty$, $x_2^t \rightarrow x_2^\infty$ such that $V'_{\tau, \xi_1}(x_2^\infty) = 0$ or $b'(x_2^\infty) = 0$.

They are in some sense also a degenerate version of periodic trajectories since for arbitrarily close initial data, one can obtain periodic orbits.

Definition 4.3. *We will denote by $\delta\mathcal{P}$ the subset of the phase space $T^*\mathbf{R}^2$ consisting in initial data corresponding to fixed points and stopping motions along x_2 .*

Using the characterization of the energy surfaces which carry such pathological motions, we can prove that $\delta\mathcal{P}$ is a codimension 1 subset of the phase space.

We first consider the energy surfaces containing a fixed point \tilde{x}_2 such that $b'(\tilde{x}_2) = 0$. We then have $\tau = \xi_1 \bar{u}_1(\tilde{x}_2)$. The corresponding set of initial data

$$\{(x_1^0, \xi_1, x_2^0, \xi_2^0) / \tilde{\tau}_R(\xi_1, x_2^0, \xi_2^0) = \xi_1 \bar{u}_1(\tilde{x}_2)\} \text{ is of codimension 1.}$$

As we have assumed that b has only a finite number of critical points, the union of these sets is still of codimension 1.

We then consider the energy surfaces containing a fixed point $(\tilde{x}_1, \tilde{\xi}_1, \tilde{x}_2, 0)$ such that $V_{\tilde{\tau}, \tilde{\xi}_1}(\tilde{x}_2) = V'_{\tilde{\tau}, \tilde{\xi}_1}(\tilde{x}_2) = 0$. We then have

$$-\bar{u}'_1(\tilde{x}_2) + \frac{2b(\tilde{x}_2)b'(\tilde{x}_2)}{(\tilde{\xi}_1^2 + b^2(\tilde{x}_2))^2} - \frac{b''(\tilde{x}_2)}{\tilde{\xi}_1^2 + b^2(\tilde{x}_2)} = 0.$$

For each \tilde{x}_2 , there are at most two values of $\tilde{\xi}_1^2$ such that the previous quantity vanishes. We therefore deduce that

$$\left\{ (x_1^0, \xi_1, x_2^0, \xi_2^0) / \tilde{\tau}_R(\xi_1, x_2^0, \xi_2^0) = \frac{b'(\tilde{x}_2)\xi_1}{\xi_1^2 + b^2(\tilde{x}_2)} + \bar{u}_1(\tilde{x}_2) \xi_1 \quad \text{and} \right.$$

$$\left. -\bar{u}'_1(\tilde{x}_2) + \frac{2b(\tilde{x}_2)b'(\tilde{x}_2)}{(\tilde{\xi}_1^2 + b^2(\tilde{x}_2))^2} - \frac{b''(\tilde{x}_2)}{\tilde{\xi}_1^2 + b^2(\tilde{x}_2)} = 0 \right\} \text{ is of codimension 1.}$$

It consists indeed of at most eight manifolds, each one of them parametrized by the real parameter \tilde{x}_2 .

In the sequel, we will avoid these pathological motions assuming that the initial data is microlocalized outside from $\delta\mathcal{P}$.

4.2.3. Asymptotic trajectories. These correspond to the case when there exists an interval $[x_{min}, x_{max}]$ of \mathbf{R} containing x_2^0 such that

- V_{τ, ξ_1} has no singularity and does not vanish on $]x_{min}, x_{max}[$;
- x_{min} and x_{max} are either zeros or singularities of V_{τ, ξ_1} ;
- as $t \rightarrow \infty$, $x_2^t \rightarrow x_2^\infty$ where $x_2^\infty \in \{x_{min}, x_{max}\}$ is a pole of multiplicity 1 of V_{τ, ξ_1} . For the sake of simplicity, we further impose that $b'(x_2^\infty) \neq 0$.

This situation therefore corresponds to a motion which is not periodic.

Depending on the sign of ξ_1 , an asymptotic trajectory will either encounter a turning point and then converge asymptotically to the singular point, or converge monotonically to the limiting point.

As x_2^∞ is such that $\bar{u}_1(x_2^\infty) = \tau/\xi_1$, one has

- either $\tau \neq 0$ and x_2^∞ belongs to the support of \bar{u}_1 ,
- or $\tau = 0$ and $\bar{u}_1(x_2^0) = -b'(x_2^0)/(\xi_1^2 + \xi_2^2 + b^2(x_2^0)) \neq 0$, meaning that x_2^0 belongs to the support of \bar{u}_1 . Therefore, x_2^∞ is either $\min\{y > x_2^0/\bar{u}_1(y) = 0\}$ or $\max\{y < x_2^0/\bar{u}_1(y) = 0\}$, in particular x_2^∞ belongs to the support of \bar{u}_1 .

Definition 4.4. We will denote by \mathcal{A} the subset of the phase space consisting of initial data corresponding to asymptotic motions along x_2 .

The same kind of arguments as in the previous paragraph allow to prove that \mathcal{A} is an open subset of the phase space $\mathbf{R} \times \mathbf{R}^* \times \mathbf{R}^2$. Consider indeed some $(\tilde{x}_1^0, \tilde{\xi}_1, \tilde{x}_2^0, \tilde{\xi}_2^0) \in \mathcal{A}$, and the corresponding asymptotic point \tilde{x}_2^∞ . As x_2^∞ is a pole of multiplicity 1 of $V_{\tau, \tilde{\xi}_1}$, the implicit function theorem shows that $V_{\tau, \tilde{\xi}_1}^{-1}$ admits locally a unique zero, which depends continuously on τ and $\tilde{\xi}_1$. Using further the continuity of the possible turning point, we get that \mathcal{A} contains a neighborhood of $(\tilde{x}_1^0, \tilde{\xi}_1, \tilde{x}_2^0, \tilde{\xi}_2^0)$.

Moreover, we can obtain bounds on the expansion (with respect to time) of any compact subset of \mathcal{A} . Here we will focus on the growth of ξ_2^t , and proves that it depends continuously on the initial data in \mathcal{A} . Without loss of generality, we can consider the case when the asymptotic point is x_{max} . Then we recall that

$$\lim_{t \rightarrow +\infty} \xi_2^t = \infty \quad \text{and} \quad \lim_{t \rightarrow +\infty} x_2^t = x_2^\infty, \quad \text{with} \quad \xi_1 \bar{u}_1(x_2^\infty) = \tau.$$

As x tends to x_2^∞ , we have (recalling that $b'(x_2^\infty) \neq 0$)

$$V_\tau(x) \sim -\frac{b'(x_2^\infty)}{\bar{u}'_1(x_2^\infty)}(x - x_2^\infty)^{-1}.$$

This implies that

$$|\xi_2^t|^2 \sim -\frac{b'(x_2^\infty)}{\bar{u}'_1(x_2^\infty)}(x - x_2^\infty)^{-1} \quad \text{and} \quad \dot{x}_2^t \sim -2b'(x_2^\infty)\xi_1 \frac{\xi_2^t}{|\xi_2^t|^4} \sim 2\frac{|\bar{u}'_1(x_2^\infty)|^{3/2}|\xi_1|}{|b'(x_2^\infty)|^{1/2}}(x_2^\infty - x)^{3/2}.$$

By integration, we get

$$x_2^t \sim x_2^\infty + C_1 t^{-2}, \quad \xi_2^t \sim C_2 t$$

where the constants C_1 and C_2 depend continuously on x_2^∞ , and consequently on the initial data.

Singular trajectories are in some sense a degenerate version of the asymptotic trajectories above: they correspond to the case when there exists an interval $[x_{min}, x_{max}]$ of \mathbf{R} containing x_2^0 such that

- V_{τ, ξ_1} has no singularity and does not vanish on $]x_{min}, x_{max}[$;
- x_{min} and x_{max} are either zeros or singularities of V_τ ;
- as $t \rightarrow \infty$, $x_2^t \rightarrow x_2^\infty$ where $x_2^\infty \in \{x_{min}, x_{max}\}$ is either a singularity of order greater than 1 of V_{τ, ξ_1} , or a singularity which is also a zero of b' .

As previously, it is easy to check that x_2^∞ is necessarily in the support of \bar{u}_1 . Furthermore, we have $\bar{u}'_1(x_2^\infty) = 0$. Indeed, in the vicinity of x_2^∞ , we have

$$V_{\tau, \xi_1}(y) \sim \frac{b'(x_2^\infty)\xi_1}{\bar{u}(x_2^\infty) - \bar{u}(y)}$$

which has a singularity of order greater than 1, or a singularity which is also a zero of b' if and only if $\bar{u}'(x_2^\infty) = 0$.

Definition 4.5. *We will denote by $\delta\mathcal{A}$ the subset of the phase space consisting of initial data corresponding to singular motions along x_2 .*

The previous condition $\bar{u}'_1(x_2^\infty) = 0$ shows that $\delta\mathcal{A}$ is included in the union of energy surfaces

$$\mathcal{E}_{\tau, \xi_1} \text{ with } \tau = \bar{u}_1(y) \text{ for some } y \in \text{supp}(\bar{u}_1) \text{ such that } \bar{u}'_1(y) = 0.$$

As \bar{u}'_1 has only a finite number of zeros in the support of \bar{u}_1 , this implies that $\delta\mathcal{A}$ is a codimension 1 subset of the phase space.

Gathering all the previous results together, we obtain the following

Proposition 4.6. *The phase space $\mathbf{R} \times \mathbf{R}^* \times \mathbf{R}^2$ admits the following partition*

$$\mathbf{R} \times \mathbf{R}^* \times \mathbf{R}^2 = \mathcal{P} \cup \mathcal{A} \cup \Sigma$$

where \mathcal{P} and \mathcal{A} are the open sets of initial data giving rise respectively to periodic motions and asymptotic motions along x_2 , and $\Sigma := \delta\mathcal{P} \cup \delta\mathcal{A}$ is the codimension 1 set of initial data giving rise to pathological motions along x_2 .

For any compact set $K \subset \mathcal{A} \cup \mathcal{P}$, and for any time $T > 0$, we further have a uniform bound on the image of K by the flow up to time T .

Proof. The first statement just tells us that all trajectories belong to one of the four categories described above. Indeed, for any initial data $(x_1^0, \xi_1, x_2^0, \xi_2^0)$, one has

$$V_{\tau, \xi_1}(x_2^0) = (\xi_2^0)^2 \geq 0, \text{ and } V_{\tau, \xi_1}(x_2) \rightarrow -\infty \text{ as } |x_2| \rightarrow \infty,$$

so that there exists an interval $[x_{min}, x_{max}]$ of \mathbf{R} containing x_2^0 such that V_{τ, ξ_1} has no singularity and does not vanish on $]x_{min}, x_{max}[$, and x_{min} and x_{max} are either zeros or singularities of V_{τ, ξ_1} .

The second statement is then a simple corollary of the continuity results established on \mathcal{P} and \mathcal{A} . \square

4.3. Analysis of the trajectories in the x_1 direction: trapping phenomenon. Proposition 4.1 states that the trajectories are always bounded in the x_2 variable, so it remains to study the x_1 variable. Our aim is to find a set $\Lambda \subset \mathbf{R} \times \mathbf{R}^* \times \mathbf{R}^2$ such that any initial data $(x_1^0, \xi_1, x_2^0, \xi_2^0)$ in Λ gives rise to a trapped trajectory, meaning that

$$\int_0^t \dot{x}_1^s ds \text{ is uniformly bounded for } t \in \mathbf{R}^+.$$

4.3.1. *The criterion of capture.* Let us prove the following result.

Proposition 4.7. *A necessary and sufficient condition for a trajectory with initial data in $\mathcal{A} \cup \mathcal{P}$ to be trapped is*

$$\lim_{t \rightarrow T} \frac{1}{t} \int_0^t \dot{x}_1^s ds = 0,$$

where

- T denotes the (finite) period of the motion along x_2 in the periodic case;
- $T = +\infty$ in the asymptotic case.

Proof. We will study separately the different situations described in the previous section, namely the case of periodic and asymptotic trajectories in (x_2, ξ_2) .

• *In the case of a periodic motion* in (x_2, ξ_2) of period $T > 0$, the function \dot{x}_1^t is also periodic, with the same period. Writing

$$x_1^t = x_1^0 + \int_0^t \left(\dot{x}_1^s - \frac{1}{T} \int_0^T \dot{x}_1^{s'} ds' \right) ds + \frac{t}{T} \int_0^T \dot{x}_1^s ds$$

we see that depending on the average of \dot{x}_1^t over $[0, T]$, x_1^t is either a periodic function, or the sum of a periodic function and a linear function. It follows that trapped trajectories are characterized by the criterion $\int_0^T \dot{x}_1^t dt = 0$.

Note that, depending on the period T , the trajectory can explore a domain in x_1 the size of which may be very large. Nevertheless the continuity statement in Proposition 4.6 shows that we have a uniform bound on this size on any compact subset of \mathcal{P} .

• For asymptotic motions, we need to check that

$$\dot{x}_1^t - \bar{u}_1(x_2^\infty)t \text{ is integrable at infinity.}$$

We have indeed

$$\dot{x}_1^t = \bar{u}_1(x_2^t) + \frac{b'(x_2^t)(-\xi_1^2 + \xi_2^{t2} + b^2(x_2^t))}{(\xi_1^2 + \xi_2^{t2} + b^2(x_2^t))^2},$$

which, together with the asymptotic expansions of x_2^t and ξ_2^t obtained in the previous section, implies that

$$\dot{x}_1^t = \bar{u}_1(x_2^\infty) + O(t^{-2}).$$

It is then clear that the trajectory is trapped if and only if

$$\bar{u}_1(x_2^\infty) = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \dot{x}_1^s ds = 0.$$

Using again the continuity statement in Proposition 4.6, we also get a uniform bound on the size of the time evolution of any compact subset of \mathcal{A} . This ends the proof of Proposition 4.7. \square

Remark 4.8. *Note that, in the case of a singular asymptotic motion along x_2 , the criterion of capture is equivalent to*

$$\bar{u}(x_2^\infty) = \tau = 0.$$

In the case of periodic motions along x_2 , this criterion - even more complicated - can also be expressed in terms of the initial data. The map x_2^t is indeed a smooth bijection from a time interval $]t_0, t_0 + \frac{T}{2}[$ to $]x_{min}, x_{max}[$, so

$$\int_0^T \dot{x}_1^t dt = 2 \int_{t_0}^{t_0+T/2} \dot{x}_1^t dt = 2 \int_{x_{min}}^{x_{max}} \dot{x}_1^t|_{x_2^t=x_2} (\dot{x}_2^t|_{x_2^t=x_2})^{-1} dx_2.$$

The trajectory is therefore trapped if and only if

$$(4.3) \quad \int_{x_{min}}^{x_{max}} \left[\xi_1^2 - \frac{(\tau/\xi_1) b'(y)}{2((\tau/\xi_1) - \bar{u}_1(y))^2} \right] V_{\tau, \xi_1}(y)^{-1/2} dy = 0$$

(notice that since we are only looking for the function to vanish, we replace ξ_2 by $\sqrt{V_{\tau, \xi_1}}$ without discussing the sign). Such a formula would be useful to investigate numerically the initial data giving rise to trapped trajectories. We shall also use it when investigating in more detail the case of the betaplane approximation, in Paragraph 4.4.

4.3.2. *Exhibiting a subset of Λ of codimension 1.* Let us prove the following proposition.

Proposition 4.9. *Let $\bar{u}_1 \in C_c^\infty(\mathbf{R})$ be a function which is not identically positive, with some zero of finite multiplicity. Then the set Λ_{sing} consisting of data in Λ giving rise to singular trajectories, is nonempty. It contains a submanifold of $\mathbf{R} \times \mathbf{R}^* \times \mathbf{R}^2$ which is of codimension 1.*

Proof. Without loss of generality, we can assume that there exist $y_1 < y_2$ where \bar{u}_1 vanishes, with $\bar{u}'_1(y_2) > 0$ and $\bar{u}_1(y) < 0$ on $]y_1, y_2[$.

As we want to study trapped asymptotic trajectories, we will restrict our attention to the case when $\tau = 0$, which is a necessary condition for asymptotic trajectories to be trapped. Extremal points of the trajectories are then defined in terms of the function

$$V_{0, \xi_1}(y) = -\frac{b'(y)}{\bar{u}_1(y)} - b(y)^2 - \xi_1^2.$$

More precisely, if we introduce the auxiliary function

$$\varrho(y) := -\frac{b'(y)}{\bar{u}_1(y)} - b(y)^2, \quad y \in]y_1, y_2[,$$

we obtain

- turning (or stopping) points y_s if $\rho(y_s) = \xi_1^2$;
- singular points y_s if $\lim_{y \rightarrow y_s} \rho(y) = +\infty$ or equivalently $\bar{u}_1(y_s) = 0$.

By definition of y_1 and y_2 , one has

$$\lim_{y \rightarrow y_1+} \varrho(y) = +\infty \quad \text{and} \quad \lim_{y \rightarrow y_2-} \varrho(y) = +\infty.$$

Let us define $N := \max(0; \inf_{y \in]y_1, y_2[} \varrho(y)) \in \mathbf{R}_+$. For ξ_1 such that $\xi_1^2 \geq N$, we then define

$$h(\xi_1) := \sup \{ y \in]-\infty, y_2[; \varrho(y) \leq \xi_1^2 \} \in]y_1, y_2[.$$

We therefore have that

$$\forall y \in]h(\xi_1), y_2[, \quad y \text{ is neither a turning point nor a singular point.}$$

As h is a decreasing function on $] -\infty, -\sqrt{N}]$, all $\xi_1 \in] -\infty, -\sqrt{N}]$ except a countable number are continuity points. Choose then some $\tilde{\xi}_1$ to be a continuity point of h and $\tilde{x}_2^0 \in]h(\tilde{\xi}_1), y_2[$. By continuity of h , there exists a neighborhood \tilde{V} of $(\tilde{x}_2^0, \tilde{\xi}_1)$ such that

$$\forall (x_2^0, \xi_1) \in \tilde{V}, \quad x_2^0 - h(\xi_1) > 0.$$

The set

$$\left\{ (x_1^0, \xi_1, x_2^0, \sqrt{\varrho(x_2^0) - \xi_1^2}); (x_1^0, \xi_1, x_2^0) \in \mathbf{R} \times \tilde{V} \right\},$$

is clearly a submanifold of $\mathbf{R} \times \mathbf{R}^* \times \mathbf{R}^2$ having codimension 1. Furthermore, for any initial data in this set, we have $\dot{x}_2|_{t=0} > 0$ and a simple connexity argument shows that x_2^t is an increasing function of time. In particular $x_2^t \rightarrow y_2$ as $t \rightarrow \infty$. This proves Proposition 4.9 \square

4.4. Some examples in the betaplane approximation.

We are concerned here with the betaplane approximation, that is when $b(x_2) \equiv \beta x_2$. One has therefore

$$\tilde{\tau}_R(\xi_1, x_2, \xi_2) = \bar{u}_1(x_2)\xi_1 + \frac{\beta\xi_1}{\xi_1^2 + \xi_2^2 + \beta^2 x_2^2}$$

and with the notation of Paragraph 4.2,

$$V_{\tau, \xi_1}(x_2) = \frac{\beta\xi_1}{\tau} - \xi_1^2 - \beta^2 x_2^2.$$

- In the absence of convection, one can characterize exactly the set Λ of initial data giving rise to trapped trajectories. Recalling that the energy surface $\mathcal{E}_{\tau, \xi_1}$ is given by

$$\mathcal{E}_{\tau, \xi_1} := \{ (x_2, \xi_2) / \tilde{\tau}_R(\xi_1, x_2, \xi_2) = \tau \},$$

one can notice that for τ such that $\tau \xi_1^{-1} \in]0, \beta \xi_1^{-2}]$, the energy surface $\mathcal{E}_{\tau, \xi_1}$ is simply the ellipse

$$\{ (x_2, \xi_2) / \xi_2^2 + \beta^2 x_2^2 = \beta \xi_1 \tau^{-1} - \xi_1^2 \}.$$

Now let us go through the previous analysis and study the various trajectories that can be produced in this situation.

One notices easily that *fixed points* correspond to $x_2(t) = \xi_2(t) = 0$, with $x_1(t) = x_1^0$.

There are no *asymptotic trajectories* (singular trajectories would correspond to $\tau = 0$, which is not possible here since $\xi_1 \neq 0$).

Finally let us consider *periodic trajectories*. In order to get trapping one must check that the criterion (4.3) is satisfied. If the energy level τ and the frequency ξ_1 are fixed, we have to test if

$$\int_{x_{min}}^{x_{max}} \frac{\beta \left(\frac{\beta \xi_1}{\tau} - 2 \xi_1^2 \right)}{\sqrt{\frac{\beta \xi_1}{\tau} - \xi_1^2 - \beta^2 x_2^2}} dx_2 = 0.$$

Let us choose for instance any $(x_1, \xi_1) \in \mathbf{R} \times \mathbf{R}^*$ and define the energy level

$$\tau = \frac{\beta}{2\xi_1}.$$

Then the integral is identically equal to zero hence the corresponding trajectory is trapped. It corresponds to $x_1^t = x_1^0$, and to x_2^t, ξ_2^t satisfying for all times

$$\xi_2^{t^2} + \beta^2 x_2^{t^2} = \xi_1^2.$$

Such an explicit characterization cannot be obtained if there is some convection, in particular in physically relevant situations. Nevertheless, we are able to prove that, under suitable assumptions, Λ is not empty, and more precisely that it contains both singular trapped trajectories and periodic trapped trajectories.

- Proposition 4.9 shows that under the assumption that $\bar{u}_1 \in C_c^\infty(\mathbf{R})$ is a function which is not identically positive, with some zero of finite multiplicity, Λ contains a subset of codimension 1 of initial data giving rise to singular trapped trajectories.

- Finally let us construct periodic trajectories. We suppose to simplify the notation that $\beta = 1$, and that \bar{u}_1 has a local maximum at zero, with, say

$$0 < \bar{u}_1(0) < 2/3 \quad \text{and} \quad \bar{u}_1''(0) < -6.$$

Then we shall prove that the set Λ_{per} is nonempty and contains a submanifold of $\mathbf{R} \times \mathbf{R}^* \times \mathbf{R}^2$ which is of codimension 1. Let τ be an energy level. We notice that for η small enough and $y \in]-\eta, \eta[$

$$\bar{u}_1(y) < \bar{u}_1(0) < \frac{\tau}{\xi_1}$$

so there is no singular point for V_{τ, ξ_1} in the interval $]-\eta, \eta[$. Now let us show that there are two turning points inside $]-\eta, \eta[$ for η small enough; this will imply that there is a periodic trajectory of energy τ . Define the function

$$H_{\xi_1}(y) := \frac{\tau}{\xi_1} - \frac{1}{\xi_1^2 + y^2} - \bar{u}_1(y).$$

Then $H'_{\xi_1}(0) = 0$ and an easy computation shows that if η is chosen small enough, then for all $(y, \xi_1) \in]-\eta, \eta[\times [1, +\infty[$,

$$H''_{\xi_1}(y) = \frac{2\xi_1^2 - 8y^2}{(\xi_1^2 + y^2)^3} - \bar{u}_1''(y) \geq 1/2 > 0.$$

Now let us choose η such that

$$\sup_{]-\eta, \eta[} \bar{u}_1''(y) < -3,$$

and let us define the number $\delta := 1 - \frac{\eta^2}{8}$. We choose from now on $\xi_1 > 0$ and the the energy level

$$\tau \equiv \tau(\xi_1) := \bar{u}_1(0)\xi_1 + \frac{\delta}{\xi_1}$$

so that $H_{\xi_1}(0) = -\frac{1-\delta}{\xi_1^2} < 0$. The function H_{ξ_1} is decreasing on the interval $]-\eta, 0[$ and increasing on the interval $]0, \eta[$; moreover

$$H_{\xi_1}(\pm\eta) \geq H_{\xi_1}(0) + \frac{\eta^2}{4} = -(1-\delta) + \frac{\eta^2}{4} \geq \frac{\eta^2}{8} > 0.$$

It follows that, for all $\xi_1 \in [1, +\infty[$, there are two points $x_{min}(\xi_1)$ (in $]-\eta, 0[$) and $x_{max}(\xi_1)$ (in $]0, \eta[$) satisfying the requirements of periodic trajectories, in the sense of Section 4.2.2.

According to the criterion (4.3), we define the following function on $[1, +\infty[$:

$$G(\xi_1) := \int_{x_{min}(\xi_1)}^{x_{max}(\xi_1)} \left[\xi_1^2 - \frac{(\tau/\xi_1)}{2((\tau/\xi_1) - \bar{u}_1(y))^2} \right] V_{\tau, \xi_1}(y)^{-1/2} dy,$$

and let us prove it vanishes. We notice that recalling that $\bar{u}_1(0) < 2/3$,

$$\begin{aligned} G(1) &\geq \int_{x_{min}(1)}^{x_{max}(1)} \left[1 - \left(\bar{u}_1(0) + \delta \right) \frac{1}{2\delta^2} \right] V_{\tau, \xi_1}(y)^{-1/2} dy \\ &\geq \frac{6\delta^2 - 2 - 3\delta}{6\delta^2} \int_{x_{min}(1)}^{x_{max}(1)} V_{\tau, \xi_1}(y)^{-1/2} dy > 0 \end{aligned}$$

since $\eta \in]0, 1[$. On the other hand, when ξ_1 goes to $+\infty$, we find that $x_{min}(\xi_1) \rightarrow 0$ and $x_{max}(\xi_1) \rightarrow 0$ so that recalling that $\tau(\xi_1) = \bar{u}_1(0)\xi_1 - \delta/\xi_1$,

$$G(\xi_1) \sim -\bar{u}_1(0) \frac{\xi_1^4}{2\delta^2} \int_{x_{min}(\xi_1)}^{x_{max}(\xi_1)} V_{\tau, \xi_1}(y)^{-1/2} dy,$$

so $\lim_{\xi_1 \rightarrow +\infty} G(\xi_1) < 0$. By construction, the function G is smooth so there is some $\tilde{\xi}_1$ belonging to $]1, +\infty[$ such that $G(\tilde{\xi}_1) = 0$. We have generically (in τ) $G'(\tilde{\xi}_1) \neq 0$. This implies that $\Lambda_{per} \neq \emptyset$, where Λ_{per} is the subset of Λ giving rise to periodic trajectories, and that Λ_{per} contains a submanifold of $\mathbf{R} \times \mathbf{R}^* \times \mathbf{R}^2$ which is of codimension 1.

5. STUDY OF THE POINCARÉ WAVES

5.1. **The strategy.** In this section we want to prove the dispersion property of the Poincaré polarisation, namely result (2) of Theorem 1.

We recall that the principal symbols τ_{\pm} are of order one, so one needs to study diffractive-type propagation, on a time scale of the order $1/\varepsilon^2$. Computing the bicharacteristics, as in the previous section, is therefore not enough to understand the classical flow. We shall instead rely on a spectral argument to prove that Poincaré waves do disperse, and escape from any compact set in the physical space.

The first step consists in taking the Fourier transform in x_1 (recalling that the problem is invariant by translations in the x_1 direction). We also recall that the data is microlocalized on a compact set \mathcal{C} such that ξ_1 is bounded away from zero. Since $\tau_{\pm} = \pm\sqrt{\xi_2^2 + b^2(x_2) + \xi_1^2}$, functional calculus implies that one can find classical pseudo-differential operators $H_{\pm}(\xi_1)$ of principal symbols $\xi_2^2 + b^2(x_2)$ such that $T_{\pm} = \pm\sqrt{H_{\pm}(\xi_1) + \xi_1^2}$.

Let us now call $\lambda_{\pm}^k(\xi_1)$ and $\varphi_{\pm}^k(\xi_1; x_2)$ the eigenvalues and eigenfunctions of $H_{\pm}(\xi_1)$. The following proposition will be proved in the next paragraph.

Proposition 5.1. *Let ϕ be an eigenfunction of $H_{\pm}(\xi_1)$, microlocalized on an energy surface which intersects \mathcal{C} . Then ϕ and its associate eigenvalue λ are C^{∞} functions of ξ_1 . Moreover $\frac{1}{\varepsilon}\partial_{\xi_1}\lambda$ is bounded on compact sets in ξ_1 .*

Assuming that proposition, the proof of result (2) of Theorem 1 is achieved through the following steps:

- (1) Take the Fourier transform of the initial data in ξ_1 , since the Hamiltonian is a Fourier multiplier in x_1 ;
- (2) In order to localize the x_1 variable (which has disappeared in the previous step), write a decomposition of the initial data into coherent states, in Fourier variables.
- (3) Finally write the solution to the evolution equation using a decomposition onto the eigenfunctions $\varphi_{\pm}^k(\xi_1; x_2)$.
- (4) Use a stationary phase lemma (using Proposition 5.1 to justify the multiple integrations by parts) to conclude.

5.2. **Proof of Theorem 1(2) assuming Proposition 5.1.** Let us now carry out this program. We consider an initial data denoted φ^0 , microlocalized in \mathcal{C} . One can take the Fourier transform in x_1 which gives

$$\varphi^0(x) = \frac{1}{\sqrt{2\pi\varepsilon}} \int \hat{\varphi}^0(\xi_1, x_2) e^{-i\frac{x_1\xi_1}{\varepsilon}} d\xi_1.$$

Now let us consider a coherent state (in Fourier variables) at (q, p) (see Appendix B), that is:

$$(5.1) \quad \varphi_{qp}(\xi_1) := \frac{1}{(\pi\varepsilon)^{\frac{1}{4}}} e^{i\frac{\xi_1 q}{\varepsilon}} e^{-\frac{(\xi_1 - p)^2}{2\varepsilon}}.$$

After decomposition onto coherent states we get

$$\varphi^0(x) = \frac{1}{\sqrt{2\pi\varepsilon}} \frac{1}{(\pi\varepsilon)^{\frac{1}{4}}} \int \hat{\varphi}^0(q, p, x_2) e^{i\frac{\xi_1(q-x_1)}{\varepsilon}} e^{-\frac{(\xi_1-p)^2}{2\varepsilon}} dq dp d\xi_1$$

where

$$\tilde{\varphi}^0(q, p, x_2) := (\varphi_{qp} |\hat{\varphi}^0(\cdot, x_2)|)_{L^2}.$$

We notice that the integral over p and q is, modulo $O(\varepsilon^\infty)$, on a compact domain due to the microlocalization assumption on φ^0 . Finally decomposing onto the eigenfunctions $\varphi_\pm^k(\xi_1; x_2)$ gives

$$\varphi^0(x) = \frac{1}{\sqrt{2\pi\varepsilon}} \frac{1}{(\pi\varepsilon)^{\frac{1}{4}}} \sum_k \int \bar{\varphi}^0(q, p; k, \xi_1, x_2) \varphi_\pm^k(\xi_1; x_2) e^{i\frac{\xi_1(q-x_1)}{\varepsilon}} e^{-\frac{(\xi_1-p)^2}{2\varepsilon}} dq dp d\xi_1$$

where

$$\bar{\varphi}^0(q, p; k, \xi_1, x_2) := \left(\varphi_\pm^k(\xi_1; \cdot) | \tilde{\varphi}^0(q, p, \cdot) \right)_{L^2}.$$

Note that the dependence of $\bar{\varphi}^0$ on ξ_1 is only through the eigenfunction φ_\pm^k , so $\bar{\varphi}^0$ depends smoothly on ξ_1 , as stated in Proposition 5.1.

The sum over k contains $O(\varepsilon^{-1})$ terms, due to the fact that $\lambda_\pm^k(\xi_1)$ remains in a finite interval (this will be made more precise in the next section, see Remark 5.3). Now it remains to propagate at time t/ε^2 this initial data, which gives rise to the following expression:

$$\frac{1}{\sqrt{2\pi\varepsilon}} \frac{1}{(\pi\varepsilon)^{\frac{1}{4}}} \sum_k \int \bar{\varphi}^0(q, p; k, \xi_1, x_2) \varphi_\pm^k(\xi_1; x_2) e^{i\frac{\xi_1(q-x_1)}{\varepsilon}} e^{-\frac{(\xi_1-p)^2}{2\varepsilon}} e^{\pm i(\lambda_\pm^k(\xi_1) + \xi_1^2) \frac{1}{2} \frac{t}{\varepsilon^2}} dq dp d\xi_1$$

The stationary phase lemma then gives that this integral is $O(\varepsilon^\infty)$ except if there exists a stationary point, given by the conditions:

$$\xi_1 = p \text{ and } \varepsilon(x_1 - q) \pm \frac{2\xi_1 + \partial_{\xi_1} \lambda_\pm^k(\xi_1)}{2\sqrt{\lambda_\pm^k(\xi_1) + \xi_1^2}} t = 0.$$

The second condition gives

$$2\sqrt{\lambda_\pm^k(\xi_1) + \xi_1^2}(x_1 - q) = \mp \frac{1}{\varepsilon} \left(2p + \partial_{\xi_1} \lambda_\pm^k(\xi_1) \right) t.$$

Therefore, since $p \neq 0$ and the λ_\pm^k 's are bounded, with $\partial_{\xi_1} \lambda_\pm^k(\xi_1) = O(\varepsilon)$, there is no critical point for x_1 in a compact set. Proposition 5.1 therefore allows to apply the stationary phase lemma and to conclude the proof of result (2) of Theorem 1.

Notice that the (fixed) losses in ε (namely the negative powers of ε appearing in the integrals and the number of k 's in the sum) are compensated by the fact that the result is $O(\varepsilon^\infty)$; it is important at this point that as noticed above, the function $\bar{\varphi}^0$ depends smoothly on ξ_1 . One could indeed be tempted to perform integrations by parts in q (since ξ_1 does not vanish) but the apparent gain in ε would automatically be lost by the integration in q of $\bar{\varphi}^0$.

5.3. Proof of Proposition 5.1. The first step of the proof consists in using the theory of normal forms in order to reduce the problem to the study of functions of the harmonic oscillator (as in Bohr-Sommerfeld quantization, paying special attention to the dependence on ξ_1). The second step then consists in checking that the eigenvectors and eigenvalues have the required dependence on ξ_1 .

In the following we shall only deal with H_+ to simplify, and we shall write $H := H_+$.

The first step relies on the following lemma.

Lemma 5.2. *Suppose that ξ_1 lies in a compact set away from zero. There is an elliptic Fourier Integral Operator $U : \mathcal{D}(\mathbf{R}) \rightarrow \mathcal{D}(\mathbf{R})$, independent of ξ_1 , and a pseudodifferential operator $V(\xi_1)$ with C^∞ symbol in ξ_1 such that microlocally in any compact set $K \subset T^*\mathbf{R}$ one has $(VU)^* = (VU)^{-1}$ and*

$$VUH(VU)^{-1} = f(-\varepsilon^2\partial_2^2 + x_2^2; \xi_1; \varepsilon) + O(\varepsilon^\infty)$$

where

$$f(I; \xi_1; \varepsilon) = f_0(I) + \sum \varepsilon^j f_j(I; \xi_1; \varepsilon)$$

while f_0 is a smooth bijection on \mathbf{R}^+ and the f_j 's are C^∞ functions of I and ξ_1 .

Proof. The proof consists in using techniques linked to the isochore Morse lemma (see [6], [30]). Actually we introduce a canonical change of variables (the corresponding operator being the FIO U) allowing to pass from the variables (x_2, ξ_2) to action-angle variables (see [1] for instance). Let us make this first step more precise : we recall that the action variables are given by

$$I := \oint \xi_2 dx_2,$$

where the integral is taken on a constant energy curve $H = h$. The angle variables are then given by solving $\theta = \partial_I S(I, \xi_2)$, where K is the hamiltonian in the new variables, which only depends on I , and where S is defined by

$$dS_{I=\text{constant}} = \xi_2 dx_2.$$

Note that this is a *global* change of variables.

It is now well known (see [7] for instance) that such a canonical change of variables is associated with an FIO U (independent of ξ_1 since the principal symbol of H does not depend on ξ_1) such that

$$(5.2) \quad UHU^{-1} = f^0(-\varepsilon^2\partial_2^2 + x_2^2) + \varepsilon F_1$$

where f^0 is a smooth, global bijection on \mathbf{R}^+ . In this case we can actually write (see [7]) the following formula for U : for any L^2 function φ

$$U\varphi(x_2) = \frac{1}{(2\pi\varepsilon)^{\frac{3}{2}}} \int e^{i\frac{S(x_2, \xi_2) - y\xi_2}{\varepsilon}} a_0(x_2, \xi_2, \varepsilon) u(y) dy d\xi_2$$

where a_0 is constructed so that U is unitary (up to $O(\varepsilon)$).

Using those new coordinates and the formula giving U as well as the formula for the principal symbol of the adjoint (which is here the inverse) given in Appendix B, it is not difficult to show that (5.2) holds.

Once the function f^0 is obtained we proceed by induction: we first look for a symbol q_1 such that $Q_1 := \text{Op}_\varepsilon^0(q_1)$ satisfies

$$e^{iQ_1} UHU^{-1} e^{-iQ_1} = f^{0,1}(-\varepsilon^2\partial_2^2 + x_2^2) + \varepsilon^2 F_2.$$

In order to compute q_1 we notice that

$$e^{iQ_1} UHU^{-1} e^{-iQ_1} = UHU^{-1} + \int_0^t \partial_t H_t dt$$

where

$$H_t := e^{itQ_1} U H U^{-1} e^{-itQ_1}.$$

One sees easily that

$$\begin{aligned} \partial_t H_t &= i[Q_1, H_t] \\ &= e^{itQ_1} [Q_1, f^0(-\varepsilon^2 \partial_2^2 + x_2^2) + \varepsilon F_1] e^{-itQ_1} \end{aligned}$$

so the principal symbol of $\partial_t H_t$ is therefore the Poisson bracket $\varepsilon\{q_1, f^0(x_2^2 + \xi_2^2)\}$. One then remarks that the equation

$$\{q, f^0(x_2^2 + \xi_2^2)\} = g$$

has a solution if and only if g has zero mean value (in action-angle variables): one has indeed necessarily

$$\partial_\theta q = \frac{I g}{\partial_I f}.$$

Note that if f and g are smooth, then so is q since $\partial_I f > 0$ everywhere. It follows that one can find q_1 simply by solving

$$\{q_1, f^0(x_2^2 + \xi_2^2)\} = f_1 - \bar{f}_1$$

where \bar{f}_1 is the average of f_1

$$\bar{f}_1(I) = \frac{1}{2\pi} \int_0^{2\pi} f_1(R_\theta(x_2, \xi_2); \xi_1) d\theta$$

where R_θ denotes the rotation of angle θ . This implies that with this choice of q_1 and writing $Q_1 := \text{Op}_\varepsilon^0(q_1)$ one has

$$e^{iQ_1} U H U^{-1} e^{-iQ_1} = (f^0 + \varepsilon \bar{f}_1)(-\varepsilon^2 \partial_2^2 + x_2^2) + \varepsilon^2 F_2.$$

One proceeds similarly at all orders. \square

The next step consists in using that lemma to check that the eigenvalues and eigenfunctions enjoy the expected smoothness properties. Actually this is rather straightforward since if (ψ, λ) satisfy

$$H\psi = \lambda\psi + O(\varepsilon^\infty)$$

then $\tilde{\psi} := VU\psi$ satisfies

$$VUH(VU)^{-1}\tilde{\psi} = \lambda\tilde{\psi} + O(\varepsilon^\infty)$$

hence

$$f(-\varepsilon^2 \partial_2^2 + x_2^2)\tilde{\psi} = \lambda\tilde{\psi} + O(\varepsilon^\infty).$$

This implies that

$$\lambda = \lambda_n = f\left(\varepsilon\left(n + \frac{1}{2}\right); \xi_1; \varepsilon\right) \quad \text{and} \quad \tilde{\psi} = \tilde{\psi}_n = \frac{1}{\varepsilon^{\frac{1}{4}}} h_n\left(\frac{x_2}{\sqrt{\varepsilon}}\right)$$

where h_n is the n -th Hermite function.

The conclusion follows recalling that $f(I; \xi_1; \varepsilon) = f_0(I) + \sum \varepsilon^j f_j(I; \xi_1; \varepsilon)$ (where f_0 does not depend on ξ_1) and that each eigenfunction is microlocalized in a compact set.

Remark 5.3. *Since we have a complete spectral description of T_\pm , with discrete spectrum, it is obvious that if the initial data is microlocalized in (ξ_1, x_2, ξ_2) , then the solution to the equation $\varepsilon^2 \partial_t \varphi = iT_\pm \varphi$ remains microlocalized in the set of energy surfaces containing (ξ_1, x_2, ξ_2) . Notice also that there are $O(1/\varepsilon)$ eigenvalues in a compact energy surface.*

6. DIAGONALIZATION

In this section we shall prove that the scalar propagators defined in Section 3 correspond indeed to a diagonalization of the original linear system (2.1). Let us prove the following proposition.

Proposition 6.1. *Consider a compact set $\mathcal{K} \subset \mathbf{R}^* \times T^*\mathbf{R}$. With the notation of Definitions 3.3 and 3.6, the operator $\Pi := (\Pi_- \Pi_R \Pi_+)$ maps continuously H_ε^∞ functions, microlocalized in $\mathbf{R} \times \mathcal{K}$ and satisfying (2.3), onto H_ε^∞ . Moreover it is left-invertible modulo ε^∞ , and its left inverse Q (modulo ε^∞) maps continuously those functions onto H_ε^∞ .*

Remark 6.2. *This proposition, along with Proposition 4.6 and Remark 5.3 showing the propagation of the microlocal support (in (ξ_1, x_2, ξ_2)) of Rossby and Poincaré modes, prove the first part of Theorem 1.*

Proof. The main step consists in showing that Π does have a left inverse, and in computing its principal symbol. The construction of a left inverse can be done symbolically as follows. We first compute the matrix-principal symbol \mathcal{P}_0 of Π . One gets

$$\mathcal{P}_0 = \begin{pmatrix} \frac{-\xi_1 \sqrt{\xi^2 + b^2} - i\xi_2 b}{\xi_1^2 + b^2} & -\frac{ib}{\xi_1} & \frac{\xi_1 \sqrt{\xi^2 + b^2} - i\xi_2 b}{\xi_1^2 + b^2} \\ 1 & -\frac{\xi_2}{\xi_1} & 1 \\ \frac{\xi_1 \xi_2 + ib \sqrt{\xi^2 + b^2}}{\xi_1^2 + b^2} & 1 & \frac{\xi_1 \xi_2 - ib \sqrt{\xi^2 + b^2}}{\xi_1^2 + b^2} \end{pmatrix}.$$

This shows that Π maps microlocalized functions in $\mathbf{R} \times \mathcal{K}$ onto H_ε^∞ . A simple computation shows that

$$|\det \mathcal{P}_0| = \frac{2(\xi^2 + b^2)^{\frac{3}{2}}}{(\xi_1^2 + b^2)|\xi_1|} \geq 2,$$

therefore $\mathcal{Q}_0 = \mathcal{P}_0^{-1}$ exists. Let us call Q_0 the matrix obtained by Weyl quantization (term by term) of \mathcal{Q}_0 . By symbolic calculus we have that:

$$Q_0 \Pi = \text{Id} + \varepsilon I_1$$

Let us call $\mathcal{Q}_1 = -\mathcal{I}_1 \mathcal{Q}_0^{-1}$, where \mathcal{I}_1 is the (matrix) principal symbol of I_1 . Again by symbolic calculus we have that

$$(Q_0 + \varepsilon Q_1) \Pi = \text{Id} + \varepsilon^2 I_2$$

where Q_1 has principal symbol \mathcal{Q}_1 . Defining now $\mathcal{Q}_2 = -\mathcal{I}_2 \mathcal{Q}_0^{-1}$ we get

$$(Q_0 + \varepsilon Q_1 + \varepsilon^2 Q_2) \Pi = \text{Id} + \varepsilon^3 I_3$$

where Q_2 has principal symbol \mathcal{Q}_2 , and so on. This allows to invert the matrix Π up to $O(\varepsilon^\infty)$, and the principal symbol of the (approximate) inverse matrix Q is given by \mathcal{Q}_0 : we have

$$\mathcal{Q}_0 = \frac{1}{2(\xi^2 + b^2)} \begin{pmatrix} ib\xi_2 - \xi_1 \sqrt{\xi^2 + b^2} & \xi_1^2 + b^2 & -ib \sqrt{\xi^2 + b^2} + \xi_1 \xi_2 \\ 2ib\xi_1 & -2\xi_1 \xi_2 & 2\xi_1^2 \\ ib\xi_2 + \xi_1 \sqrt{\xi^2 + b^2} & \xi_1^2 + b^2 & ib \sqrt{\xi^2 + b^2} + \xi_1 \xi_2 \end{pmatrix}.$$

Note that each term of the expansion of the symbol of Q is a polynomial of increasing order in ξ , but that is not a problem due to the microlocalization assumption. The operator Q therefore clearly maps continuously H_ε^∞ onto itself. The proposition is proved. \square

7. CONTROL OF THE NONLINEAR TERMS FOR A WEAK COUPLING

We are now interested in describing the behaviour of our initial nonlinear system, which includes the effect of the convection by the unknown (u_1, u_2) . We shall prove Theorem 2 in this section. We recall that the system reads as follows:

$$(7.1) \quad \varepsilon^2 \partial_t U + A(x, \varepsilon D_x)U + \varepsilon^{3+\eta} S_1(U) \varepsilon \partial_1 U + \varepsilon^{3+\eta} S_2(U) \varepsilon \partial_2 U = 0, \quad \eta \geq 0$$

with

$$(7.2) \quad S_1(U) = \begin{pmatrix} u_1 & \frac{1}{2}u_0 & 0 \\ \frac{1}{2}u_0 & u_1 & 0 \\ 0 & 0 & u_1 \end{pmatrix}, \quad S_2(U) = \begin{pmatrix} u_2 & 0 & \frac{1}{2}u_0 \\ 0 & u_2 & 0 \\ \frac{1}{2}u_0 & 0 & u_2 \end{pmatrix}$$

and $U = (u_0, u_1, u_2)$. The usual theory of symmetric hyperbolic systems provides the local existence of a solution to (7.1) in $H^s(\mathbf{R}^2)$ for $s > 2$, on a time interval depending on ε a priori. Because of the semiclassical framework (fixed by the form of the initial data), it is actually natural to rather consider ε -derivatives. Moreover, as the derivative with respect to x_2 does not commute with the singular perturbation, we expect even the semiclassical Sobolev norms to grow like $\exp\left(\frac{Ct}{\varepsilon}\right)$, and therefore the life span of the solutions to (7.1) to be non uniform with respect to ε . That is the reason why the W_ε^s spaces were introduced in (1.8).

7.1. Propagation of regularity for the linear singular perturbation problem.

Let us first remark that derivatives with respect to x_1 do commute with $A(x, \varepsilon D_x)$, so we can propagate as much regularity in x_1 as needed. Extending a result by Dutrifoy, Majda and Schochet [10] obtained in the particular case when $b(x_2) = \beta x_2$, we will actually prove that there is an operator of principal symbol $(\xi_2^2 + b^2)\text{Id}$ which ‘‘almost commutes’’ with $A(x, \varepsilon D_x)$ in the semiclassical regime.

- The first step, as in [10], is to perform the following orthogonal change of variable

$$\tilde{U} := \left(\frac{u_0 + u_1}{\sqrt{2}}, \frac{u_0 - u_1}{\sqrt{2}}, u_2 \right)$$

in order to produce the generalized creation and annihilation operators

$$L_\pm := \frac{1}{\sqrt{2}}(\varepsilon \partial_2 \mp b).$$

The system (7.1) can indeed be rewritten

$$\varepsilon^2 \partial_t \tilde{U} + \tilde{A}(x, \varepsilon D_x) \tilde{U} + \varepsilon^3 \tilde{S}_1(\tilde{U}) \varepsilon \partial_1 \tilde{U} + \varepsilon^3 \tilde{S}_2(\tilde{U}) \varepsilon \partial_2 \tilde{U} = 0$$

with

$$\tilde{A}(x, \varepsilon D_x) := \begin{pmatrix} \varepsilon \bar{u}_1 \varepsilon \partial_1 + \varepsilon \partial_1 & 0 & L_+ + \frac{\varepsilon^2}{\sqrt{2}} \bar{u}'_1 \\ 0 & \varepsilon \bar{u}_1 \varepsilon \partial_1 - \varepsilon \partial_1 & L_- - \frac{\varepsilon^2}{\sqrt{2}} \bar{u}'_1 \\ L_- & L_+ & \varepsilon \bar{u}_1 \varepsilon \partial_1 \end{pmatrix},$$

and

$$\tilde{S}_1(\tilde{U}) := \begin{pmatrix} \frac{3\tilde{u}_0 - \tilde{u}_1}{2\sqrt{2}} & 0 & 0 \\ 0 & \frac{\tilde{u}_0 - 3\tilde{u}_1}{2\sqrt{2}} & 0 \\ 0 & 0 & \frac{\tilde{u}_0 - \tilde{u}_1}{\sqrt{2}} \end{pmatrix}, \quad \tilde{S}_2(\tilde{U}) := \begin{pmatrix} \tilde{u}_2 & 0 & \frac{\tilde{u}_0 + \tilde{u}_1}{4} \\ 0 & \tilde{u}_2 & \frac{\tilde{u}_0 + \tilde{u}_1}{4} \\ \frac{\tilde{u}_0 + \tilde{u}_1}{4} & \frac{\tilde{u}_0 + \tilde{u}_1}{4} & \tilde{u}_2 \end{pmatrix}.$$

• Next, remarking that

$$[\varepsilon^2 \partial_2^2 - b^2, \varepsilon \partial_2 \pm b] = \pm 2\varepsilon b' (\varepsilon \partial_2 \pm b) \pm \varepsilon^2 b'',$$

we introduce the operator

$$D_\varepsilon := \begin{pmatrix} \varepsilon^2 \partial_2^2 - b^2 + 2\varepsilon b' & 0 & 0 \\ 0 & \varepsilon^2 \partial_2^2 - b^2 - 2\varepsilon b' & 0 \\ 0 & 0 & \varepsilon^2 \partial_2^2 - b^2 \end{pmatrix}.$$

We notice that D_ε is a scalar operator at leading order. Moreover one can compute the commutator $[D_\varepsilon, \tilde{A}(x, \varepsilon D_x)]$: we find

$$[D_\varepsilon, \tilde{A}(x, \varepsilon D_x)] = \begin{pmatrix} [\varepsilon^2 \partial_2^2, \varepsilon \bar{u}_1] \varepsilon \partial_1 & 0 & \frac{\varepsilon^2}{\sqrt{2}} ([\varepsilon^2 \partial_2^2, \bar{u}'_1] + 2\varepsilon b' \bar{u}'_1 - b'') \\ 0 & [\varepsilon^2 \partial_2^2, \varepsilon \bar{u}_1] \varepsilon \partial_1 & -\frac{\varepsilon^2}{\sqrt{2}} ([\varepsilon^2 \partial_2^2, \bar{u}'_1] - 2\varepsilon b' \bar{u}'_1 - b'') \\ -\frac{\varepsilon^2}{\sqrt{2}} b'' & \frac{\varepsilon^2}{\sqrt{2}} b'' & [\varepsilon^2 \partial_2^2, \varepsilon \bar{u}_1] \varepsilon \partial_1 \end{pmatrix}$$

• At this stage, we have proved that

$$(7.3) \quad [D_\varepsilon, \tilde{A}(x, \varepsilon D_x)] = O(\varepsilon^2 (\text{Id} - \varepsilon^2 \partial_1^2 - D_\varepsilon))$$

meaning that the commutator $[D_\varepsilon, \tilde{A}(x, \varepsilon D_x)]$ is of order $O(\varepsilon^2)$ with respect to the elliptic operator $\text{Id} - \varepsilon^2 \partial_1^2 - D_\varepsilon$. That implies that the regularity of the solution to the linear equation

$$\varepsilon^2 \partial_t V + \tilde{A}(x, \varepsilon D_x) V = 0$$

can be controlled by an application of Gronwall's lemma: one has

$$\varepsilon^2 \|(\text{Id} - \varepsilon^2 \partial_1^2 - D_\varepsilon) V(t)\|_{L^2}^2 \leq \varepsilon^2 \|(\text{Id} - \varepsilon^2 \partial_1^2 - D_\varepsilon) V_0\|_{L^2}^2 + C \varepsilon^2 \int_0^t \|(\text{Id} - \varepsilon^2 \partial_1^2 - D_\varepsilon) V(s)\|_{L^2}^2 ds,$$

where C depends on the $W^{2,\infty}$ norms of \bar{u}_1 and b , so

$$\varepsilon^2 \|(\text{Id} - \varepsilon^2 \partial_1^2 - D_\varepsilon) V(t)\|_{L^2}^2 \leq \varepsilon^2 \|(\text{Id} - \varepsilon^2 \partial_1^2 - D_\varepsilon) V_0\|_{L^2}^2 e^{C\varepsilon^2 t}.$$

7.2. Uniform a priori estimates for the nonlinear equation.

It remains then to handle the nonlinear terms, namely to establish product laws in some convenient weighted spaces defined in terms of D_ε .

Since the extended harmonic oscillator controls two derivatives in x_2 , we get a control on the Lipschitz norm of U of the type

$$(7.4) \quad \|\varepsilon \partial_j U\|_{L^\infty} \leq \frac{C}{\varepsilon} (\|D_\varepsilon^2 U\|_{L^2} + \|\varepsilon^4 \partial_1^4 U\|_{L^2} + \|U\|_{L^2}).$$

As D_ε is a scalar differential operator at leading order in ε , the antisymmetry of the higher order nonlinear term is preserved. More precisely, we have, using the Leibniz formula,

$$\begin{aligned} & \varepsilon^2 \partial_t D_\varepsilon \tilde{U} + \tilde{A}(x, \varepsilon D_x) D_\varepsilon \tilde{U} + \varepsilon^3 \tilde{S}_1(\tilde{U}) \varepsilon \partial_1 D_\varepsilon \tilde{U} + \varepsilon^3 \tilde{S}_2(\tilde{U}) \varepsilon \partial_2 D_\varepsilon \tilde{U} \\ &= -[D_\varepsilon, \tilde{A}(x, \varepsilon D_x)] \tilde{U} - \varepsilon^3 \tilde{S}_2(\tilde{U}) [D_\varepsilon, \varepsilon \partial_2] \tilde{U} - 2\varepsilon^3 \varepsilon \partial_2 \tilde{S}_j(\tilde{U}) \varepsilon^2 \partial_{j_2}^2 \tilde{U} - \varepsilon^3 (\varepsilon \partial_2)^2 \tilde{S}_j(\tilde{U}) \varepsilon \partial_j \tilde{U} \end{aligned}$$

as well as

$$\begin{aligned} & \varepsilon^2 \partial_t D_\varepsilon^2 \tilde{U} + \tilde{A}(x, \varepsilon D_x) D_\varepsilon^2 \tilde{U} + \varepsilon^3 \tilde{S}_1(\tilde{U}) \varepsilon \partial_1 D_\varepsilon^2 \tilde{U} + \varepsilon^3 \tilde{S}_2(\tilde{U}) \varepsilon \partial_2 D_\varepsilon^2 \tilde{U} = -[D_\varepsilon^2, \tilde{A}(x, \varepsilon D_x)] \tilde{U} \\ & \quad - \varepsilon^3 \tilde{S}_2(\tilde{U}) [D_\varepsilon, \varepsilon \partial_2] D_\varepsilon \tilde{U} - 2\varepsilon^3 \varepsilon \partial_2 \tilde{S}_j(\tilde{U}) \varepsilon^2 \partial_{j_2}^2 D_\varepsilon \tilde{U} - \varepsilon^3 (\varepsilon \partial_2)^2 \tilde{S}_j(\tilde{U}) \varepsilon \partial_j D_\varepsilon \tilde{U} \\ & \quad + D_\varepsilon \left(-\varepsilon^3 \tilde{S}_2(\tilde{U}) [D_\varepsilon, \varepsilon \partial_2] \tilde{U} - 2\varepsilon^3 \varepsilon \partial_2 \tilde{S}_j(\tilde{U}) \varepsilon^2 \partial_{j_2}^2 \tilde{U} - \varepsilon^3 (\varepsilon \partial_2)^2 \tilde{S}_j(\tilde{U}) \varepsilon \partial_j \tilde{U} \right). \end{aligned}$$

and in the same way, for $1 \leq \ell \leq 4$,

$$\begin{aligned} & \varepsilon^2 \partial_t (\varepsilon \partial_1)^\ell \tilde{U} + \tilde{A}(x, \varepsilon D_x) (\varepsilon \partial_1)^\ell \tilde{U} + \varepsilon^3 \tilde{S}_1(\tilde{U}) (\varepsilon \partial_1)^{\ell+1} \tilde{U} + \varepsilon^3 \tilde{S}_2(\tilde{U}) \varepsilon \partial_2 (\varepsilon \partial_1)^\ell \tilde{U} \\ &= -\varepsilon^4 \sum_{k=1}^{\ell} C_4^\ell (\varepsilon \partial_1)^\ell \tilde{S}_j(\tilde{U}) \varepsilon \partial_j (\varepsilon \partial_1)^{\ell-k} \tilde{U}. \end{aligned}$$

In all cases, the terms of higher order disappear by integration in x and the other terms are controlled with the following trilinear estimate (writing generically $\tilde{Q}(\tilde{U})$ for all the nonlinearities): for all $0 \leq k \leq 2$ and all $0 \leq \ell \leq 4$

$$(7.5) \quad \begin{aligned} & | \langle D_\varepsilon^k \tilde{U} | D_\varepsilon^k \tilde{Q}(\tilde{U}) \rangle | + | \langle (\varepsilon \partial_1)^\ell \tilde{U} | (\varepsilon \partial_1)^\ell \tilde{Q}(\tilde{U}) \rangle | \\ & \leq C \|\tilde{U}\|_{W_\varepsilon^{1,\infty}} (\|D_\varepsilon^2 \tilde{U}\|_{L^2} + \|(\varepsilon \partial_1)^4 \tilde{U}\|_{L^2} + \|\tilde{U}\|_{L^2})^2 \\ & \leq \frac{C}{\varepsilon} \left(\|D_\varepsilon^2 \tilde{U}\|_{L^2} + \|(\varepsilon \partial_1)^4 \tilde{U}\|_{L^2} + \|\tilde{U}\|_{L^2} \right)^3. \end{aligned}$$

Remark 7.1. *Note that because of the bad embedding inequality*

$$\|\nabla U\|_{L^\infty} \leq \frac{1}{\varepsilon} \|U\|_{W_\varepsilon^4},$$

we lose one power of ε , which seems not to be optimal considering for instance the fast oscillating functions

$$x_2 \mapsto \exp\left(\frac{ik_2 x_2}{\varepsilon}\right).$$

A challenging question in order to apply semiclassical methods to nonlinear problems is to determine appropriate functional spaces (in the spirit of [23]) which measures on the one hand the Sobolev regularity of the amplitudes, and on the other hand the oscillation frequency.

We are finally able **to obtain a uniform life span** for the weakly nonlinear system, thus proving result (1) of Theorem 2. Indeed combining the trilinear estimate (7.5) and the commutator estimate (7.3), we obtain the following Gronwall inequality

$$\varepsilon^2 \frac{d}{dt} \left(\|D_\varepsilon^2 \tilde{U}\|_{L^2}^2 + \|(\varepsilon \partial_1)^4 \tilde{U}\|_{L^2}^2 + \|\tilde{U}\|_{L^2}^2 \right) \leq C\varepsilon^2 \left(1 + \|D_\varepsilon^2 \tilde{U}\|_{L^2} + \|(\varepsilon \partial_1)^4 \tilde{U}\|_{L^2} + \|\tilde{U}\|_{L^2} \right)^3$$

from which we deduce the uniform a priori estimate

$$\|D_\varepsilon^2 \tilde{U}\|_{L^2}^2 + \|(\varepsilon \partial_1)^4 \tilde{U}\|_{L^2}^2 + \|\tilde{U}\|_{L^2}^2 \leq (C_0 - Ct)^{-2}$$

where C_0 depends only on the initial data.

Such an estimate shows that the life span of the solutions to (7.1) is at least $T^* = C_0/C$.

7.3. Approximation by the linear dynamics. In this paragraph we shall prove results (2) and (3) of Theorem 2. The proof of both results relies on standard energy estimates. We have

$$\varepsilon^2 \partial_t (U_\varepsilon - V_\varepsilon) + A(x, \varepsilon D_x)(U_\varepsilon - V_\varepsilon) + \varepsilon^{3+\eta} \tilde{S}_1(U_\varepsilon) \varepsilon \partial_1 U_\varepsilon + \varepsilon^{3+\eta} \tilde{S}_2(U_\varepsilon) \varepsilon \partial_2 U_\varepsilon = 0$$

• If $\eta = 0$ and $\varepsilon V_\varepsilon \rightarrow 0$ in L^∞ , we use the decomposition

$$\varepsilon^2 \partial_t (U_\varepsilon - V_\varepsilon) + A(x, \varepsilon D_x)(U_\varepsilon - V_\varepsilon) + \varepsilon^3 (\tilde{S}_j(U_\varepsilon) - \tilde{S}_j(V_\varepsilon)) \varepsilon \partial_j U_\varepsilon + \varepsilon^3 \tilde{S}_j(V_\varepsilon) \varepsilon \partial_j U_\varepsilon = 0$$

and obtain the following L^2 estimate

$$\begin{aligned} \frac{\varepsilon^2}{2} \frac{d}{dt} \|U_\varepsilon - V_\varepsilon\|_{L^2}^2 &\leq 3\varepsilon^3 \|\varepsilon \partial_j U_\varepsilon\|_{L^\infty} \|U_\varepsilon - V_\varepsilon\|_{L^2}^2 + 3\varepsilon^3 \|V_\varepsilon\|_{L^\infty} \|\varepsilon \partial_j U_\varepsilon\|_{L^2} \|U_\varepsilon - V_\varepsilon\|_{L^2} \\ &\leq C\varepsilon^2 (\|\varepsilon \partial_j U_\varepsilon\|_{L^\infty} + \|\varepsilon \partial_j U_\varepsilon\|_{L^2}^2) \|U_\varepsilon - V_\varepsilon\|_{L^2}^2 + C\varepsilon^2 (\|\varepsilon \|V_\varepsilon\|_{L^\infty})^2 \end{aligned}$$

from which we conclude by Gronwall's lemma

$$\|U_\varepsilon - V_\varepsilon\|_{L^2}^2 \leq C \int_0^t (\|\varepsilon \|V_\varepsilon(s)\|_{L^\infty})^2 \exp C \left(\int_s^t (\|\varepsilon \partial_j U_\varepsilon\|_{L^\infty} + \|\varepsilon \partial_j U_\varepsilon\|_{L^2}^2) d\sigma \right) ds$$

on $[0, T^*]$, and that proves result (2).

• If $\eta > 0$, the same arguments show that the life span of the solutions to (2.5) tends to infinity as $\varepsilon \rightarrow 0$

$$T_\varepsilon \geq C\varepsilon^{-\eta},$$

and that these solutions are uniformly bounded in W_ε^4 on any finite time interval.

Furthermore, on any finite time interval $[0, T]$, the previous energy estimate gives

$$\frac{\varepsilon^2}{2} \frac{d}{dt} \|U_\varepsilon - V_\varepsilon\|_{L^2}^2 \leq C\varepsilon^{3+\eta} \|\varepsilon \partial_j U_\varepsilon\|_{L^\infty} \|U_\varepsilon\|_{L^2}^2,$$

from which we deduce

$$\|U_\varepsilon - V_\varepsilon\|_{L^2}^2 \leq C\varepsilon^\eta \int_0^t \|\varepsilon \partial_j U_\varepsilon(s)\|_{L^\infty} \|U_\varepsilon(s)\|_{L^2}^2 ds.$$

Result (3) of Theorem 2 is proved.

APPENDIX A. A DIAGONALIZATION THEOREM

In this appendix we shall state and prove the crucial theorem allowing to diagonalize semi-classically the matrix of pseudo-differential operators $A(x_2, \varepsilon D)$. The construction of Rossby and Poincaré modes (see Propositions 3.2 and 3.5 in Section 3) are direct corollaries of that theorem. We refer to Appendix B for the notation and results of semi-classical analysis used in this paragraph.

We consider an order function g on \mathbf{R}^{2d} , and a symbol $h(X; \varepsilon, \tau)$ in $S_{2d}(g)$, depending polynomially on ε and τ , where we have defined $X = (x, \xi) \in \mathbf{R}^{2d}$. The statement is the following.

Theorem 3. *Let g be an order function on \mathbf{R}^{2d} , and let $h(X; \varepsilon, \tau)$ be a classical symbol in $S_{2d}(g)$, depending polynomially on ε and τ : there is an integer N_0 such that*

$$(A.1) \quad \forall X \in \mathbf{R}^{2d}, \quad h(X; \varepsilon, \tau) = \sum_{j, \ell=0}^{N_0} h^{j, \ell}(X) \varepsilon^\ell \tau^j,$$

where the symbols $h^{j, \ell}(X)$ belong to $S_{2d}(g)$. Let $\tilde{\tau}_\varepsilon = \tilde{\tau}_\varepsilon(X)$ be a root of the polynomial $h(X; \varepsilon, \cdot)$ which can be written for some $\nu \in \mathbf{N}$,

$$\tilde{\tau}_\varepsilon(X) = \sum_{k=0}^{\infty} \varepsilon^{\nu+k} \tau_k(X) + O(\varepsilon^\infty), \quad \text{with } \tau_0 \neq 0$$

and where τ_0 is a symbol. Finally let $h_0(X, \tau)$ be the principal symbol of h , satisfying the following assumption:

$$(A.2) \quad \exists C > 0, \quad \forall X \in \mathbf{R}^{2d}, \quad \forall \varepsilon \in]0, 1[, \quad |\partial_\tau h_0(X; \varepsilon^\nu \tau_0)| \geq C.$$

Let \mathcal{K} be a compact subset of \mathbf{R}^{2d} . Then there is a pseudo-differential operator T of principal symbol $\varepsilon^\nu \tau_0$ such that if $\psi \in \mathcal{S}'(\mathbf{R}^d)$ is microlocalized in \mathcal{K} and satisfies $i\partial_s \psi = T\psi$ then

$$(A.3) \quad \text{Op}_\varepsilon^0(h(\varepsilon, i\partial_s))\psi = O(\varepsilon^\infty) \quad \text{in } L^2(\mathbf{R}^d).$$

Proof. The idea of the proof of the theorem is the following. Let us define a smooth function χ , compactly supported in \mathbf{R}^{2d} , identically equal to one on \mathcal{K} . Then for any integer N , we shall compute recursively the coefficients of the symbol

$$\tau_\varepsilon^N := \varepsilon^\nu \tau_0 + \sum_{k=1}^N \varepsilon^{\nu+k} \tau_k^-$$

so that $T_\chi := \text{Op}_\varepsilon^1(\tau_\varepsilon^N)$ satisfies the required property, where τ_ε^N , unique up to $O(\varepsilon^\infty)$, is given by

$$\tau_\varepsilon^N := \varepsilon^\nu \tau_0 + \sum_{k=1}^{\infty} \varepsilon^{\nu+k} \chi \tau_k^- + O(\varepsilon^\infty).$$

This allows to replace the root $\tilde{\tau}_\varepsilon$ by an actual symbol.

The above strategy will be achieved in the following way. We notice that if $i\partial_s \psi = T_\chi \psi$ then of course

$$i\partial_s \psi = \text{Op}_\varepsilon^1(\tau_\varepsilon^{N, N})\psi + O(\varepsilon^{N+\nu+1}), \quad \text{where} \quad \tau_\varepsilon^{N, N} := \varepsilon^\nu \tau_0 + \sum_{k=1}^N \varepsilon^{\nu+k} \chi \tau_k^-.$$

If moreover ψ is microlocalized in \mathcal{K} , then by definition of χ one has using (B.3-B.4) and the fact that χ is identically equal to one over \mathcal{K} (so in particular its derivatives vanish on \mathcal{K}),

$$i\partial_s\psi = \text{Op}_\varepsilon^1(\tau_\varepsilon^N)\psi + O(\varepsilon^{N+\nu+1}).$$

This means one can (and shall) compute recursively $(\tau_k^-)_{1 \leq k \leq N}$ so that

$$\text{Op}_\varepsilon^0(h(\varepsilon, i\partial_s))\psi = O(\varepsilon^{N+1}), \quad \text{when} \quad i\partial_s\psi = \text{Op}_\varepsilon^1(\tau_\varepsilon^N)\psi + O(\varepsilon^{N+1}).$$

Note that it is convenient in the computations to compute T_χ as the ‘‘right’’-quantization (see Appendix B) of the symbol τ_ε^X ; recall that the principal symbol of an operator does not depend on the choice of quantization.

Now let us carry out the algebraic computations allowing to achieve the result. We shall start by dealing with the case when $\nu \neq 0$, as the computations can be carried out in an easier way, and then we shall discuss the case when $\nu = 0$. Recalling that $h(X; \varepsilon, \tilde{\tau}_\varepsilon(X)) = 0$ by definition of $\tilde{\tau}_\varepsilon$, then in particular the first term of the expansion in powers of ε vanishes so we infer that $h^{0,\ell} \equiv 0$ if $\ell < \nu$ and

$$(A.4) \quad h^{0,\nu}(x, \xi) + h^{1,0}(x, \xi)\tau_0(x, \xi) = 0.$$

Now we recall that for any ψ in $\mathcal{S}'(\mathbf{R}^d)$, one has

$$\text{Op}_\varepsilon^0(h(\varepsilon, i\partial_s))\psi(x) = (2\pi\varepsilon)^{-d} \int_{\mathbf{R}^{2d}} e^{i\frac{(x-y)\cdot\xi}{\varepsilon}} h(x, \xi; \varepsilon, i\partial_s)\psi(y) dyd\xi.$$

The above integral, as all the ones appearing in this proof, is to be understood in the distributional sense. So if τ_ε and ψ are such that $\text{Op}_\varepsilon^1(\tau_\varepsilon^X)\psi = i\partial_s\psi$ with ψ microlocalized in \mathcal{K} , then as explained above

$$(A.5) \quad \begin{aligned} \text{Op}_\varepsilon^0(h(\varepsilon, i\partial_s))\psi(x) &= (2\pi\varepsilon)^{-d} \int_{\mathbf{R}^{2d}} e^{i\frac{(x-y)\cdot\xi}{\varepsilon}} h(x, \xi; \varepsilon, \text{Op}_\varepsilon^1(\tau_\varepsilon^X))\psi(y) dyd\xi \\ &= (2\pi\varepsilon)^{-d} \int_{\mathbf{R}^{2d}} e^{i\frac{(x-y)\cdot\xi}{\varepsilon}} h(x, \xi; \varepsilon, \text{Op}_\varepsilon^1(\tau_\varepsilon^N))\psi(y) dyd\xi + O(\varepsilon^{N+1}). \end{aligned}$$

Now we need to compute $h(x, \xi; \varepsilon, \text{Op}_\varepsilon^1(\tau_\varepsilon^N))$. First of all we use the fact that h is polynomial in τ to write

$$\begin{aligned} h(x, \xi; \varepsilon, \text{Op}_\varepsilon^1(\tau_\varepsilon^N)) &= \sum_{j,\ell=0}^{N_0} h^{j,\ell}(x, \xi)\varepsilon^\ell (\text{Op}_\varepsilon^1(\tau_\varepsilon^N))^j \\ &=: \text{Op}_\varepsilon^1(\tilde{h}(x, \xi, y, \eta; \varepsilon)) \end{aligned}$$

where using compositions rules recalled in Appendix B, the symbol $\tilde{h}(x, \xi, y, \eta; \varepsilon)$ (in the (y, η) variable) can be expanded as

$$\tilde{h}(x, \xi, y, \eta; \varepsilon) = \varepsilon^\nu \sum_{k=0}^{\infty} \tilde{h}^k(x, \xi, y, \eta)\varepsilon^k + O(\varepsilon^\infty)$$

where the principal symbol of $\tilde{h}(x, \xi, y, \eta; \varepsilon)$ is

$$(A.6) \quad \varepsilon^\nu \tilde{h}^0(x, \xi, y, \eta) = \varepsilon^\nu (h^{0,\nu}(x, \xi) + h^{1,0}(x, \xi)\tau_0(y, \eta)).$$

We recall indeed that according to Appendix B, one has

$$(\text{Op}_\varepsilon^1(\tau_\varepsilon^N))^j =: \text{Op}_\varepsilon^1(m_j(y, \eta; \varepsilon)),$$

where $m_1(y, \eta; \varepsilon) = \tau_\varepsilon^N(y, \eta)$ and

$$m_j(y, \eta; \varepsilon) = \sum_{k \geq 0} \frac{(i\varepsilon)^k}{k!} \partial_y^k m_{j-1}(y, \eta; \varepsilon) \partial_\eta^k \tau_\varepsilon^N(y, \eta) + O(\varepsilon^\infty).$$

In particular the principal symbol of $(\text{Op}_\varepsilon^1(\tau_\varepsilon))^j$ is $\varepsilon^{\nu j} \tau_0^j(y, \eta)$, which yields (A.6). We notice that due to (A.4), (A.6) implies that

$$(A.7) \quad \tilde{h}^0(x, \xi, x, \xi) = 0,$$

so that $\tilde{h}(x, \xi, x, \xi; \varepsilon) = O(\varepsilon^{\nu+1})$. More generally, plugging the expansion of m_j into the formula defining \tilde{h} and noticing that $h^{1,0}(x, \xi) = \partial_\tau h_{0|\tau=0}(x, \xi)$, one finds that there is $h^k(x, \xi, y, \eta)$, depending on the symbol coefficients of $h(x, \xi)$, and in the (y, η) variables on the symbols $\tau_0(y, \eta), \tau_1^-(y, \eta), \dots, \tau_{k-1}^-(y, \eta)$ only, such that

$$(A.8) \quad \forall k \geq 1, \quad \tilde{h}^k(x, \xi, y, \eta) = h^k(x, \xi, y, \eta) + \partial_\tau h_{0|\tau=0}(x, \xi) \tau_k^-(y, \eta).$$

Finally going back to (A.5) we find that

$$\text{Op}_\varepsilon^0(h(\varepsilon, \lambda))\psi(x) = (2\pi\varepsilon)^{-2d} \int_{\mathbf{R}^{4d}} e^{i\frac{(x-y)\cdot\xi}{\varepsilon}} e^{i\frac{(y-y')\cdot\eta}{\varepsilon}} \tilde{h}(x, \xi, y', \eta; \varepsilon) \psi(y') dy dy' d\xi d\eta + O(\varepsilon^N).$$

We can first perform the integration in the y variable, which creates a Dirac mass at $\xi - \eta$, and therefore we have

$$(A.9) \quad \text{Op}_\varepsilon^0(h(\varepsilon, \lambda))\psi(x) = (2\pi\varepsilon)^{-d} \int_{\mathbf{R}^{2d}} e^{i\frac{(x-y')\cdot\xi}{\varepsilon}} \tilde{h}(x, \xi, y', \xi; \varepsilon) \psi(y') dy' d\xi + O(\varepsilon^N).$$

Now we shall construct τ_1^- ensuring that the order of $\text{Op}_\varepsilon^0(h(\varepsilon, \lambda))\psi(x)$ is $O(\varepsilon^{\nu+1})$ instead of $O(\varepsilon^\nu)$. The argument will easily be adaptable by induction to $\text{Op}_\varepsilon^0(h(\varepsilon, \lambda))\psi(x) = O(\varepsilon^N)$ for any $N \in \mathbf{N}$, by a convenient choice of τ_j^- , for $j \leq N - 1$. We notice that in (A.9), the quantity $\tilde{h}(x, \xi, y', \xi; \varepsilon)$ can easily be replaced by $\tilde{h}(x, \xi, x, \xi; \varepsilon)$ by Taylor's formula: more precisely we write

$$(A.10) \quad \tilde{h}(x, \xi, y', \xi; \varepsilon) = \tilde{h}(x, \xi, x, \xi; \varepsilon) + (y' - x) \cdot (\nabla_{y'} \tilde{h})(x, \xi, x, \xi; \varepsilon) + O(|y' - x|^2)$$

which gives after integrations by parts

$$\begin{aligned} \text{Op}_\varepsilon^0(h(\varepsilon, \lambda))\psi(x) &= \int_{\mathbf{R}^{2d}} e^{i\frac{(x-y')\cdot\xi}{\varepsilon}} \tilde{h}(x, \xi, x, \xi; \varepsilon) \psi(y') \frac{dy' d\xi}{(2\pi\varepsilon)^d} \\ &\quad - i\varepsilon \int_{\mathbf{R}^{2d}} \nabla_\xi e^{i\frac{(x-y')\cdot\xi}{\varepsilon}} \cdot (\nabla_{y'} \tilde{h})(x, \xi, x, \xi; \varepsilon) \psi(y') \frac{dy' d\xi}{(2\pi\varepsilon)^d} + O(\varepsilon^{\nu+2}) \\ &= \int_{\mathbf{R}^{2d}} e^{i\frac{(x-y')\cdot\xi}{\varepsilon}} \tilde{h}(x, \xi, x, \xi; \varepsilon) \psi(y') \frac{dy' d\xi}{(2\pi\varepsilon)^d} \\ &\quad + i\varepsilon \int_{\mathbf{R}^{2d}} e^{i\frac{(x-y')\cdot\xi}{\varepsilon}} \nabla_\xi \cdot (\nabla_{y'} \tilde{h})(x, \xi, x, \xi; \varepsilon) \psi(y') \frac{dy' d\xi}{(2\pi\varepsilon)^d} + O(\varepsilon^{\nu+2}). \end{aligned}$$

Now using (A.7), it remains to choose τ_1^- so that

$$\int_{\mathbf{R}^{2d}} e^{i\frac{(x-y')\cdot\xi}{\varepsilon}} (\tilde{h}(x, \xi, x, \xi; \varepsilon) + i\varepsilon \nabla_\xi \cdot (\nabla_{y'} \tilde{h})(x, \xi, x, \xi; \varepsilon)) \psi(y') \frac{dy' d\xi}{(2\pi\varepsilon)^d} = O(\varepsilon^{\nu+2}).$$

That is possible simply by looking at formula (A.8) and choosing

$$\tau_1^-(x, \xi) = -\frac{\tilde{h}^1(x, \xi, x, \xi) + i\nabla_\xi \cdot (\nabla_y \tilde{h}^0)(x, \xi, x, \xi)}{\partial_\tau h_0|_{\tau=0}(x, \xi)}.$$

Note that Assumption (A.2) guarantees that τ_1^- is well defined. The argument may be pursued at the next order simply replacing (A.10) by

$$\begin{aligned} \tilde{h}(x, \xi, y', \xi; \varepsilon) &= \tilde{h}(x, \xi, x, \xi; \varepsilon) + (y' - x) \cdot (\nabla_{y'} \tilde{h})(x, \xi, x, \xi; \varepsilon) \\ &\quad + \frac{1}{2}(y' - x) \otimes (y' - x) : (\nabla_{y'}^2 \tilde{h})(x, \xi, x, \xi; \varepsilon) + O(|y' - x|^3) \end{aligned}$$

and using integrations by parts again. Then the choice

$$\tau_2^-(x, \xi) = -\frac{\tilde{h}^2(x, \xi, x, \xi) + i\nabla_\xi \cdot (\nabla_y \tilde{h}^1)(x, \xi, x, \xi) - \frac{1}{2}\nabla_\xi^2 : (\nabla_y^2 \tilde{h}^0)(x, \xi, x, \xi)}{\partial_\tau h_0|_{\tau=0}(x, \xi)}$$

gives that $\text{Op}_\varepsilon^0(h(\varepsilon, \lambda))\psi(x) = O(\varepsilon^{\nu+3})$. We leave the rest of the induction argument to the reader.

To end the proof of Theorem 3 we need to consider the case $\nu = 0$. The argument is similar to the case $\nu \neq 0$ treated above, though the formulas are slightly more complicated. We shall use the same notation as in the previous case. We start by noticing that by definition of $\tilde{\tau}_\varepsilon$ we have in particular

$$\forall (x, \xi) \in \mathbf{R}^{2d}, \quad \sum_{j=0}^{N_0} h^{j,0}(x, \xi) \tau_0^j(x, \xi) = 0.$$

Then as before let us write

$$h(x, \xi; \varepsilon, \text{Op}_\varepsilon^1(\tau_\varepsilon^N)) =: \text{Op}_\varepsilon^1(\tilde{h}(x, \xi, y, \eta; \varepsilon)).$$

One computes easily that the principal symbol of $\tilde{h}(x, \xi, y, \eta; \varepsilon)$ is (unlike the case $\nu \neq 0$)

$$(A.11) \quad \tilde{h}^0(x, \xi, y, \eta) = \sum_{j=0}^{N_0} h^{j,0}(x, \xi) \tau_0^j(y, \eta).$$

Note that as above one has

$$\tilde{h}^0(x, \xi, x, \xi) = 0,$$

so that $\tilde{h}(x, \xi, x, \xi; \varepsilon) = O(\varepsilon)$. One can also compute the next orders, and as in the case $\nu \neq 0$ they can be written in the following form:

$$\forall k \geq 1, \quad \tilde{h}^k(x, \xi, y, \eta) = h^k(x, \xi, y, \eta) + \tau_k^-(y, \eta) \sum_{j=1}^{N_0} h^{j,0}(x, \xi) j \tau_0^{j-1}(y, \eta)$$

where $h^k(x, \xi, y, \eta)$ depends on the symbol coefficients of $h(x, \xi)$, and in the (y, η) variables on $\tau_0(y, \eta), \tau_1^-(y, \eta), \dots, \tau_{k-1}^-(y, \eta)$ only. In particular we notice that

$$(A.12) \quad \forall k \geq 1, \quad \tilde{h}^k(x, \xi, x, \xi) = h^k(x, \xi, x, \xi) + \tau_k^-(y, \eta) \partial_\tau h_0|_{\tau=\tau_0}.$$

Now that these formulas have been established, it remains to go through exactly the same computations as in the case $\nu = 0$, and we find

$$\tau_1^-(x, \xi) = -\frac{\tilde{h}^1(x, \xi, x, \xi) + i\nabla_\xi \cdot (\nabla_y \tilde{h}^0)(x, \xi, x, \xi)}{\partial_\tau h_0|_{\tau=\tau_0}(x, \xi)}$$

which is well defined thanks to Assumption (A.2). The other orders are obtained exactly as in the case $\nu = 0$.

This ends the proof of Theorem 3. \square

APPENDIX B. SOME WELL-KNOWN FACTS IN SEMI-CLASSICAL ANALYSIS

In this section we recollect some well-known facts in semi-classical analysis, which have been used throughout the paper. Most of the material is taken from [7], [20], [25], [30] and [29], which we refer to for details.

B.1. Semi-classical symbols and operators.

B.1.1. *Definitions.* We recall that an **order function** is any function $g \in C^\infty(\mathbf{R}^d; \mathbf{R}^+ \setminus \{0\})$ such that there is a constant C satisfying

$$\forall X \in \mathbf{R}^d, \forall \alpha \in \mathbf{N}^d, \quad |\partial^\alpha g(X)| \leq Cg(X).$$

For instance $g(x, \xi) = (1 + |\xi|^2)^{\frac{1}{2}} =: \langle \xi \rangle$ is an order function. Note that the variable X usually refers to a point (x, ξ) in the cotangent space $T^*\mathbf{R}^n \equiv \mathbf{R}^{2n}$, or to a point of the type (x, y, ξ) with $y \in \mathbf{R}^n$. A **semi-classical symbol** in the class $S_d(g)$ is then a function $a = a(X; \varepsilon)$ defined on $\mathbf{R}^d \times]0, \varepsilon_0]$ for some $\varepsilon_0 > 0$, which depends smoothly on X and such that for any $\alpha \in \mathbf{N}^d$, there is a constant C such that for any $(X, \varepsilon) \in \mathbf{R}^d \times]0, \varepsilon_0]$,

$$|\partial^\alpha a(X, \varepsilon)| \leq Cg(X).$$

If $(a_j)_{j \in \mathbf{N}}$ is a family of semi-classical symbols in the class $S_d(g)$, we write that

$$a = \sum_{j=0}^{\infty} \varepsilon^j a_j + O(\varepsilon^\infty)$$

if for any $N \in \mathbf{N}$ and for any $\alpha \in \mathbf{N}^d$, there are ε_0 and C such that

$$\forall X \in \mathbf{R}^d, \forall \varepsilon \in]0, \varepsilon_0] \quad \left| \partial^\alpha \left(a(X, \varepsilon) - \sum_{j=0}^N \varepsilon^j a_j(X, \varepsilon) \right) \right| \leq C\varepsilon^N g(X).$$

Conversely for any sequence $(a_j)_{j \in \mathbf{N}}$ of symbols in $S_d(g)$, there is $a \in S_d(g)$ (unique up to $O(\varepsilon^\infty)$) such that

$$a = \sum_{j=0}^{\infty} \varepsilon^j a_j + O(\varepsilon^\infty).$$

An ε -**pseudodifferential operator** is defined as follows: if a belongs to $S_{3n}(g)$, and u is in $\mathcal{D}(\mathbf{R}^n)$, then

$$\left(\text{Op}_\varepsilon(a) \right) u(x) := \frac{1}{(2\pi\varepsilon)^n} \int e^{i(x-y) \cdot \xi / \varepsilon} a(x, y, \xi) u(y) dy d\xi.$$

B.1.2. *Changes of quantization.* If $a \in S_{2n}(g)$ and $t \in [0, 1]$ then $a^t(x, y, \xi) := a((1-t)x + ty, \xi)$ belongs to $S_{3n}(g)$, and one defines

$$\text{Op}_\varepsilon^t(a) := \text{Op}_\varepsilon(a^t).$$

When $t = 0$ this corresponds to the **classical**, or “**left**” quantization, when $t = 1/2$ this is known as the **Weyl quantization** (and is usually denoted by $\text{Op}_\varepsilon^W(a) = \text{Op}_\varepsilon^{\frac{1}{2}}(a)$), while when $t = 1$ one refers to the “**right**” quantization. Furthermore, if a belongs to $S_{3n}(\langle \xi \rangle^m)$ for some integer m (or more generally if a belongs to $S_{3n}(g)$ where g is a Hörmander metric [20]), then there is a unique symbol a_t belonging to $S_{2n}(\langle \xi \rangle^m)$ (resp. $a_t \in S_{3n}(g)$) such that $\text{Op}_\varepsilon^t(a_t) = \text{Op}_\varepsilon(a)$, and one has

$$a_t(x, \xi) = \sum_{\alpha} \frac{(i\varepsilon)^\alpha}{\alpha!} \partial_\xi^\alpha \partial_\eta^\alpha a(x + t\eta, x - (1-t)\eta, \xi)|_{\eta=0} + O(\varepsilon^\infty).$$

A **classical symbol** is a symbol a in $S_{2n}(\langle \xi \rangle^m)$ such that

$$a(x, \xi; \varepsilon) = \sum_{j=0}^{\infty} \varepsilon^j a_j(x, \xi) + O(\varepsilon^\infty)$$

with a_0 not identically zero, and $a_j \in S_{2n}(\langle \xi \rangle^m)$ independent of ε . For any real number ν , the term $\varepsilon^\nu a_0$ is the **principal symbol** of the classical pseudo-differential operator $A = \varepsilon^\nu \text{Op}_\varepsilon^t(a)$ (and this does not depend on the quantization). On the other hand $\varepsilon^{\nu+1} a_1$ is the **subprincipal symbol** of $A = \varepsilon^\nu \text{Op}_\varepsilon^W(a)$ (in the Weyl quantization only). In the following we shall denote by $\sigma_t(A)$ the symbol of an operator $A = \text{Op}_\varepsilon^t(a)$ (in other words $a = \sigma_t(A)$), and by $\sigma_P(A)$ its principal symbol.

B.1.3. *Microlocal support and ε -oscillation.* If u is an ε -dependent function in a ball of $L^2(\mathbf{R}^n)$, its ε -**frequency set** (or **microlocal support**) is the complement in \mathbf{R}^{2n} of the points (x_0, ξ_0) such that there is a function $\chi_0 \in S_{2n}(1)$ equal to one at (x_0, ξ_0) , satisfying

$$\|\text{Op}_\varepsilon^W(\chi_0 u)\|_{L^2(\mathbf{R}^{2n})} = O(\varepsilon^\infty).$$

We say that an ε -dependent function f_ε bounded in $L^2(\mathbf{R}^n)$ is ε -**oscillatory** if for every continuous, compactly supported function φ on \mathbf{R}^n ,

$$(B.1) \quad \limsup_{\varepsilon \rightarrow 0} \int_{|\xi| \geq R/\varepsilon} |\varphi \widehat{f}_\varepsilon(\xi)|^2 d\xi \rightarrow 0 \quad \text{as } R \rightarrow \infty.$$

On the other hand an ε -dependent function f_ε bounded in $L^2(\mathbf{R}^n)$ is said to be **compact at infinity** if

$$(B.2) \quad \limsup_{\varepsilon \rightarrow 0} \int_{x \geq R} |f_\varepsilon(x)|^2 dx \rightarrow 0 \quad \text{as } R \rightarrow \infty.$$

B.1.4. *Adjoint and composition.* Let a be a symbol in $S_{3n}(g)$, where g is a Hörmander metric [20], and define

$$a^*(x, y, \xi) := \overline{a(y, x, \xi)}.$$

Then the operator $(\text{Op}_\varepsilon(a))^* := \text{Op}_\varepsilon(a^*)$ satisfies for all u, v in $\mathcal{S}(\mathbf{R}^n)$

$$\left((\text{Op}_\varepsilon(a))^* u, v \right)_{L^2} = \left(u, (\text{Op}_\varepsilon(a))^* v \right)_{L^2}$$

and is therefore called the **formal adjoint** of $\text{Op}_\varepsilon(a)$. In particular $\text{Op}_\varepsilon^{\frac{1}{2}}(a)$ is **formally self-adjoint** if a is real.

Let a and b be two symbols in $S_{2n}(g_1)$ and $S_{2n}(g_2)$ respectively, where g_j are Hörmander metrics. For all $t \in [0, 1]$, there is a unique symbol c_t in $S_{2n}(g_1 g_2)$ such that

$$\text{Op}_\varepsilon^t(a) \circ \text{Op}_\varepsilon^t(b) = \text{Op}_\varepsilon^t(c_t).$$

Moreover one has

$$(B.3) \quad c_t(x, \xi; \varepsilon) = e^{i\varepsilon[\partial_u \partial_\xi - \partial_\eta \partial_v]} (a((1-t)x + tu, \eta) b((1-t)v + tx, \xi)) \Big|_{\substack{u=v=x \\ \eta=\xi}} =: a \#^t b.$$

This can be also written

$$c_t(x, \xi; \varepsilon) = \sum_{k \geq 0} \frac{\varepsilon^k}{i^k k!} (\partial_\eta \partial_v - \partial_\xi \partial_u)^k (a((1-t)x + tu, \eta) b((1-t)v + tx, \xi)) \Big|_{\substack{u=v=x \\ \eta=\xi}} + O(\varepsilon^\infty).$$

In particular one has

$$\sigma_t(A \circ B) = \sigma_t(A) \sigma_t(B) + O(\varepsilon).$$

For example in the case when $t = 0$ then $\text{Op}_\varepsilon(a) \circ \text{Op}_\varepsilon(b) = \text{Op}_\varepsilon(c)$, with

$$(B.4) \quad c(x, \xi) = a \# b = \sum_{\alpha} \frac{\varepsilon^\alpha}{i^\alpha \alpha!} \partial_\xi^\alpha a(x, \xi) \partial_x^\alpha b(x, \xi) + O(\varepsilon^\infty).$$

In particular if a and b are two classical symbols in the sense described above, and if one defines $A := \text{Op}_\varepsilon^t(a)$ and $B = \text{Op}_\varepsilon^t(b)$, then the principal symbols satisfy (if $\sigma_P(A) \sigma_P(B)$ does not vanish identically)

$$\sigma_P(AB) = \sigma_P(A) \sigma_P(B).$$

B.2. Semiclassical operators, Wigner transforms and propagation of energy. One of the main interests of the semiclassical setting is that it allows a precise description of the propagation of the energy, on times of the order of $O(\varepsilon)$. We refer for instance to [14] (Section 6) for the proof of the following property (actually in the more general setting of matrix-valued operators): consider a scalar symbol $\tau_\varepsilon(x, \xi)$ defined on \mathbf{R}^{2n} , belonging to the class $S_{2n}(\langle \xi \rangle^\sigma)$ for some $\sigma \in \mathbf{R}$ (or more generally to $S_{2n}(g)$ where g is a Hörmander metric). We assume moreover that $\text{Op}_\varepsilon^W(\tau_\varepsilon)$ is essentially skew-self-adjoint on $L^2(\mathbf{R}^n)$. Then consider f_ε^0 an ε -oscillatory initial data in the sense of (B.1), bounded in $L^2(\mathbf{R}^n)$ and compact at infinity in the sense of (B.2), and the PDE

$$\varepsilon \partial_t f_\varepsilon + \text{Op}_\varepsilon^W(\tau_\varepsilon) f_\varepsilon = 0, \quad f_\varepsilon|_{t=0} = f_\varepsilon^0.$$

Then the Wigner transform $W_\varepsilon(t, x, \xi)$ of $f_\varepsilon(t)$ defined by

$$W_\varepsilon(t, x, \xi) := (2\pi)^{-n} \int_{\mathbf{R}^n} e^{iv \cdot \xi} f_\varepsilon(x - \frac{\varepsilon}{2} v) \bar{f}_\varepsilon(x + \frac{\varepsilon}{2} v) dv$$

converges, locally uniformly in t , to the solution W of

$$\partial_t W + \{\tau_0, W\} = 0$$

where τ_0 is the principal symbol of τ_ε , and where the Poisson bracket is given by

$$\{\tau_0, W\} := \nabla_\xi \tau_0 \cdot \nabla_x W - \nabla_x \tau_0 \cdot \nabla_\xi W.$$

Notice that the interest of Wigner transforms lies in particular in the fact that under the assumptions made on f_ε^0 , for any compact set $K \subset \mathbf{R}^n$ one has

$$\int_K |f_\varepsilon(t, x)|^2 dx = W_\varepsilon(t, K \times \mathbf{R}^n)$$

due to the fact that $|f_\varepsilon(t, x)|^2 = \int_{\mathbf{R}^n} W_\varepsilon(t, x, \xi) d\xi$.

B.3. Coherent states. A **coherent state** centered at $(p, q) \in \mathbf{R}^{2n}$ is the function

$$\Phi_{p,q}(y) := (\pi\varepsilon)^{-\frac{n}{4}} e^{i\frac{(y-q)\cdot p}{\varepsilon}} e^{-\frac{(y-q)^2}{2\varepsilon}}$$

and any tempered distribution u defined on \mathbf{R}^n may be written

$$u(y) = (2\pi\varepsilon)^{-\frac{n}{2}} \int Tu(p, q) \Phi_{p,q}(y) dpdq,$$

where T is the *FBI* (for Fourier-Bros-Iagolnitzer) transform

$$Tu(p, q) := 2^{-\frac{n}{2}} (\pi\varepsilon)^{-\frac{3n}{4}} \int e^{i\frac{(q-y)\cdot p}{\varepsilon}} e^{-\frac{(y-q)^2}{2\varepsilon}} u(y) dy.$$

This transformation maps isometrically $L^2(\mathbf{R}^n)$ to $L^2(\mathbf{R}^{2n})$. The above formula simply translates the fact that $u = T^*Tu$.

B.4. Fourier Integral Operators. A Fourier Integral Operator (FIO) is an operator U which can be written, for any $f \in L^2(\mathbf{R}^n)$

$$Uf(x) = \frac{1}{(2\pi\varepsilon)^{\frac{3n}{2}}} \int_{\mathbf{R}^{2n}} e^{i\Phi(x,y,\tau)/\varepsilon} a(x, y, \tau) f(y) dyd\tau$$

where a is a symbol of order 0, compactly supported in x and y , Φ is real valued and homogeneous of degree 1 in τ , smooth for $\tau \neq 0$. One requires also a non degeneracy condition on the phase (see [29], Chapter 9 Paragraph 6.11) on the support of a . Then U is continuous over $L^2(\mathbf{R}^n)$.

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