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Sheaf quantization of Hamiltonian isotopies and applications to non displaceability problems

Stéphane Guillermou, Masaki Kashiwara and Pierre Schapira

May 5, 2010

Abstract

Let I be an open interval, M be a real manifold, T^*M its cotangent bundle with the zero-section removed and $\Phi = \{\varphi_t\}_{t \in I}$ a homogeneous Hamiltonian isotopy of T^*M . Let $\Lambda \subset T^*M \times T^*M \times T^*I$ be the conic Lagrangian submanifold associated with Φ . We prove the existence and unicity of a sheaf K on $M \times M \times I$ whose microsupport is contained in the union of Λ and the zero-section and whose restriction to $t = 0$ is the constant sheaf on the diagonal of $M \times M$. We give applications of this result to problems of non displaceability in contact and symplectic topology. In particular we prove that some strong Morse inequalities are stable by Hamiltonian isotopies and we also give results of non displaceability for positive isotopies in the contact setting.

Introduction

The microlocal theory of sheaves has been introduced and systematically developed in [9, 10], the central idea being that of the microsupport of sheaves. More precisely, consider a real manifold M of class C^∞ and a field \mathbf{k} . Denote by $\mathbf{D}^b(\mathbf{k}_M)$ the bounded derived category of sheaves of \mathbf{k} -modules on M . The microsupport $\text{SS}(F)$ of an object F of $\mathbf{D}^b(\mathbf{k}_M)$ is a closed subset of the cotangent bundle T^*M , conic for the action of \mathbb{R}^+ on T^*M and co-isotropic. Hence, this theory is “conic”, that is, it is invariant by the \mathbb{R}^+ -action and is related to the homogeneous symplectic structure better than the symplectic structure of T^*M .

In order to treat non homogeneous symplectic problems, a classical trick is to add a variable which replaces the homogeneity. This is performed for complex symplectic manifolds in [15] and later in the real case by D. Tamarkin in [16] who adapts the microlocal theory of sheaves to the non homogeneous situation and deduce a new and very original proof of the classical non-displaceability theorem conjectured by Arnold. Note that a link between sheaf theory and symplectic topology already appeared in [14].

In this paper, we will also find a new proof of the non-displaceability theorem and other related results, but when remaining in the homogeneous symplectic framework, which makes the use of sheaf theory much easier. In other words, instead of adapting microlocal sheaf theory to treat non homogeneous geometrical problems, we translate these geometrical problems to homogeneous ones and apply the classical microlocal sheaf theory. Note that the converse is not always possible: there are interesting geometrical problems, for example those related to the notion of positive Hamiltonian isotopies, which make sense in the homogeneous case and which have no counterpart in the purely symplectic case.

Our main tool is, following the title of this paper, a quantization of Hamiltonian isotopies in the category of sheaves. More precisely, consider a homogeneous Hamiltonian isotopy $\Phi = \{\varphi_t\}_{t \in I}$ of \dot{T}^*M (the complementary of the zero-section of T^*M) defined on an open interval I of \mathbb{R} containing 0 such that $\varphi_0 = \text{id}$ and $\varphi_t = \text{id}$ outside of $A \times_M T^*M$ for a compact subset A of M . Denoting by $\Lambda \subset \dot{T}^*M \times \dot{T}^*M \times T^*I$ the conic Lagrangian submanifold associated with Φ , we prove that there exists a unique $K \in \mathbf{D}^b(\mathbf{k}_{M \times M \times I})$ whose microsupport is contained in the union of Λ and the zero-section of $T^*(M \times M \times I)$ and whose restriction to $t = 0$ is the constant sheaf on the diagonal of $M \times M$.

We give a few applications of this result to problems of non displaceability in symplectic and contact geometry. The classical non displaceability conjecture of Arnold says that, on the cotangent bundle to a compact manifold M , the image of the zero-section of T^*M by an Hamiltonian isotopy always intersects the zero-section. This conjecture (and its refinements, using Morse inequalities) have been proved by Chaperon [1] who treated the case of the torus using the methods of Conley and Zehnder [5], then by Hofer [7] and Laudenbach and Sikorav[13]. For related results in the contact case, let us quote in particular Chaperon [2], Chekanov [3] and Ferrand [6].

In this paper we recover the non displaceability result in the symplectic case as well as its refinement using Morse inequalities. Indeed, we deduce

these results from their homogeneous counterparts which are easy corollaries of our theorem of quantization of homogeneous Hamiltonian isotopies. We also study positive Hamiltonian isotopies (which make sense only in the contact setting): we prove that the conormal bundle to a point cannot be interchanged with the conormal bundle to another point by such an isotopy, as soon as M is not compact, a variant of a result of [4].

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We are also extremely grateful to Emmanuel Ferrand who pointed out to us the crucial fact that the Arnold non displaceability problem could be treated through homogeneous symplectic methods.

1 Microlocal theory of sheaves, after [10]

In this section, we recall some definitions and results from [10], following its notations with the exception of slight modifications. We consider a real manifold M of class C^∞ .

Some geometrical notions ([10, § 4.2, § 6.2])

A C^1 -map $f: M \rightarrow N$ is called *smooth* if its differential $d_x f: T_x M \rightarrow T_{f(x)} N$ is surjective for any $x \in M$. For a locally closed subset A of M , one denotes by $\text{Int}(A)$ its interior and by \overline{A} its closure. One denotes by Δ_M or simply Δ the diagonal of $M \times M$.

One denotes by $\tau_M: TM \rightarrow M$ and $\pi_M: T^*M \rightarrow M$ the tangent and cotangent bundles to M . If $L \subset M$ is a (smooth) submanifold, one denotes by $T_L M$ its normal bundle and $T_L^* M$ its conormal bundle. They are defined by the exact sequences

$$\begin{aligned} 0 \rightarrow TL \rightarrow L \times_M TM \rightarrow T_L M \rightarrow 0, \\ 0 \rightarrow T_L^* M \rightarrow L \times_M T^* M \rightarrow T^* L \rightarrow 0. \end{aligned}$$

One identifies M with the zero-section $T_M^* M$ of $T^* M$. One sets $\dot{T}^* M := T^* M \setminus T_M^* M$ and one denotes by $\dot{\pi}_M: \dot{T}^* M \rightarrow M$ the projection.

Let $f: M \rightarrow N$ be a morphism of real manifolds. To f are associated the tangent morphisms

$$(1.1) \quad \begin{array}{ccccc} TM & \xrightarrow{f'} & M \times_N TN & \xrightarrow{f_\tau} & TN \\ \downarrow \tau_M & & \downarrow \tau_M & & \downarrow \tau_N \\ M & \xlongequal{\quad} & M & \xrightarrow{f} & N. \end{array}$$

By duality, we deduce the diagram:

$$(1.2) \quad \begin{array}{ccccc} T^*M & \xleftarrow{f_d} & M \times_N T^*N & \xrightarrow{f_\pi} & T^*N \\ \downarrow \pi_M & & \downarrow \pi_N & & \downarrow \pi_N \\ M & \xlongequal{\quad} & M & \xrightarrow{f} & N. \end{array}$$

One sets

$$T_M^*N := \text{Ker } f_d = f_d^{-1}(T_M^*M).$$

Note that, denoting by Γ_f the graph of f in $M \times N$, the projection $T^*(M \times N) \rightarrow M \times T^*N$ identifies $T_{\Gamma_f}^*(M \times N)$ and $M \times_N T^*N$.

Now consider the homogeneous symplectic manifold T^*M : it is endowed with the Liouville 1-form given in a local homogeneous symplectic coordinate system $(x; \xi)$ on T^*M by

$$\alpha_M = \langle \xi, dx \rangle.$$

The antipodal map a_M is defined by:

$$(1.3) \quad a_M: T^*M \rightarrow T^*M, \quad (x; \xi) \mapsto (x; -\xi).$$

If A is a subset of T^*M , we denote by A^a instead of $a_M(A)$ its image by the antipodal map.

We shall use the Hamiltonian isomorphism $H: T^*T^*M \xrightarrow{\simeq} TT^*M$ given in a local symplectic coordinate system $(x; \xi)$ by

$$H(\langle \lambda, dx \rangle + \langle \mu, d\xi \rangle) = -\langle \lambda, \partial_\xi \rangle + \langle \mu, \partial_x \rangle.$$

Microsupport ([10, § 5.1, 6.5])

For simplicity, we consider a field \mathbf{k} , but this assumption could be weakened by assuming that \mathbf{k} is a commutative ring with finite global dimension (*e.g.* $\mathbf{k} = \mathbb{Z}$). We denote by $\mathbf{D}^b(\mathbf{k}_M)$ the bounded derived category of sheaves of \mathbf{k} -modules on M . We denote by $\omega_M \in \mathbf{D}^b(\mathbf{k}_M)$ the dualizing complex on M . Recall that ω_M is isomorphic to the orientation sheaf shifted by the dimension. Recall the definition of the microsupport (or singular support) $\text{SS}(F)$ of a sheaf F ([10, Def. 5.1.2]).

Definition 1.1. Let $F \in \mathbf{D}^b(\mathbf{k}_M)$ and let $p \in T^*M$. One says that $p \notin \text{SS}(F)$ if there exists an open neighborhood U of p such that for any $x_0 \in M$ and any real C^1 -function φ on M defined in a neighborhood of x_0 with $(x_0; d\varphi(x_0)) \in U$, one has $\text{R}\Gamma_{\{x; \varphi(x) \geq \varphi(x_0)\}}(F)_{x_0} \simeq 0$.

In other words, $p \notin \text{SS}(F)$ if the sheaf F has no cohomology supported by “half-spaces” whose conormals are contained in a neighborhood of p .

- By its construction, the microsupport is \mathbb{R}^+ -conic, that is, invariant by the action of \mathbb{R}^+ on T^*M .
- Denoting by T_M^*M the zero-section of T^*M , identified with M , the intersection of $\text{SS}(F)$ with T_M^*M coincides with the support of F .
- The microsupport is additive: if $F_1 \rightarrow F_2 \rightarrow F_3 \xrightarrow{+1}$ is a distinguished triangle in $\mathbf{D}^b(\mathbf{k}_M)$, then $\text{SS}(F_i) \subset \text{SS}(F_j) \cup \text{SS}(F_k)$ for all $i, j, k \in \{1, 2, 3\}$ with $j \neq k$.

In the sequel, for a locally closed subset Z of M , we denote by \mathbf{k}_Z the constant sheaf with stalk \mathbf{k} on Z , extended by 0 on $M \setminus Z$.

Example 1.2. (i) If F is a local system on M , then $\text{SS}(F) = T_M^*M \cap \pi_M^{-1}(\text{Supp}(F))$.

(ii) If N is a smooth closed submanifold of M and $F = \mathbf{k}_N$, then $\text{SS}(F) = T_N^*M$, the conormal bundle to N in M .

(iii) Let φ be C^1 -function with $d\varphi(x) \neq 0$ when $\varphi(x) = 0$. Let $U = \{x \in M; \varphi(x) > 0\}$ and let $Z = \{x \in M; \varphi(x) \geq 0\}$. Then

$$\begin{aligned} \text{SS}(\mathbf{k}_U) &= U \times_M T_M^*M \cup \{(x; \lambda d\varphi(x)); \varphi(x) = 0, \lambda \leq 0\}, \\ \text{SS}(\mathbf{k}_Z) &= Z \times_M T_M^*M \cup \{(x; \lambda d\varphi(x)); \varphi(x) = 0, \lambda \geq 0\}. \end{aligned}$$

For a precise definition of being involutive (or co-isotropic), we refer to [10, Def 6.5.1]

Theorem 1.3. *Let $F \in \mathbf{D}^b(\mathbf{k}_M)$. Then its microsupport $\mathrm{SS}(F)$ is involutive.*

Localization ([10, § 6.1])

Now let A be a subset of T^*M and let $Z = T^*M \setminus A$. The full subcategory $\mathbf{D}_Z^b(\mathbf{k}_M)$ of $\mathbf{D}^b(\mathbf{k}_M)$ consisting of sheaves F such that $\mathrm{SS}(F) \subset Z$ is triangulated. One sets

$$\mathbf{D}^b(\mathbf{k}_M; A) := \mathbf{D}^b(\mathbf{k}_M) / \mathbf{D}_Z^b(\mathbf{k}_M),$$

the localization of $\mathbf{D}^b(\mathbf{k}_M)$ by $\mathbf{D}_Z^b(\mathbf{k}_M)$. Hence, the objects of $\mathbf{D}^b(\mathbf{k}_M; A)$ are those of $\mathbf{D}^b(\mathbf{k}_M)$ but a morphism $u: F_1 \rightarrow F_2$ in $\mathbf{D}^b(\mathbf{k}_M)$ becomes an isomorphism in $\mathbf{D}^b(\mathbf{k}_M; A)$ if, after embedding this morphism in a distinguished triangle $F_1 \rightarrow F_2 \rightarrow F_3 \xrightarrow{+1}$, one has $\mathrm{SS}(F_3) \cap A = \emptyset$.

When $A = \{p\}$ for some $p \in T^*M$, one simply writes $\mathbf{D}^b(\mathbf{k}_M; p)$ instead of $\mathbf{D}^b(\mathbf{k}_M; \{p\})$.

Functorial operations ([10, § 5.4])

Let M and N be two real manifolds. We denote by q_i ($i = 1, 2$) the i -th projection defined on $M \times N$ and by p_i ($i = 1, 2$) the i -th projection defined on $T^*(M \times N) \simeq T^*M \times T^*N$.

Definition 1.4. Let $f: M \rightarrow N$ be a morphism of manifolds and let $\Lambda \subset T^*N$ be a closed \mathbb{R}^+ -conic subset. One says that f is non-characteristic for Λ (or else, Λ is non-characteristic for f , or f and Λ are transversal) if

$$f_\pi^{-1}(\Lambda) \cap T_M^*N \subset M \times_N T_N^*N.$$

A morphism $f: M \rightarrow N$ is non characteristic for a closed \mathbb{R}^+ -conic subset Λ if and only if $f_d: M \times_N T^*N \rightarrow T^*M$ is proper on $f_\pi^{-1}(\Lambda)$ and in this case $f_d f_\pi^{-1}(\Lambda)$ is closed and \mathbb{R}^+ -conic in T^*M .

Theorem 1.5. (See [10, § 5.4].) Let $f: M \rightarrow N$ be a morphism of manifolds, let $F \in \mathbf{D}^b(\mathbf{k}_M)$ and let $G \in \mathbf{D}^b(\mathbf{k}_N)$.

- (i) Assume that f is proper on $\mathrm{Supp}(F)$. Then $\mathrm{SS}(Rf_!F) \subset f_\pi f_d^{-1} \mathrm{SS}(F)$.

- (ii) Assume that f is non characteristic with respect to $\text{SS}(G)$. Then $\text{SS}(f^{-1}G) \subset f_d f_\pi^{-1} \text{SS}(G)$.

The corollary below is a particular case of the microlocal Morse lemma (see [10, Cor. 5.4.19]) and follows immediately from Theorem 1.5 (ii). The classical theory corresponds to the constant sheaf $F = \mathbf{k}_M$.

Proposition 1.6. *Let $F \in \text{D}^b(\mathbf{k}_M)$, let $\varphi: M \rightarrow \mathbb{R}$ be a function of class C^1 and assume that φ is proper on $\text{supp}(F)$. For $t \in \mathbb{R}$, set $M_t = \varphi^{-1}(] - \infty, t[)$. Let $a < b$ in \mathbb{R} and assume that $d\varphi(x) \notin \text{SS}(F)$ for $a \leq \varphi(x) < b$. Then the natural morphism $\text{R}\Gamma(M_b; F) \rightarrow \text{R}\Gamma(M_a; F)$ is an isomorphism.*

There exist estimates of the microsupport for characteristic inverse images and (in some special situations) for non proper direct images but we shall not use them here.

Kernels([10, § 3.6])

Let M_i ($i = 1, 2, 3$) be manifolds. For short, we write $M_{ij} := M_i \times M_j$ ($1 \leq i, j \leq 3$) and $M_{123} = M_1 \times M_2 \times M_3$. We denote by q_i the projection $M_{ij} \rightarrow M_i$ or the projection $M_{123} \rightarrow M_i$ and by q_{ij} the projection $M_{123} \rightarrow M_{ij}$. Similarly, we denote by p_i the projection $T^*M_{ij} \rightarrow T^*M_i$ or the projection $T^*M_{123} \rightarrow T^*M_i$ and by p_{ij} the projection $T^*M_{123} \rightarrow T^*M_{ij}$. We also need to introduce the map p_{12}^a , the composition of p_{12} and the antipodal map on T^*M_2 .

Let $\Lambda_1 \subset T^*M_{12}$ and $\Lambda_2 \subset T^*M_{23}$. We set

$$\Lambda_1 \circ \Lambda_2 := p_{13}(p_{12}^a{}^{-1} \Lambda_1 \cap p_{23}^{-1} \Lambda_2).$$

We consider the operation of convolution of kernels:

$$\begin{aligned} \circ_{M_2} : \text{D}^b(\mathbf{k}_{M_{12}}) \times \text{D}^b(\mathbf{k}_{M_{23}}) &\rightarrow \text{D}^b(\mathbf{k}_{M_{13}}) \\ (K_1, K_2) &\mapsto K_1 \circ_{M_2} K_2 := \text{R}q_{13!}(q_{12}^{-1} K_1 \otimes q_{23}^{-1} K_2). \end{aligned}$$

Let $\Lambda_i = \text{SS}(K_i) \subset T^*M_{i,i+1}$ and assume that

$$(1.4) \quad \begin{cases} \text{(i)} & q_{13} \text{ is proper on } q_{12}^{-1} \text{supp}(K_1) \cap q_{23}^{-1} \text{supp}(K_2), \\ \text{(ii)} & p_{12}^a{}^{-1} \Lambda_1 \cap p_{23}^{-1} \Lambda_2 \cap (T_{M_1}^* M_1 \times T^* M_2 \times T_{M_3}^* M_3) \\ & \subset T_{M_1 \times M_2 \times M_3}^*(M_1 \times M_2 \times M_3). \end{cases}$$

It follows from Theorem 1.5 that under the assumption (1.4) we have:

$$\mathrm{SS}(K_1 \circ_{M_2} K_2) \subset \Lambda_1 \circ \Lambda_2.$$

If there is no risk of confusion, we write \circ instead of \circ_{M_2} .

We will also use a relative version of the convolution of kernels. For a manifold I , $K_1 \in \mathbf{D}^b(\mathbf{k}_{M_{12} \times I})$ and $K_2 \in \mathbf{D}^b(\mathbf{k}_{M_{23} \times I})$ we set

$$(1.5) \quad K_1 \circ_I K_2 := \mathrm{R}q_{13|I} (q_{12|I}^{-1} K_1 \otimes q_{23|I}^{-1} K_2),$$

where $q_{ij|I}$ is the projection $M_{123} \times I \rightarrow M_{ij} \times I$. The above results extend to the relative case.

Quantized contact transformations ([10, § 7.2])

Consider two manifolds M and N , two conic open subsets $U \subset T^*M$ and $V \subset T^*N$ and a homogeneous contact transformation χ :

$$(1.6) \quad T^*M \supset U \xrightarrow[\chi]{\simeq} V \subset T^*N.$$

Denote by V^a the image of V by the antipodal map a_N on T^*N and by Λ the image of the graph of φ by $\mathrm{id}_U \times a_N$. Hence Λ is a conic Lagrangian submanifold of $U \times V^a$. A quantized contact transformation (a QCT, for short) above χ is a kernel $K \in \mathbf{D}^b(\mathbf{k}_{M \times N})$ such that $\mathrm{SS}(K) \cap (U \times V^a) \subset \Lambda$ and satisfying some technical properties that we do not recall here so that the kernel K induces an equivalence of categories

$$(1.7) \quad K: \mathbf{D}^b(\mathbf{k}_N; V) \xrightarrow{\simeq} \mathbf{D}^b(\mathbf{k}_M; U).$$

Given χ and $p \in U$, $q = \chi(p) \in V$, there exists such a QCT after replacing U and V by sufficiently small neighborhoods of p and q .

The functor $\mu\mathrm{hom}$ ([10, § 4.4, 7.2])

The functor of microlocalization along a submanifold has been introduced by Mikio Sato in the 70's and has been at the origin of what is now called "microlocal analysis". A variant of this functor, the bifunctor

$$(1.8) \quad \mu\mathrm{hom}: \mathbf{D}^b(\mathbf{k}_M)^{\mathrm{op}} \times \mathbf{D}^b(\mathbf{k}_M) \rightarrow \mathbf{D}^b(\mathbf{k}_{T^*M})$$

has been constructed in [10]. Since $\text{Supp}(\mu\text{hom}(F, F')) \subset \text{SS}(F) \cap \text{SS}(F')$, (1.8) induces a bifunctor for any open subset U of T^*M :

$$\mu\text{hom}: \mathbf{D}^b(\mathbf{k}_M; U)^{\text{op}} \times \mathbf{D}^b(\mathbf{k}_M; U) \rightarrow \mathbf{D}^b(\mathbf{k}_U).$$

Let us only recall the properties of this functor that we shall use. Consider a smooth function $\psi: M \rightarrow \mathbb{R}$ defined in a neighborhood of $x_0 \in M$ and set $N := \{x \in M; \psi(x) = \psi(x_0)\}$. Then, setting $p = d\psi(x_0)$, we have for any $F \in \mathbf{D}^b(\mathbf{k}_M)$,

$$\text{R}\Gamma_{\{\psi(x) \geq \psi(x_0)\}}(F)_{x_0} \simeq \mu\text{hom}(\mathbf{k}_N, F)_p.$$

Moreover if χ is a contact transform as in (1.6) and if K is a QCT as in (1.7), then K induces a natural isomorphism for any $F, G \in \mathbf{D}^b(\mathbf{k}_N; V)$

$$(1.9) \quad \chi^{-1}(\mu\text{hom}(F, G)|_V) \xrightarrow{\simeq} \mu\text{hom}(K \circ F, K \circ G)|_U.$$

Simple sheaves ([10, § 7.5])

Let $\Lambda \subset \dot{T}^*M$ be a locally closed conic Lagrangian submanifold and let $p \in \Lambda$. Simple sheaves along Λ at p are defined in [10, Def. 7.5.4].

When Λ is the conormal bundle to a smooth submanifold $N \subset M$, that is, when the projection $\pi_M|_\Lambda: \Lambda \rightarrow M$ has constant rank, then an object $F \in \mathbf{D}^b(\mathbf{k}_M)$ is simple along Λ at p if $F \simeq \mathbf{k}_N[d]$ in $\mathbf{D}^b(\mathbf{k}_M; p)$ for some shift $d \in \mathbb{Z}$.

If $\text{SS}(F)$ is contained in Λ on a neighborhood of Λ , Λ is connected and F is simple at some point of Λ , then F is simple at every point of Λ .

If $\Lambda_1 \subset T^*M_{12}$ and $\Lambda_2 \subset T^*M_{23}$ are locally closed conic smooth Lagrangian submanifolds and if $K_i \in \mathbf{D}^b(\mathbf{k}_{M_{i,i+1}})$ ($i = 1, 2$) are simple along Λ_i , then $K_1 \circ K_2$ is simple along $\Lambda_1 \circ \Lambda_2$ under some conditions (see [10, Th. 7.5.11]). In particular, simple sheaves are stable by QCT.

Now, let M and N be two manifolds with the same dimension. Let $F \in \mathbf{D}^b(\mathbf{k}_{M \times N})$. Set

$$(1.10) \quad F^{-1} = v^{-1} \text{R}\mathcal{H}om(F, \omega_M \boxtimes \mathbf{k}_N) \in \mathbf{D}^b(\mathbf{k}_{N \times M}),$$

where $v: N \times M \rightarrow M \times N$ is the swap. Let q_{ij} be the (i, j) -th projection from $N \times M \times N$. Then we have $F^{-1} \circ F = \text{R}q_{13!}(q_{12}^{-1}F^{-1} \otimes q_{23}^{-1}F)$. Let $\delta: N \rightarrow N \times N$ be the diagonal embedding. Then we have $\delta^{-1}(F^{-1} \circ F) \simeq \text{R}q_{2!}(F \otimes$

$R\mathcal{H}om(F, \omega_M \boxtimes \mathbf{k}_N)$). Hence $\delta^{-1}(F^{-1} \circ F) \simeq Rq_{2!}(F \otimes R\mathcal{H}om(F, q_2^! \mathbf{k}_N)) \rightarrow Rq_{2!}(q_2^! \mathbf{k}_N) \rightarrow \mathbf{k}_N$ gives a morphism

$$F^{-1} \circ F \rightarrow \mathbf{k}_{\Delta_N}.$$

Proposition 1.7 ([10, Proposition 7.1.8, Proposition 7.1.9, Theorem 7.2.1]). *Let $p_M \in \dot{T}^*M$ and $p_N \in \dot{T}^*N$, and we assume the following conditions.*

- (i) $\text{Supp}(F) \rightarrow N$ is proper,
- (ii) F is cohomologically constructible (see [10, Def. 3.4.1]),
- (iii) $\text{SS}(F) \cap (\dot{T}^*M \times T_N^*N) = \emptyset$,
- (iv) $\text{SS}(F) \cap (T^*M \times \{p_N^a\}) = \{(p_M, p_N^a)\}$,
- (v) $\text{SS}(F)$ is a Lagrangian submanifold of $T^*(M \times N)$ on a neighborhood of (p_M, p_N^a) ,
- (vi) F is simple along $\text{SS}(F)$ at (p_M, p_N^a) ,
- (vii) $\text{SS}(F) \rightarrow T^*N$ is a local isomorphism at (p_M, p_N^a) .

Then the morphism $F^{-1} \circ F \rightarrow \mathbf{k}_{\Delta_N}$ is an isomorphism in $D^b(\mathbf{k}_{N \times N}; (p_N, p_N^a))$.

2 Deformation of the conormal to the diagonal

As usual, we denote by Δ_M or simply Δ the diagonal of $M \times M$. We denote by p_1 and p_2 the first and second projection from $T^*(M \times M)$ to T^*M and by p_2^a the composition of p_2 and the antipodal map on T^*M .

Consider a C^∞ -function $f(x, y)$ defined on an open neighborhood $\Omega_0 \subset M \times M$ of the diagonal Δ_M . We assume that

- (i) $f|_{\Delta_M} \equiv 0$,
- (ii) $f(x, y) > 0$ for $(x, y) \in \Omega_0 \setminus \Delta_M$,
- (iii) the Hessian $\frac{\partial^2 f}{\partial x_i \partial x_j}(x, y)$ is positive-definite for $(x, y) \in \Delta_M$.

Such a pair (Ω_0, f) exists.

Proposition 2.1. *Assume that (Ω_0, f) satisfies the conditions (i)–(iii). Let U be a relatively compact open subset of M . Then there exist an $\varepsilon > 0$ and an open subset Ω of $M \times M$ satisfying the following conditions:*

- (a) $\Delta_U \subset \Omega \subset \Omega_0 \cap (M \times U)$,
- (b) $Z_\varepsilon := \{(x, y) \in \Omega; f(x, y) \leq \varepsilon\}$ is proper over U by the map induced by the second projection,
- (c) for any $y \in U$, the open subset $\{x \in M; (x, y) \in \Omega, f(x, y) < \varepsilon\}$ is homeomorphic to \mathbb{R}^n ,
- (d) $d_x f(x, y) \neq 0, d_y f(x, y) \neq 0$ for $(x, y) \in \Omega \setminus \Delta_M$,
- (e) setting $\Gamma_{Z_\varepsilon} = \{(x, y; \xi, \eta) \in T^*(\Omega); f(x, y) = \varepsilon, (\xi, \eta) = \lambda df(x, y), \lambda < 0\}$, the projection $p_2^a: T^*(M \times U) \rightarrow T^*U$ induces an isomorphism $\Gamma_{Z_\varepsilon} \xrightarrow[p_2^a]{\simeq} \dot{T}^*U$ and the projection $p_1: T^*(M \times U) \rightarrow T^*M$ induces an open embedding $\Gamma_{Z_\varepsilon} \hookrightarrow \dot{T}^*M$.

Proof. Replacing Ω_0 with the open subset

$$\Delta_M \cup \{(x, y) \in \Omega_0; d_x f(x, y) \neq 0, d_y f(x, y) \neq 0\},$$

we may assume from the beginning that Ω_0 satisfies (d).

Let $F: \Omega_0 \rightarrow T^*M$ be the map $(x, y) \mapsto d_y f(x, y)$. This map sends Δ_M to T_M^*M and is a local isomorphism. Then there exists an open neighborhood $\Omega' \subset \Omega_0$ of Δ_M such that $F|_{\Omega'}: \Omega' \rightarrow T^*M$ is an open embedding. Hence by identifying Ω' as its image, we can reduce the proposition to the following lemma. Q.E.D.

Lemma 2.2. *Let $p: E \rightarrow X$ be a vector bundle of rank n , $i: X \rightarrow E$ the zero-section, $SE = (E \setminus i(X))/\mathbb{R}_{>0}$ the associated sphere bundle and $q: E \setminus i(X) \rightarrow SE$ the projection. Let f be a C^∞ -function on a neighborhood Ω of the zero-section $i(X)$ of E . Assume the following conditions:*

- (i) $f(z) = 0$ for $z \in i(X)$,
- (ii) $f(z) > 0$ for $z \in \Omega \setminus i(X)$,
- (iii) for any $x \in X$ the Hessian of $f|_{p^{-1}(x)}$ at the origin is positive-definite.

Then, for any relatively compact open subset U of X , there exist $\varepsilon > 0$ and an open subset $\Omega' \subset \Omega \cap p^{-1}(U)$ containing $i(U)$ that satisfy the following conditions:

- (a) $\{z \in \Omega' ; f(z) \leq \varepsilon\}$ is proper over U ,
- (b) $\{z \in \Omega' ; 0 < f(z) < \varepsilon\} \rightarrow (SE \times_X U) \times]0, \varepsilon[$ given by $z \mapsto (q(z), f(z))$ is an isomorphism,
- (c) for any $x \in X$ and $t \in]0, \varepsilon[$, the set $\{z \in \Omega' \cap p^{-1}(x) ; f(z) < t\}$ is homeomorphic to \mathbb{R}^n .

Since the proof is elementary, we omit it.

Recall (1.10).

Theorem 2.3. *We keep the notations in Proposition 2.1 and assume further that U is connected. Set $L = \mathbf{k}_{Z_\varepsilon} \in \mathbf{D}^b(\mathbf{k}_{M \times U})$. Then $\text{SS}(L) \subset \Gamma_{Z_\varepsilon} \cup Z_\varepsilon$ and $L^{-1} \circ L \simeq \mathbf{k}_{\Delta_U}$.*

Proof. Set $Z = Z_\varepsilon$. We have $\text{SS}(L^{-1} \circ L) \subset T_{\Delta_U}^*(U \times U) \cup T_{U \times U}^*(U \times U)$. By Proposition 1.7, there exists a morphism $L^{-1} \circ L \rightarrow \mathbf{k}_{\Delta_U}$ which is an isomorphism in $\mathbf{D}^b(\mathbf{k}_{U \times U}; \dot{T}^*(U \times U))$. Hence if $K \rightarrow L^{-1} \circ L \rightarrow \mathbf{k}_{\Delta_U} \xrightarrow{+1}$ is a distinguished triangle, then $\text{SS}(K) \subset T_{U \times U}^*(U \times U)$ and hence K has locally constant cohomologies. Let $\delta: U \rightarrow U \times U$ be the diagonal embedding. Then we have $\delta^{-1}(L^{-1} \circ L) \simeq \text{R}q_{2!}(L \otimes \text{R}\mathcal{H}om(L, \mathbf{k}_{M \times U}) \otimes q_2^! \mathbf{k}_U)$. Since $L \simeq \mathbf{k}_Z$ and $\text{R}\mathcal{H}om(L, \mathbf{k}_{M \times U}) \simeq \mathbf{k}_{\text{Int}(Z)}$, we have $\delta^{-1}(L^{-1} \circ L) \simeq \text{R}q_{2!}(\mathbf{k}_{\text{Int}(Z)} \otimes q_2^! \mathbf{k}_U)$. Since the fibers of $\text{Int}(Z) \rightarrow U$ are homeomorphic to \mathbb{R}^n , we have $\text{R}q_{2!}(\mathbf{k}_{\text{Int}(Z)} \otimes q_2^! \mathbf{k}_U) \simeq \mathbf{k}_U$. Thus we obtain that $\delta^{-1}(L^{-1} \circ L) \simeq \mathbf{k}_U$, and hence $\delta^{-1}K \simeq 0$. Since K has locally constant cohomologies and $U \times U$ is connected, we conclude that $K \simeq 0$. Q.E.D.

3 Quantization of homogeneous Hamiltonian isotopies

Let I be an open interval of \mathbb{R} containing the origin and let $\Phi: \dot{T}^*M \times I \rightarrow \dot{T}^*M$ be a map such that $\varphi_t := \Phi(\cdot, t): \dot{T}^*M \rightarrow \dot{T}^*M$ is a homogeneous symplectic isomorphism for each $t \in I$ and is the identity for $t = 0$. We

denote by Λ_t the Lagrangian submanifold of $\dot{T}^*M \times \dot{T}^*M$ associated with φ_t , that is, the image by the antipodal map of the second factor of $\dot{T}^*M \times \dot{T}^*M$ of the graph of φ_t :

$$\Lambda_t = \left\{ (\varphi_t(v), v^a); v \in \dot{T}^*M \right\}.$$

Consider the differential

$$\frac{\partial \Phi}{\partial t}: \dot{T}^*M \times I \rightarrow T\dot{T}^*M \simeq T^*\dot{T}^*M.$$

We make the hypothesis

$$(3.1) \quad \left\{ \begin{array}{l} \text{there exists a function } f: \dot{T}^*M \times I \rightarrow \mathbb{R} \text{ homogeneous of} \\ \text{degree 1 such that } \frac{\partial \Phi}{\partial t} = H_f, \text{ where } H_f \text{ is the Hamiltonian} \\ \text{vector field of } f. \end{array} \right.$$

After identifying T^*I with $I \times \mathbb{R}$, we can define the Lagrangian submanifold:

$$(3.2) \quad \Lambda := \left\{ (\Phi(v, t), v^a, t, -f(\Phi(v, t), t)); v \in \dot{T}^*M, t \in I \right\} \\ \subset \dot{T}^*M \times \dot{T}^*M \times T^*I,$$

and we have

$$\Lambda_t = \Lambda \circ T_t^*I.$$

Lemma 3.1. *The set $\Lambda \cup T_{M \times M \times I}^*(M \times M \times I)$ is closed in $T^*(M \times M \times I)$.*

Proof. In local homogeneous symplectic coordinates $(x, y; \xi, \eta) \in T^*(M \times M)$, $(t; \tau) \in T^*I$, the construction of Λ implies that for any compact set $C \subset M \times M \times I$ there exists $D > 0$ such that $|\tau| \leq D|\xi|$, $|\xi| \leq D|\eta|$ and $|\eta| \leq D|\xi|$ for any $(x, y, t; \xi, \eta, \tau) \in \Lambda \cap \pi_{M \times M \times I}^{-1}(C)$. Hence the same inequalities hold on the closure $\overline{\Lambda}$ of Λ . Hence if $(x, y, t; \xi, \eta, \tau) \in \overline{\Lambda} \setminus (\dot{T}^*M \times \dot{T}^*M \times T^*I)$, then $\xi = \eta = 0$ and $\tau = 0$, and hence it belongs to the zero-section. Q.E.D.

In the course of the proof of Theorem 3.3 below we shall need an elementary lemma that we state without proof.

Lemma 3.2. *Let X be a smooth manifold, $U \subset X$ an open subset with smooth boundary ∂U and locally on one side of its boundary, let $Y \subset X \times I$ be a smooth hypersurface satisfying*

- (a) Y is transversal to $X \times \{0\}$,
- (b) $Y_0 = \partial U$, where $Y_t = Y \cap (X \times \{t\})$,
- (c) there exists a compact $C \subset X$ such that $Y \cap ((X \setminus C) \times I) = (\partial U \setminus C) \times I$.

Then there exists $\varepsilon > 0$ and an open subset $V \subset X \times]-\varepsilon, \varepsilon[$ with smooth boundary ∂V and locally on one side of its boundary such that $\partial V = Y \cap (X \times]-\varepsilon, \varepsilon[)$.

Recall that for $F \in \mathbf{D}^b(\mathbf{k}_{M \times N})$, the object F^{-1} is defined in (1.10). For an object $K \in \mathbf{D}^b(\mathbf{k}_{M \times M \times I})$ and $t_0 \in I$, we set

$$K_{t_0} = K|_{t=t_0} \simeq K \circ \mathbf{k}_{t=t_0} \in \mathbf{D}^b(\mathbf{k}_{M \times M}).$$

We introduce the hypothesis on Φ :

$$(3.3) \quad \begin{cases} \Phi \text{ is the identity outside of } \dot{\pi}_M^{-1}(A) \times I, \text{ where } A \text{ is a compact} \\ \text{subset of } M \text{ and } \dot{\pi}_M: \dot{T}^*M \rightarrow M \text{ is the projection.} \end{cases}$$

Theorem 3.3. *Assume that Φ satisfies hypotheses (3.1) and (3.3). Let us consider the following conditions on $K \in \mathbf{D}^b(\mathbf{k}_{M \times M \times I})$:*

- (a) $\text{SS}(K) \subset \Lambda \cup T_{M \times M \times I}^*(M \times M \times I)$,
- (b) $K_0 \simeq \mathbf{k}_\Delta$,
- (c) $K \simeq \mathbf{k}_{\Delta_{M \times I}}$ on $((M \times M) \setminus (A \times A)) \times I$. In particular, the two projections $\text{Supp}(K) \rightrightarrows M \times I$ are proper,
- (d) $K_t \circ K_t^{-1} \simeq K_t^{-1} \circ K_t \simeq \mathbf{k}_\Delta$.

Then we have

- (i) The conditions (a) and (b) imply the other two conditions (c) and (d).
- (ii) There exists K satisfying (a)–(d).
- (iii) Moreover such a K satisfying the conditions (a)–(d) is unique up to a unique isomorphism.

We shall call K a *quantization* of Φ on I , or a quantization of $\{\varphi_t\}_{t \in I}$.

Proof. (A) We shall prove first that (a) and (b) imply the other conditions. We have outside of $A \times A \times I$:

$$(3.4) \quad \begin{aligned} \text{SS}(K) &\subset T_{\Delta_{M \times I}}^*(M \times M \times I) \cup T_{M \times M \times I}^*(M \times M \times I) \\ &\subset T^*(M \times M) \times T_I^*I. \end{aligned}$$

Hence K is constant on the fibers of $M \times M \times I \rightarrow M \times M$ outside of $A \times A \times I$. Since $K|_{t=0} \simeq \mathbf{k}_{\Delta_M}$, we obtain (c).

Let us prove (d). Set $F = K \circ|_I K^{-1}$ (notation (1.5)). Then $\text{SS}(F)$ satisfies on $M \times M \times I$:

$$(3.5) \quad \begin{aligned} \text{SS}(F) &\subset T_{\Delta_{M \times I}}^*(M \times M \times I) \cup T_{M \times M \times I}^*(M \times M \times I) \\ &\subset T^*(M \times M) \times T_I^*I. \end{aligned}$$

Let $i: M \times M \rightarrow M \times M \times I$ be the inclusion associated to $\{t = 0\} \subset I$. Then we have

$$\begin{aligned} i^! \text{R}\mathcal{H}om(K, \mathbf{k}_{M \times M \times I}) &\simeq \text{R}\mathcal{H}om(i^{-1}K, i^! \mathbf{k}_{M \times M \times I}) \\ &\simeq \text{R}\mathcal{H}om(K_0, \mathbf{k}_{M \times M}) \otimes i^! \mathbf{k}_{M \times M \times I}. \end{aligned}$$

On the other hand, the condition on $\text{SS}(K)$ implies

$$i^! \text{R}\mathcal{H}om(K, \mathbf{k}_{M \times M \times I}) \simeq i^{-1} \text{R}\mathcal{H}om(K, \mathbf{k}_{M \times M \times I}) \otimes i^! \mathbf{k}_{M \times M \times I}.$$

Therefore $i^{-1} \text{R}\mathcal{H}om(K, \mathbf{k}_{M \times M \times I}) \simeq \text{R}\mathcal{H}om(K_0, \mathbf{k}_{M \times M})$ which gives the isomorphism $i^{-1}K^{-1} \simeq K_0^{-1}$. Thus we obtain $i^{-1}F \simeq K_0 \circ K_0^{-1} \simeq \mathbf{k}_{\Delta}$ and the isomorphism $F \simeq \mathbf{k}_{\Delta \times I}$ follows since F is constant on the fibers of $M \times M \times I \rightarrow M \times M$.

(B) Unicity. Assume that we have two kernels K and K' satisfying the conditions (a)–(d) and set $L = K \circ|_I K'^{-1}$. Then

$$\begin{aligned} \text{SS}(L) &\subset T_{\Delta \times I}^*(M \times M \times I) \cup T_{M \times M \times I}^*(M \times M \times I) \\ &\subset T^*(M \times M) \times T_I^*I, \\ L|_{t=0} &\simeq \mathbf{k}_{\Delta}. \end{aligned}$$

The first estimates imply that L is constant along the fibers of $M \times M \times I \rightarrow M \times M$ and hence the last one implies that $L \simeq \mathbf{k}_{\Delta \times I}$.

(C) Local existence. We shall prove that there exists a quantization of $\{\varphi_t\}_{t \in]-\varepsilon, \varepsilon[}$ for $0 < \varepsilon \ll 1$. We may assume that M is connected. We use the results and notations of Proposition 2.1 and Theorem 2.3. Take a relatively compact open subset U of M such that U contains A . We may assume that U is connected. Take ε as in Proposition 2.1. Then $L := \mathbf{k}_{Z_\varepsilon} \in \mathbf{D}^b(\mathbf{k}_{M \times U})$ satisfies $\text{SS}(L) = \Gamma_{Z_\varepsilon} \cup Z_\varepsilon$, and $L^{-1} \circ |_I L \simeq \mathbf{k}_{\Delta_U}$.

Set $\Lambda' = \Lambda \cap \dot{T}^*(U \times U \times I)$ and $\Lambda'_t = \Lambda_t \cap \dot{T}^*(U \times U)$. Set

$$\tilde{\Lambda} := \Gamma_{Z_\varepsilon} \circ |_I \Lambda' \subset \dot{T}^*M \times \dot{T}^*U \times T^*I,$$

$$\tilde{\Lambda}_t := \Gamma_{Z_\varepsilon} \circ \Lambda'_t \subset \dot{T}^*M \times \dot{T}^*U.$$

Since $\Lambda'_0 = \dot{T}^*_\Delta(U \times U)$ and Γ_{Z_ε} is (a half of) the conormal bundle to a hypersurface, the Lagrangian manifold $\tilde{\Lambda}_0$ is (a half of) the conormal to the hypersurface ∂Z_ε of $M \times U$. Hence $\tilde{\Lambda}$ is also (a half of) the conormal to a hypersurface Y of $M \times U \times I'$ for a small interval I' containing 0. We apply Lemma 3.2 with $X = M \times U$, U the interior of Z_ε and $C = A \times A$. We get an interval I'' containing 0 and an open subset $V \subset M \times U \times I''$ such that setting $\tilde{L} = \mathbf{k}_{\tilde{V}}$, the object $\tilde{L} \in \mathbf{D}^b(\mathbf{k}_{M \times U \times I''})$ satisfies:

(a) $\text{SS}(\tilde{L}) \subset (\tilde{\Lambda} \times_I I'') \cup T^*_{M \times U \times I''}(M \times U \times I'')$,

(b) $\tilde{L}_0 \simeq L$,

(c) the projection $M \times U \times I'' \rightarrow U \times I''$ is proper on $\text{supp } \tilde{L}$.

Now, set $K = L^{-1} \circ |_I \tilde{L} \in \mathbf{D}^b(\mathbf{k}_{U \times U \times I''})$. Then K will satisfy the properties (a)–(b) of Theorem 3.3 when replacing X and I with U , I'' . Since

$$K|_{((U \times U) \setminus (A \times A)) \times I''} \simeq \mathbf{k}_{\Delta_M}|_{((U \times U) \setminus (A \times A)) \times I''},$$

K extends to $\tilde{K} \in \mathbf{D}^b(\mathbf{k}_{X \times X \times I''})$ with

$$\tilde{K}|_{((X \times X) \setminus (A \times A)) \times I''} \simeq \mathbf{k}_{\Delta_M}|_{((X \times X) \setminus (A \times A)) \times I''}.$$

Then \tilde{K} is a quantization of $\{\varphi_t\}_{t \in I''}$.

(D) Gluing (a). Assume K^{t_0, t_1} is a quantization of the isotopy $\{\varphi_t\}_{t \in]t_0, t_1[}$ for an open interval $]t_0, t_1[$ of I containing the origin. We shall denote by

$K_t^{t_0, t_1}$ the restriction of K^{t_0, t_1} at $t \in]t_0, t_1[$. Assuming $t_1 \in I$, we shall show that there exists a quantization of the isotopy $\{\varphi_t\}_{t \in]t_0, t_4[}$ for an open interval $]t_0, t_4[\subset I$ for some $t_4 > t_1$.

By applying the result of (C) to the isotopy $\{\varphi_t \circ \varphi_{t_1}^{-1}\}_{t \in I}$, there exists $t_0 < t_3 < t_1 < t_4$ with $t_4 \in I$ and a quantization L^{t_3, t_4} of the isotopy $\{\varphi_t \circ \varphi_{t_1}^{-1}\}_{t \in]t_3, t_4[}$. Choose t_2 with $t_3 < t_2 < t_1$ and set

$$\begin{aligned} F &= (K^{t_0, t_1}|_{]t_3, t_1[}) \circ (K_{t_2}^{t_0, t_1})^{-1}, \\ F' &= (L^{t_3, t_4}|_{]t_3, t_1[}) \circ (L_{t_2}^{t_3, t_4})^{-1}. \end{aligned}$$

Then both F and F' are a quantization of the isotopy $\{\varphi_t \circ \varphi_{t_2}^{-1}\}_{t \in]t_3, t_1[}$. Using (C), F and F' are isomorphic and hence we have an isomorphism

$$K^{t_3, t_4}|_{]t_3, t_1[} \simeq K^{t_0, t_1}|_{]t_3, t_1[} \text{ in } \mathbf{D}^b(\mathbf{k}_{M \times M \times]t_3, t_1[}).$$

where $K^{t_3, t_4} = L^{t_3, t_4} \circ (L_{t_2}^{t_3, t_4})^{-1} \circ K_{t_2}^{t_0, t_1} \in \mathbf{D}^b(\mathbf{k}_{M \times M \times]t_3, t_4[})$. Therefore, there exists $K^{t_0, t_4} \in \mathbf{D}^b(\mathbf{k}_{M \times M \times]t_0, t_4[})$ such that $K^{t_0, t_4}|_{M \times M \times]t_0, t_1[} \simeq K^{t_0, t_1}$ and $K^{t_0, t_4}|_{M \times M \times]t_3, t_4[} \simeq K^{t_3, t_4}$. Then K^{t_0, t_4} is a quantization of the isotopy $\{\varphi_t\}_{t \in]t_0, t_4[}$.

(E) Gluing (b). Consider an increasing sequence of open intervals $I_n \subset I$ and assume we have constructed quantizations K_n of $\{\varphi_t\}_{t \in I_n}$. Hence, $K_{n+1}|_{M \times M \times I_n} \simeq K_n$. Set $J = \bigcup_n I_n$ and denote by i_n the embedding $M \times M \times I_n \hookrightarrow M \times M \times J$. We have morphisms

$$\iota_n: \mathbf{R}i_{n!}K_n \rightarrow \mathbf{R}i_{n+1!}K_{n+1},$$

hence an inductive system $\{(\mathbf{R}i_{n!}K_n, \iota_n)\}_n$ in $\mathbf{D}^b(\mathbf{k}_{M \times M \times J})$. Denote by K_J a homotopy colimit of this inductive system. Then K_J is a quantization of $\{\varphi_t\}_{t \in J}$.

(F) Gluing (c). Consider the set of pairs (J, K_J) where J is an open interval contained in I and containing 0 and K_J is a quantization of $\{\varphi_t\}_{t \in J}$. This set, ordered by inclusion, is inductively ordered by (E). Let (J, K_J) be a maximal element. It follows from (D) that $J = I$. Q.E.D.

Example 3.4. Let $M = \mathbb{R}^n$ and denote by $(x; \xi)$ the homogeneous symplectic coordinates on $T^*\mathbb{R}^n$. Consider the isotopy $\varphi_t(x; \xi) = (x - t \frac{\xi}{|\xi|}; \xi)$, $t \in I = \mathbb{R}$. Then

$$\begin{aligned} \Lambda_t &= \{(x, y, \xi, \eta); |x - y| = |t|, \xi = -\eta = s(x - y), st < 0\} \quad \text{for } t \neq 0, \\ \Lambda_0 &= \hat{T}_\Delta^*(M \times M). \end{aligned}$$

The isomorphisms

$$\begin{aligned} \mathcal{R}\mathcal{H}om(\mathbf{k}_{\Delta \times \{t=0\}}, \mathbf{k}_{M \times M \times \mathbb{R}}) &\simeq \mathbf{k}_{\Delta \times \{t=0\}}[-n-1] \\ \mathcal{R}\mathcal{H}om(\mathbf{k}_{\{|x-y| \leq -t\}}, \mathbf{k}_{M \times M \times \mathbb{R}}) &\simeq \mathbf{k}_{\{|x-y| < -t\}} \end{aligned}$$

together with the morphism $\mathbf{k}_{\{|x-y| \leq -t\}} \rightarrow \mathbf{k}_{\Delta \times \{t=0\}}$ induce the morphism $\mathbf{k}_{\Delta \times \{t=0\}}[-n-1] \rightarrow \mathbf{k}_{\{|x-y| < -t\}}$. Hence we obtain

$$\mathbf{k}_{\{|x-y| \leq t\}} \rightarrow \mathbf{k}_{\Delta \times \{t=0\}} \rightarrow \mathbf{k}_{\{|x-y| < -t\}}[n+1].$$

Therefore there exists a distinguished triangle in $\mathbf{D}^b(\mathbf{k}_{M \times M \times I})$:

$$\mathbf{k}_{\{|x-y| < -t\}}[n] \rightarrow K \rightarrow \mathbf{k}_{\{|x-y| \leq t\}} \xrightarrow{+1}.$$

We can verify that K satisfies the properties (a)–(d) of Theorem 3.3. From this distinguished triangle, we deduce the isomorphisms in $\mathbf{D}^b(\mathbf{k}_{M \times M})$: $K_t \simeq \mathbf{k}_{\{|x-y| \leq t\}}$ for $t \geq 0$ and $K_t \simeq \mathbf{k}_{\{|x-y| < -t\}}[n]$ for $t < 0$.

Lemma 3.5. *Assume that Φ satisfies hypothesis (3.1) and let U be a relatively compact open subset of M , $I' \subset I$ a relatively compact open interval containing 0. Then there exists $\Phi' : \dot{T}^*M \times I \rightarrow \dot{T}^*M$ satisfying hypothesis (3.1) and (3.3) such that*

$$(3.6) \quad \Phi'|_{\dot{T}^*U \times I'} = \Phi|_{\dot{T}^*U \times I'}.$$

Proof. Since Φ is homogeneous, the set $B = \dot{\pi}_M(\Phi(\dot{T}^*U \times I'))$ has a compact closure in M . We choose a compact neighborhood A of B and a function $g : M \rightarrow \mathbb{R}$ which vanishes outside A and takes value 1 on a neighborhood of B . We define $f' : \dot{T}^*M \times I \rightarrow \mathbb{R}$ by $f'(x, \xi, t) = g(x)f(x, \xi, t)$.

Then f' is homogeneous of degree 1 with support contained in $\dot{\pi}^{-1}(A) \times I$. Hence its Hamiltonian flow Φ' satisfies (3.1) and (3.3). Moreover we have by construction

$$H_{f'}|_{\dot{\pi}_M^{-1}(B) \times I} = H_f|_{\dot{\pi}_M^{-1}(B) \times I}$$

and this implies (3.6). Q.E.D.

4 Applications

We denote by $\Phi = \{\varphi_t\}_{t \in I} : \dot{T}^*M \times I \rightarrow \dot{T}^*M$ a homogeneous symplectic isotopy as in Theorem 3.3. Hence, Φ satisfies hypotheses (3.1) and (3.3).

Let $F_0 \in \mathbf{D}^b(\mathbf{k}_M)$. We assume

$$(4.1) \quad F_0 \text{ has a compact support.}$$

Let $K \in \mathbf{D}^b(\mathbf{k}_{M \times M \times I})$ be the quantization of Φ on I constructed in Theorem 3.3. We set:

$$\begin{aligned} F &= K \circ F_0 \in \mathbf{D}^b(\mathbf{k}_{M \times I}), \\ F_{t_0} &= F|_{\{t=t_0\}} \in \mathbf{D}^b(\mathbf{k}_M) \quad \text{for } t_0 \in I. \end{aligned}$$

Then

$$(4.2) \quad \begin{cases} F_t \text{ has a compact support in } M, \\ \text{SS}(F_t) \cap \dot{T}^*M = \varphi_t(\text{SS}(F_0) \cap \dot{T}^*M). \end{cases}$$

Lemma 4.1. *We have an isomorphism $\text{R}\Gamma(M; F_t) \simeq \text{R}\Gamma(M; F_0)$ for any $t \in I$.*

Proof. Denote by $q: M \times I \rightarrow I$ the projection. Then q is proper on $\text{supp}(F)$ and $\text{SS}(\text{R}q_*F) \subset T_I^*I$, so that $\text{R}q_*F$ is a constant sheaf on I . Then $\text{R}\Gamma(M; F_s) \simeq (\text{R}q_*F)_s \simeq \text{R}\Gamma(M; F_0)$. Q.E.D.

Application to non displaceability

We consider a C^1 -map $\psi: M \rightarrow \mathbb{R}$ and we assume that

$$(4.3) \quad \text{the differential } d\psi(x) \text{ never vanishes.}$$

Hence the map $\psi_d: M \times_{\mathbb{R}} T^*\mathbb{R} \rightarrow T^*M$ is an embedding and we set

$$(4.4) \quad \begin{aligned} T_{\pm}^*\mathbb{R} &= \{(t, \tau) \in T^*\mathbb{R}; \pm\tau > 0\}, \\ \Lambda_{\psi} &= \{(x; d\psi(x))\}. \end{aligned}$$

Theorem 4.2. *Assume that $\Phi = \{\varphi_t\}_{t \in I}$ satisfies (3.1) and (3.3) and $\psi: M \rightarrow \mathbb{R}$ satisfies (4.3). Let $F_0 \in \mathbf{D}^b(\mathbf{k}_M)$ satisfying (4.1) and also assume that $\text{R}\Gamma(M; F_0) \neq 0$. Then for any $t \in I$, $\varphi_t(\text{SS}(F_0) \cap \dot{T}^*M) \cap \Lambda_{\psi} \neq \emptyset$.*

Proof. It follows from Lemma 4.1 that $\text{R}\Gamma(\mathbb{R}; \text{R}\psi_*F_t) \simeq \text{R}\Gamma(M; F_t) \neq 0$ for any $t \in I$. By the assumption, $\text{R}\psi_*F_t$ has compact support. Hence, if $\Lambda_{\psi} \cap \text{SS}(F_t) = \emptyset$, then Proposition 1.6 implies that $\text{R}\Gamma(M; F_t) \simeq 0$, which is a contradiction. Q.E.D.

Corollary 4.3. *Let $\Phi = \{\varphi_t\}_{t \in I}$ and $\psi: M \rightarrow \mathbb{R}$ be as in Theorem 4.2. Let N be a non empty compact submanifold of M . Then for any $t \in I$, $\varphi_t(\dot{T}_N^*M) \cap \Lambda_{\psi} \neq \emptyset$.*

Application to Morse inequalities

The classical Morse inequalities are extended to sheaves in [10, Prop. 5.4.20]. Let us first briefly recall this result.

For a bounded complex E with finite-dimensional cohomologies, we set

$$b_j(E) = \dim H^j(E), \quad b_l^*(E) = (-1)^l \sum_{j \leq l} (-1)^j b_j(E).$$

We consider a map $\psi: M \rightarrow \mathbb{R}$ of class C^1 and define Λ_ψ as above. Note that we do not ask ψ to be smooth. Assume that

$$(4.5) \quad \text{the set } \Lambda_\psi \cap \text{SS}(F_0) \text{ is finite, say } \{p_1, \dots, p_N\}$$

and, setting

$$(4.6) \quad x_i = \pi(p_i), \quad V_i := \text{R}\Gamma_{\{\psi(x) \geq \psi(x_i)\}}(M; F_0)_{x_i},$$

also assume that

$$(4.7) \quad \text{the cohomologies of } V_i \text{ are finite-dimensional } \mathbf{k}\text{-vector spaces.}$$

Set

$$b_j(F_0) = \dim H^j(\text{R}\Gamma(M; F_0)),$$

Assume (4.1) (4.5) and (4.7). Then the Morse inequalities for sheaves are stated as:

$$(4.8) \quad b_l^*(F_0) \leq \sum_{i=1}^N b_l^*(V_i).$$

In fact, assumption (4.1) may be weakened, see loc. cit.

Notice that (4.8) immediately implies

$$(4.9) \quad b_j(F_0) \leq \sum_{i=1}^N b_j(V_i) \quad \text{for any } j.$$

In the sequel, we set

$$S_0 = \text{SS}(F_0) \cap \dot{T}^*M.$$

Now we assume that

$$(4.10) \quad \left\{ \begin{array}{l} \text{(i) } \psi \text{ is of class } C^2 \text{ and the differential } d\psi(x) \text{ never vanishes,} \\ \text{(ii) } S_{0,\text{reg}} \text{ is an open dense subset of } S_0 \text{ such that } S_{0,\text{reg}} \text{ is a} \\ \text{Lagrangian submanifold of class } C^1 \text{ and } F_0 \text{ is a simple} \\ \text{sheaf along } S_{0,\text{reg}}. \end{array} \right.$$

Lemma 4.4. *Let Λ be a smooth Lagrangian manifold defined in a neighborhood of $p \in \dot{T}^*M$, let $G \in \mathbf{D}^b(\mathbf{k}_M)$ and assume G is simple along Λ at p . Assume that Λ and Λ_ψ intersect transversally at p . Set $x_0 = \pi(p)$. Then*

$$\sum_j \dim H^j(\mathbf{R}\Gamma_{\{\psi(x) \geq \psi(x_0)\}}(G))_{x_0} = 1.$$

Proof. By the definition ([10, Definition 7.5.4]), $\mathbf{R}\Gamma_{\{\psi(x) \geq \psi(x_0)\}}(G)_{x_0}$ is concentrated in a single degree and its cohomology in this degree has rank one. Q.E.D.

In the sequel, for a finite set A , we denote by $\#A$ its cardinal.

Theorem 4.5. *Let $\Phi = \{\varphi_t\}_{t \in I}$, F_t and $\psi: M \rightarrow \mathbb{R}$ be as above and assume (4.1) and (4.10). Let $t_0 \in I$. Assume that $\Lambda_\psi \cap \varphi_{t_0}(S_0)$ is contained in $\Lambda_\psi \cap \varphi_{t_0}(S_{0,\text{reg}})$ and the intersection is finite and transversal. Then*

$$\#(\varphi_{t_0}(S_0) \cap \Lambda_\psi) \geq \sum_j b_j(F_0).$$

Proof. It follows from Lemma 4.1 that $b_j(F_t) = b_j(F_0)$ for all $j \in \mathbb{Z}$ and all $t \in I$.

Let $\{q_1, \dots, q_L\} = \Lambda_\psi \cap \varphi_{t_0}(S_0)$, $y_i = \pi(q_i)$ and set

$$W_i := \mathbf{R}\Gamma_{\{\psi(x) \geq \psi(y_i)\}}(F_{t_0})_{y_i}.$$

By Lemma 4.4, W_i is a bounded complex with finite-dimensional cohomologies and it follows from the Morse inequalities that

$$\sum_j b_j(F_{t_0}) \leq \sum_j \sum_i b_j(W_i).$$

Moreover

$$\sum_j \dim H^j(\mathbb{R}\Gamma_{\{\psi(x) \geq \psi(y_i)\}}(F_{t_0})_{y_i}) = 1 \quad \text{for any } i,$$

and it implies

$$\sum_j \sum_i b_j(W_i) = \#(\text{SS}(F_{t_0}) \cap \Lambda_\psi) = \#(\varphi_{t_0}(\text{SS}(F_0) \cap \dot{T}^*M) \cap \Lambda_\psi).$$

Q.E.D.

Corollary 4.6. *Let $\Phi = \{\varphi_t\}_{t \in I}$, F_t and $\psi: M \rightarrow \mathbb{R}$ be as above and let N be a compact submanifold of M . Let $t_0 \in I$. Assume that $\varphi_{t_0}(\dot{T}^*N)$ and Λ_ψ intersect transversally. Then*

$$\#(\varphi_{t_0}(\dot{T}^*N) \cap \Lambda_\psi) \geq \sum_j \dim H^j(N; \mathbb{C}).$$

Remark 4.7. Corollaries 4.3 and 4.6 extend to the case where N is replaced with a compact submanifold with boundary or even with corners. In this case, one has to replace the conormal bundle T_N^*M with the microsupport of the constant sheaf \mathbf{k}_N on M . Note that this microsupport is easily calculated. For Morse inequalities on manifolds with boundaries, see the recent paper [12].

Positive Hamiltonian isotopies

Consider as above a manifold M and $\Phi: \dot{T}^*M \times I \rightarrow \dot{T}^*M$ a homogeneous Hamiltonian isotopy associated to a function $f: \dot{T}^*M \times I \rightarrow \mathbb{R}$ homogeneous of degree 1 satisfying (3.1). We also define $\Lambda \subset \dot{T}^*M \times \dot{T}^*M \times T^*I$ as in (3.2). The following definition has been communicated to us by Emmanuel Ferrand and is used in [4] where the authors prove (a variant of) Corollary 4.10 below and other related results.

Definition 4.8. The isotopy Φ is said to be positive if $\langle \alpha_M, H_f \rangle \geq 0$.

Let eu_M be the Euler vector field on T^*M . Then $\langle \alpha_M, H_f \rangle = \text{eu}_M(f)$ and since f is of degree 1 we have $\text{eu}_M(f) = f$. Hence Φ is positive if and only if f is a non-negative valued function. We let (t, τ) be the coordinates on T^*I . Then by (3.2) this condition is also equivalent to

$$\Lambda \subset \{\tau \leq 0\}.$$

Proposition 4.9. *Let N be a manifold, $I =]a, b[$ an open interval of \mathbb{R} . Let $F \in \mathbf{D}^b(\mathbf{k}_{N \times I})$ and, for $t \in I$, set $F_t = F|_{N \times \{t\}} \in \mathbf{D}^b(\mathbf{k}_N)$. Assume that*

- (i) $\mathrm{SS}(F) \subset \{\tau \leq 0\}$,
- (ii) $\mathrm{SS}(F) \cap (T_N^*N \times T^*I) \subset T_{N \times I}^*(N \times I)$,
- (iii) $\mathrm{Supp}(F) \rightarrow I$ is proper.

Then we have:

- (a) for all $s \leq t \in I$ there are natural morphisms $r_{t,s}: F_s \rightarrow F_t$,
- (b) $r_{t,s}$ induces a commutative diagram with isomorphisms

$$\begin{array}{ccc} \mathrm{R}\Gamma(N \times I; F) & & \\ \downarrow \wr & \searrow \sim & \\ \mathrm{R}\Gamma(N; F_s) & \xrightarrow{\sim r_{t,s}} & \mathrm{R}\Gamma(N; F_t). \end{array}$$

Proof. (i) By a similar argument to Lemma 4.1, (ii) and (iii) implies that $\mathrm{R}\Gamma(N \times I; F) \rightarrow \mathrm{R}\Gamma(N; F_t)$ is an isomorphism for any $t \in I$.

(ii) Take b' such that $a < b' < b$, and set $F' = F \otimes \mathbf{k}_{N \times]a, b'}$. Then F' also satisfies (i). Hence [10, Prop. 5.2.3] implies that $F' \simeq F' \circ \mathbf{k}_D$, where

$$D = \{(s, t) \in I \times I; t \leq s \leq b'\}.$$

We deduce the isomorphisms, for any $t_0 \in I' :=]a, b'[$:

$$(4.11) \quad F_{t_0} \simeq F' \circ \mathbf{k}_{\{t_0\}} \simeq F' \circ \mathbf{k}_D \circ \mathbf{k}_{\{t_0\}} \simeq F' \circ \mathbf{k}_{[t_0, b']} \simeq F' \circ \mathbf{k}_{[t_0, b']}.$$

The morphism $r_{t,s}$ ($a < s \leq t < b'$) is then induced by the morphism $\mathbf{k}_{[s, b']} \rightarrow \mathbf{k}_{[t, b']}$. Hence we obtain a commutative diagram

$$\begin{array}{ccc} & \mathrm{R}\Gamma(N \times I; F) & \\ & \swarrow \sim & \searrow \sim \\ \mathrm{R}\Gamma(N \times [s, b']; F) & \xrightarrow{\quad} & \mathrm{R}\Gamma(N \times [t, b']; F) \\ \downarrow \wr & & \downarrow \wr \\ \mathrm{R}\Gamma(N; F \circ \mathbf{k}_{[s, b']}) & \xrightarrow{\quad} & \mathrm{R}\Gamma(N; F \circ \mathbf{k}_{[t, b']}) \\ \downarrow \wr & & \downarrow \wr \\ \mathrm{R}\Gamma(N; F_s) & \xrightarrow{\quad r_{t,s} \quad} & \mathrm{R}\Gamma(N; F_t). \end{array}$$

Q.E.D.

Corollary 4.10. *Let M be a connected and non compact manifold, and let x, y be two distinct points on M . Then there does not exist any positive homogeneous Hamiltonian isotopy $\Phi: \dot{T}^*M \times I \rightarrow \dot{T}^*M$ such that $\varphi_t(\dot{T}_x^*M) = \dot{T}_y^*M$ for some $t \in I$.*

Proof. (i) Assume we have such Φ and t . We set $F_0 = \mathbf{k}_{\{x\}} \in \mathbf{D}^b(\mathbf{k}_M)$. Let $K \in \mathbf{D}^b(\mathbf{k}_{M \times M \times I})$ be the quantization of Φ on I constructed in Theorem 3.3. Then $\text{SS}(K) \subset \Lambda \cup T_{M \times M \times I}^*(M \times M \times I)$, so that $\text{SS}(K) \subset \{\tau \leq 0\}$. We set:

$$F = K \circ F_0, \quad F_{t_0} = F \circ \mathbf{k}_{\{t=t_0\}} \quad (t_0 \in I).$$

Then we also have $\text{SS}(F) \subset \{\tau \leq 0\}$. Moreover $\text{supp}(F) \rightarrow I$ is proper and $\text{SS}(F) \cap (T_M^*M \times T^*I) \subset T_{M \times I}^*(M \times I)$. To check the last condition we note that outside the zero-section $\text{SS}(F) \subset \Lambda \circ T^*M = p_{13}(\Lambda)$. But $\Lambda \subset \dot{T}^*M \times T^*M \times T^*I$ and it follows that $\text{SS}(F) \subset (\dot{T}^*M \times T^*I) \cup T_{M \times I}^*(M \times I)$.

(ii) Hence the hypotheses of Proposition 4.9 are satisfied and we deduce that there exists a morphism $F_t \rightarrow F_0$ if $t < 0$ or $F_0 \rightarrow F_t$ if $t > 0$ which induces an isomorphism on global sections, and so is non-zero since $\text{R}\Gamma(M; F_0) \simeq \mathbf{k}$.

(iii) On the other hand we have

$$\text{SS}(F_t) \cap \dot{T}^*M = \varphi_t(\text{SS}(F_0) \cap \dot{T}^*M) = \varphi_t(\dot{T}_x^*M) = \dot{T}_y^*M.$$

In particular, the cohomologies of F_t are locally constant sheaves on $M \setminus \{y\}$. Since M is connected and non compact and $\text{Supp}(F_t)$ is compact, this implies that $F_t|_{M \setminus \{y\}} \simeq 0$. In particular, any morphism $F_t \rightarrow F_0$ or $F_0 \rightarrow F_t$ vanishes and this contradicts (ii). Q.E.D.

Non homogeneous case

In this subsection we recall the link between non homogeneous symplectic geometry and homogeneous symplectic geometry with an extra variable.

For a manifold M we define the map

$$\rho = \rho_M: T^*M \times \dot{T}^*\mathbb{R} \rightarrow T^*M, \quad (x, \xi, s, \sigma) \mapsto (x, \xi/\sigma).$$

We consider a Hamiltonian isotopy $\Phi: T^*M \times I \rightarrow T^*M$ as in Section 3 but we do not assume anymore it is homogeneous. Recall what this means:

- (i) $\varphi_0 = \text{id}_{T^*M}$, where $\varphi_t := \Phi(\cdot, t): T^*M \rightarrow T^*M$,
- (ii) there exists a function $f: T^*M \times I \rightarrow \mathbb{R}$ such that $\frac{\partial \Phi}{\partial t} = H_f$, where H_f is the Hamiltonian vector field of f .

We also assume that there exists a compact set $C \subset T^*M$ such that f vanishes on $(T^*M \setminus C) \times I$. This implies that $\varphi_t|_{T^*M \setminus C} = \text{id}$ for all $t \in I$.

We may define $\tilde{f}: \dot{T}^*(M \times \mathbb{R}) \times I \rightarrow \mathbb{R}$ by

$$\tilde{f}(x, \xi, s, \sigma, t) = \begin{cases} \sigma f(x, \xi/\sigma, t) & \text{if } (x, \xi, s, \sigma) \in T^*M \times \dot{T}^*\mathbb{R}, \\ 0 & \text{if } (x, \xi, s, \sigma) \in \dot{T}^*M \times T^*\mathbb{R}. \end{cases}$$

Indeed for a given $(x, \xi) \in \dot{T}^*M$, ξ/σ goes to infinity when σ goes to 0. Hence, $\tilde{f}(x, \xi, s, \sigma, t) = 0$ for $|\sigma|$ small enough.

Lemma 4.11. *The Hamiltonian flow of \tilde{f} is defined on $\dot{T}^*(M \times \mathbb{R}) \times I$. We denote it by $\tilde{\Phi}: \dot{T}^*(M \times \mathbb{R}) \times I \rightarrow \dot{T}^*(M \times \mathbb{R})$. We have the commutative diagram*

$$\begin{array}{ccc} T^*M \times \dot{T}^*\mathbb{R} \times I & \xrightarrow{\tilde{\Phi}} & T^*M \times \dot{T}^*\mathbb{R} \\ \rho \times \text{id}_I \downarrow & & \rho \downarrow \\ T^*M \times I & \xrightarrow{\Phi} & T^*M. \end{array}$$

More precisely there exists a function $u: \dot{T}^*(M \times \mathbb{R}) \times I \rightarrow \mathbb{R}$ homogeneous of degree 0 such that

$$(4.12) \quad \tilde{\Phi}(x, \xi, s, \sigma, t) = (\sigma \cdot \varphi_t(x, \xi/\sigma), u(x, \xi, s, \sigma, t), \sigma),$$

where $\sigma \cdot$ denotes the multiplicative action of \mathbb{R} on T^*M .

Proof. We have to describe the Hamiltonian vector field $H_{\tilde{f}}$ of \tilde{f} . We denote by $p: T^*M \times T^*\mathbb{R} \rightarrow T^*\mathbb{R}$ the projection $(x, \xi, s, \sigma) \mapsto (s, \sigma)$. For a point $q = (x, \xi, s, \sigma) \in T^*M \times \dot{T}^*\mathbb{R}$ we use the derivatives of ρ and p to decompose $T_q(T^*M \times T^*\mathbb{R})$:

$$(4.13) \quad d\rho_q \times dp_q: T_q(T^*M \times T^*\mathbb{R}) \xrightarrow{\sim} T_{(x, \xi/\sigma)}(T^*M) \oplus T_{(s, \sigma)}(T^*\mathbb{R}).$$

Setting $\sigma_M = d\alpha_M$ where α_M is the Liouville form on T^*M we have

$$\begin{aligned} \alpha_{M \times \mathbb{R}}|_{T^*M \times \dot{T}^*\mathbb{R}} &= \sigma \rho^*(\alpha_M) + p^*(\alpha_{\mathbb{R}}), \\ \sigma_{M \times \mathbb{R}}|_{T^*M \times \dot{T}^*\mathbb{R}} &= \sigma \rho^*(\sigma_M) + p^*(\sigma_{\mathbb{R}}) + p^*(d\sigma) \wedge \rho^*(\alpha_M). \end{aligned}$$

In the sequel we fix t and write \tilde{f} for $\tilde{f}(\cdot, t)$ for short. Then $H_{\tilde{f}}$ is determined by $\iota_{H_{\tilde{f}}}(\sigma_{M \times \mathbb{R}}) = -d\tilde{f}$. We decompose $(H_{\tilde{f}})_q = \theta_M + \theta_{\mathbb{R}}$ according to (4.13) and we also use the decomposition of $T_q^*(T^*M \times T^*\mathbb{R})$ induced by (4.13). We find

$$\iota_{H_{\tilde{f}}}(\sigma_{M \times \mathbb{R}}) = (\sigma \iota_{\theta_M}(\sigma_M) + \langle \theta_{\mathbb{R}}, d\sigma \rangle \alpha_M) + (\iota_{\theta_{\mathbb{R}}}(\sigma_{\mathbb{R}}) - \langle \theta_M, \alpha_M \rangle d\sigma).$$

Since $d\tilde{f} = \sigma \rho^* df + fp^* d\sigma$ we obtain

$$-d\tilde{f} = \iota_{\theta_M}(\sigma_M) + \sigma^{-1} \langle \theta_{\mathbb{R}}, d\sigma \rangle \alpha_M, \quad -f d\sigma = \iota_{\theta_{\mathbb{R}}}(\sigma_{\mathbb{R}}) - \langle \theta_M, \alpha_M \rangle d\sigma.$$

The second equality gives $\theta_{\mathbb{R}} = a \partial / \partial s$ for some function a . Then the first one gives $\theta_M = H_f$ and hence $a = (f - \langle H_f, \alpha_M \rangle) \circ \rho = (f - \text{eu}_M(f)) \circ \rho$. Finally

$$(H_{\tilde{f}})_{(x, \xi, s, \sigma, t)} = (H_f)_{(x, \xi / \sigma, t)} + (f - \text{eu}_M(f))(x, \xi / \sigma, t) \frac{\partial}{\partial s}.$$

This shows that $H_{\tilde{f}}$ projects to H_f through ρ and that the flow $\tilde{\Phi}$ of \tilde{f} keeps the σ coordinate fixed. Hence (4.12) holds as soon as $\tilde{\Phi}$ is defined. Since $\tilde{\Phi}$ is homogeneous the function u is homogeneous of degree 0. Since $\tilde{\Phi}$ is the identity map on $\dot{T}^*M \times T_{\mathbb{R}}^*\mathbb{R}$ we may also extend u to $\dot{T}^*(M \times \mathbb{R}) \times I$ by $u = s$ on $\dot{T}^*M \times T_{\mathbb{R}}^*\mathbb{R} \times I$.

To see that $\tilde{\Phi}$ is defined on $\dot{T}^*(M \times \mathbb{R}) \times I$ we just need to check that the s coordinate remains bounded when $t \in J$ for any given compact interval $J \subset I$. But f and $\text{eu}_M(f)$ are bounded on $T^*M \times J$ hence the coefficient a of $\partial / \partial s$ introduced above also is bounded on $T^*M \times \dot{T}^*\mathbb{R} \times J$. It follows that the s coordinate remains bounded as required. Q.E.D.

Theorem 4.12. *Let N be a compact manifold and assume that N is not empty. Let $\Phi: T^*N \times I \rightarrow T^*N$ be a Hamiltonian isotopy as above. We let $c = \sum_j \dim H^j(N; \mathbb{C})$, the sum of the Betti numbers of N . Then for any $t \in I$ the intersection $\varphi_t(T_N^*N) \cap T_N^*N$ is never empty. Moreover its cardinality is at least c whenever the intersection is transversal.*

Proof. (i) We set $M = N \times \mathbb{R}$ and identify N with $N \times \{0\}$. We let $\tilde{\Phi}: \dot{T}^*M \times I \rightarrow \dot{T}^*M$ be the isotopy given by Lemma 4.11 and we set $\tilde{\varphi}_t = \tilde{\Phi}(\cdot, t)$.

For a given t Lemma 3.5 gives the existence of a Hamiltonian isotopy $\hat{\Phi}: \dot{T}^*M \times I \rightarrow \dot{T}^*M$ which satisfies (3.1) and (3.3) and such that

$$\tilde{\varphi}_t(\dot{T}_N^*M) = \hat{\varphi}_t(\dot{T}_N^*M).$$

We apply Theorem 4.2 and 4.5 to $M = N \times \mathbb{R}$, $\widehat{\Phi}$, $F_0 = \mathbf{k}_N$ and $\psi = t$, the projection from M to \mathbb{R} . We obtain that the intersection $\widetilde{\varphi}_t(\dot{T}_N^*M) \cap \Lambda_\psi$ is a non empty set whose cardinality is at least c whenever the intersection is transversal.

(ii) Now we compare $\widetilde{\varphi}_t(\dot{T}_N^*M) \cap \Lambda_\psi$ with the intersection considered in the theorem. To study the transversality of the intersections we use the following remark.

Let $p: E \rightarrow X$ be a smooth morphism, let A, B be submanifolds of X and A' a submanifold of E . We assume that p induces a diffeomorphism $p|_{A'}: A' \xrightarrow{\sim} A$. We set $B' = p^{-1}(B)$. Then

$$(4.14) \quad p \text{ induces a bijection } A' \cap B' \xrightarrow{\sim} A \cap B,$$

$$(4.15) \quad A' \text{ and } B' \text{ intersect transversally if and only if } A \text{ and } B \text{ intersect transversally.}$$

We will apply this remark to the cases (a) and (b) described below ($t \in I$ is fixed).

(a) $X = T^*N$, $E = T^*N \times \mathbb{R}^\times$, $p(x, \xi, \sigma) = (x, \xi/\sigma)$, $A = T_N^*N$, $B = \varphi_t(T_N^*N)$ and $A' = T_N^*N \times \{1\} \subset E$. We set $\Sigma_t := B' = p^{-1}(B)$ and we find

$$\Sigma_t = \{(\sigma \cdot \varphi_t(x, 0), \sigma) \in T^*N \times \mathbb{R}^\times; x \in N, \sigma \in \mathbb{R}^\times\}.$$

(b) $X = T^*N \times \mathbb{R}^\times$, $E = T^*N \times \dot{T}^*\mathbb{R}$, $p(x, \xi, s, \sigma) = (x, \xi, \sigma)$, $A = \Sigma_t$, $B = T_N^*N \times \{1\}$ and $A' = \widetilde{\varphi}_t(\dot{T}_N^*M)$. In view of (4.12) we have

$$\widetilde{\varphi}_t(\dot{T}_N^*M) = \{(\sigma \cdot \varphi_t(x, 0), u(x, 0, 0, \sigma, t), \sigma); x \in N, \sigma \in \mathbb{R}^\times\}.$$

Hence the restriction of p to $\widetilde{\varphi}_t(\dot{T}_N^*M)$ induces an isomorphism $\widetilde{\varphi}_t(\dot{T}_N^*M) \xrightarrow{\sim} \Sigma_t$ as required. We find that $B' = p^{-1}(B)$ is nothing but Λ_ψ .

Then the theorem follows from (4.14) and (4.15) applied to (a) and (b) and the results on $\widetilde{\varphi}_t(\dot{T}_N^*M) \cap \Lambda_\psi$ obtained in part (i) of the proof. Q.E.D.

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