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# An estimation of the stability and the localisability functions of multistable processes

R. Le Guével

*Université Nantes, Laboratoire de Mathématiques Jean Leray UMR CNRS 6629*

*2 Rue de la Houssinière - BP 92208 - F-44322 Nantes Cedex 3, France*

*ronan.leguevel@univ-nantes.fr*

## Abstract

Multistable processes are tangent at each point to a stable process, but where the index of stability and the index of localisability varies along the path. In this work, we give two estimators of the stability and the localisability functions, and we prove the consistency of those two estimators. We illustrate these convergences with two classical examples, the Levy multistable process and the Linear Multifractional Multistable Motion.

## 1 Introduction

Multifractional multistable processes have been recently introduced as models for phenomena where the regularity and the intensity of jumps are non constant. Multistable processes are stochastic processes which are locally stable, but where the index of stability varies with “time”, and therefore is a function. They were constructed in [4, 5, 6, 8] using respectively moving averages, sums over Poisson processes, multistable measures, and the Ferguson-Klass-LePage series representation, this last definition being the representation used here after. These processes are, under general assumptions, localisable, that is they are locally self-similar, with an index of self-similarity which is also a function. The aim of this paper is to introduce an estimator for each function, the index of stability and the self-similarity function.

Let us recall the definition of a *localisable process* [2, 3]:  $Y = \{Y(t) : t \in \mathbf{R}\}$  is said to be localisable at  $u$  if there exists an  $h(u) \in \mathbf{R}$  and a non-trivial limiting process  $Y'_u$  such that

$$\lim_{r \rightarrow 0} \frac{Y(u + rt) - Y(u)}{r^{h(u)}} = Y'_u(t), \quad (1.1)$$

where the convergence is in finite dimensional distributions. When the limit exists,  $Y'_u = \{Y'_u(t) : t \in \mathbf{R}\}$  is termed the *local form* or tangent process of  $Y$  at  $u$ , and when the convergence is in distribution, the process is called *strongly h-localisable*.

### Ferguson-Klass-LePage series representation

We define now the multistable processes using the Ferguson-Klass-LePage series representation, that are defined as “diagonals” of random fields that we described below. In

the sequel,  $(E, \mathcal{E}, m)$  will be a measure space, and  $U$  an open interval of  $\mathbf{R}$ . We will assume that  $m$  is a finite measure as well as a  $\sigma$ -finite measure. Let  $\alpha$  be a  $C^1$  function defined on  $U$  and ranging in  $[c, d] \subset (0, 2)$ . Let  $f(t, u, \cdot)$  be a family of functions such that, for all  $(t, u) \in U^2$ ,  $f(t, u, \cdot) \in \mathcal{F}_{\alpha(u)}(E, \mathcal{E}, m)$ . We define also  $r : E \rightarrow \mathbb{R}_+$  such that  $\hat{m}(dx) = \frac{1}{r(x)}m(dx)$  is a probability measure.  $(\Gamma_i)_{i \geq 1}$  will be a sequence of arrival times of a Poisson process with unit arrival time and  $(\gamma_i)_{i \geq 1}$  a sequence of i.i.d. random variables with distribution  $P(\gamma_i = 1) = P(\gamma_i = -1) = 1/2$ . Let  $(V_i)_{i \geq 1}$  a sequence of i.i.d. random variables with distribution  $\hat{m}$  on  $E$  and we assume that the three sequences  $(\Gamma_i)_{i \geq 1}$ ,  $(V_i)_{i \geq 1}$ , and  $(\gamma_i)_{i \geq 1}$  are independent. As in [8], we will consider the following random field:

$$X(t, u) = C_{\alpha(u)}^{1/\alpha(u)} \sum_{i=1}^{\infty} \gamma_i \Gamma_i^{-1/\alpha(u)} r(V_i)^{1/\alpha(u)} f(t, u, V_i), \quad (1.2)$$

where  $C_\eta = \left(\int_0^\infty x^{-\eta} \sin(x) dx\right)^{-1}$ .

Note that when the function  $\alpha$  is constant, then (1.2) is just the Ferguson - Klass - LePage series representation of a stable random variable (see [1, 7, 10, 11, 13] and [14, Theorem 3.10.1] for specific properties of this representation).

### Multistable processes

Multistable processes are obtained by taking diagonals on  $X$  defined in (1.2), *i.e.*

$$Y(t) = X(t, t). \quad (1.3)$$

Indeed, as shown in Theorems 3.3 and 4.5 of [8], provided some conditions are satisfied both by  $X$  and by the function  $f$ ,  $Y$  will be a localisable process whose local form is a stable process.

The aim of this work is to estimate both functions of stability  $\alpha$  and localisability  $h$ . Given one trajectory of a multistable process, we provide an estimator for each function and we obtain the convergence in all the spaces  $L^p$  for the two estimators. We illustrate these convergences with two classical examples, the Levy multistable process and the Linear Multifractional Multistable Motion.

## 2 Construction of the estimators

Let  $Y$  be a multistable process defined in (1.3). The estimation of the localisability function and the stability function is based on the increments  $(Y_{k,N})$  of  $Y$ . Define the sequence  $(Y_{k,N})_{k \in \mathbb{Z}, N \in \mathbb{N}}$  by

$$Y_{k,N} = Y\left(\frac{k+1}{N}\right) - Y\left(\frac{k}{N}\right).$$

Let  $t_0 \in \mathbb{R}$  fixed. We introduce an estimator of  $H(t_0)$  with

$$\hat{H}_N(t_0) = -\frac{1}{n(N) \log N} \sum_{k=[Nt_0] - \frac{n(N)}{2}}^{[Nt_0] + \frac{n(N)}{2} - 1} \log |Y_{k,N}|$$

where  $(n(N))_{N \in \mathbb{N}}$  is a sequence taking even integer values. We expect the sequence  $(\hat{H}_N(t_0))_N$  to converge to  $H(t_0)$  thanks to the localisability of the process  $Y$ . For the

integers  $k$  and  $N$  such that  $\frac{k}{N}$  is close to  $t_0$ ,  $\frac{Y_{k,N}}{(\frac{1}{N})^{H(t_0)}}$  is asymptotically distributed as  $Y'_{t_0}(1)$ . We have then  $-\frac{\log |Y_{k,N}|}{\log N} = H(t_0) + \frac{Z_{k,N}}{\log N}$  where  $(Z_{k,N})_{k,N}$  converge weakly to  $-\log |Y'_{t_0}(1)|$  when  $N$  tends to infinity and  $\frac{k}{N}$  tends to  $t_0$ . We regulate the sequence  $(Z_{k,N})$  near  $t_0$  using the mean  $\frac{1}{n(N)} \sum_{k=[Nt_0]-\frac{n(N)}{2}}^{[Nt_0]+\frac{n(N)}{2}-1} Z_{k,N}$  and we can expect this sum will be bounded in the  $L^r$  spaces to obtain the convergence with a rate  $\frac{1}{\log N}$ . The convergence is proved in Theorem 3.1.

Let  $p_0 > 0$  and  $\gamma \in (0, 1)$ . With the increments of the process, we define the empirical moments  $S_N(p)$  by

$$S_N(p) = \left( \frac{1}{n(N)} \sum_{k=[Nt_0]-\frac{n(N)}{2}}^{[Nt_0]+\frac{n(N)}{2}-1} |Y_{k,N}|^p \right)^{\frac{1}{p}}.$$

Let

$$R_{\text{exp}}(p) = \frac{S_N(p_0)}{S_N(p)} \text{ and } R_\alpha(p) = \frac{(\mathbb{E}|Z|^{p_0})^{1/p_0}}{(\mathbb{E}|Z|^p)^{1/p}} \mathbf{1}_{p < \alpha}$$

where  $Z$  is a standard symmetric  $\alpha$ -stable random variable (written  $Z \sim S_\alpha(1, 0, 0)$  as in [14]), i.e  $\mathbb{E}|Z|^p = \frac{2^{p-1}\Gamma(1-\frac{p}{\alpha})}{p \int_0^{+\infty} u^{-p-1} \sin^2(u) du}$ .

We define now an estimator of  $\alpha(t_0)$  by

$$\hat{\alpha}_N(t_0) = \arg \min_{\alpha \in [0, 2]} \left( \int_{p_0}^2 |R_{\text{exp}}(p) - R_\alpha(p)|^\gamma dp \right)^{1/\gamma}.$$

Under the conditions of Theorem 3.2,  $Y$  is  $H(t_0)$ -localisable and  $Y'_{t_0}(1) \sim S_{\alpha(t_0)}(1, 0, 0)$  so  $\frac{|Y_{k,N}|^p}{(\frac{1}{N})^{pH(t_0)}}$  converge weakly to  $|Y'_{t_0}(1)|^p$  and with a meaning effect,  $N^{H(t_0)}S_N(p)$  tends to  $(\mathbb{E}|Y'_{t_0}(1)|^p)^{1/p}$  in probability, which is the result of Theorem 3.2. Without more conditions,  $\int_{p_0}^2 |R_{\text{exp}}(p) - R_\alpha(p)|^\gamma dp$  tends to  $\int_{p_0}^2 |R_{\alpha(t_0)}(p) - R_\alpha(p)|^\gamma dp$ . Naturally,  $\alpha(t_0)$  is the only solution of  $\arg \min_{\alpha \in [0, 2]} \int_{p_0}^2 |R_{\alpha(t_0)}(p) - R_\alpha(p)|^\gamma dp$  and this leads to the definition of  $\hat{\alpha}_N(t_0)$ . The convergence is proved in Theorem 3.3.

### 3 Main results

The three following theorems apply to a diagonal process  $Y$  defined from the field  $X$  given by (1.2). For convenience, the conditions required on  $X$  and the function  $f$  that appears in (1.2), denoted (C1), ..., (C14), are gathered in Section 6. Theorem 3.1 lead to the convergence in the  $L^r$  spaces of the estimator of the localisability function  $H$ , while the two Theorems 3.2 and 3.3 draw to the convergence of the estimator of the stability function  $\alpha$ .

### 3.1 Approximation of the localisability function

**Theorem 3.1** *Let  $Y$  a multistable process. Assume the conditions (C1), (C2), (C3) (or (C1), (Cs2), (Cs3) and (Cs4) in the  $\sigma$ -finite space case), and that there exists a function  $H$  such that (C5)-(C14) hold. Assume in addition that  $\lim_{N \rightarrow +\infty} \frac{N}{n(N)} = +\infty$ .*

*Then, for all  $t_0 \in U$  and all  $r > 0$ ,*

$$\lim_{N \rightarrow +\infty} \mathbf{E} \left| \hat{H}_N(t_0) - H(t_0) \right|^r = 0.$$

**Proof**

See Section 5.

**Remark:** Under the conditions (C1), (C2), (C3) and (C5) listed in the theorem, Theorems 3.3 and 4.5 of [8] imply that  $Y$  is  $H(t_0)$ -localisable at  $t_0$ .

### 3.2 Approximation of the stability function

We first give conditions for the convergence in probability of  $S_N(p)$  in Theorem 3.2, which is useful to establish the consistency of the estimator  $\hat{\alpha}_N(t_0)$ .

**Theorem 3.2** *Let  $Y$  a multistable process. Assume the conditions (C1), (C2), (C3) (or (C1), (Cs2), (Cs3) and (Cs4) in the  $\sigma$ -finite space case). Assume in addition that:*

- $\lim_{N \rightarrow +\infty} n(N) = +\infty$ .
- $\lim_{N \rightarrow +\infty} \frac{N}{n(N)} = +\infty$ .
- *The process  $X(\cdot, t_0)$  is  $H(t_0)$ -self-similar with stationary increments and  $H(t_0) < 1$ .*
- *(C\*) There exists  $\epsilon_1 > 0$  and  $j_0 \in \mathbb{N}$  such that for all  $j \geq j_0$ ,*

$$\int_E |h_{0,t_0}(x) h_{j,t_0}(x)|^{\frac{\alpha(t_0)}{2}} m(dx) \leq (1 - \epsilon_1) \|h_{0,t_0}\|_{\alpha(t_0)}^{\alpha(t_0)},$$

where  $h_{j,u}(x) = f(j+1, u, x) - f(j, u, x)$ .

- $\lim_{j \rightarrow +\infty} \int_E |h_{0,t_0}(x) h_{j,t_0}(x)|^{\frac{\alpha(t_0)}{2}} m(dx) = 0$ .

*Then, for all  $p \in [p_0, \alpha(t_0))$ ,*

$$N^{H(t_0)} S_N(p) \xrightarrow[N \rightarrow +\infty]{} (\mathbf{E}|Z|^p)^{1/p}$$

*where the convergence is in probability and  $Z \sim S_{\alpha(t_0)}(1, 0, 0)$ .*

**Proof**

See Section 5.

**Theorem 3.3** *Let  $Y$  a multistable process. Assume the conditions of Theorem 3.2, then, for all  $t_0 \in U$  and  $r > 0$ ,*

$$\lim_{N \rightarrow +\infty} \mathbf{E} |\hat{\alpha}_N(t_0) - \alpha(t_0)|^r = 0.$$

**Proof**

See Section 5.

## 4 Examples and simulations

In this section, we consider the “multistable versions” of some classical processes: the  $\alpha$ -stable Lévy motion and the Linear Fractional Stable Motion.

We first recall some definitions. In the sequel,  $M$  will denote a symmetric  $\alpha$ -stable ( $0 < \alpha < 2$ ) random measure on  $\mathbf{R}$  with control measure Lebesgue measure  $\mathcal{L}$ . We will write

$$L_\alpha(t) := \int_0^t M(dz)$$

for  $\alpha$ -stable Lévy motion, and we will use the Ferguson-Klass-LePage representation,

$$\forall t \in (0, 1), \quad L_\alpha(t) = C_\alpha^{1/\alpha} \sum_{i=1}^{\infty} \gamma_i \Gamma_i^{-1/\alpha} \mathbf{1}_{[0,t]}(V_i).$$

The following process is called *linear fractional  $\alpha$ -stable motion*:

$$L_{\alpha,H,b^+,b^-}(t) = \int_{-\infty}^{\infty} f_{\alpha,H}(b^+, b^-, t, x) M(dx)$$

where  $t \in \mathbb{R}$ ,  $H \in (0, 1)$ ,  $b^+, b^- \in \mathbb{R}$ , and

$$\begin{aligned} f_{\alpha,H}(b^+, b^-, t, x) &= b^+ \left( (t-x)_+^{H-1/\alpha} - (-x)_+^{H-1/\alpha} \right) \\ &\quad + b^- \left( (t-x)_-^{H-1/\alpha} - (-x)_-^{H-1/\alpha} \right). \end{aligned}$$

When  $b^+ = b^- = 1$ , this process is called well-balanced linear fractional  $\alpha$ -stable motion and denoted  $L_{\alpha,H}$ .

The localisability of Lévy motion and linear fractional  $\alpha$ -stable motion simply stems from the fact that they are  $1/\alpha$ -self-similar with stationary increments [3].

We now apply our results to the multistable versions of these processes, that were defined in [4, 5].

### 4.1 Symmetric multistable Lévy motion

Let  $\alpha : [0, 1] \rightarrow [c, d] \subset (1, 2)$  be continuously differentiable. Define

$$X(t, u) = C_{\alpha(u)}^{1/\alpha(u)} \sum_{i=1}^{\infty} \gamma_i \Gamma_i^{-1/\alpha(u)} \mathbf{1}_{[0,t]}(V_i) \tag{4.4}$$

and the symmetric multistable Lévy motion

$$Y(t) = X(t, t).$$

**Proposition 4.4** *If  $\lim_{N \rightarrow +\infty} n(N) = +\infty$  and  $\lim_{N \rightarrow +\infty} \frac{N}{n(N)} = +\infty$ , then for all  $r > 0$ ,*

$$\lim_{N \rightarrow +\infty} \mathbf{E} \left| \hat{H}_N(t_0) - \frac{1}{\alpha(t_0)} \right|^r = 0 \quad \text{and} \quad \lim_{N \rightarrow +\infty} \mathbf{E} |\hat{\alpha}_N(t_0) - \alpha(t_0)|^r = 0.$$

**Proof**

We know from [9] that all the conditions (C1)-(C14) are satisfied. We deduce from Theorem 3.1 that  $\lim_{N \rightarrow +\infty} \mathbf{E} \left| \hat{H}_N(t_0) - \frac{1}{\alpha(t_0)} \right|^r = 0$ . Since the process  $X(\cdot, t_0)$  is a Lévy motion  $\alpha(t_0)$ -stable,  $X(\cdot, t_0)$  is  $\frac{1}{\alpha(t_0)}$ -self-similar with stationary increments [14]. We then prove that the condition (C\*) is satisfied.

$$h_{j,t_0}(x) = \mathbf{1}_{[j, j+1[}(x)$$

so for  $j \geq 1$ ,

$$\int_{\mathbf{R}} |h_{0,t_0}(x)h_{j,t_0}(x)|^{\frac{\alpha(t_0)}{2}} dx = 0.$$

We conclude with Theorem 3.3 ■

We display on Figure 1 some examples of estimations for various functions  $\alpha$ , the function  $H$  satisfying the relation  $H(t) = \frac{1}{\alpha(t)}$ . The trajectories have been simulated using the field (4.4). For each  $u \in (0, 1)$ ,  $X(\cdot, u)$  is a  $\alpha(u)$ -stable Lévy Motion. It is then an  $\alpha(u)$ -stable process with independent increments. We have generated these increments using the RSTAB program available in [15] or in [14], and then taken the diagonal  $X(t, t)$ .

Each function is pretty well-evaluated. We are able to recreate with the estimators the shape of the functions. However, we notice a significant bias on Figure 1 in the estimation of  $H$ . It seems to decrease when  $H$  is getting values close to 1. We observe this phenomenon with most trajectories, while the estimator  $\hat{\alpha}$  seems to be unbiased. We have displayed the product  $\hat{\alpha}\hat{H}$  in order to show the link between the estimators. We actually find again the asymptotic relationship  $H(t) = \frac{1}{\alpha(t)}$ .

We observe on Figure 2 an evolution of the variance in the estimation of  $\alpha$ . It seems to increase when the function  $\alpha$  is decreasing, and we conjecture that the variance at the point  $t_0$  depends on the value  $\alpha(t_0)$  in this way. In fact, the increments  $Y_{k,N}$  are asymptotically distributed as an  $\alpha(t_0)$ -stable variable, so we expect that  $S_N$  and  $R_{\text{exp}}$  have a variance increasing when  $\alpha$  is decreasing.

We have increased the resolution on Figure 3, taking more points for the discretization. The distance observed on Figure 2.b for  $\alpha$  near 1 is then corrected.

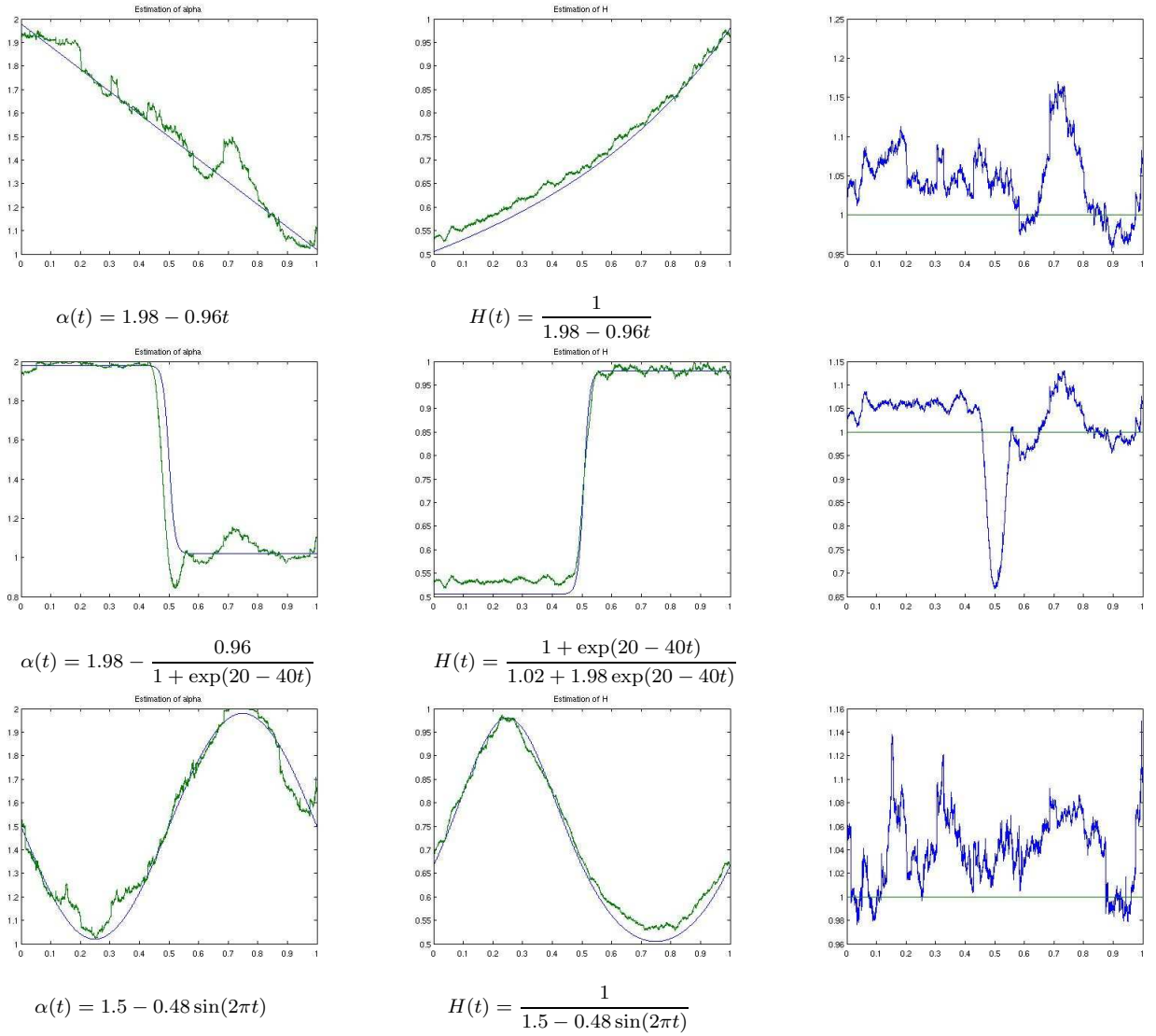


Figure 1: Trajectories on  $(0, 1)$  with  $N = 20000$  points,  $n(N) = 2042$  points for the estimator  $\hat{\alpha}$ , and  $n(N) = 500$  for  $\hat{H}$ .  $\alpha$  and  $\hat{\alpha}$  are represented in the first column,  $H$  and  $\hat{H}$  in the second column, and in the last column, we have drawn the product  $\hat{\alpha}\hat{H}$ .

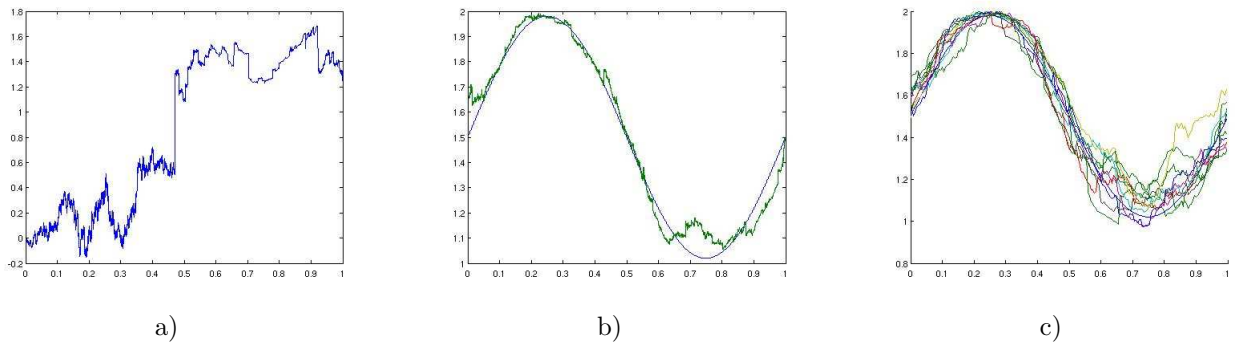
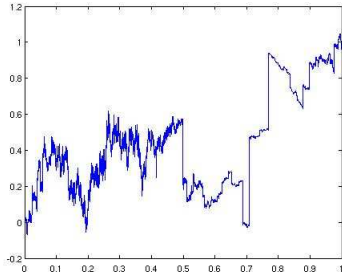
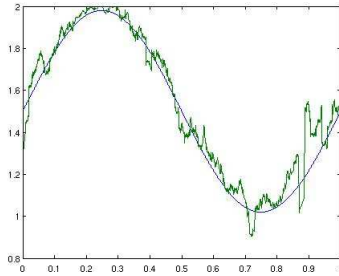


Figure 2: Trajectory of a Levy process with  $\alpha(t) = 1.5 + 0.48 \sin(2\pi t)$  in figure a), and the corresponding estimation of  $\alpha$  in figure b) with  $n(N) = 2042$ . The figure c) represents various estimations of  $\alpha$  for the same function  $\alpha(t) = 1.5 + 0.48 \sin(2\pi t)$ , with different trajectories.





d)



e)

Figure 3: Trajectory with  $N = 200000$  in figure d), and the estimation with  $n(N) = 3546$  in figure e).

## 4.2 Linear multistable multifractional motion

Let  $\alpha : \mathbf{R} \rightarrow [c, d] \subset (0, 2)$  and  $H : \mathbf{R} \rightarrow (0, 1)$  be continuously differentiable. Define

$$X(t, u) = C_{\alpha(u)}^{1/\alpha(u)} \left( \frac{\pi^2 j^2}{3} \right)^{1/\alpha(u)} \sum_{i,j=1}^{\infty} \gamma_i \Gamma_i^{-1/\alpha(u)} (|t - V_i|^{H(u)-1/\alpha(u)} - |V_i|^{H(u)-1/\alpha(u)}) \mathbf{1}_{[-j, -j+1] \cup [j-1, j]}(V_i) \quad (4.5)$$

and the linear multistable multifractional motion

$$Y(t) = X(t, t).$$

**Proposition 4.5** *Assume that  $H - \frac{1}{\alpha}$  is a non-negative function,  $\lim_{N \rightarrow +\infty} n(N) = +\infty$  and  $\lim_{N \rightarrow +\infty} \frac{N}{n(N)} = +\infty$ . Then for all  $r > 0$ ,*

$$\lim_{N \rightarrow +\infty} \mathbf{E} \left| \hat{H}_N(t_0) - H(t_0) \right|^r = 0 \text{ and } \lim_{N \rightarrow +\infty} \mathbf{E} |\hat{\alpha}_N(t_0) - \alpha(t_0)|^r = 0.$$

### Proof

We know from [9] that all the conditions (C1)-(C14) are satisfied. We deduce from Theorem 3.1 that  $\lim_{N \rightarrow +\infty} \mathbf{E} \left| \hat{H}_N(t_0) - H(t_0) \right|^r = 0$ . Since the process  $X(\cdot, t_0)$  is a  $(H(t_0), \alpha(t_0))$  linear fractional stable motion,  $X(\cdot, t_0)$  is  $H(t_0)$ -self-similar with stationary increments [14]. We write  $h_{j,t_0}(x) = g(j-x)$  with  $g(u) = |1+u|^{H(t_0)-\frac{1}{\alpha(t_0)}} - |u|^{H(t_0)-\frac{1}{\alpha(t_0)}}$  so with Proposition 2.2 of [12], the condition (C\*) is satisfied. Let us show that  $\lim_{j \rightarrow +\infty} \int_{\mathbf{R}} |h_{0,t_0}(x)h_{j,t_0}(x)|^{\frac{\alpha(t_0)}{2}} dx = 0$  and conclude with Theorem 3.3.

Let  $\epsilon > 0$ . Let  $c_0 > 0$  such that  $\int_{|x|>c_0} |h_{0,t_0}(x)|^{\alpha(t_0)} dx \leq \frac{\epsilon}{2}$ . Since  $\forall x \in [-c_0, c_0]$ ,  $\lim_{j \rightarrow +\infty} |h_{0,t_0}(x)h_{j,t_0}(x)|^{\frac{\alpha(t_0)}{2}} = 0$  and  $(h_{j,t_0}(x))_j$  is uniformly bounded on  $[-c_0, c_0]$ ,

$$\lim_{j \rightarrow +\infty} \int_{|x| \leq c_0} |h_{0,t_0}(x)h_{j,t_0}(x)|^{\frac{\alpha(t_0)}{2}} dx = 0 \quad \blacksquare$$

We show on Figure 4 some paths of Lmmm, with the two corresponding estimations of  $\alpha$  and  $H$ . To simulate the trajectories, we have used the field (4.5). All the increments of  $X(\cdot, u)$  are  $(H(u), \alpha(u))$ -linear fractional stable motions, generated using the LFSN program of [15]. After we have taken the diagonal process  $X(t, t)$ .

These estimates are overall further than the estimates in the case of the Levy process, because of greater correlations between the increments of the process. However, the estimation of  $H$  does not seem to be disturbed by those correlations. The shape of the function  $H$  is kept. For  $\alpha$ , we notice some disruptions when the function is close to 1. We finally show an example where the estimation of  $\alpha$  is not good enough in the last line of Figure 4. The trajectory, Figure 4.a), seems to have a big jump, which leads to decrease the estimator  $\hat{\alpha}$ , represented on Figure 4.b), while the jump is taken account in the  $n(N)$  points. The estimation of  $H$ , represented on Figure 4.c), does not seem to be affected by this phenomenon.

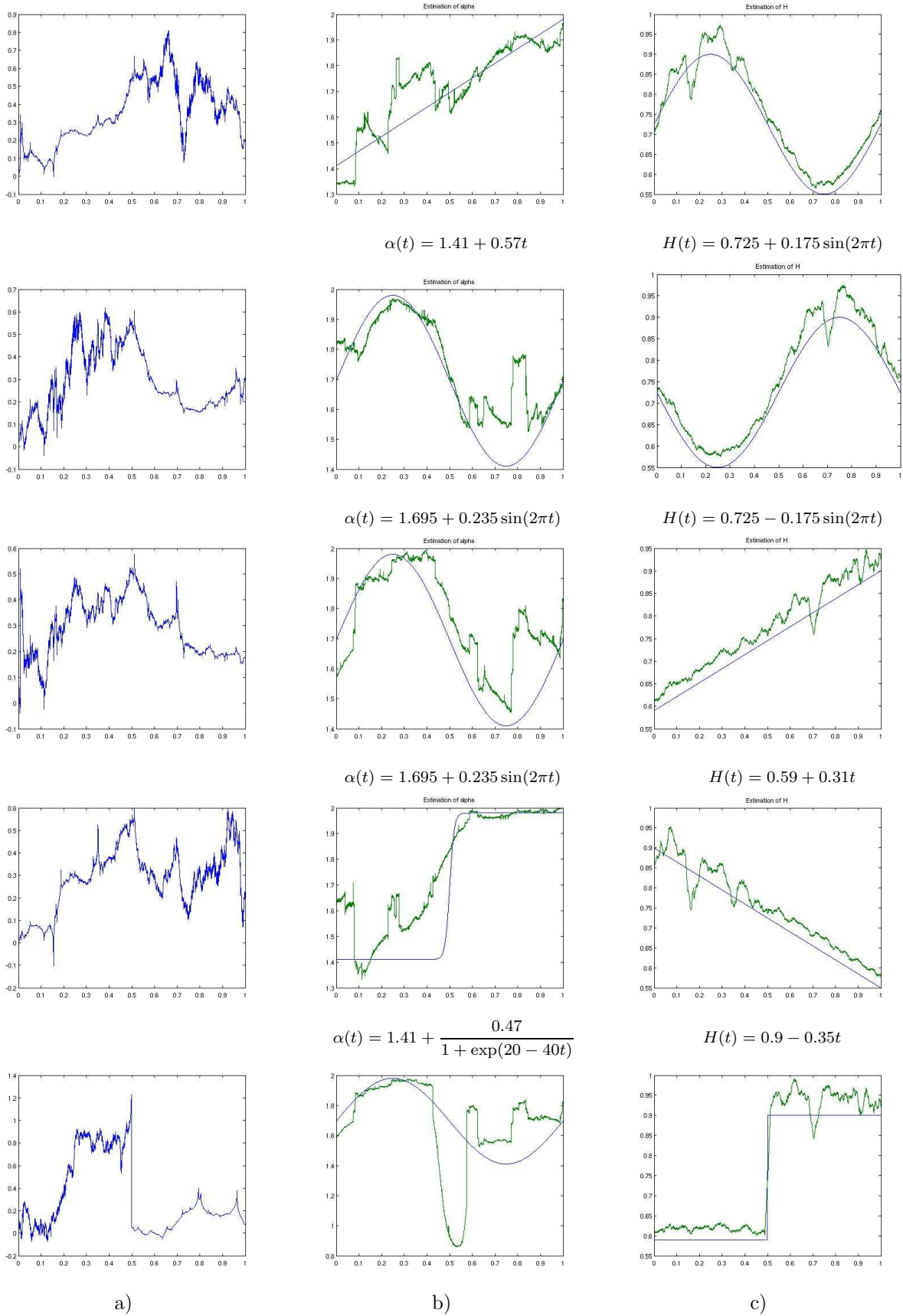


Figure 4: Trajectories with  $N = 20000$  in the first column, the estimations of  $\alpha$  with  $n(N) = 3000$  points in the second column, and in the last one, the estimations of  $H$  with  $n(N) = 500$  points.

## 5 Proofs

### Proof of Theorem 3.1

Note that it is sufficient to prove the result of Theorem 3.1 for  $r \geq 1$  since the convergence in  $L^p$  implies the convergence in  $L^q$  for all  $q < p$ . Let  $r \geq 1$ . Let  $H$  satisfying the condition **(C5)**. We write

$$\begin{aligned}\hat{H}_N(t_0) - H(t_0) &= -\frac{1}{n(N) \log N} \sum_{k=[Nt_0] - \frac{n(N)}{2}}^{[Nt_0] + \frac{n(N)}{2} - 1} \log \left| \frac{Y_{k,N}}{(\frac{1}{N})^{H(t_0)}} \right| \\ &= -\frac{N}{n(N) \log N} \int_{\frac{[Nt_0]}{N} - \frac{n(N)}{2N}}^{\frac{[Nt_0]}{N} + \frac{n(N)}{2N}} \log \left| \frac{Y(\frac{[Nt]+1}{N}) - Y(\frac{[Nt]}{N})}{(\frac{1}{N})^{H(t)}} \right| dt.\end{aligned}$$

Let  $\delta_N(dt) = \frac{N}{n(N)} \mathbf{1}_{\{\frac{[Nt_0]}{N} - \frac{n(N)}{2N} \leq t < \frac{[Nt_0]}{N} + \frac{n(N)}{2N}\}} dt$  and  $f_N(t) = \log \left| \frac{Y(\frac{[Nt]+1}{N}) - Y(\frac{[Nt]}{N})}{(\frac{1}{N})^{H(t)}} \right|$ .

Since  $\int_0^1 \delta_N(dt) = 1$ , we obtain

$$\hat{H}_N(t_0) - H(t_0) = -\frac{1}{\log N} \int_0^1 f_N(t) \delta_N(dt) + \int_0^1 (H(t) - H(t_0)) \delta_N(dt).$$

Then, there exists a constant  $K_r \in \mathbf{R}$  depending on  $r$  such that

$$\mathbf{E} \left[ |\hat{H}_N(t_0) - H(t_0)|^r \right] \leq K_r \frac{\mathbf{E} \left( \left| \int_0^1 f_N(t) \delta_N(dt) \right|^r \right)}{|\log N|^r} + K_r \left| \int_0^1 (H(t) - H(t_0)) \delta_N(dt) \right|^r.$$

$H$  is continuously differentiable and  $\lim_{N \rightarrow +\infty} \frac{N}{n(N)} = +\infty$  so

$$\lim_{N \rightarrow +\infty} \int_0^1 (H(t) - H(t_0)) \delta_N(dt) = 0.$$

To conclude, it is sufficient to show that there exists a constant  $K \in \mathbf{R}$  depending on  $t_0$  and  $r$  such that for all  $N \in \mathbb{N}$ ,  $\mathbf{E} \left( \left| \int_0^1 f_N(t) \delta_N(dt) \right|^r \right) \leq K$ . Let  $U$  an open interval satisfying all the conditions **(C1)**-**(C14)**, and  $t_0 \in U$ . We can fix  $N_0 \in \mathbb{N}$  and  $V \subset U$  an open interval depending on  $t_0$  such that for all  $N \geq N_0$  and all  $t \in V$ ,  $\frac{[Nt]+1}{N} \in U$ ,  $\frac{[Nt]}{N} \in U$  and  $\int_0^1 f_N(t) \delta_N(dt) = \int_V f_N(t) \delta_N(dt)$ . With the Jensen inequality,

$$\mathbf{E} \left( \left| \int_0^1 f_N(t) \delta_N(dt) \right|^r \right) \leq \int_V \mathbf{E} |f_N(t)|^r \delta_N(dt).$$

We consider  $\mathbf{E} |f_N(t)|^r = \int_0^{+\infty} \mathbf{P} (|f_N(t)|^r > x) dx$ .

$$\begin{aligned}
\mathbb{E}|f_N(t)|^r &= \int_0^{+\infty} \mathbb{P}(f_N(t) > x^{1/r}) dx + \int_0^{+\infty} \mathbb{P}(f_N(t) < -x^{1/r}) dx \\
&= \int_0^{+\infty} \mathbb{P}\left(\left|Y\left(\frac{[Nt]+1}{N}\right) - Y\left(\frac{[Nt]}{N}\right)\right| > \frac{e^{x^{1/r}}}{NH(t)}\right) dx \\
&\quad + \int_0^{+\infty} \mathbb{P}\left(\left|Y\left(\frac{[Nt]+1}{N}\right) - Y\left(\frac{[Nt]}{N}\right)\right| < \frac{e^{-x^{1/r}}}{NH(t)}\right) dx.
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}\left(\left|Y\left(\frac{[Nt]+1}{N}\right) - Y\left(\frac{[Nt]}{N}\right)\right| > \frac{e^{x^{1/r}}}{NH(t)}\right) &\leq \mathbb{P}\left(\left|Y\left(\frac{[Nt]+1}{N}\right) - Y(t)\right| \geq \frac{e^{x^{1/r}}}{2NH(t)}\right) \\
&\quad + \mathbb{P}\left(\left|Y(t) - Y\left(\frac{[Nt]}{N}\right)\right| \geq \frac{e^{x^{1/r}}}{2NH(t)}\right)
\end{aligned}$$

so

$$\mathbb{E}|f_N(t)|^r \leq \mathbf{I}_N^1(t) + \mathbf{I}_N^2(t) + \mathbf{I}_N^3(t)$$

with

$$\mathbf{I}_N^1(t) = \int_0^{+\infty} \mathbb{P}\left(\left|Y\left(\frac{[Nt]+1}{N}\right) - Y(t)\right| \geq \frac{e^{x^{1/r}}}{2NH(t)}\right) dx,$$

$$\mathbf{I}_N^2(t) = \int_0^{+\infty} \mathbb{P}\left(\left|Y\left(\frac{[Nt]}{N}\right) - Y(t)\right| \geq \frac{e^{x^{1/r}}}{2NH(t)}\right) dx$$

and

$$\mathbf{I}_N^3(t) = \int_0^{+\infty} \mathbb{P}\left(\left|Y\left(\frac{[Nt]+1}{N}\right) - Y\left(\frac{[Nt]}{N}\right)\right| < \frac{e^{-x^{1/r}}}{NH(t)}\right) dx.$$

We consider first  $\mathbf{I}_N^1(t)$ .

$$\begin{aligned}
\mathbf{I}_N^1(t) &\leq \int_0^{+\infty} \mathbb{P}\left(\left|X\left(\frac{[Nt]+1}{N}, \frac{[Nt]+1}{N}\right) - X\left(\frac{[Nt]+1}{N}, t\right)\right| \geq \frac{e^{x^{1/r}}}{4NH(t)}\right) dx \\
&\quad + \int_0^{+\infty} \mathbb{P}\left(\left|X\left(\frac{[Nt]+1}{N}, t\right) - X(t, t)\right| \geq \frac{e^{x^{1/r}}}{4NH(t)}\right) dx.
\end{aligned}$$

With the conditions **(C1)**, **(C2)** and **(C3)** (or **(C1)**, **(Cs2)**, **(Cs3)** and **(Cs4)** in the  $\sigma$ -finite space case) we can apply Proposition 4.9 or 4.10 of [9] : there exists  $K_U > 0$  such that for all  $(u, v) \in U^2$  and  $x > 0$ ,

$$\mathbf{P}(|X(v, v) - X(v, u)| > x) \leq K_U \left( \frac{|v - u|^d}{x^d} (1 + |\log \frac{|v - u|}{x}|^d) + \frac{|v - u|^c}{x^c} (1 + |\log \frac{|v - u|}{x}|^c) \right) \quad (5.6)$$

so there exists  $K_U > 0$  such that for all  $N \geq N_0$  and all  $t \in V$ ,

$$\begin{aligned} & \mathbf{P} \left( \left| X\left(\frac{[Nt] + 1}{N}, \frac{[Nt] + 1}{N}\right) - X\left(\frac{[Nt] + 1}{N}, t\right) \right| \geq \frac{e^{x^{1/r}}}{4N^{H(t)}} \right) \\ & \leq K_U \left( \frac{(\log N)^c}{N^{c(1-H(t))} e^{cx^{1/r}}} \right) + K_U \left( \frac{x^{c/r}}{N^{c(1-H(t))} e^{cx^{1/r}}} \right) \\ & + K_U \left( \frac{(\log N)^d}{N^{d(1-H(t))} e^{dx^{1/r}}} \right) + K_U \left( \frac{x^{d/r}}{N^{d(1-H(t))} e^{dx^{1/r}}} \right). \end{aligned}$$

Since  $H_+ < 1$ , we conclude that

$$\lim_{N \rightarrow +\infty} \int_0^{+\infty} \sup_{t \in U} \mathbf{P} \left( \left| X\left(\frac{[Nt] + 1}{N}, \frac{[Nt] + 1}{N}\right) - X\left(\frac{[Nt] + 1}{N}, t\right) \right| \geq \frac{e^{x^{1/r}}}{4N^{H(t)}} \right) dx = 0.$$

With the same arguments,

$$\lim_{N \rightarrow +\infty} \int_0^{+\infty} \sup_{t \in U} \mathbf{P} \left( \left| X\left(\frac{[Nt]}{N}, \frac{[Nt]}{N}\right) - X\left(\frac{[Nt]}{N}, t\right) \right| \geq \frac{e^{x^{1/r}}}{4N^{H(t)}} \right) dx = 0. \quad (5.7)$$

Let  $\eta < c$ . The Markov inequality gives

$$\mathbf{P} \left( \left| X\left(\frac{[Nt] + 1}{N}, t\right) - X(t, t) \right| \geq \frac{e^{x^{1/r}}}{4N^{H(t)}} \right) \leq \frac{4^\eta N^{\eta H(t)}}{e^{\eta x^{1/r}}} \mathbf{E} \left[ \left| X\left(\frac{[Nt] + 1}{N}, t\right) - X(t, t) \right|^\eta \right]$$

and Property 1.2.17 of [14]

$$\mathbf{E} \left[ \left| X\left(\frac{[Nt] + 1}{N}, t\right) - X(t, t) \right|^\eta \right] = c_{\alpha(t), 0}(\eta)^\eta \left( \int_E (|f\left(\frac{[Nt] + 1}{N}, t, x\right) - f(t, t, x)|^{\alpha(t)}) m(dx) \right)^{\eta/\alpha(t)}.$$

With the condition **(C9)**, there exists  $K > 0$  such that for all  $N \geq N_0$  and all  $t \in V$ ,

$$\int_0^{+\infty} \mathbf{P} \left( \left| X\left(\frac{[Nt] + 1}{N}, t\right) - X(t, t) \right| \geq \frac{e^{x^{1/r}}}{4N^{H(t)}} \right) dx \leq K.$$

Finally, there exists  $K > 0$  such that for all  $N \geq N_0$  and all  $t \in V$ ,

$$\mathbf{I}_N^1(t) \leq K.$$

Using the equation (5.7) and the condition (C9), we obtain that there exists  $K > 0$  such that for all  $N \geq N_0$  and all  $t \in V$ ,

$$\mathbf{I}_N^2(t) \leq K.$$

Thanks to the conditions (C1), (C6), (C7), (C8), (C10), (C11), (C12), (C13) and (C14), we conclude for  $\mathbf{I}_N^3(t)$  using Proposition 4.11 and 4.8 of [9] : there exists  $K > 0$  such that for all  $N \geq N_0$  and all  $t \in V$ ,

$$\mathbf{P} \left( \left| Y\left(\frac{[Nt] + 1}{N}\right) - Y\left(\frac{[Nt]}{N}\right) \right| < \frac{e^{-x^{1/r}}}{N^{H(t)}} \right) \leq K \frac{e^{-x^{1/r}}}{N^{H(t)}} N^{H(\frac{[Nt]}{N})}$$

so there exists  $K > 0$  such that for all  $N \geq N_0$  and all  $t \in V$ ,

$$\mathbf{I}_N^3(t) \leq K \quad \blacksquare$$

### Proof of Theorem 3.2

Let  $p \in [p_0, \alpha(t_0))$ . We define

$$A_N(p) = \frac{N^{pH(t_0)}}{n(N)} \sum_{k=[Nt_0] - \frac{n(N)}{2}}^{[Nt_0] + \frac{n(N)}{2} - 1} \left| X\left(\frac{k+1}{N}, \frac{k+1}{N}\right) - X\left(\frac{k+1}{N}, t_0\right) \right|^p,$$

$$B_N(p) = \frac{N^{pH(t_0)}}{n(N)} \sum_{k=[Nt_0] - \frac{n(N)}{2}}^{[Nt_0] + \frac{n(N)}{2} - 1} \left| X\left(\frac{k}{N}, \frac{k}{N}\right) - X\left(\frac{k}{N}, t_0\right) \right|^p$$

and

$$C_N(p) = \frac{N^{pH(t_0)}}{n(N)} \sum_{k=[Nt_0] - \frac{n(N)}{2}}^{[Nt_0] + \frac{n(N)}{2} - 1} \left| X\left(\frac{k+1}{N}, t_0\right) - X\left(\frac{k}{N}, t_0\right) \right|^p.$$

We have, for  $p \leq 1$ ,

$$\begin{aligned} \mathbf{P} \left( |N^{pH(t_0)} S_N^p(p) - \mathbf{E}|Z|^p| > x \right) &\leq \mathbf{P} \left( |N^{pH(t_0)} S_N^p(p) - C_N(p)| \geq \frac{x}{2} \right) + \mathbf{P} \left( |\mathbf{E}|Z|^p - C_N(p)| \geq \frac{x}{2} \right) \\ &\leq \mathbf{P} \left( |\mathbf{E}|Z|^p - C_N(p)| \geq \frac{x}{2} \right) + \mathbf{P} \left( A_N(p) + B_N(p) \geq \frac{x}{2} \right) \end{aligned}$$

and for  $p \geq 1$ ,

$$\begin{aligned} \mathbf{P} \left( |N^{H(t_0)} S_N(p) - (\mathbf{E}|Z|^p)^{\frac{1}{p}}| > x \right) &\leq \mathbf{P} \left( |N^{H(t_0)} S_N(p) - C_N^{\frac{1}{p}}(p)| \geq \frac{x}{2} \right) \\ &\quad + \mathbf{P} \left( |C_N^{\frac{1}{p}}(p) - (\mathbf{E}|Z|^p)^{\frac{1}{p}}| \geq \frac{x}{2} \right) \\ &\leq \mathbf{P} \left( |(\mathbf{E}|Z|^p)^{\frac{1}{p}} - C_N^{\frac{1}{p}}(p)| \geq \frac{x}{2} \right) + \mathbf{P} \left( A_N^{\frac{1}{p}}(p) + B_N^{\frac{1}{p}}(p) \geq \frac{x}{2} \right). \end{aligned}$$

To prove Theorem 3.2, it is enough to show that  $A_N(p) \xrightarrow{\mathbf{P}} 0$ ,  $B_N(p) \xrightarrow{\mathbf{P}} 0$  and  $C_N(p) \xrightarrow{\mathbf{P}} \mathbf{E}|Z|^p$ , with  $Z \sim S_{\alpha(t_0)}(1, 0, 0)$ .

We consider first  $A_N(p) \xrightarrow{P} 0$ . Let  $\delta_N(dt) = \frac{N}{n(N)} \mathbf{1}_{\{\frac{[Nt_0]}{N} - \frac{n(N)}{2N} \leq t < \frac{[Nt_0]}{N} + \frac{n(N)}{2N}\}} dt$ . Let  $U$  an open interval satisfying the conditions of the theorem and  $t_0 \in U$ . We can fix  $N_0 \in \mathbb{N}$  and  $V \subset U$  an open interval depending on  $t_0$  such that for all  $N \geq N_0$  and all  $t \in V$ ,  $\frac{[Nt]+1}{N} \in U$ ,  $\frac{[Nt]}{N} \in U$ ,  $\int_0^1 \delta_N(dt) = \int_V \delta_N(dt)$ , and such that the inequality (5.6) holds.

$$\begin{aligned} \mathbf{P}(A_N(p) > x) &= \mathbf{P}\left(\int_0^1 \left| \frac{X(\frac{[Nt]+1}{N}, \frac{[Nt]+1}{N}) - X(\frac{[Nt]+1}{N}, t_0)}{(1/N)^{H(t_0)}} \right|^p \delta_N(dt) > x\right) \\ &\leq \frac{1}{x} \int_0^1 \mathbf{E} \left[ \left| \frac{X(\frac{[Nt]+1}{N}, \frac{[Nt]+1}{N}) - X(\frac{[Nt]+1}{N}, t_0)}{(1/N)^{H(t_0)}} \right|^p \right] \delta_N(dt) \\ &= \frac{1}{x} \int_V \mathbf{E} \left[ \left| \frac{X(\frac{[Nt]+1}{N}, \frac{[Nt]+1}{N}) - X(\frac{[Nt]+1}{N}, t_0)}{(1/N)^{H(t_0)}} \right|^p \right] \delta_N(dt) \end{aligned}$$

Let  $t \in V$ .

$$\mathbf{E} \left[ \left| \frac{X(\frac{[Nt]+1}{N}, \frac{[Nt]+1}{N}) - X(\frac{[Nt]+1}{N}, t_0)}{(1/N)^{H(t_0)}} \right|^p \right] = \int_0^\infty \mathbf{P} \left( \left| \frac{X(\frac{[Nt]+1}{N}, \frac{[Nt]+1}{N}) - X(\frac{[Nt]+1}{N}, t_0)}{(1/N)^{H(t_0)}} \right| > u^{1/p} \right) du.$$

Let  $u > 0$ . We know from (5.6) that there exists  $K_U > 0$  such that for all  $t \in V$ ,

$$\mathbf{P} \left( \left| \frac{X(\frac{[Nt]+1}{N}, \frac{[Nt]+1}{N}) - X(\frac{[Nt]+1}{N}, t_0)}{(1/N)^{H(t_0)}} \right| > u^{1/p} \right) \leq K_U \frac{((\log N)^c + |\log u|^c)}{N^{c(1-H(t_0))} u^{c/p}} + K_U \frac{((\log N)^d + |\log u|^d)}{N^{d(1-H(t_0))} u^{d/p}},$$

so, with the assumption  $H(t_0) < 1$ ,

$$\lim_{N \rightarrow +\infty} \mathbf{P} \left( \left| \frac{X(\frac{[Nt]+1}{N}, \frac{[Nt]+1}{N}) - X(\frac{[Nt]+1}{N}, t_0)}{(1/N)^{H(t_0)}} \right| > u^{1/p} \right) = 0.$$

There exists  $K_{U,p} > 0$  such that

$$\mathbf{P} \left( \left| \frac{X(\frac{[Nt]+1}{N}, \frac{[Nt]+1}{N}) - X(\frac{[Nt]+1}{N}, t_0)}{(1/N)^{H(t_0)}} \right| > u^{1/p} \right) \leq \mathbf{1}_{u < 1} + K_{U,p} \left( \frac{|\log u|^d}{u^{d/p}} + \frac{|\log u|^c}{u^{c/p}} \right) \mathbf{1}_{u \geq 1}. \quad (5.8)$$

Since  $\alpha$  is a continuous function, we can fix  $U$  small enough such that  $c = \inf_{t \in U} \alpha(t) > p$ . We deduce from the dominated convergence theorem that for all  $t \in U$ ,

$$\lim_{N \rightarrow +\infty} \mathbf{E} \left[ \left| \frac{X(\frac{[Nt]+1}{N}, \frac{[Nt]+1}{N}) - X(\frac{[Nt]+1}{N}, t_0)}{(1/N)^{H(t_0)}} \right|^p \right] = 0.$$

With the inequality (5.8),

$$\mathbf{E} \left[ \left| \frac{X(\frac{[Nt]+1}{N}, \frac{[Nt]+1}{N}) - X(\frac{[Nt]+1}{N}, t_0)}{(1/N)^{H(t_0)}} \right|^p \right] \leq 1 + \int_1^{+\infty} K_{U,p} \left( \frac{|\log u|^d}{u^{d/p}} + \frac{|\log u|^c}{u^{c/p}} \right) du$$



and again with the dominated convergence theorem,

$$\lim_{N \rightarrow +\infty} \mathbf{P}(A_N(p) > x) = 0.$$

The same inequalities holds with  $B_N(p)$  so we obtain  $B_N(p) \xrightarrow{\mathbf{P}} 0$ . We conclude proving  $C_N(p) \xrightarrow{\mathbf{P}} \mathbf{E}|Z|^p$ . Let  $c_0 > 0$ . We use the decomposition

$$\begin{aligned} C_N(p) - \mathbf{E}|Z|^p &= \frac{1}{n(N)} \sum_{k=[Nt_0] - \frac{n(N)}{2}}^{[Nt_0] + \frac{n(N)}{2} - 1} \left| \frac{X(\frac{k+1}{N}, t_0) - X(\frac{k}{N}, t_0)}{(1/N)^{H(t_0)}} \right|^p \mathbf{1}_{\left| \frac{X(\frac{k+1}{N}, t_0) - X(\frac{k}{N}, t_0)}{(1/N)^{H(t_0)}} \right| > c_0} - \mathbf{E}|Z|^p \mathbf{1}_{|Z| > c_0} \\ &+ \frac{1}{n(N)} \sum_{k=[Nt_0] - \frac{n(N)}{2}}^{[Nt_0] + \frac{n(N)}{2} - 1} \left| \frac{X(\frac{k+1}{N}, t_0) - X(\frac{k}{N}, t_0)}{(1/N)^{H(t_0)}} \right|^p \mathbf{1}_{\left| \frac{X(\frac{k+1}{N}, t_0) - X(\frac{k}{N}, t_0)}{(1/N)^{H(t_0)}} \right| \leq c_0} - \mathbf{E}|Z|^p \mathbf{1}_{|Z| \leq c_0}. \end{aligned}$$

Let  $\epsilon > 0$  and  $x > 0$ . By Markov's inequality, we have

$$\begin{aligned} \mathbf{P}_1 &= \mathbf{P} \left( \frac{1}{n(N)} \sum_{k=[Nt_0] - \frac{n(N)}{2}}^{[Nt_0] + \frac{n(N)}{2} - 1} \left| \frac{X(\frac{k+1}{N}, t_0) - X(\frac{k}{N}, t_0)}{(1/N)^{H(t_0)}} \right|^p \mathbf{1}_{\left| \frac{X(\frac{k+1}{N}, t_0) - X(\frac{k}{N}, t_0)}{(1/N)^{H(t_0)}} \right| > c_0} > \frac{x}{4} \right) \\ &\leq \frac{4}{xn(N)} \sum_{k=[Nt_0] - \frac{n(N)}{2}}^{[Nt_0] + \frac{n(N)}{2} - 1} \mathbf{E} \left[ \left| \frac{X(\frac{k+1}{N}, t_0) - X(\frac{k}{N}, t_0)}{(1/N)^{H(t_0)}} \right|^p \mathbf{1}_{\left| \frac{X(\frac{k+1}{N}, t_0) - X(\frac{k}{N}, t_0)}{(1/N)^{H(t_0)}} \right| > c_0} \right]. \end{aligned}$$

Since  $X(\cdot, t_0)$  is  $H(t_0)$ -self-similar with stationary increments,

$$\mathbf{P}_1 \leq \frac{4}{x} \mathbf{E} [ |X(1, t_0)|^p \mathbf{1}_{|X(1, t_0)| > c_0} ]$$

and

$$\mathbf{E}|Z|^p \mathbf{1}_{|Z| \leq c_0} = \frac{1}{n(N)} \sum_{k=[Nt_0] - \frac{n(N)}{2}}^{[Nt_0] + \frac{n(N)}{2} - 1} \mathbf{E} \left[ \left| \frac{X(\frac{k+1}{N}, t_0) - X(\frac{k}{N}, t_0)}{(1/N)^{H(t_0)}} \right|^p \mathbf{1}_{\left| \frac{X(\frac{k+1}{N}, t_0) - X(\frac{k}{N}, t_0)}{(1/N)^{H(t_0)}} \right| \leq c_0} \right].$$

We fix  $c_0$  large enough such that for all  $N \in \mathbb{N}$ ,  $\mathbf{P}_1 \leq \frac{\epsilon}{2}$  and  $\mathbf{E}|Z|^p \mathbf{1}_{|Z| > c_0} < \frac{\epsilon}{4}$ . Writing  $K(x) = |x|^p \mathbf{1}_{|x| \leq c_0}$  and  $\Delta X_{k, t_0} = X(k+1, t_0) - X(k, t_0)$ , using Chebyshev's inequality, we get

$$\begin{aligned} \mathbf{P}(|C_N(p) - \mathbf{E}|Z|^p| > x) &\leq \frac{\epsilon}{2} + \frac{4}{x^2 n(N)^2} \sum_{k, j=[Nt_0] - \frac{n(N)}{2}}^{[Nt_0] + \frac{n(N)}{2} - 1} \text{Cov}(K(\Delta X_{k, t_0}), K(\Delta X_{j, t_0})) \\ &\leq \frac{\epsilon}{2} + \frac{4}{x^2} \frac{\text{Var}(K(\Delta X_{0, t_0}))}{n(N)} + \frac{4}{x^2} \frac{1}{n(N)} \sum_{j=1}^{n(N)-1} \text{Cov}(K(\Delta X_{0, t_0}), K(\Delta X_{j, t_0})). \end{aligned}$$

Under the condition **(C\*)**, we can apply Theorem 2.1 of [12]: there exists a positive constant  $C$  such that

$$\text{Cov}(K(\Delta X_{0,t_0}), K(\Delta X_{j,t_0})) \leq C \|K\|_1^2 \int_E |h_{0,t_0}(x)h_{j,t_0}(x)|^{\frac{\alpha(t_0)}{2}} m(dx).$$

Since the process  $X(\cdot, t_0)$  is  $H(t_0)$ -self-similar with stationary increments, the constant  $C$  does not depend on  $k, j$ . We then obtain the existence of a positive constant  $C_{p,c_0}$  depending on  $p, c_0$  and  $x$  such that

$$\mathbb{P}(|C_N(p) - \mathbb{E}|Z|^p| > x) \leq \frac{\epsilon}{2} + \frac{C_{p,c_0}}{n(N)} \int_E |h_{0,t_0}(x)|^{\alpha(t_0)} m(dx) + \frac{C_{p,c_0}}{n(N)} \sum_{j=1}^{n(N)-1} \int_E |h_{0,t_0}(x)h_{j,t_0}(x)|^{\frac{\alpha(t_0)}{2}} m(dx).$$

Since  $\lim_{N \rightarrow +\infty} n(N) = +\infty$  and  $\lim_{j \rightarrow +\infty} \int_E |h_{0,t_0}(x)h_{j,t_0}(x)|^{\frac{\alpha(t_0)}{2}} m(dx) = 0$ , we conclude with Cesaro's theorem that there exists  $N_0 \in \mathbb{N}$  such that for all  $N \geq N_0$ ,

$$\frac{C_{p,c_0}}{n(N)} \int_E |h_{0,t_0}(x)|^{\alpha(t_0)} m(dx) + \frac{C_{p,c_0}}{n(N)} \sum_{j=1}^{n(N)-1} \int_E |h_{0,t_0}(x)h_{j,t_0}(x)|^{\frac{\alpha(t_0)}{2}} m(dx) \leq \frac{\epsilon}{2}$$

and

$$\mathbb{P}(|C_N(p) - \mathbb{E}|Z|^p| > x) \leq \epsilon \quad \blacksquare$$

### Proof of Theorem 3.3

Since  $x \rightarrow x^\gamma$  is an increasing function on  $\mathbf{R}_+$ ,

$$\hat{\alpha}_N(t_0) = \arg \min_{\alpha \in [0,2]} \int_{p_0}^2 |R_{\text{exp}}(p) - R_\alpha(p)|^\gamma dp.$$

Let  $g_N(\alpha) = \int_{p_0}^2 |R_{\text{exp}}(p) - R_\alpha(p)|^\gamma dp$  and  $g(\alpha) = \int_{p_0}^2 |R_{\alpha(t_0)}(p) - R_\alpha(p)|^\gamma dp$ .

$g$  is a continuous function on  $(0, 2]$ , with  $g(0) > 0, g(2) > 0$ . The only solution of the equation  $g(\alpha) = 0$  is  $\alpha(t_0)$ . Moreover,  $\lim_{\alpha \rightarrow \alpha(t_0)} \frac{|g(\alpha) - g(\alpha(t_0))|}{|\alpha - \alpha(t_0)|^\gamma} > 0$ .

Then, there exists  $K_{\alpha(t_0)}$  a positive constant depending only on  $\alpha(t_0)$  such that:

$$\forall \alpha \in (0, 2), |g(\alpha)| \geq K_{\alpha(t_0)} |\alpha - \alpha(t_0)|. \quad (5.9)$$

We estimate now  $|g(\hat{\alpha}_N(t_0))|$ .

$$\begin{aligned} |g(\hat{\alpha}_N(t_0))| &\leq |g(\hat{\alpha}_N(t_0)) - g_N(\hat{\alpha}_N(t_0))| + |g_N(\hat{\alpha}_N(t_0))| \\ &\leq |g(\hat{\alpha}_N(t_0)) - g_N(\hat{\alpha}_N(t_0))| + g_N(\alpha(t_0)), \end{aligned}$$

and

$$\begin{aligned} |g(\hat{\alpha}_N(t_0)) - g_N(\hat{\alpha}_N(t_0))| &= \left| \int_{p_0}^2 (|R_{\alpha(t_0)}(p) - R_{\hat{\alpha}_N(t_0)}(p)|^\gamma - |R_{\text{exp}}(p) - R_{\hat{\alpha}_N(t_0)}(p)|^\gamma) dp \right| \\ &\leq \int_{p_0}^2 |R_{\alpha(t_0)}(p) - R_{\text{exp}}(p)|^\gamma dp \\ &= g_N(\alpha(t_0)). \end{aligned}$$

From (5.9),

$$\begin{aligned} |\hat{\alpha}_N(t_0) - \alpha(t_0)| &\leq \frac{1}{K_{\alpha(t_0)}} g(\hat{\alpha}_N(t_0)) \\ &\leq \frac{2}{K_{\alpha(t_0)}} g_N(\alpha(t_0)). \end{aligned}$$

Let us show that  $\forall r > 0$ ,  $\lim_{N \rightarrow +\infty} \mathbb{E} |g_N(\alpha(t_0))|^r = 0$ . Let  $r > 0$ . One has, using the inequality  $S_N(p) \leq S_N(q)$  for  $p \leq q$ ,

$$\begin{aligned} g_N(\alpha(t_0)) &= \int_{p_0}^{\alpha(t_0)} |R_{\text{exp}}(p) - R_{\alpha(t_0)}(p)|^\gamma dp + \int_{\alpha(t_0)}^2 |R_{\text{exp}}(p)|^\gamma dp \\ &\leq \int_{p_0}^{\alpha(t_0)} |R_{\text{exp}}(p) - R_{\alpha(t_0)}(p)|^\gamma dp + (2 - \alpha(t_0)) \left| \frac{S_N(p_0)}{S_N(\alpha(t_0))} \right|^\gamma. \end{aligned}$$

For the first term, we use Theorem 3.2 : for all  $p \in [p_0, \alpha(t_0))$ ,

$$N^{H(t_0)} S_N(p) \xrightarrow{\mathbb{P}} (\mathbb{E}|Z|^p)^{1/p} \quad (5.10)$$

where  $Z \sim S_{\alpha(t_0)}(1, 0, 0)$ . It is clear that  $\forall p \in [p_0, \alpha(t_0))$ ,

$$(N^{H(t_0)} S_N(p_0), N^{H(t_0)} S_N(p)) \xrightarrow{\mathbb{P}} ((\mathbb{E}|Z|^{p_0})^{1/p_0}, (\mathbb{E}|Z|^p)^{1/p}),$$

and

$$R_{\text{exp}}(p) = \frac{S_N(p_0)}{S_N(p)} \xrightarrow{\mathbb{P}} R_{\alpha(t_0)}(p). \quad (5.11)$$

Note that  $\forall N \in \mathbb{N}$ ,  $\forall p \in [p_0, \alpha(t_0))$ ,  $|R_{\text{exp}}(p)| \leq 1$  so there exists a positive constant  $K$  depending on  $\gamma r$ ,  $\alpha(t_0)$  and  $p$  such that

$$\mathbb{E} |R_{\text{exp}}(p) - R_{\alpha(t_0)}(p)|^{\gamma r} = \int_0^K \mathbb{P} (|R_{\text{exp}}(p) - R_{\alpha(t_0)}(p)|^{\gamma r} > x) dx.$$

Finally, with (5.11),  $\forall p \in [p_0, \alpha(t_0))$ ,  $\mathbb{E} |R_{\text{exp}}(p) - R_{\alpha(t_0)}(p)|^{\gamma r} \xrightarrow{N \rightarrow +\infty} 0$ . With the inequality  $\mathbb{E} |R_{\text{exp}}(p) - R_{\alpha(t_0)}(p)|^{\gamma r} \leq 2C_{\gamma r}$  where  $C_{\gamma r}$  is a positive constant depending on  $\gamma r$ , by the dominating convergence theorem,

$$\lim_{N \rightarrow +\infty} \int_{p_0}^{\alpha(t_0)} \mathbb{E} |R_{\text{exp}}(p) - R_{\alpha(t_0)}(p)|^{\gamma r} dp = 0.$$

To conclude we show that  $\left| \frac{S_N(p_0)}{S_N(\alpha(t_0))} \right|^\gamma \xrightarrow{L^r} 0$ . Since  $\forall N \in \mathbb{N}$ ,  $\left| \frac{S_N(p_0)}{S_N(\alpha(t_0))} \right|^\gamma \leq 1$ , it is enough to show  $\frac{S_N(p_0)}{S_N(\alpha(t_0))} \xrightarrow{\mathbb{P}} 0$ . Let  $p < \alpha(t_0)$ .

$$\mathbb{P} \left( \frac{1}{|N^{H(t_0)} S_N(\alpha(t_0))|} > x \right) \leq \mathbb{P} \left( \frac{1}{|N^{H(t_0)} S_N(p)|} > x \right).$$

So,

$$\begin{aligned}
\limsup_{N \rightarrow +\infty} \mathbb{P}\left(\frac{1}{|N^{H(t_0)} S_N(\alpha(t_0))|} > x\right) &\leq \limsup_{N \rightarrow +\infty} \mathbb{P}\left(\frac{1}{|N^{H(t_0)} S_N(p)|} > x\right) \\
&= \lim_{N \rightarrow +\infty} \mathbb{P}\left(\frac{1}{|N^{H(t_0)} S_N(p)|} > x\right) \\
&= \mathbb{P}\left(\frac{1}{(\mathbb{E}|Z|^p)^{1/p}} > x\right),
\end{aligned}$$

with (5.10). Since  $\lim_{p \rightarrow \alpha(t_0)} \mathbb{P}\left(\frac{1}{(\mathbb{E}|Z|^p)^{1/p}} > x\right) = 0$ , we have  $\limsup_{N \rightarrow +\infty} \mathbb{P}\left(\frac{1}{|N^{H(t_0)} S_N(\alpha(t_0))|} > x\right) = 0$  and  $\frac{1}{N^{H(t_0)} S_N(\alpha(t_0))} \xrightarrow{\mathbb{P}} 0$ . Using the convergence  $N^{H(t_0)} S_N(p_0) \xrightarrow{\mathbb{P}} (\mathbb{E}|Z|^{p_0})^{1/p_0}$ , we obtain  $\frac{S_N(p_0)}{S_N(\alpha(t_0))} \xrightarrow{\mathbb{P}} 0$  ■

## 6 Assumptions

This section gathers the various conditions required on the considered processes so that our results hold.

- (C1) The family of functions  $v \rightarrow f(t, v, x)$  is differentiable for all  $(v, t)$  in  $U^2$  and almost all  $x$  in  $E$ . The derivatives of  $f$  with respect to  $v$  are denoted by  $f'_v$ .
- (C2) There exists  $\delta > \frac{d}{c} - 1$  such that :

$$\sup_{t \in U} \int_{\mathbf{R}} \left[ \sup_{w \in U} (|f(t, w, x)|^{\alpha(w)}) \right]^{1+\delta} \hat{m}(dx) < \infty.$$

- (Cs2) There exists  $\delta > \frac{d}{c} - 1$  such that :

$$\sup_{t \in U} \int_{\mathbf{R}} \left[ \sup_{w \in U} (|f(t, w, x)|^{\alpha(w)}) \right]^{1+\delta} r(x)^\delta m(dx) < \infty.$$

- (C3) There exists  $\delta > \frac{d}{c} - 1$  such that :

$$\sup_{t \in U} \int_{\mathbf{R}} \left[ \sup_{w \in U} (|f'_v(t, w, x)|^{\alpha(w)}) \right]^{1+\delta} \hat{m}(dx) < \infty.$$

- (Cs3) There exists  $\delta > \frac{d}{c} - 1$  such that :

$$\sup_{t \in U} \int_{\mathbf{R}} \left[ \sup_{w \in U} (|f'_v(t, w, x)|^{\alpha(w)}) \right]^{1+\delta} r(x)^\delta m(dx) < \infty.$$

- (Cs4) There exists  $\delta > \frac{d}{c} - 1$  such that :

$$\sup_{t \in U} \int_{\mathbf{R}} \left[ \sup_{w \in U} \left[ |f(t, w, x) \log(r(x))|^{\alpha(w)} \right] \right]^{1+\delta} r(x)^\delta m(dx) < \infty.$$

- (C5)  $X(t, u)$  (as a process in  $t$ ) is localisable at  $u$  with exponent  $H(u) \in (H_-, H_+) \subset (0, 1)$ , with local form  $X'_u(t, u)$ , and  $u \mapsto H(u)$  is a  $C^1$  function .

- (C6) There exists  $K_U > 0$  such that  $\forall v \in U, \forall u \in U, \forall x \in \mathbf{R}$ ,

$$|f(v, u, x)| \leq K_U.$$

- (C7) There exists  $K_U > 0$  such that  $\forall v \in U, \forall u \in U, \forall x \in \mathbf{R}$ ,

$$|f'_v(v, u, x)| \leq K_U.$$

- (C8) There exists  $K_U > 0$  and a function  $H$  defined on  $U$  such that  $\forall v \in U, \forall u \in U, \forall x \in \mathbf{R}$ ,

$$\frac{1}{|v - u|^{H(u)-1/\alpha(u)}} |f(v, u, x) - f(u, u, x)| \leq K_U.$$

- (C9) There exists  $\varepsilon_0 > 0, K_U > 0$  and a function  $H$  defined on  $U$  such that  $\forall r < \varepsilon_0, \forall t \in U$ ,

$$\frac{1}{r^{H(t)\alpha(t)}} \int_{\mathbf{R}} |f(t+r, t, x) - f(t, t, x)|^{\alpha(t)} m(dx) \leq K_U.$$

- (C10) There exists  $p \in (d, 2), p \geq 1, K_U > 0$  and a function  $H$  defined on  $U$  such that  $\forall v \in U, \forall u \in U$ ,

$$\frac{1}{|v - u|^{1+p(H(u)-\frac{1}{\alpha(u)})}} \int_{\mathbf{R}} |f(v, u, x) - f(u, u, x)|^p m(dx) \leq K_U.$$

- (C11) There exists  $K_U > 0$  such that  $\forall v \in U, \forall u \in U$ ,

$$\int_{\mathbf{R}} |f(v, u, x)|^2 m(dx) \leq K_U.$$

- (C12)

$$\inf_{v \in U} \int_{\mathbf{R}} f(v, v, x)^2 m(dx) > 0.$$

- (C13) There exists a positive function  $g$  and a function  $H$  defined on  $U$  such that

$$\limsup_{r \rightarrow 0} \sup_{t \in U} \left| \frac{1}{r^{1+2(H(t)-1/\alpha(t))}} \int_{\mathbf{R}} (f(t+r, t, x) - f(t, t, x))^2 m(dx) - g(t) \right| = 0.$$

- (C14)  $\exists K_U > 0$  such that,  $\forall v \in U, \forall u \in U$ ,

$$\frac{1}{|v - u|^2} \int_{\mathbf{R}} |f(v, v, x) - f(v, u, x)|^2 m(dx) \leq K_U.$$

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