Watching systems in graphs: an extension of identifying codes

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Abstract

We introduce the notion of watching systems in graphs, which is a generalization of that of identifying codes. We give some basic properties of watching systems, an upper bound on the minimum size of a watching system, and results on the graphs which achieve this bound; we also study the cases of the paths and cycles, and give complexity results.

Key Words: Graph theory, Complexity, Identifying codes, Watching systems, Paths, Cycles

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1 Introduction and definitions

1.1 Identifying systems

Many search problems, either mathematical problems or 'real life' issues, come down to determine whether a particular item lies in a given set X of possible locations, and locate it if this is the case, by asking questions about its location. Let us suggest a simple model for this, in the so-called *non-adaptive* case, when all questions must be prepared in advance, before getting the answers.

Consider a finite set X and assume that we can only query whether the item lies in certain sets $S \subseteq X$ that belong to a given family $\mathcal{S}$ of subsets of X. For $x \in X$, the *$\mathcal{S}$-identifying set*, or *$\mathcal{S}$-label* (or simply *label* if there is no ambiguity) of x is the set

$$L_{\mathcal{S}}(x) = \{S \in \mathcal{S} : x \in S\}.$$

We say that $\mathcal{S}$ is an *identifying system* of X if the labels of the elements of X are all nonempty and pairwise distinct.

In this case, we can simply ask if the item belongs to S for every $S \in \mathcal{S}$: either all the answers will be negative and the item cannot be in X, or the set of questions with positive answers will correspond to the label $L_{\mathcal{S}}(x)$ of the location x where the item is located. Since $|\mathcal{S}|$ can be much larger than the minimum number of questions required to always succeed, an interesting problem is to find an identifying system $\mathcal{S}'$ with $\mathcal{S}' \subseteq \mathcal{S}$ and with minimum size.

Let us mention graph theoretical problems that are particular instances of this general framework. Karpovsky, Chakrabarty and Levitin introduced the notion of *identifying codes* in [21]; here, with the previous notation, X is the set of vertices of a finite, (in general) undirected graph and $\mathcal{S}$ is the set of all the closed neighbourhoods of the vertices of the graph (see the next section for details). More generally, with the so-called $(r, \leq \ell)$-identifying codes, one can identify sets of vertices within a certain distance (see for instance [23], [24] or [25]). Honkala, Karpovsky and Litsyn, as well as Rosendahl, studied the identification of vertices and edges of a graph using cycles (see [17], [18], [26], [27]). In [19], Honkala and Lobstein considered the identification of vertices in Z^2, using arbitrary subsets of Z^2: we shall see below that this approach is quite close to the notion of watching system. Charbit, Charon, Cohen, Hudry and Lobstein studied the general problem of identifying systems in a bipartite graph framework ([7], [8], [10], [11]). In this paper, we will introduce a new problematics in graphs, which extends the concept of identifying codes, and which can be thought of as identifying vertices with *subsets* of the closed neighbourhoods of the vertices of the graph.

1.2 Notation

We use standard notation: by *graph* we mean a simple, finite, undirected, generally connected, graph (if the graph is not connected, we can consider separately its connected components). If G is a graph, we denote its vertex set by $V(G)$ and its edge set by $E(G)$. The *closed neighbourhood* $N_G[v]$ of a vertex v consists of v and its neighbours in G. For $r \geq 0$ and $v \in V(G)$, the *ball of radius r and centre v* is the set $B_G(v, r)$ of all vertices $x \in V(G)$ satisfying $d_G(v, x) \leq r$, where d_G is the usual distance in G. Obviously, $B_G(v, 1) = N_G[v]$. For standard notions such as degree, diameter, spanning tree, etc., we refer to [4] or [6], whereas for the notion of *NP*-completeness and general background about algorithmic complexity we refer to [3] or [15].

1.3 Identifying codes

Identifying codes were introduced in [21] in 1998 with the original motivation of fault detection in multiprocessor systems. If G is a graph, an *identifying code* is a subset $\mathcal{C} \subseteq V(G)$ such that the family

$$\{N_G[v] : \ v \in \mathcal{C}\}$$

is an identifying system of $V(G)$. The elements of $\mathcal{C}$ are usually called *codewords*.

Of course, such a system will exist if and only if the family of all closed neighbourhoods $\{N_G[v] : v \in V(G)\}$ is itself identifying, which means in this case that distinct vertices must have distinct closed neighbourhoods; a graph with this property is called *twin-free* or *identifiable*.

As aforementioned, in the original motivation the graph models a finite network of processors, and codewords correspond to processors equipped with a monitor able to detect a faulty processor in the closed neighbourhood of its location. Then, if there is at most one fault in the network and if every monitor sends a one-bit message referring to whether it detects a fault or not, we will be able to tell if there is a faulty processor in the graph, and locate it. See the graph G_1 on Figure 1 for an example; one can check that a minimum identifying code in this graph has five codewords. Another example is the graph G_2, depicted on Figure 2, which is a star on 15 vertices. One can check that the minimum size of an identifying code in G_2 is 14.

1.4 Watching systems

The graph G_1 (Figure 1) is slightly pathological, because requiring five codewords to monitor six vertices is very much to ask (in fact, $n-1$ codewords is the maximum that can be required for a graph on n vertices, see [9] or [16]). The reason why we need so many codewords is that the closed neighbourhoods of two distinct vertices only differ by at most two vertices (the same phenomenon is also true for the leaves in G_2 on Figure 2), and in this context a codeword has no choice but to check its whole closed neighbourhood, so that two distinct codewords check almost the same sets of vertices.

There are problems in which this situation is close to reality. For instance, consider a smoke detector: it has no choice but to detect smoke, regardless of the direction where it came from. So an identifying code is a good model for a fire-monitoring system in a building.

On the other hand, for instance in fault detection in multiprocessor systems, it seems plausible that we could easily assign a smaller control area to every detector by simply *not connecting it* to some adjacent vertices. Let us use the term *watcher* instead of codeword for this generalization.

First, let us define it informally with two examples. Assume that an edge between two vertices a and b denotes the possibility for a watcher in a to watch out what happens in b, but that we can choose not to use this possibility: thus we can assign to a watcher located at a vertex v a *watching zone*, which will be any subset of $N_G[v]$.

Let us try this on G_1 and check out Figure 3: we only need three watchers with this protocol (the locations of the watchers 1, 2 and 3 are written down in squares, whereas the label of each vertex, i.e., the set of watchers watching it, is written down in italics nearby, so that the watching zone of each watcher can be retrieved), when five codewords were needed previously. All we have to check is that the labels of all vertices are nonempty and different, and that each watcher only watches vertices in the closed neighbourhood of its location.

What we can also do is to place several watchers at the same location, with distinct watching zones. For instance consider G_2 (see Figure 4): only four watchers are needed whereas 14 codewords were necessary. This can be thought of as a single detector in the centre of the star, but needing four bits instead of one to send information, since it has 15 different vertices to watch. Thus watchers also enable us to model a monitoring system where monitors could simply tell where they detect a fault, but where the cost of a monitor is proportional to the number of bits needed to send this information.

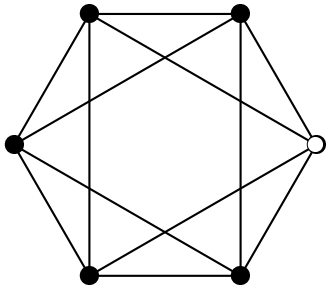

Figure 1: The graph G_1 and a minimum identifying code, of size five. Codewords are in black.

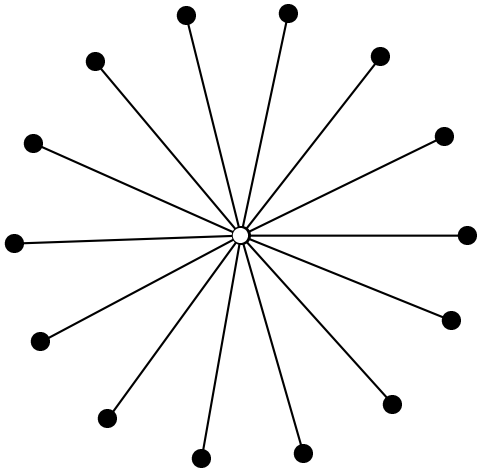

Figure 2: The graph G_2 and a minimum identifying code, of size 14. Codewords are in black.

Let us define this formally:

Definition 1. *A watching system in a graph $G = (V(G), E(G))$ is a finite set*

$$\mathcal{W} = \{w_1, w_2, \ldots, w_k\}$$

where each w_i is a couple $w_i = (v_i, Z_i)$, where v_i is a vertex and $Z_i \subseteq N_G[v_i]$, such that $\{Z_1, \ldots, Z_k\}$ is an identifying system.

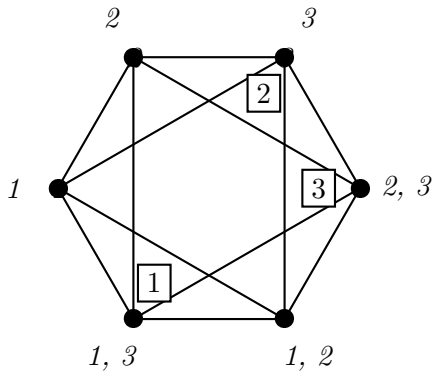

Figure 3: The graph G_1 with a minimum watching system, of size three. Watchers' locations are written down inside squares and labels nearby vertices, in italics.

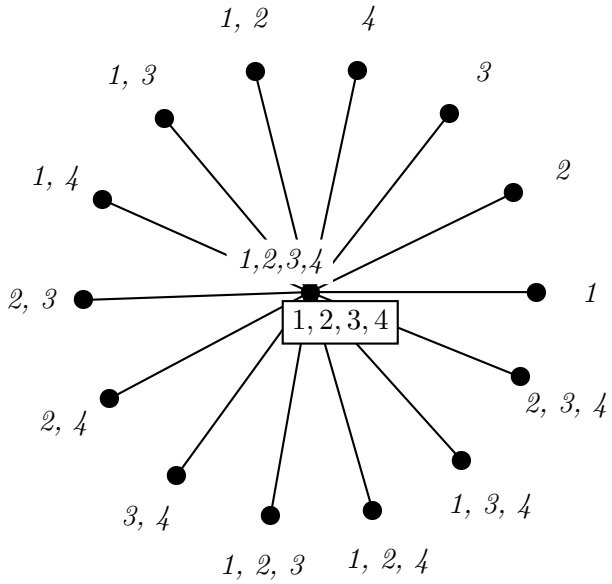

Figure 4: A minimum watching system in G_2, of size four.

We will often represent watchers simply by integers, as we did in Figures 3 and 4. Note that any graph G admits the trivial watching system $\{(v, \{v\}) : v \in V(G)\}$.

If $\mathcal{W}$ is a watching system in G and $w = (v, Z) \in \mathcal{W}$ is a watcher, we will say that v is the *location* of w, or that w is *located* at v. The set Z is the *watching zone*, or *watching area*, of w, and if $x \in Z$ we say that w *covers* x, or that x is *covered by* w. We say that w *separates* the vertices x and y (or x from y) if w covers x and does not cover y, or the other way round. Therefore, $\mathcal{W}$ is a watching system of G if every vertex is covered by at least one watcher in $\mathcal{W}$ and any two distinct vertices are separated by at least one watcher in $\mathcal{W}$. Let us define the $\mathcal{W}$-*label*, or $\mathcal{W}$-*identifying set*, or simply *label*, of a vertex v as the set $L_{\mathcal{W}}(v)$ of watchers covering v. We will say that a vertex v is *identified* by $\mathcal{W}$ if its label $L_{\mathcal{W}}(v)$ is nonempty (v is covered by one watcher at least) and there is no other vertex in G with the same label. Thus another way to express the fact that $\mathcal{W}$ is a watching system is to say that all vertices in G are identified by $\mathcal{W}$.

2 First properties of watching systems

Let us recall that a *dominating set* in G is a subset Γ of $V(G)$ such that every vertex not in Γ is adjacent to at least one element in Γ. Let respectively $w(G)$, $\gamma(G)$ and $i(G)$ denote the minimum sizes of a watching system, of a dominating set and, when it exists, of an identifying code in G. These parameters will be called *watching number*, *domination number*, and *identifying number*, respectively.

If we have k questions to be answered by yes or no, there are $2^k - 1$ possibilities to answer all these questions without answering always by the negative, so we get a trivial lower bound for the size of a watching system. It is known that this bound also holds for identifying codes (see [21]). Noticing that an identifying code, when it exists, defines a watching system in an obvious way, we have the following relationship involving $|V(G)|$ and the watching and identifying numbers:

Theorem 2. *For any graph G, we have:*

$$\lceil \log_2(|V(G)| + 1) \rceil \leq w(G).$$

For any twin-free graph G, we have:

$$w(G) \leq i(G).$$

We now compare the watching and domination numbers of a graph, with the following result, where $\Delta(G)$ denotes the maximum degree of G:

Theorem 3. *For any graph G, we have:*

$$\gamma(G) \leq w(G) \leq \gamma(G) \cdot \lceil \log_2(\Delta(G) + 2) \rceil .$$

Proof. If $\mathcal{W}$ is a watching system, then the set of the watchers' locations in $\mathcal{W}$ is a dominating set, so we have the left-hand inequality. On the other hand, if we have a dominating set $\Gamma \subseteq V(G)$ of size $\gamma(G)$, we can identify all vertices simply by locating enough watchers at every vertex of Γ. One just has to notice that in order to identify a vertex v and its (at most) $\Delta(G)$ neighbours, we need at most $p := \lceil \log_2(\Delta(G) + 2) \rceil$ watchers, since a set with p elements has at least $\Delta(G) + 1$ nonempty subsets. □

3 An upper bound for the watching number

It is known that $i(G) \leq |V(G)| - 1$ for any connected twin-free graph with at least three vertices (see [9], [16]), and that this bound is reached, for instance, by the star, cf. Figure 2. We prove a much better upper bound for watching systems, namely $2n/3$, in Theorem 7, the proof of which will use the following three lemmata.

Lemma 4. *Let G be a graph and H be a partial graph of G, i.e., with $V(H) = V(G)$ and $E(H) \subseteq E(G)$. Then*

$$w(H) \geq w(G).$$

Proof. If $\mathcal{W}$ is a watching system for H, then the same $\mathcal{W}$ is a watching system for G, since two adjacent vertices in H are also adjacent in G. □

Note that this monotony property does not hold in general for identifying codes.

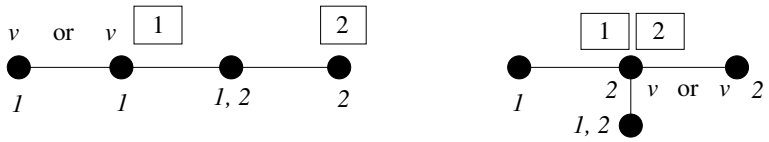

Figure 5: Trees with four vertices.

Lemma 5. *Let T be a tree, x be a leaf of T, and y be the neighbour of x.*

(a) *There exists a minimum watching system for T with one watcher located at y.*

(b) *If y has degree 2, there exists a minimum watching system for T with one watcher located at z, the second neighbour of y.*

Proof. (a) A watching system must cover x, so there is a watcher w_1 located at x or y, with $x \in Z$. If $w_1 = (x, Z)$, then we can replace it by $w_2 = (y, Z)$, since $N_G[y] \supseteq N_G[x]$.

(b) If $y \notin Z$, then one other watcher must cover y, and if $y \in Z$, then one must separate x and y, since $x \in Z$. In both cases, the task can be done by a watcher located at z. □

Lemma 6. *Let T be a tree with four vertices, and let v be a vertex of T; there exists a set $\mathcal{W}$ of two watchers such that*

- *the vertices in $V(T) \setminus \{v\}$ are covered and pairwise separated by $\mathcal{W}$ — in this case, we shall say, with a slight abuse of notation, that $\mathcal{W}$ is a watching system of $V(T) \setminus \{v\}$;*
- *the vertex v is covered by at least one watcher.*

Proof. On Figure 5, we give all possibilities: the two trees with four vertices, and for each of them, the two possible locations for v (v is a leaf, or v is not a leaf). □

Theorem 7. *Let G be a connected graph of order n, i.e., with n vertices.*

- *If $n = 1$, $w(G) = 1$.*
- *If $n = 2$ or $n = 3$, $w(G) = 2$.*
- *If $n = 4$ or $n = 5$, $w(G) = 3$.*
- *If $n \notin \{1, 2, 4\}$, $w(G) \leq \frac{2n}{3}$.*

Proof. For $n = 1$, $n = 2$, or $n = 3$, the result is direct. For $n = 4$, it is necessary to have at least $\lceil \log_2(5) \rceil = 3$ watchers and it is easy to verify that this is sufficient. For $n = 5$, all possibilities are given by Figure 6 and we can see that we always have $w(G) = 3$.

We proceed by induction on n. We assume that $n \geq 6$ and that the theorem is true for any connected graph of order less than n.

Let G be a connected graph of order n. Let T be a spanning tree of G; we will prove that $w(T) \leq \frac{2n}{3}$ and then the theorem will result from Lemma 4. We denote by D the diameter of T and we consider a path $v_0, v_1, v_2, \ldots, v_{D-1}, v_D$ of T, with length D.

We distinguish between four cases, according to some conditions on the degrees of v_{D-1} and v_{D-2}.

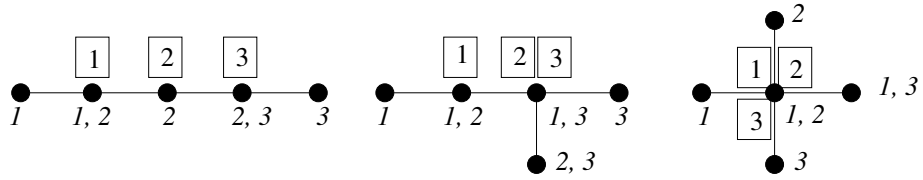

Figure 6: The case n=5 in Theorem 7.

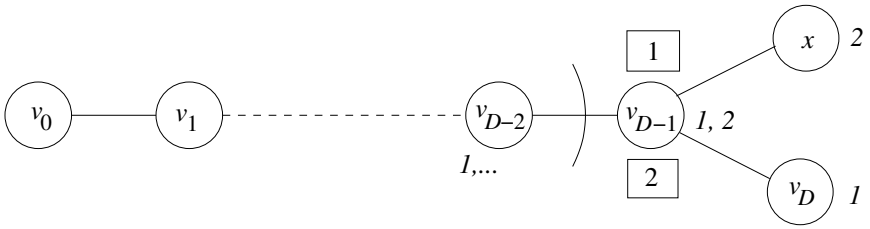

Figure 7: First case of Theorem 7: the degree of v_{D-1} is equal to 3.

• *First case: the degree of v_{D-1} is equal to 3*
The vertex v_{D-1} is adjacent to a vertex x other than v_{D-2} and v_D; because D is the diameter, clearly x and v_D are leaves of T (see Figure 7). We consider the tree obtained by removing x, v_{D-1} and v_D from T; this new tree T' has order $n-3$.

If $n \geq 8$ or if $n = 6$, we consider a minimum watching system $\mathcal{W}$ for T'; if $n = 7$, then T' is of order 4, and, using Lemma 6, we choose a set $\mathcal{W}$ of two watchers which is a watching system for $V(T') \setminus \{v_{D-2}\}$ *and* covers the vertex v_{D-2}.

Then for T, in both cases, we add to $\mathcal{W}$ two watchers $w_1 = (v_{D-1}, \{v_{D-2}, v_{D-1}, v_D\})$ and $w_2 = (v_{D-1}, \{v_{D-1}, x\})$. On Figure 7, we rename 1 and 2 these watchers. Then $\mathcal{W} \cup \{w_1, w_2\}$ is a watching system for T. So, $w(T) \leq |\mathcal{W}| + 2 \leq w(T') + 2$.

Now we use the induction hypothesis: if $n \geq 8$ or $n = 6$, then $w(T) \leq \frac{2}{3}(n-3) + 2 = \frac{2n}{3}$; and if $n = 7$, then $w(T) \leq 2 + 2 = 4 < \frac{2}{3} \times 7$.

• *Second case: the degrees of v_{D-1} and v_{D-2} are equal to 2*
The neighbours of v_{D-1} are v_{D-2} and v_D, the neighbours of v_{D-2} are v_{D-3} and v_{D-1} (see Figure 8). We consider the tree obtained by removing v_{D-2}, v_{D-1} and v_D from T; this new tree T' has order $n-3$.

If $n \geq 8$ or if $n = 6$, we consider a minimum watching system $\mathcal{W}$ for T'; if $n = 7$, T' is of order 4; again using Lemma 6, we choose a set $\mathcal{W}$ of two watchers which is a watching system

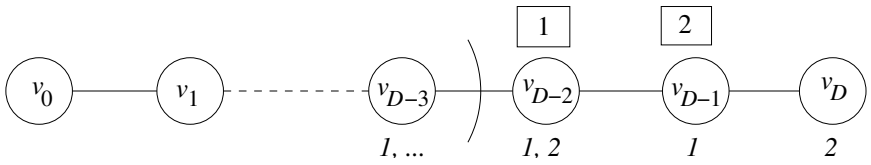

Figure 8: Second case of Theorem 7: the degrees of v_{D-1} and v_{D-2} are equal to 2.

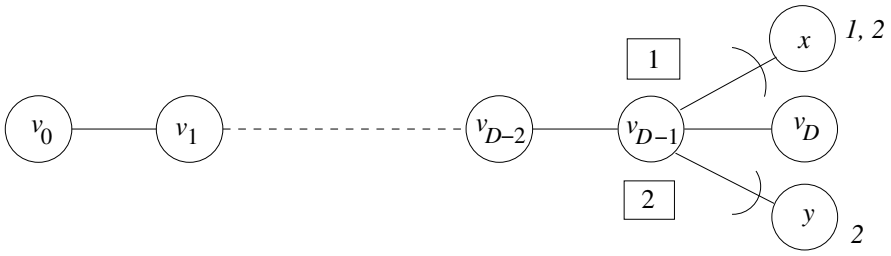

Figure 9: Third case of Theorem 7: the degree of v_{D-1} is at least 4.

for $V(T')\setminus\{v_{D-3}\}$ and covers the vertex v_{D-3}. As in the first case, we add to $\mathcal{W}$ two watchers: $w_1 = (v_{D-2}, \{v_{D-3}, v_{D-2}, v_{D-1}\})$ and $w_2 = (v_{D-1}, \{v_{D-2}, v_D\})$, and obtain a watching system for T. So, $w(T) \leq |\mathcal{W}|+2 \leq w(T')+2$. The end of this case is the same as in the first case.

• *Third case: the degree of v_{D-1} is at least 4*
The vertex v_{D-1} is adjacent to at least two vertices other than v_{D-2} and v_D: let x and y be two neighbours of v_{D-1} distinct from v_{D-2} and v_D; these two vertices are leaves of T (see Figure 9). We consider the tree T' obtained by removing x and y from T. By Lemma 5, there exists a minimum watching system $\mathcal{W}$ of T' with a watcher w_1 located at v_{D-1}. For T, we take the set $\mathcal{W}$ and add the watcher $w_2 = (v_{D-1}, \{x, y\})$; we also add the vertex x to the watching zone of w_1. The set $\mathcal{W}$ being a watching system for T', the set $\mathcal{W} \cup \{w_2\}$ is a watching system for T. So, $w(T) \leq w(T') + 1$.

If $n \geq 7$, the order of T' is at least 5 and, using the induction hypothesis, $w(T) \leq \frac{2}{3}(n-2)+1 < \frac{2n}{3}$.

If $n = 6$, then $n - 2 = 4$ and $w(T) \leq 3 + 1 = 4 = \frac{2}{3} \times 6$.

• *Fourth case: the degree of v_{D-1} is equal to 2 and the degree of v_{D-2} is at least 3*
The neighbours of v_{D-1} are v_{D-2} and v_D. The vertex v_{D-2} is adjacent to v_{D-3} and v_{D-1} but also to at least one other vertex x (see Figure 10); if the degree of x is at least 3, using the fact that the diameter of T is equal to D, we can use the first or third case to conclude, with x playing the part of v_{D-1}.

So, we assume that the degree of x is 1 or 2; if its degree is 2, it has a neighbour y other than v_{D-2}.

We consider the tree T' of order $n-2$ obtained by removing v_{D-1} and v_D from T. By Lemma 5, there exists a minimum watching system $\mathcal{W}$ of T' with a watcher w_1 located at v_{D-2}. For T, we take the set $\mathcal{W}$ and add the watcher $w_2 = (v_{D-1}, \{v_{D-1}, v_D\})$; we also add the vertex v_{D-1} to the watching zone of w_1. Then $\mathcal{W} \cup \{w_2\}$ is a watching system for T.

The end of this case is exactly the same as in the previous case. □

Moreover, we can almost characterize the graphs for which this bound is tight: in [2], we characterize the trees T with n vertices and $w(T) = \lfloor \frac{2n}{3} \rfloor$, then we characterize the graphs G with n vertices and $w(G) = \lfloor \frac{2n}{3} \rfloor$ in the cases $n = 3k$, $k \geq 1$, and $n = 3k+2$, $k \geq 1$; the case $n = 3k+1$ is more complex, and we are only able to state a conjecture for $k \geq 6$.

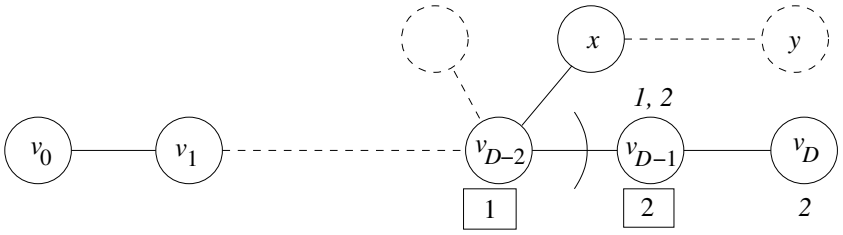

Figure 10: Fourth case of Theorem 7: the degree of v_{D-1} is equal to 2 and the degree of v_{D-2} is at least 3.

4 Watching systems in paths and cycles

Let us call a watching system $\mathcal{W}$ *compressed* if for every vertex $v \in V(G)$ and for every set A such that $\emptyset \subsetneq A \subsetneq L_{\mathcal{W}}(v)$, there is $v' \in V(G)$ such that $A = L_{\mathcal{W}}(v')$. See for example Figures 3 and 4.

If a watching system $\mathcal{W}$ is not compressed, then we can find v and A satisfying $\emptyset \subsetneq A \subsetneq L_{\mathcal{W}}(v)$ such that A is not the label of any vertex in G. Then if for every watcher (x, Z) in $L_{\mathcal{W}}(v) \setminus A$ we redefine this watcher by $(x, Z \setminus \{v\})$, we obtain another watching system of G where the labels of all vertices are the same as before, except for v that has been assigned label A. Clearly, if we do this repeatedly we get a compressed watching system of G with the same size as $\mathcal{W}$, and thus we can always require a watching system to be compressed.

The following lemma is easy but will prove useful:

Lemma 8. *Let G be a graph and $\mathcal{W}$ be a compressed watching system in G. Then for all $v \in V(G)$, we have:*

$$2^{|L_{\mathcal{W}}(v)|} - 1 \leq |B_G(v, 2)|\,.$$

Proof. Since $\mathcal{W}$ is compressed, all the $2^{|L_{\mathcal{W}}(v)|} - 1$ nonempty labels that can be formed using the watchers in $L_{\mathcal{W}}(v)$ must be attributed to vertices in G. The watchers in $L_{\mathcal{W}}(v)$ having their locations in $N_G[v]$, these labels can be attributed only inside $B_G(v, 2)$. □

The path P_n on n vertices is the graph whose vertex set is $\{1, 2, \ldots, n\}$ and whose edge set is $\{\{i, i+1\} : 1 \leq i \leq n-1\}$. We prove:

Theorem 9. *For all $n \geq 1$, we have:*

$$w(P_n) = \left\lceil \frac{n+1}{2} \right\rceil.$$

Proof. First let us prove that $w(P_n) \geq \frac{n+1}{2}$. The small cases are easy to handle, and we assume that $n \geq 6$. Let $\mathcal{W}$ be a minimum compressed watching system of P_n and let i be such that $1 \leq i \leq n$. By Lemma 8, since $|B_{P_n}(i, 2)| \leq 5$, we deduce that $|L_{\mathcal{W}}(i)| \leq 2$. Let us show that the vertices having a label of size 2 can be assumed to be nonadjacent.

First, assume that two adjacent vertices i and $i+1$ have respective labels ab and cd where a, b, c and d are distinct watchers. Since $\mathcal{W}$ is compressed, the four vertices around i and $i+1$ must be labeled by a, b, c and d (and thus we must also have $i > 2$ and $i+1 < n-2$). Without adding watchers, we can change $\mathcal{W}$ into a new compressed watching system where, without loss of generality, the labels from $i-2$ to $i+3$ are $a - ab - b - c - cd - d$.

Now assume that the labels of i and $i+1$ are ab and ac; then the vertices with labels a, b and c must be in $i-2$, $i-1$, $i+2$ or $i+3$. If, for instance, the labels in this order are $b-a-ab-ac-c$, then we can replace them by $b-ab-a-ac-c$. It is not difficult to see that in all cases, we can get a watching system with the same size as $\mathcal{W}$, where the vertices with labels of size 2 are nonadjacent.

Let us also note that we can assume that the vertices 1 and n do not belong to this set: for instance, if the labels of 1, 2 and 3 are $ab-a-b$, we can replace them by $a-ab-b$, and a similar observation can be made for the vertices $n-2$, $n-1$, n.

Once $\mathcal{W}$ is modified, the set of vertices with size-2 labels is an independent set in the path $2, 3, \ldots, n-1$ and thus has size at most $\left\lfloor \frac{n-1}{2} \right\rfloor$, and so the set of vertices with labels of size 1, whose cardinality is the same as $\mathcal{W}$, has size at least

$$n - \left\lfloor \frac{n-1}{2} \right\rfloor,$$

which is equal to $\left\lceil \frac{n+1}{2} \right\rceil$.

Constructions proving that $\left\lceil \frac{n+1}{2} \right\rceil$ is an upper bound are easy to find; actually it is sufficient to use identifying codes (cf. [5]): on the paths, watching systems are no better than identifying codes, except for $n=2$, when no identifying code exists. □

The following result on cycles is obtained in a similar way. Let C_n denote the cycle of length n, with vertices $1, 2, \ldots, n$, and edges $\{i, i+1\}$ for $i \in \{1, 2, \ldots, n-1\}$, and $\{n, 1\}$.

Theorem 10. *We have $w(C_4) = 3$, and for $n = 3$ and all $n \geq 5$:*

$$w(C_n) = \left\lceil \frac{n}{2} \right\rceil.$$

Proof. For the lower bound, using the same argument as in the proof of Theorem 9, we modify $\mathcal{W}$ so that the set of vertices with size-2 labels is an independent set in C_n, thus having size at most $\left\lfloor \frac{n}{2} \right\rfloor$. Constructions proving the upper bound are easy to find. □

If we compare to identifying codes, we can see that the cycle of length three admits no identifying code and that $i(C_4) = i(C_5) = 3$; then $i(C_n) = \frac{n}{2}$ when n is even, $n \geq 6$ (see [5]), and $i(C_n) = \frac{n+3}{2}$ when n is odd, $n \geq 7$, see [12]. So $i(C_n) = w(C_n)$ when $n = 5$ or n is even, $n \geq 4$, and $i(C_n) = w(C_n) + 1$ when n is odd, $n \geq 7$.

5 Computational complexity

Let us recall what is a *vertex cover* in a graph G. An edge $e = xy \in E(G)$ is said to be *covered* by a vertex $v \in V(G)$ if v and e are incident, i.e., if $v = x$ or $v = y$. A *vertex cover* in G is a set of vertices $\mathcal{C} \subseteq V(G)$ such that every edge of G is covered by a at least one element $c \in \mathcal{C}$. Equivalently, $\mathcal{C}$ is a vertex cover if

$$\forall e = xy \in E(G), \quad x \in \mathcal{C} \text{ or } y \in \mathcal{C}.$$

It is well known that the problem of finding the minimum cardinality of a vertex cover in a given graph is *NP*-hard (see [20]); furthermore, it was proved in [14] that this problem remains *NP*-hard when restricted to the class of planar graphs whose maximum degree is at most 3, class which we denote by Π_3. For our proof we need to go a little further. In all graphs, a vertex of degree one is never an issue when we are looking for a vertex cover, since it is easy to prove the following lemma (see [1] for instance):

Lemma 11. *Let G be a graph and $xy \in E(G)$ be an edge such that the degree of the vertex x is 1, and let G' be the graph obtained by removing x, y and all their incident edges from G. Then the minimum cardinality of a vertex cover in G equals the minimum cardinality of a vertex cover in G' plus 1.*

Let Π'_3 be the class of all planar graphs where every vertex has degree 2 or 3. In addition to the aforementioned result from [14], Lemma 11 proves that the following decision problem is *NP*-complete:

MIN VERTEX COVER IN Π'_3
- INSTANCE: A graph $G \in \Pi'_3$ and an integer k;
- QUESTION: Is there a vertex cover for G with size at most k ?

We will use this *NP*-complete problem in order to study the computational complexity of the following decision problem:

MIN WATCHING SYSTEM IN Π_3
- INSTANCE: A planar graph G', with maximum degree at most 3, and an integer k';
- QUESTION: Is there a watching system for G' with size at most k' ?

We prove the following:

Theorem 12. MIN WATCHING SYSTEM IN Π_3 *is NP-complete.*

Proof. Let us observe that MIN WATCHING SYSTEM IN Π_3 belongs to *NP*: given a watching system, it is polynomial with respect to the size of the instance, which can be taken as the order of the graph, to compute the labels of all vertices and check that they are nonempty and distinct. Now, using a polynomial reduction from the problem MIN VERTEX COVER IN Π'_3, we will show that our problem is *NP*-complete.

Consider a graph G and an integer k, an instance of the MIN VERTEX COVER IN Π'_3 problem. Denote respectively by n and m the number of vertices and edges of G. We construct a graph G' by replacing every edge xy of G by the structure S_{xy} depicted on Figure 11, consisting of 4 vertices (including x and y) and 3 edges. Thus G' has $n+2m$ vertices and $3m$ edges and clearly the construction of G' from G can be done in polynomial time. Moreover, if $G \in \Pi'_3$, we clearly have $G' \in \Pi_3$. We set $k' = k + m$. The reduction will be complete if we prove that for all $k \geq 0$:

G admits a vertex cover of size at most k if and only if G' admits a watching system of size at most k'.

Consider an edge xy of G and the structure S_{xy} replacing xy in G', and let $V'_{xy} = \{a_{xy}, b_{xy}\}$.

Assume first that $\mathcal{C}$ is a vertex cover of G. We define a watching system $\mathcal{W}$ in G' as follows:

- for every vertex x of $V(G)$ such that $x \in \mathcal{C}$, we add the watcher $(x, N_{G'}[x])$ to $\mathcal{W}$;
- for every edge $e = xy$ of G, we add the watcher $(a_{xy}, N_{G'}[a_{xy}])$ to $\mathcal{W}$.

It is easy to see that $\mathcal{W}$ is a watching system in G'. Consider a vertex x in G; since it has degree at least 2 in G, it is adjacent to at least two vertices y_1 and y_2 in G. So the corresponding vertex x in G' is covered by, at least, the two watchers located at a_{xy_1} and a_{xy_2}, belonging respectively to the structures S_{xy_1} and S_{xy_2}, and thus x is identified by $\mathcal{W}$. Also note that for every edge $e = xy$ of G, since either x or y belong to the vertex cover $\mathcal{C}$, there is a watcher in $\mathcal{W}$ that separates a_{xy} from b_{xy}. Thus G' admits a watching system with size $|\mathcal{C}| + m \leq k'$.

Conversely, assume that $\mathcal{W}$ is a watching system of G' of size at most k'. Consider an edge $xy \in E(G)$ and the watchers located in the structure S_{xy} of G'. Then:

- if no watcher is located at x nor y, there must be at least two watchers located in V'_{xy};
- if at least one watcher is located at x or y, we still need at least one watcher in V'_{xy}.

So if we denote by $\mathcal{C}$ the set of vertices $x \in V(G)$ such that $\mathcal{W}$ contains a watcher located at x, and by p the number of edges xy of G with $x \notin \mathcal{C}$ and $y \notin \mathcal{C}$, we have

$$|\mathcal{C}| \leq |\mathcal{W}| - 2p - (m - p) \leq k' - m - p \leq k - p.$$

Therefore if we add to $\mathcal{C}$ one vertex for every uncovered edge of G, we get a vertex cover of G of size at most k. □

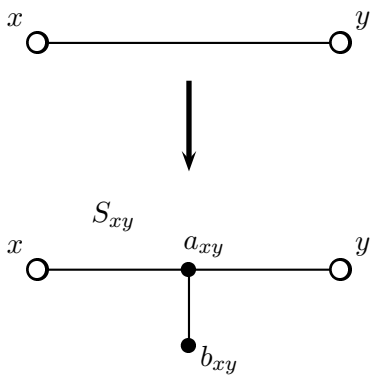

Figure 11: The structure S_{xy} replacing every edge xy of G in the transformation.

6 Distance-identification of sets of vertices

6.1 Definitions

Let us now turn to the problem of identifying several vertices within a certain distance, using a watching system. For $r \geq 1$ and $\ell \geq 1$, we define the notion of $(r, \leq \ell)$-watching systems which extends the notion of $(r, \leq \ell)$-identifying codes.

Define a r-watcher w in a graph G as in the case of a watcher $w = (v, Z)$ except for the watching zone Z that can now be any subset of the ball $B_G(v, r)$ centred at the location v of w and with radius r; thus a 1-watcher is simply a watcher. We extend in an obvious way the notions of covering, label, separation, identification, ... to r-watchers.

Let now $\mathcal{W}$ be a set of r-watchers in G. If $A \subset V(G)$, we define the $\mathcal{W}$-label of A as

$$L_{\mathcal{W}}(A) = \bigcup_{v \in A} L_{\mathcal{W}}(v),$$

and we say that $\mathcal{W}$ is a $(r, \leq \ell)$-*watching system* if all the labels of the subsets A of $V(G)$ with $1 \leq |A| \leq \ell$ are nonempty and distinct.

Note that a $(r, \leq \ell)$-watching system is a $(r', \leq \ell')$-watching system if $\ell' \leq \ell$ and $r' \geq r$.

Let $\mathcal{S} = \{S_1, S_2, \ldots, S_k\}$ be a finite family of distinct nonempty subsets of a set X and $\ell \geq 1$. We say that $\mathcal{S}$ is a ℓ-*superimposed family* on X if, whenever we consider two distinct sets I, J included in $\{1, \ldots, k\}$ with $1 \leq |I| \leq \ell$ and $1 \leq |J| \leq \ell$, we have:

$$\bigcup_{i \in I} S_i \neq \bigcup_{j \in J} S_j.$$

This is the notion of ℓ-superimposed code in a set-system version. These codes were introduced in [22]. They are related to $(r, \leq \ell)$-identifying codes, as was observed in [21] and [13]. They are also related to watching systems since, with our definition, if $\mathcal{W}$ is a $(r, \leq \ell)$-watching system in a graph G, then the family of all $\mathcal{W}$-labels of the vertices of G is a ℓ-superimposed family on $\mathcal{W}$. Note that the family of singletons of X is always a ℓ-superimposed family of X for all $\ell \geq 1$, and so every graph G admits a $(r, \leq \ell)$-watching system for all $r \geq 1$ and $\ell \geq 1$, consisting of the watchers $(v, \{v\})$ for all $v \in V(G)$.

Observe that if $\ell \geq 2$ and $i \neq j$, then $S_i \subseteq S_j$ is impossible in a ℓ-superimposed family. From this follows that if $|L_{\mathcal{W}}(x)| = 1$ for a vertex x in the graph with watching system $\mathcal{W}$, then if $\ell \geq 2$ the watcher covering x must cover only x: we will call such a watcher a *hermit*. Without loss of generality, we can suppose that this watcher is $(x, \{x\})$, since its location does not matter.

6.2 The case of $(1, \leq 2)$-watching systems in paths and cycles

Let us start with the following lemma.

Lemma 13. *For $1 \leq k \leq 4$, the only 2-superimposed family on a set with k elements with at least k subsets is the family of k singletons.*

Proof. The result is obvious if $1 \leq k \leq 3$, so we just check the case $k = 4$. Let S_1, S_2, S_3, S_4 be a 2-superimposed family on $\{1, 2, 3, 4\}$. If there is a singleton in the family, say $S_1 = \{1\}$, then we have $S_i \subset \{2, 3, 4\}$ for $i > 1$ and we use the case $k = 3$ to conclude.

If an element, say 1, is in at least three different sets, say S_1, S_2 and S_3, then by intersecting these sets with $\{2, 3, 4\}$ we get a 2-superimposed family of size 3 on $\{2, 3, 4\}$, so, using the case $k = 3$, we must have (up to permutations) $S_1 = \{1, 2\}$, $S_2 = \{1, 3\}$ and $S_3 = \{1, 4\}$. Then S_4 cannot contain 1, and the remaining possibilities for S_4 all lead to contradictions.

If all the elements are in at most two sets and there are no singletons, then by a simple counting argument we see that all the sets must be pairs, and so the family must be (up to permutations) $\{\{1, 2\}, \{1, 3\}, \{3, 4\}, \{2, 4\}\}$, which is not 2-superimposed. □

In other words, with k watchers, $1 \leq k \leq 4$, we can produce k valid labels, which will be singletons, and not more.

From now on until the end of Section 6.2, the vertices of the paths or cycles are denoted by $x_1, x_2, \ldots$ and the watchers by $1, 2, \ldots$ or $w_1, w_2, \ldots$, depending on the context.

Theorem 14. *For all $n \geq 1$, the minimum size of a $(1, \leq 2)$-watching system in the path P_n is*

$$\min(n, \left\lceil \frac{5(n+1)}{6} \right\rceil), \text{ which is equal to } \begin{cases} n & \text{if } n \leq 10, \\ \left\lceil \frac{5(n+1)}{6} \right\rceil & \text{if } n \geq 5. \end{cases}$$

Proof. Let $\lambda_n = \left\lceil \frac{5(n+1)}{6} \right\rceil$ and $\Lambda_n = \min(n, \lambda_n)$: we have to prove that Λ_n is the minimum size of a $(1, \leq 2)$-watching system in P_n. The proof for the lower bound works by induction on n.

If $1 \leq n \leq 4$, the result comes directly from Lemma 13: we need n watchers since the labels of the vertices form a 2-superimposed family of size n. So from now on, we assume that $n \geq 5$, that the lower bound holds for all $n' < n$, and that $\mathcal{W}$ is a $(1, \leq 2)$-watching system in P_n. We call a *hole* any vertex at which no watcher is located.

If there is no hole, then $|\mathcal{W}| \geq n$, and so $|\mathcal{W}| \geq \Lambda_n$ for all $n \geq 5$; therefore, we can assume that at least one hole exists in P_n.

Case 1) Suppose first that holes exist only on x_1 or x_n.
Without loss of generality, we can assume that x_1 is a hole. If x_1 is covered by only one watcher w, this watcher is located at x_2 and cannot cover any other vertex x_i, $i \in \{2, 3\}$, since this would imply that the sets $\{x_1, x_i\}$ and $\{x_i\}$ cannot be separated by $\mathcal{W}$. So $w = (x_2, \{x_1\})$; then

$$\mathcal{W}' = \big(\mathcal{W} \setminus \{w\}\big) \cup \{(x_1, \{x_1\})\}$$

is still a $(1, \leq 2)$-watching system in P_n, and

$$\mathcal{W}'' = \mathcal{W}' \setminus \{(x_1, \{x_1\})\}$$

is a $(1, \leq 2)$-watching system in $P_{n-1} = x_2, x_3, \ldots, x_n$. By the induction hypothesis, we have $|\mathcal{W}''| \geq \Lambda_{n-1}$, from which we immediately derive that $|\mathcal{W}| = |\mathcal{W}''| + 1 \geq \Lambda_n$.

If now x_1 is covered by at least two watchers w_i, $i = 1, 2, \ldots$, located at x_2, then we consider the path $x_3, x_4, \ldots, x_n$ and the set $\mathcal{W}_1 = \mathcal{W} \setminus \{w_i : i = 1, 2, \ldots\}$. We then have three subcases:

• if x_n is not a hole, then x_1 is the only hole in P_n, and $|\mathcal{W}_1| \geq n - 2$, implying $|\mathcal{W}| \geq n \geq \Lambda_n$;

• if x_n is a hole covered by only one watcher, we use the same argument as used in this case for x_1, and we are done;

• if x_n is a hole covered by at least two watchers w'_j, $j = 1, 2, \ldots$, located at x_{n-1}, then we consider the path $P_{n-4} = x_3, x_4, \ldots, x_{n-2}$ and the set $\mathcal{W}_2 = \mathcal{W}_1 \setminus \{w'_j : j = 1, 2, \ldots\}$; by hypothesis, P_{n-4} has no holes, so $|\mathcal{W}_2| \geq n - 4$ and again $|\mathcal{W}| \geq n \geq \Lambda_n$. In all three cases, we are done, which ends Case 1.

Case 2) There is a hole x_ℓ, $\ell \in \{2, 3, \ldots, n-1\}$.
Without loss of generality, we can assume that $\ell \leq \lfloor \frac{n+1}{2} \rfloor$, so that there are more vertices to the right of x_ℓ than to its left: $\ell - 1 \leq n - \ell$. We distinguish between three subcases:

• $\ell - 1 \leq 4$ and there are at least ℓ watchers located at $x_1, x_2, \ldots, x_{\ell-1}$; since $r = 1$, there is no influence by these ℓ watchers on $x_{\ell+1}$, and we can apply the induction hypothesis to $x_{\ell+1}, \ldots, x_n$, and obtain:

$$|\mathcal{W}| \geq \ell + \Lambda_{n-\ell} = \ell + \min\big(n - \ell, \left\lceil \frac{5(n-\ell+1)}{6} \right\rceil\big) = \min\big(n, \left\lceil \frac{5(n+1)+\ell}{6} \right\rceil\big) \geq \Lambda_n.$$

• $\ell - 1 \leq 4$ and there are exactly $\ell - 1$ hermits located at $x_1, x_2, \ldots, x_{\ell-1}$ (by Lemma 13, and without loss of generality as far as the hermits' locations are concerned, this is the only

possibility left when $\ell - 1 \leq 4$); since these hermits cannot interfere with x_ℓ, we can apply the induction hypothesis to $x_\ell, \ldots, x_n$, and obtain this time:

$$|\mathcal{W}| \geq \ell - 1 + \Lambda_{n-\ell+1} = \ell - 1 + \min\big(n-\ell+1, \Big\lceil \frac{5(n-\ell+2)}{6} \Big\rceil\big) = \min\big(n, \Big\lceil \frac{5(n+1)+\ell-1}{6} \Big\rceil\big) \geq \Lambda_n.$$

• $\ell - 1 \geq 5$ (and so, $n - \ell \geq 5$); again, $x_{\ell-1}$ cannot interfere with $x_{\ell+1}$, and we can apply the induction hypothesis to $x_1, x_2, \ldots, x_{\ell-1}$ and to $x_{\ell+1}, \ldots, x_n$, and obtain:

$$|\mathcal{W}| \geq \Big\lceil \frac{5\ell}{6} \Big\rceil + \Big\lceil \frac{5(n-\ell+1)}{6} \Big\rceil \geq \Big\lceil \frac{5(n+1)}{6} \Big\rceil \geq \Lambda_n,$$

which ends the proof for the lower bound.

We give a construction that matches the lower bound. For $n \in \{1, 2, \ldots, 10\}$, we need at least n watchers, and we can do it with n hermits. For $n = 11$, see Figure 12.

When $n = 6k - 1$, $k \geq 3$, for which at least $5k$ watchers are necessary, use Figure 12, add the pattern of the last six vertices to the right of the right-most vertex, x_{11}, change, for these new vertices, $6, 7, 8, 9, 10$ into $11, 12, 13, 14, 15$, and so on.

When $n = 6k + i$, $k \geq 2$, $i \in \{0, 1, 2, 3, 4\}$, for which at least $5k + i + 1$ watchers are necessary, use the construction for $6k - 1$, where the watchers $5k - 4, \ldots, 5k$ are located at the last five vertices, and where the label of the last vertex, x_{6k-1}, is $\{5k - 1, 5k\}$. Now add $n - 6k + 1 = i + 1$ vertices to the right of x_{6k-1}, and locate $i + 1$ watchers $5k + 1, \ldots, 5k + i + 1$ at the vertices $x_{6k}, \ldots, x_{6k+i}$. Change the label of x_{6k-1} to $\{5k - 1, 5k + 1\}$ and assign the labels $\{5k + j, 5k + j + 2\}$ to x_{6k+j}, for $j \in \{0, \ldots, i - 1\}$, and the label $\{5k + i, 5k + i + 1\}$ to $x_{6k+i} = x_n$.

We let the reader check that this is indeed a $(1, \leq 2)$-watching system; see Figure 13 for an example. □

Observe also that no $(1, \leq 2)$-identifying code (and more generally, no $(1, \leq \ell)$-identifying code) exists in the path P_n, because, since $N_{P_n}[x_1] \subseteq N_{P_n}[x_2]$, the sets of vertices $\{x_2\}$ and $\{x_1, x_2\}$ cannot be separated.

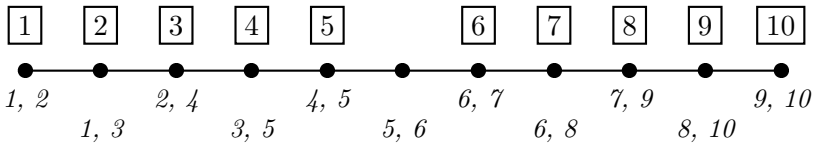

Figure 12: An optimal $(1, \leq 2)$-watching system in the path P_{11}.

In the case of cycles, we have the following result.

Theorem 15. *For all $n \geq 3$, the minimum size of a $(1, \leq 2)$-watching system in the cycle C_n is $\lceil \frac{5}{6}n \rceil$, except for $n = 6$, for which it is 6.*

Proof. The small cases, up to $n = 5$, are easy to handle. The proof of the case $n = 6$ is cumbersome and is not given here. Now consider a $(1, \leq 2)$-watching system $\mathcal{W}$ in C_n, $n \geq 7$.

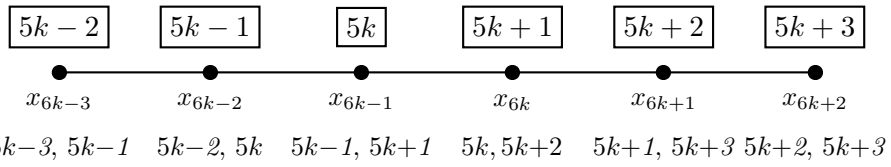

Figure 13: The right end of the path P_n for $n = 6k + 2$.

Remembering that a hole is a vertex at which no watcher is located, we can see that if there is no hole, we are done; so we consider a hole x_ℓ in C_n, and the path on $n-1$ vertices obtained from C_n by removing x_ℓ. Obviously, $\mathcal{W}$ is a $(1, \leq 2)$-watching system for this path, and so $|\mathcal{W}| \geq \Lambda_{n-1} = \lceil \frac{5((n-1)+1)}{6} \rceil$, and the lower bound is proved. Constructions meeting the lower bound are easy: take a path with $n-1$ vertices together with the construction of an optimal $(1, \leq 2)$-watching system described in the proof of Theorem 14. Add a vertex x_n which is linked to x_1 and x_{n-1}, and assign to x_n the label $\{w_1, w_2\}$, where w_1 is located at x_1 and w_2 is located at x_{n-1} (in our construction, there are always watchers located at each end of the path). You obtain a $(1, \leq 2)$-watching system for C_n, of size $\Lambda_{n-1} = \lceil \frac{5n}{6} \rceil$, for $n \geq 7$. The reason why this construction does not work for $n = 6$ is that there would be three labels, $\{1, 3\}$, $\{3, 5\}$ and $\{1, 5\}$, whose pairwise unions are equal to $\{1, 3, 5\}$. □

If we compare to $(1, \leq 2)$-identifying codes in C_n, we can see that, because for all i the sets $B_{C_n}(x_i, 1)$ and $B_{C_n}(x_i, 1) \cup B_{C_n}(x_{i+1}, 1)$ differ by only one vertex, x_{i+2}, this vertex, hence by symmetry all vertices, must belong to the code. Starting from $n = 7$, the only $(1, \leq 2)$-identifying code in the cycle C_n is $V(C_n)$.

6.3 The case of $(1, \leq \ell)$-watching systems in paths and cycles for $\ell \geq 3$

Like every graph, the path P_n and the cycle C_n admit, for all $\ell \geq 3$, a $(1, \leq \ell)$-watching system, which is the trivial watching system consisting of all the hermits. In the case of P_n and C_n, this is the best we can do:

Theorem 16. *For all $n \geq 1$ (respectively, $n \geq 3$) and $\ell \geq 3$, the minimum size of a $(1, \leq \ell)$-watching system in the path P_n (respectively, the cycle C_n) is n.*

Proof. Consider a $(1, \leq \ell)$-watching system $\mathcal{W}$ for P_n or C_n, where $\ell \geq 3$. Let $\mathcal{H} \subseteq \mathcal{W}$ be the set of hermits in $\mathcal{W}$, and let $V_\mathcal{H} \subset V(G)$ be the set of vertices covered by these hermits (as aforementioned, $V_\mathcal{H}$ can be taken, without loss of generality, as the set of the locations of the hermits). Now assume that there is a vertex x in $V(G) \setminus V_\mathcal{H}$; then we have $|L_\mathcal{W}(x)| > 1$.

Suppose that $L_\mathcal{W}(x) = \{w_1, w_2\}$: then w_1 and w_2 are not hermits and so there exist a vertex $v_1 \neq x$ covered by w_1 and a vertex $v_2 \neq x$ covered by w_2 (though we may have $v_1 = v_2$). But in this case we have $L_\mathcal{W}(\{v_1, v_2\}) = L_\mathcal{W}(\{v_1, v_2, x\})$ (or $L_\mathcal{W}(\{v_1\}) = L_\mathcal{W}(\{v_1, x\})$ if $v_1 = v_2$), and so $\mathcal{W}$ cannot be a $(1, \leq \ell)$-watching system if $\ell \geq 3$.

Therefore, all vertices in $V(G) \setminus V_\mathcal{H}$ are covered by at least three watchers from $\mathcal{W} \setminus \mathcal{H}$; since a watcher can only cover at most three vertices, we clearly have $|\mathcal{W} \setminus \mathcal{H}| \geq |V(G) \setminus V_\mathcal{H}|$. Since we also have $|\mathcal{H}| = |V_\mathcal{H}|$, the result follows. □

Finally, observe that for $\ell \geq 3$, no $(1, \leq \ell)$-identifying code exists in the cycle C_n, because the sets of vertices $\{1, 3\}$ and $\{1, 2, 3\}$ (or more generally, $\{x, x+2\}$ and $\{x, x+1, x+2\}$) cannot be separated.

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