

AUSLANDER-REITEN CONJECTURE FOR SYMMETRIC ALGEBRAS OF POLYNOMIAL GROWTH

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ABSTRACT. This paper studies self-injective algebras of polynomial growth. We prove that the derived equivalence classification of weakly symmetric algebras of domestic type coincides with the classification up to stable equivalences (of Morita type). As for weakly symmetric non-domestic algebras of polynomial growth, up to some scalar problems, the derived equivalence classification coincides with the classification up to stable equivalences of Morita type. As a consequence, we get the validity of the Auslander-Reiten conjecture for stable equivalences of Morita type between weakly symmetric algebras of polynomial growth.

INTRODUCTION

Drozd [7] showed that finite dimensional algebras A over an algebraically closed field K are either of finite representation type, or of tame representation type or of wild representation type. Here, an algebra is of finite representation type if there are only a finite number of isomorphism classes of indecomposable modules. An algebra is of wild representation type if all module categories are full subcategories of the module category of the algebra. An algebra is of tame representation type if for every positive integer d there are $A - K[X]$ -bimodules $M_i(d)$ for $i \in \{1, \dots, n_d\}$ so that $M_i(d)$ is free as $K[X]$ -module and so that all but finite number of isomorphism classes of indecomposable d -dimensional A -modules are isomorphic to $M_i(d) \otimes_{K[X]} K[X]/(X - \lambda)$ for $\lambda \in K$ and $i \in \{1, \dots, n_d\}$. If n_d is minimal with respect to d satisfying this property, A is of polynomial growth if there is an integer m so that $\lim_{d \rightarrow \infty} \frac{n_d}{d^m} = 0$, and A is domestic if there is an integer m so that $n_d \leq d \cdot m$ for all d .

Bocian, Holm and Skowroński [5, 6, 12] classified all weakly symmetric algebras of domestic type up to derived equivalence and Białkowski, Holm and Skowroński [3, 4, 13] did the same thing for all weakly symmetric non-domestic algebras of polynomial growth. The result is a finite number of families of algebras given by quivers and relations, depending on certain parameters. In the case of domestic algebras the classification is complete, whereas in case of weakly symmetric algebras of polynomial growth it is not known if certain choices of parameters may lead to the same derived equivalence class. This question if certain parameters lead to derived equivalent algebras will be called the scalar problem in the sequel.

The main result of this paper is that the derived equivalence classification of weakly symmetric algebras of polynomial growth coincides (up to some scalar problems) with the classification up to stable equivalences of Morita type as given in [5, 6, 12] and [3, 4, 13]. As a consequence, the Auslander-Reiten conjecture holds for a stable equivalence of Morita type between two weakly symmetric algebras of polynomial growth.

For weakly symmetric algebras of domestic type we can show more. We prove that the derived equivalence classification coincides with the classification up to stable equivalences (no scalar problem occurs in this case) and that the Auslander-Reiten conjecture holds for stable equivalences between weakly symmetric algebras of domestic types.

As mentioned above the classification of symmetric algebras of polynomial growth up to derived equivalence leaves open if for certain choices of parameters the algebras are derived equivalent or not.

Date: May 3, 2010.

2000 Mathematics Subject Classification. 16G10, 16E40, 20C20.

Key words and phrases. Auslander-Reiten conjecture; Derived equivalence; Reynolds ideal; Self-injective algebras of polynomial growth; Stable equivalence; Stable equivalence of Morita type.

The same questions concerning the same choices of parameters remain open also for the classification up to stable equivalences of Morita type.

Moreover, using our result [26], we show that for tame symmetric algebras with periodic modules, the derived equivalence classification coincides with the classification up to stable equivalence of Morita type, up to some scalar problem, and that the Auslander-Reiten conjecture holds for stable equivalences between tame symmetric algebras with periodic modules.

The paper is organised as follows. In Section 1 we recall basic definitions and recall the stable invariants we use in the sequel. In Section 2 we prove the main result Theorem 2.9 for weakly symmetric algebras of domestic representation type and in Section 3 we show the main result Theorem 3.7 for stable equivalences of Morita type for weakly symmetric algebras of polynomial growth.

As soon as we use statements on representation type we shall always assume that K is an algebraically closed field.

1. STABLE CATEGORIES; DEFINITIONS, NOTATIONS AND INVARIANTS

1.1. The definitions. Let K be a field and let A be a finite dimensional K -algebra. The stable category $A\text{-mod}$ has as objects all finite dimensional A -modules and morphisms from an A -module X to another A -module Y are the equivalence classes of A -linear homomorphisms $\text{Hom}_A(X, Y)$, where two morphisms of A -modules are declared to be equivalent if their difference factors through a projective A -module.

Two algebras are called *stably equivalent* if $A\text{-mod} \simeq B\text{-mod}$ as additive categories. Let M be an $A\text{-}B$ -bimodule and let N be a $B\text{-}A$ -bimodule. Following Broué the couple (M, N) defines a *stable equivalence of Morita type* if

- M is projective as A -module and is projective as B -module
- N is projective as B -module and is projective as A -module
- $M \otimes_B N \simeq A \oplus P$ as $A\text{-}A$ -bimodules, where P is a projective $A\text{-}A$ -bimodule.
- $N \otimes_A M \simeq B \oplus Q$ as $B\text{-}B$ -bimodules, where Q is a projective $B\text{-}B$ -bimodule.

Of course, if (M, N) defines a stable equivalence of Morita type, then $M \otimes_B - : B\text{-mod} \simeq A\text{-mod}$ is an equivalence. However, it is known that there are algebras which are stably equivalent but for which there is no stable equivalence of Morita type.

1.2. Invariants under stable equivalences. Auslander and Reiten conjectured that if A and B are stably equivalent, then the number of isomorphism classes of non-projective simple A -modules is equal to the number of isomorphism classes of non-projective simple B -modules. The conjecture is open in general, but we shall use in this article the following positive result [22] of Pogorzały: If A is stably equivalent to B and if A is selfinjective special biserial that is not Nakayama, then B is self-injective special biserial as well and the Auslander-Reiten conjecture holds in this case.

Krause and Zvara [17] showed that stable equivalences preserve the representation type. Suppose A and B are two finite dimensional stably equivalent K -algebras. If A is of domestic representation type (respectively of polynomial growth, respectively of tame representation type), then so is B .

Reiten [23] show that being selfinjective is almost invariant under stable equivalences. More precisely, let A be a selfinjective finite dimensional K -algebra which is stably equivalent to a finite dimensional indecomposable K -algebra B . Then B is either selfinjective or a radical squared zero Nakayama algebra.

Finally, if A and B are stably equivalent finite dimensional algebras with Loewy length at least three, then the stable Auslander-Reiten quivers are isomorphic as translation quivers.

1.3. Invariants under stable equivalence of Morita type. Much more is known for stable equivalences of Morita type. We shall cite only those properties used in the sequel.

Xi has shown [25] that if two finite dimensional K -algebras A and B are stably equivalent of Morita type, then the absolute value of the Cartan determinants are equal. Even better, the elementary divisors, including their multipliers, of the Cartan matrices different from ± 1 coincide for A and for B .

Liu showed [19] that if A is an indecomposable finite dimensional K -algebra, and B is a finite dimensional K -algebra stably equivalent of Morita type to A , then B is indecomposable as well.

If K is an algebraically closed field of characteristic $p > 0$ and A a finite dimensional K -algebra. Define $[A, A]$ the K -space generated by all elements $ab - ba \in A$ for all $a, b \in A$. Then,

$$\{x \in A \mid x^{p^n} \in [A, A]\} =: T_n(A)$$

is a K -subspace of A containing $[A, A]$. Liu and the authors have shown [20] that

$$\dim_K(T_n(A)/[A, A]) = \dim_K(T_n(B)/[B, B])$$

for all $n \in \mathbb{N}$. If A and B are symmetric and stably equivalent of Morita type, then taking the orthogonal space with respect to the symmetrising form, observing that $[A, A]^\perp = Z(A)$, Liu, König and the first author have shown [15] that

$$Z(A)/T_n(A)^\perp \simeq Z(B)/T_n(B)^\perp$$

as rings.

2. WEAKLY SYMMETRIC ALGEBRAS OF DOMESTIC REPRESENTATION TYPE

Selfinjective algebras of domestic type split into two subclasses: standard algebras and non-standard ones.

A selfinjective algebra of tame representation type is called *standard* if its basic algebra admit simply connected Galois coverings. A selfinjective algebra of tame representation type is called *nonstandard* if it is not standard.

2.1. Standard domestic weakly symmetric algebras.

Theorem 2.1. [5, Theorem 1] *For an algebra A the following statements are equivalent:*

- (1) *A is representation-infinite domestic weakly symmetric algebras having simply connected Galois coverings and the Cartan matrix C_A is singular.*
- (2) *A is derived equivalent to the trivial extension $T(C)$ of a canonical algebra C of Euclidean type.*
- (3) *A is stably equivalent to the trivial extension $T(C)$ of a canonical algebra C of Euclidean type.*

Moreover, the trivial extensions $T(C)$ and $T(C')$ of two canonical algebras C and C' of Euclidean type are derived equivalent (respectively, stably equivalent) if and only if the algebras C and C' are isomorphic.

From this theorem, we know that a standard weakly symmetric algebras of domestic representation type with *singular Cartan matrix* is symmetric.

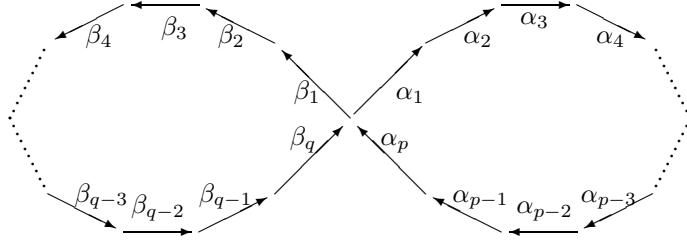
Corollary 2.2. *The class of indecomposable standard weakly symmetric algebras of domestic representation type with singular Cartan matrices is closed under stable equivalences.*

Proof In fact, let A be an indecomposable algebra stably equivalent to an indecomposable standard weakly symmetric algebras of domestic representation type. Then by the preceding theorem, A is stably equivalent to the trivial extension $T(C)$ of a canonical algebra C of Euclidean type. Again by the preceding theorem, A is standard weakly symmetric algebras of domestic representation type. □

For standard weakly symmetric algebras of domestic representation type with *nonsingular Cartan matrices*, we have the following derived norm forms.

$$\begin{array}{ccc}
 A(\lambda) & \begin{array}{c} \beta \quad \bigcirc \quad \bullet \quad \bigcirc \quad \alpha \\ \text{---} \end{array} & \\
 \lambda \in K \setminus \{0\} & & \alpha^2 = 0, \beta^2 = 0, \alpha\beta = \lambda\beta\alpha
 \end{array}$$

$A(p, q)$
 $1 \leq p \leq q$
 $p + q \geq 3$



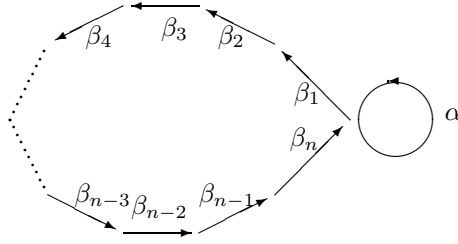
$$\alpha_1 \alpha_2 \cdots \alpha_p \beta_1 \beta_2 \cdots \beta_q = \beta_1 \beta_2 \cdots \beta_q \alpha_1 \alpha_2 \cdots \alpha_p$$

$$\alpha_p \alpha_1 = 0, \beta_q \beta_1 = 0$$

$$\alpha_i \alpha_{i+1} \cdots \alpha_p \beta_1 \cdots \beta_q \alpha_1 \cdots \alpha_{i-1} \alpha_i = 0, 2 \leq i \leq p$$

$$\beta_j \beta_{j+1} \cdots \beta_q \alpha_1 \cdots \alpha_p \beta_1 \cdots \beta_{j-1} \beta_j = 0, 2 \leq j \leq q$$

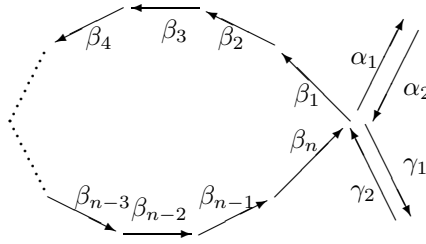
$\Lambda(n)$
 $n \geq 2$



$$\alpha^2 = (\beta_1 \beta_2 \cdots \beta_n)^2, \alpha \beta_1 = 0, \beta_n \alpha = 0$$

$$\beta_j \beta_{j+1} \cdots \beta_n \beta_1 \cdots \beta_n \beta_1 \cdots \beta_{j-1} \beta_j = 0, 2 \leq j \leq n$$

$\Gamma(n)$
 $n \geq 1$



$$\alpha_1 \alpha_2 = (\beta_1 \beta_2 \cdots \beta_n)^2 = \gamma_1 \gamma_2,$$

$$\alpha_2 \beta_1 = 0, \gamma_2 \beta_1 = 0, \beta_n \alpha_1 = 0$$

$$\beta_n \gamma_1 = 0, \alpha_2 \gamma_1 = 0, \gamma_2 \alpha_1 = 0$$

$$\beta_j \beta_{j+1} \cdots \beta_n \beta_1 \cdots \beta_n \beta_1 \cdots \beta_{j-1} \beta_j = 0, 2 \leq j \leq n$$

Theorem 2.3. [5, Theorem 2] *For a domestic standard self-injective algebra A , the following statements are equivalent:*

- (1) A is weakly symmetric and the Cartan matrix C_A is nonsingular.
- (2) A is derived equivalent to an algebra of the form $A(\lambda), A(p, q), \Lambda(n), \Gamma(n)$.
- (3) A is stably equivalent to an algebra of the form $A(\lambda), A(p, q), \Lambda(n), \Gamma(n)$.

Moreover, two algebras of the forms $A(\lambda), A(p, q), \Lambda(n)$ or $\Gamma(n)$ are derived equivalent (respectively, stably equivalent) if and only if they are isomorphic.

All these algebras except

$$A(\lambda) = k\langle X, Y \rangle / (X^2, Y^2, XY - \lambda YX), \lambda \notin \{0, 1\}$$

are symmetric. Remark that except $\Gamma(n)$, all algebras are special biserial algebras.

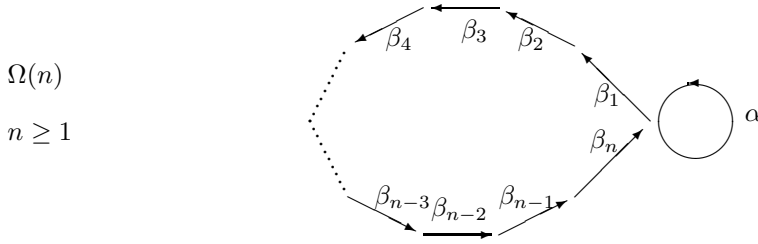
Since the class of indecomposable standard weakly symmetric algebras of domestic representation type is closed under stable equivalences, we have the following

Corollary 2.4. *Two standard weakly symmetric algebras of domestic representation type are stably equivalent if and only if they are derived equivalent.*

2.2. Non standard domestic weakly symmetric algebras. Nonstandard self-injective algebras of domestic representation type are classified up to derived and stable equivalences in [6].

Theorem 2.5. [6, Theorem 1] *Any nonstandard representation-infinite selfinjective algebra of domestic type is derived equivalent (resp. stably equivalent) to an algebra $\Omega(n)$ with $n \geq 1$. Moreover, two algebras $\Omega(n)$ and $\Omega(m)$ are derived equivalent (respectively, stably equivalent) if and only if $m = n$.*

The quiver with relations of $\Omega(n)$ is as follows.



$$\alpha^2 = \alpha\beta_1\beta_2 \cdots \beta_n, \alpha\beta_1\beta_2 \cdots \beta_n + \beta_1\beta_2 \cdots \beta_n\alpha = 0,$$

$$\beta_n\beta_1 = 0, \beta_j\beta_{j+1} \cdots \beta_n\beta_1 \cdots \beta_n\beta_1 \cdots \beta_j\beta_{j-1}\beta_j = 0, 2 \leq j \leq n$$

Remark that the algebra $\Omega(n)$ is always weakly symmetric, but it is symmetric only when the characteristic of the base field is 2. Note that $\Omega(n)$ is not special biserial.

2.3. Domestic standard type versus domestic non standard type.

Theorem 2.6. [12, Theorem 1.1] *A standard symmetric algebra of domestic representation type cannot be derived equivalent to a non-standard symmetric one.*

In course of the proof of the above theorem, one needs to compare the algebras $\Omega(n)$ with $A(1, n)$ in case the characteristic of K is 2. It is proved in [12] that the dimensions of the centre modulo the first Külshammer ideal, for $\Omega(n)$ and $A(1, n)$, are different, but this dimension is in fact invariant under stable equivalences of Morita type ([20]). So a standard symmetric algebra of domestic representation type cannot be stably equivalent of Morita type to a non-standard symmetric one. In fact, we can even prove more.

Lemma 2.7. *A standard weakly symmetric algebra of domestic type cannot be stably equivalent to a nonstandard one.*

Proof Stably equivalent algebras of Loewy length at least 3 have isomorphic stable Auslander-Reiten quivers ([2, Corollary X. 1.9]). Notice that the stable Auslander-Reiten quiver of $A(\lambda)$ for $\lambda \neq 0$ consists of an Euclidean component of type $\mathbb{Z}\tilde{A}_1$ and a $\mathbb{P}_1(K)$ -family of homogenous tubes. Since all algebras in the above list of Theorems 2.1, 2.3, 2.5 are of Loewy length at least 3, by comparing the shapes of the stable Auslander-Reiten quivers as in [12, Section 4], we know that a standard weakly symmetric algebra of domestic type A in the list of Theorems 2.1, 2.3 is stably equivalent to a nonstandard one $B = \Omega(n)$ only when $A = A(1, n)$ and $B = \Omega(n)$ for $n \geq 1$. However, notice that $A(1, n)$ is special biserial and $\Omega(n)$ is not. By the result of Pogorzały [22, Theorem 7.3], they can never be stably equivalent. □

2.4. Stable equivalence classification of domestic weakly symmetric algebras.

Proposition 2.8. (1) *Two symmetric algebras of domestic representation type are derived equivalent if and only if they are stably equivalent of Morita type if and only if they are stably equivalent.*
 (2) *The class of symmetric algebras of domestic representation type is closed under stable equivalences, hence is closed under stably equivalences of Morita type and derived equivalences.*

Proof Corollary 2.4 shows that for standard weakly symmetric algebras the notions of derived equivalence and of stable equivalence are the same. Lemma 2.7 shows that standard weakly symmetric algebras cannot be stably equivalent to non standard weakly symmetric algebras. Theorem 2.5 shows that for non standard domestic weakly symmetric algebras derived and stable equivalence are the same notion. This proves (1).

Now let A be an indecomposable algebra stably equivalent to an indecomposable symmetric algebra B of domestic representation type in the list of algebras in Theorems 2.1, 2.3 and 2.5. Then by a result of Idun Reiten [23, Theorem 2.6] (see also [18, Theorem]), A itself is also self-injective or A is isomorphic to a radical square zero algebra whose quiver is of type A with linear orientation. But the latter algebra has finite representation type, and thus cannot be stably equivalent to a representation-infinite algebra. We infer that A is self-injective and now by [5, Proposition 5.2], A is weakly symmetric. By [17, Corollary 2 and the discussions afterwards], A is of domestic type. We thus proved that A is weakly symmetric of domestic type.

To finish the proof of the second assertion, one needs to show that A is symmetric and thus to exclude the cases where A is stably equivalent to $A(\lambda)$ with $\lambda \in K \setminus \{0, 1\}$ or to $\Omega(n)$ with $n \geq 1$ in case of $\text{char}K \neq 2$, since these are the only cases where non symmetric algebras may occur.

If A is stably equivalent to $A(\lambda)$ with $\lambda \in K \setminus \{0, 1\}$, then by [22, Theorem 7.3], B is special biserial. Since the Auslander-Reiten conjecture is proved for self-injective special biserial algebras ([22, Theorem 0.1]), B is also local and is thus necessarily isomorphic to $A(1)$. However, $A(1)$ is never stably equivalent to $A(\lambda)$ with $\lambda \neq 1$ by an unpublished result of Rickard.

If $\text{char}(K) \neq 2$ and A is stably equivalent to $A = \Omega(n)$ with $n \geq 1$, then by Lemma 2.7, B is necessarily nonstandard and $B = \Omega(n)$ for some n . But the fact that B is symmetric implies that $\text{char}(K) = 2$ which is a contradiction. □

In fact we can extend (at least partially) the above result to all weakly symmetric algebras of domestic representation type.

Theorem 2.9. (1) *Two weakly symmetric algebras of domestic representation type are derived equivalent if and only if they are stably equivalent of Morita type if and only if they are stably equivalent.*
 (2) *The class of weakly symmetric algebras of standard domestic representation type is closed under stable equivalences, hence under stably equivalences of Morita type, and derived equivalences.*

Proof Since by Lemma 2.7, a standard weakly symmetric algebra of domestic type cannot be stably equivalent to a nonstandard one, Theorems 2.1, 2.3 and 2.5 imply the first assertion.

For the second assertion, let A be an indecomposable algebra stably equivalent to a weakly symmetric standard algebra B of domestic representation type in the list of algebras in Theorems 2.1 and 2.3. By [17, Corollary 2 and the discussions afterwards], A is of domestic type. As in the proof of Proposition 2.8, A is still self-injective. Since by Proposition 2.8(2), the class of symmetric algebras of domestic type is closed under stable equivalences, one can assume that B is not symmetric. Now $B = A(\lambda)$ with $\lambda \notin \{0, 1\}$. Since $B = A(\lambda)$ is a local special biserial algebra, so is A by [22, Theorems 0.1 and 7.3]. The algebra A is thus weakly symmetric. □

- Remark 2.10.**
- (1) If the class of self-injective nonstandard algebras of domestic type is closed under stable equivalences (hence under stably equivalences of Morita type and derived equivalences), so is the class of weakly symmetric algebras of domestic representation type.
 - (2) The second assertion of the above theorem answers a question of [5, Page 47].
 - (3) If we have a complete Morita equivalence classification of all self-injective algebras of domestic type, the classification up to stable equivalence (of Morita type) might be feasible as well.

As a consequence, the Auslander-Reiten conjecture holds for this class of algebras.

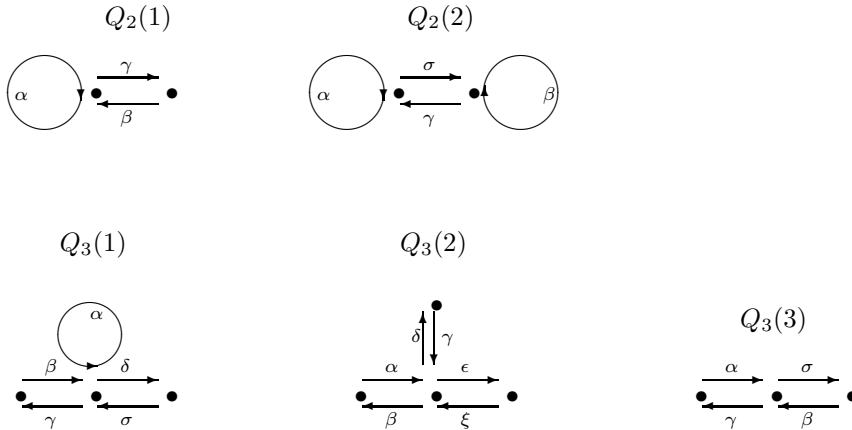
- Corollary 2.11.**
- (1) Let A be an indecomposable algebra stably equivalent to an indecomposable symmetric algebra B of domestic type. Then A has the same number of simple modules as B .
 - (2) Let A be an indecomposable algebra stably equivalent to an indecomposable weakly symmetric standard algebra B of domestic type. Then A has the same number of simple modules as B .

Remark 2.12. If the class of self-injective nonstandard algebras of domestic type is closed under stable equivalences, then Auslander-Reiten conjecture holds for a stable equivalence between two weakly symmetric algebras of domestic type.

3. NON-DOMESTIC SELF-INJECTIVE ALGEBRAS OF POLYNOMIAL GROWTH

The derived equivalence classification of the standard (resp. non-standard) non-domestic symmetric algebras of polynomial growth is achieved in [3, Page 653 Theorem] (resp. [4, Theorem 3.1]). They give a complete list of representatives of derived equivalence classes in terms of quiver with relations. We recall the derived norm forms following [13].

We have five quivers and various relations.



Define for any $\lambda \in K \setminus \{0, 1\}$ the algebras

$$\begin{aligned}
\Lambda_2 &:= KQ_2(1)/(\alpha^2\gamma, \beta\alpha^2, \gamma\beta\gamma, \beta\gamma\beta, \beta\gamma - \beta\alpha\gamma, \alpha^3 - \gamma\beta) \\
\Lambda'_2 &:= KQ_2(1)/(\alpha^2\gamma, \beta\alpha^2, \beta\gamma, \alpha^3 - \gamma\beta) \\
\Lambda_3(\lambda) &:= KQ_2(2)/(\alpha^4, \gamma\alpha^2, \alpha^2\sigma, \alpha^2 - \sigma\gamma - \alpha^3, \lambda\beta^2 - \gamma\sigma, \gamma\alpha - \beta\gamma, \sigma\beta - \alpha\sigma) \\
\Lambda'_3(\lambda) &:= KQ_2(2)/(\alpha^2 - \sigma\gamma, \lambda\beta^2 - \gamma\sigma, \gamma\alpha - \beta\gamma, \sigma\beta - \alpha\sigma) \\
\Lambda_5 &:= KQ_3(1)/(\alpha^2 - \gamma\beta, \alpha^3 - \delta\sigma, \beta\delta, \sigma\gamma, \alpha\delta, \sigma\alpha, \gamma\beta\gamma, \beta\gamma\beta, \beta\gamma - \beta\alpha\gamma) \\
\Lambda'_5 &:= KQ_3(1)/(\alpha^2 - \gamma\beta, \alpha^3 - \delta\sigma, \beta\delta, \sigma\gamma, \alpha\delta, \sigma\alpha, \beta\gamma) \\
\Lambda_9 &:= KQ_3(2)/(\beta\alpha + \delta\gamma + \epsilon\xi, \gamma\delta, \xi\epsilon, \alpha\beta\alpha, \beta\alpha\beta, \alpha\beta - \alpha\delta\gamma\beta) \\
\Lambda'_9 &:= KQ_3(2)/(\beta\alpha + \delta\gamma + \epsilon\xi, \gamma\delta, \xi\epsilon, \alpha\beta) \\
A_1(\lambda) &:= KQ_3(3)/(\alpha\gamma\alpha - \alpha\sigma\beta, \beta\gamma\alpha - \lambda \cdot \beta\sigma\beta, \gamma\alpha\gamma - \sigma\beta\gamma, \gamma\alpha\sigma - \lambda\sigma\beta\sigma) \\
A_4 &:= KQ_3(2)/(\beta\alpha + \delta\gamma + \epsilon\xi, \alpha\beta, \gamma\epsilon, \xi\delta)
\end{aligned}$$

Further, denote the trivial extensions of tubular canonical algebras as usual, in particular

$$\begin{aligned}
\Lambda(2, 2, 2, 2, \lambda) &= T(C(2, 2, 2, \lambda)) \text{ for } \lambda \in K \setminus \{0, 1\} \\
\Lambda(3, 3, 3) &= T(C(3, 3, 3)) \\
\Lambda(2, 4, 4) &= T(C(2, 4, 4)) \\
\Lambda(2, 3, 6) &= T(C(2, 3, 6)).
\end{aligned}$$

The precise quivers with relations of the trivial extensions of tubular canonical algebras in question are displayed in [13]. We will refrain from presenting them here since we do not really need this information in such details.

3.1. Weakly symmetric standard non-domestic polynomial growth algebras.

Theorem 3.1. [3, Page 653 Theorem] *Let A be an indecomposable standard non-domestic weakly symmetric algebra of polynomial growth. Then A is derived equivalent to one of the following algebras:*

- two simple modules: Λ'_2 and $\Lambda'_3(\lambda)$, $\lambda \in K \setminus \{0, 1\}$;
- three simple modules: Λ'_5 and $A_1(\lambda)$, $\lambda \in K \setminus \{0, 1\}$;
- four simple modules: Λ'_9 and A_4 ;
- six simple modules: $\Lambda(2, 2, 2, 2, \lambda)$, $\lambda \in K \setminus \{0, 1\}$;
- eight simple modules: $\Lambda(3, 3, 3)$;
- nine simple modules: $\Lambda(2, 4, 4)$;
- ten simple modules: $\Lambda(2, 3, 6)$.

The above classification is complete up to the scalar problems in $\Lambda'_3(\lambda)$ and in $A_1(\lambda)$. Remark that except Λ'_9 in case the characteristic of the base field K is different from 2, all algebras are symmetric.

We shall prove

Proposition 3.2. *The classification of indecomposable standard non-domestic weakly symmetric algebras of polynomial growth up to stable equivalences of Morita type coincide with the derived equivalence classification, modulo the scalar problems in $\Lambda'_3(\lambda)$, $\lambda \in K \setminus \{0, 1\}$ and in $A_1(\lambda)$, $\lambda \in K \setminus \{0, 1\}$.*

Proof Since a derived equivalence between self-injective algebras induces a stable equivalence of Morita type ([14], [24]), we only need to show that two algebras from the above list which are not derived equivalent are not stably equivalent of Morita type, either.

Since the property of being symmetric is invariant under a stable equivalence of Morita type ([19, Corollary 2.4]), in case the characteristic of the base field is different from 2, Λ'_9 cannot be stably equivalent of Morita type to any of the other algebras. We can concentrate on the remaining symmetric algebras in the list.

Now the algebras of at least six simple modules are trivial extensions of canonical algebras of tubular type $(2, 2, 2, 2)$, $(3, 3, 3)$, $(2, 4, 4)$, $(2, 3, 6)$ respectively. They have singular Cartan matrices,

while all other algebras have non-singular Cartan matrix. Since, by a result of Xi [25, Proposition 5.1] the absolute value of the Cartan determinant is preserved by a stable equivalence of Morita type, we can consider separately those algebras of at most four simple modules and those of at least six simple modules.

For trivial extension cases, by [3, Propositions 5.1 and 5.2], two algebras of this type are stably equivalent if and only if they are derived equivalent. So the number of simple modules, which is an invariant under derived equivalence, distinguishes them.

Now suppose two indecomposable standard non-domestic weakly symmetric algebras of polynomial growth are stably equivalent of Morita type and have non-singular Cartan matrices. Then the algebras are algebras in the above list. The following table gives some stable invariants of Morita type which distinguish two algebras in the above list.

Algebra A	Λ'_2	$\Lambda'_3(\lambda)$	Λ'_5	$A_1(\lambda)$	Λ'_9	A_4
$\det C_A$	6	12	6	16	4	12
$\dim(Z(A)/R(A))$	3	4	2	2	1	2

□

3.2. Selfinjective non-standard non-domestic polynomial growth algebras. Now we turn to non-domestic non-standard selfinjective algebras of polynomial growth. Recall the following

Theorem 3.3. (*Białkowski, Holm, Skowroński* [4, Theorem 3.1]) *Let A be an indecomposable non-standard non-domestic self-injective algebra of polynomial growth. Then we get the following.*

- *If the base field is of characteristic 3, then A is derived equivalent to Λ_2 .*
- *else the base field is of characteristic 2 and then*
 - *if A has two simple modules, A is derived equivalent to $\Lambda_3(\lambda)$, $\lambda \in K \setminus \{0, 1\}$*
 - *if A has three simple modules, A is derived equivalent to Λ_5*
 - *if A has four simple modules, A is derived equivalent to Λ_9*
 - *else A has five simple modules and then A is derived equivalent to Λ_{10}*

If the base field is of characteristic 2 and $\lambda \neq \lambda'$, we do not know if $\Lambda_3(\lambda)$ is derived equivalent to $\Lambda_3(\lambda')$ or not. We call this question the scalar problem.

The above classification is complete up to the scalar problem in case the base field is of characteristic 2 for the algebra $\Lambda_3(\lambda)$, $\lambda \in K \setminus \{0, 1\}$.

Remark that the algebra Λ_{10} is not symmetric (even not weakly symmetric), so it is not stably equivalent of Morita type to any other algebra in the above list.

We can now prove

Proposition 3.4. *The classification of indecomposable non-standard non-domestic self-injective algebras of polynomial growth up to stable equivalences of Morita type coincides with the derived equivalence classification, modulo the scalar problem in $\Lambda_3(\lambda)$.*

Proof We only need to consider the case of characteristic two, since if the characteristic of K is 3, there is only one algebra. But notice $\det C_{\Lambda_3(\lambda)} = 12$, $\det C_{\Lambda_5} = 6$ and $\det C_{\Lambda_9} = 4$.

□

3.3. Standard algebras versus non-standard algebras. Now one can prove that a non-standard algebra cannot be stably equivalent to a standard one. The case of a derived equivalence is proved by Holm-Skowroński in [13].

Theorem 3.5. [13, Main Theorem] *Let A be a standard self-injective algebra of polynomial growth and let Λ be an indecomposable, nonstandard, non-domestic, symmetric algebra of polynomial growth. Then A and Λ are not derived equivalent.*

Proposition 3.6. *Let A be an indecomposable standard weakly symmetric algebra and Λ be an indecomposable nonstandard non-domestic self-injective algebra of polynomial growth. Then A and Λ are not stably equivalent of Morita type.*

Proof Suppose that A and Λ are stably equivalent of Morita type. By Krause [16, Page 605 Corollary], A is also non-domestic of polynomial growth.

We only need to consider symmetric algebras. In fact, the only non-symmetric nonstandard, non-domestic self-injective algebra of polynomial growth Λ_{10} is only defined when the characteristic of the base field is 2, but in case of characteristic 2, all standard non-domestic weakly symmetric algebras of polynomial growth are symmetric, as Λ'_9 is non-symmetric if and only if characteristic of the base field is different from 2.

Now we consider symmetric algebras. In fact, the method of the proof of [13, Main Theorem] works. Since two algebras which are stably equivalent of Morita type have isomorphic stable Auslander-Reiten quivers, by the shape of the stable Auslander-Reiten quivers (see the second paragraph of Section 4 of Holm-Skowroński [13]), we proceed by remarking the following facts.

- (1) If K is of characteristic 3, the algebras Λ_2 and Λ'_2 are not stably equivalent of Morita type. Indeed Holm-Skowroński [13] have shown that

$$\dim Z(\Lambda'_2)/T_1(\Lambda'_2)^\perp = 3 \neq 2 = \dim Z(\Lambda_2)/T_1(\Lambda_2)^\perp.$$

- (2) If K is of characteristic 2 and $\lambda, \mu \in K \setminus \{0, 1\}$, the algebras $\Lambda_3(\lambda)$ and $\Lambda_3(\mu)'$ are not stably equivalent of Morita type. Indeed Holm-Skowroński [13] have shown that

$$\dim Z(\Lambda'_3(\lambda))/T_1(\Lambda'_3(\lambda))^\perp = 3 \neq 2 = \dim Z(\Lambda_3(\mu))/T_1(\Lambda_3(\mu))^\perp.$$

- (3) If K is of characteristic 2, the algebras Λ_5 and Λ'_5 are not stably equivalent of Morita type. Indeed Holm-Skowroński [13] have shown that

$$\dim Z(\Lambda'_5)/T_1(\Lambda'_5)^\perp = 0 \neq \dim Z(\Lambda_5)/T_1(\Lambda_5)^\perp.$$

- (4) If K is of characteristic 2, the algebras Λ_9 and Λ'_9 are not stably equivalent of Morita type. Indeed Holm-Skowroński [13] have shown that

$$\dim HH^2(\Lambda_9) = 4 \neq 3 = \dim HH^2(\Lambda'_9).$$

Since $Z(A)/T_1(A)^\perp$ and $HH^2(A)$ are invariants under stable equivalences of Morita type, this completes the proof. □

3.4. Stable equivalence classification of weakly symmetric non-domestic polynomial growth algebras.

Combining Propositions 3.2, 3.4 and 3.6, we have

Theorem 3.7. *The classification of indecomposable non-domestic weakly symmetric algebras of polynomial growth up to stable equivalences of Morita type coincides with the derived equivalence classification, up to the above mentioned scalar problems.*

As a consequence, we can prove a special case of Auslander-Reiten conjecture.

Theorem 3.8. (1) *Let A be an indecomposable algebra which is stably equivalent of Morita type to an indecomposable non-domestic symmetric algebra Λ of polynomial growth. Then A and Λ have the same number of simple modules.*

- (2) *Let A and Λ be two indecomposable algebras which are both non-domestic weakly symmetric algebra of polynomial growth or which are both non-standard self-injective algebras of polynomial growth. If they are stably equivalent of Morita type, then A and Λ have the same number of simple modules.*

Proof (1). By Krause [16, Page 605 Corollary] and Liu [19, Corollary 2.4], A is also symmetric, non-domestic, and of polynomial growth. Then one can apply Theorem 3.7, by noticing that the existing scalar problems all occur in families of algebras for which different scalars yield algebras having the same number of simple modules.

- (2) is a direct consequence of Theorem 3.7 and Proposition 3.4. □

3.5. Periodic algebras. Recently, Karin Erdmann and Andrzej Skowroński [9] have shown that a non-simple indecomposable symmetric algebra A is tame with periodic modules if and only if A belongs to one of the following classes of algebras: a representation-finite symmetric algebra, a non domestic symmetric algebra of polynomial growth, or an algebra of quaternion type (in the sense of [8]). Note that these three classes of algebras are closed under stable equivalences of Morita type, by Krause [16, Page 605 Corollary] and Liu [19, Corollary 2.4].

In [1] Asashiba, in combination with Holm-Skowroński [12], the statement displayed in [10, Theorem 3.4, Theorem 3.5, Theorem 3.6] proved that for the class of representation-finite selfinjective indecomposable algebras are derived equivalent if and only if they are stable equivalent. In our recent paper [26, Theorem 7.1], we proved that for the class of algebras of quaternion type, derived equivalence classification coincide with the classification up to stable equivalences of Morita type (up to some scalar problems). Therefore, combining the above Theorem 3.8, we showed the following

Theorem 3.9. (1) *The class of tame symmetric algebras with periodic modules is closed under stable equivalences of Morita type, in particular under derived equivalences;*
 (2) *For tame symmetric algebras with periodic modules derived equivalence classification coincide with the classification up to stable equivalences of Morita type (up to some scalar problems);*
 (3) *The Auslander-Reiten conjecture holds for a stable equivalence of Morita type between two tame symmetric algebra with periodic modules.*

4. CONCLUDING REMARKS

In Section 3 we classified symmetric algebras of polynomial growth only up to stable equivalences of Morita type, whereas the results in Section 2 concern stable equivalences in general. It would be most interesting to get a stable equivalence classification for general selfinjective algebras of polynomial growth, in particular the validity of the Auslander-Reiten conjecture in general. The crucial point for this more general statement would be to first classify algebras which are stably equivalent (of Morita type) to Λ'_9 in case the characteristic of K is different from 2 or to Λ_{10} , since these are the selfinjective polynomial growth algebras which are not symmetric. The second step is to classify weakly symmetric algebras of polynomial growth up to stable equivalences.

If the for the class of selfinjective algebras of polynomial growth the derived equivalence classification would coincide with the classification up to stable equivalences (of Morita type) the Auslander-Reiten conjecture for this class of algebras should follow.

Another point concerns the work of Martinez-Villa ([21]) of the reduction of the Auslander-Reiten conjecture to selfinjective algebras. If we can prove that this reduction preserves representation type: domestic type, polynomial growth etc, then the conjectural result of the preceding paragraph would imply the validity of the Auslander-Reiten conjecture for algebras of polynomial growth.

The results of this paper present a second occurrence of a phenomenon of the same kind: In our previous paper [26] we also obtained that the classification of a class of tame symmetric algebras up to stable equivalence coincides with the derived equivalence classification. One might ask if two indecomposable tame symmetric algebras are stably equivalent of Morita type if and only if they are derived equivalent.

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