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LIFSHITZ TAILS FOR ALLOY TYPE MODELS IN A CONSTANT MAGNETIC FIELD

FRÉDÉRIC KLOPP

Dedicated to the memory of Pierre Duclos.

ABSTRACT. In this note, we study Lifshitz tails for a 2D Landau Hamiltonian perturbed by a random alloy-type potential constructed with single site potentials decaying at least at a Gaussian speed. We prove that, if the Landau level stays preserved as a band edge for the perturbed Hamiltonian, at the Landau levels, the integrated density of states has a Lifshitz behavior of the type $e^{-\log^2 |E-2bq|}$.

RÉSUMÉ. Dans cette note, nous démontrons qu'en dimension 2, la densité d'états intégrée d'un opérateur de Landau avec un potentiel aléatoire non négatif de type Anderson dont le potentiel de simple site décroît au moins aussi vite qu'une fonction gaussienne admet en chaque niveau de Landau, disons, $2bq$, si celui-ci est un bord du spectre, une asymptotique de Lifshitz du type $e^{-\log^2 |E-2bq|}$.

0. INTRODUCTION

Consider the Landau Hamiltonian

$$H_0 = H_0(b) := (-i\nabla - A)^2 - b.$$

It is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$. Here $A = (-\frac{bx_2}{2}, \frac{bx_1}{2})$ is the magnetic potential, and $b > 0$ is the constant scalar magnetic field. It is well-known that the spectrum $\sigma(H_0)$ of the operator $H_0(b)$ consists of the so-called Landau levels $2bq$, $q \in \mathbb{Z}_+$, and each Landau level is an eigenvalue of infinite multiplicity.

Consider now a random \mathbb{Z}^2 -ergodic alloy-type electric potential

$$V(x) = V_\omega(x) := \sum_{\gamma \in \mathbb{Z}^2} \omega_\gamma u(x - \gamma), \quad x \in \mathbb{R}^2.$$

We assume that

- **H₁**: The single-site potential u satisfies the estimates

$$\frac{1}{C} \mathbf{1}_{\{x \in \mathbb{R}^2; |x-x_0| < 1/C\}} \leq u(x) \leq C e^{-|x|^2/C}$$

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for some $C > 0$, $x_0 \in \mathbb{R}^2$.

- **H₂**: The coupling constants $\{\omega_\gamma\}_{\gamma \in \mathbb{Z}^2}$ are non-trivial, almost surely bounded i.i.d. random variables.

These two assumptions guarantee V_ω is almost surely bounded. On the domain of H_0 , define the operator $H = H_\omega := H_0(b) + V_\omega$. The integrated density of states (IDS) for the operator H is defined as a non-decreasing left-continuous function $\mathcal{N}_b : \mathbb{R} \rightarrow [0, \infty)$ which, almost surely, satisfies

$$\int_{\mathbb{R}} \varphi(E) d\mathcal{N}_b(E) = \lim_{R \rightarrow \infty} R^{-2} \text{Tr} (\mathbf{1}_{\Lambda_R} \varphi(H) \mathbf{1}_{\Lambda_R}), \quad \forall \varphi \in C_0^\infty(\mathbb{R}).$$

Here and in the sequel $\mathbf{1}_{\mathcal{O}}$ denotes the characteristic function of the set \mathcal{O} , and $\Lambda_R := (-\frac{R}{2}, \frac{R}{2})^2$. By the Pastur-Shubin formula (see e.g. [9, Section 2]), we have

$$\int_{\mathbb{R}} \varphi(E) d\mathcal{N}_b(E) = \mathbb{E} (\text{Tr} (\mathbf{1}_{\Lambda_1} \varphi(H) \mathbf{1}_{\Lambda_1})), \quad \forall \varphi \in C_0^\infty(\mathbb{R}),$$

where \mathbb{E} denotes the mathematical expectation. Moreover, there exists a set $\Sigma \subset \mathbb{R}$ such that $\sigma(H_\omega) = \Sigma$ almost surely, and $\text{supp } d\mathcal{N}_b = \Sigma$. The aim of the present article is to study the asymptotic behavior of \mathcal{N}_b near the edges of Σ . It is well known that, for many random models, this behavior is characterized by a very fast decay which goes under the name of ‘‘Lifshitz tails’’. It was studied extensively (see e.g. [6, 9, 4] and references therein).

In order to fix the picture of the almost sure spectrum $\sigma(H_\omega)$, we assume $b > 0$, and make the following two additional assumptions:

- **H₃**: the support of the random variables ω_γ , $\gamma \in \mathbb{Z}^2$, consists of the interval $[\omega_-, \omega_+]$ with $\omega_- < \omega_+$ and $\omega_- \omega_+ \leq 0$.
- **H₄**: $M_+ - M_- < 2b$ where

$$\pm M_\pm := \text{ess-sup}_\omega \sup_{x \in \mathbb{R}^2} (\pm V_\omega(x)).$$

Assumptions **H₁** – **H₄** imply $M_- M_+ \leq 0$. Moreover, the union $\cup_{q=0}^\infty [2bq + M_-, 2bq + M_+]$ which contains Σ , is disjoint. Let W be the bounded \mathbb{Z}^2 -periodic potential defined by

$$W(x) := \sum_{\gamma \in \mathbb{Z}^2} u(x - \gamma), \quad x \in \mathbb{R}^2.$$

On the domain of H_0 , define the operators $H^\pm := H_0 + \omega_\pm W$. It is easy to see that

$$\sigma(H^-) \subseteq \cup_{q=0}^\infty [2bq + M_-, 2bq], \quad \sigma(H^+) \subseteq \cup_{q=0}^\infty [2bq, 2bq + M_+],$$

and

$$\sigma(H^-) \cap [2bq + M_-, 2bq] \neq \emptyset, \quad \sigma(H^+) \cap [2bq, 2bq + M_+] \neq \emptyset, \quad \forall q \in \mathbb{Z}_+.$$

Set

$$\{E_q^-\} := \partial\sigma(H^-) \cap [2bq + M_-, 2bq], \quad \{E_q^+\} := \partial\sigma(H^+) \cap [2bq, 2bq + M_+].$$

The standard characterization of the almost sure spectrum (see also [6, Theorem 5.35]) yields

$$\Sigma = \cup_{q=0}^\infty [E_q^-, E_q^+],$$

i.e. Σ is represented as a disjoint union of compact intervals, and each interval $[E_q^-, E_q^+]$ contains exactly one Landau level $2bq$, $q \in \mathbb{Z}_+$.

We describe the case $E_q^- = 2bq$, $q \in \mathbb{Z}_+$. This is the case if and only if $\omega_- = 0$; in this case, the random variables ω_γ , $\gamma \in \mathbb{Z}^2$, are non-negative which we assume from now on. We describe the behavior of the integrated density of states \mathcal{N}_b near E_q^- , $q \in \mathbb{Z}_+$; mutandi mutandis, the same results hold for E_q^+ .

In [5] (see Theorem 2.1), the authors describe the logarithmic asymptotics of $\mathcal{N}_b(2bq + E) - \mathcal{N}_b(2bq)$ when $E \rightarrow 0^+$ if u does not decay as fast as in assumption \mathbf{H}_1 . In the case of assumption \mathbf{H}_1 , the authors obtained the correct logarithmic upper bound and a lower bound that they deemed not to be optimal. In our main result, we obtain the correct lower bound.

Theorem 1. *Let $b > 0$ and assumptions $\mathbf{H}_1 - \mathbf{H}_4$ hold. Suppose that $\omega_- = 0$, and that*

$$(1) \quad \mathbb{P}(\omega_0 \leq E) \sim CE^\kappa, \quad E \downarrow 0,$$

for some $C > 0$ and $\kappa > 0$. Fix the Landau level $2bq = E_q^-$, $q \in \mathbb{Z}_+$. Then,

$$(2) \quad \lim_{E \downarrow 0} \frac{\ln |\ln (\mathcal{N}_b(2bq + E) - \mathcal{N}_b(2bq))|}{\ln |\ln E|} = 2.$$

In [5], the easy half of (2) is proved under less restrictive assumptions; indeed, Theorem 5.1 of [5] states in particular that, under our assumptions,

$$\limsup_{E \downarrow 0} \frac{\ln |\ln (\mathcal{N}_b(2bq + E) - \mathcal{N}_b(2bq))|}{\ln |\ln E|} \leq 2.$$

So it suffices to prove

$$(3) \quad \liminf_{E \downarrow 0} \frac{\ln |\ln (\mathcal{N}_b(2bq + E) - \mathcal{N}_b(2bq))|}{\ln |\ln E|} \geq 2.$$

The improvement over the results in [5] is obtained through a different analysis that borrows ideas and estimates from [1]. The basic idea is to show that, for energies at a distance E from $2bq$, the single site potential can be replaced by an effective potential that has a support of size approximately $|\log E|^{1/2}$ (see section 2 and Lemma 3 therein). This can then be used to estimate the probability of the occurrence of such energies.

1. PERIODIC APPROXIMATION

We now recall some useful results from [5]. Pick $a > 0$ such that $\frac{ba^2}{2\pi} \in \mathbb{N}$. Set $L := (2n+1)a/2$, $n \in \mathbb{N}$, and define the random $2L\mathbb{Z}^2$ -periodic potential

$$(4) \quad V_{L,\omega}^{\text{per}}(x) = V_{L,\omega}^{\text{per}}(x) := \sum_{\gamma \in 2L\mathbb{Z}^2} (V_\omega \mathbf{1}_{\Lambda_{2L}})(x + \gamma), \quad x \in \mathbb{R}^2.$$

On the domain of H_0 , define the operator $H_{L,\omega}^{\text{per}} := H_0 + V_{L,\omega}^{\text{per}}$. For brevity set $\mathbb{T}_{2L} := \mathbb{R}/(2L\mathbb{Z}^2)$, $\mathbb{T}_{2L}^* := \mathbb{T}_{2L}^*$. On the domain of h_0 define the operator

$$h(\theta) = h^{\text{per}}(\theta) := h_0(\theta) + V_{L,\omega}^{\text{per}}, \quad \theta \in \mathbb{T}_{2L}^*,$$

and set

$$\mathcal{H}^{\text{per}} := \int_{\Lambda_{2L}^*} \oplus h^{\text{per}}(\theta) d\theta.$$

As above, the operators H_0 and $H_{L,\omega}^{\text{per}}$ are unitarily equivalent to the operators \mathcal{H}_0 and \mathcal{H}^{per} respectively. Set

$$(5) \quad \mathcal{N}_{L,\omega}^{\text{per}}(E) = \mathcal{N}_{L,\omega}^{\text{per}}(E) := (2\pi)^{-2} \int_{\Lambda_{2L}^*} N(E; h^{\text{per}}(\theta)) d\theta, \quad E \in \mathbb{R}.$$

Here and in the sequel, for E a real number and T , a self-adjoint operator with compact resolvent, $N(E; T)$ denotes the number of the eigenvalues of T less than E counted with the multiplicities.

One has

Theorem 2 ([5]). *Assume that hypotheses \mathbf{H}_1 and \mathbf{H}_2 hold. Let $q \in \mathbb{Z}_+$, $\eta > 0$. Then there exist $\nu > 0$ and $E_0 > 0$ such that for $E \in (0, E_0]$ and $n \geq E^{-\nu}$ we have*

$$\begin{aligned} \mathbb{E} \left(\mathcal{N}_{L,\omega}^{\text{per}}(2bq + E/2) - \mathcal{N}_{L,\omega}^{\text{per}}(2bq - E/2) \right) &- e^{-E^{-\eta}} \\ &\leq \mathcal{N}_b(2bq + E) - \mathcal{N}_b(2bq - E) \\ &\leq \mathbb{E} \left(\mathcal{N}_{L,\omega}^{\text{per}}(2bq + 2E) - \mathcal{N}_{L,\omega}^{\text{per}}(2bq - 2E) \right) + e^{-E^{-\eta}}. \end{aligned}$$

Denote by Π_q , $q \in \mathbb{Z}_+$, the orthogonal projection onto $\text{Ker}(H_0 - 2bq)$. One can define the bounded operator $\Pi_q V_\omega \Pi_q$ acting on $\Pi_q L^2(\mathbb{R}^2)$. It is ergodic with respect to the group of magnetic translations (see e.g. [5] for details and references). It also admits an integrated density of states that we denote by ρ_q .

For $\theta \in \mathbb{T}_{2L}^*$, we have $\sigma(h(\theta)) = \cup_{q=0}^\infty \{2bq\}$, and $\dim \text{Ker}(h(\theta) - 2bq) = 2bL^2/\pi$ for each $q \in \mathbb{Z}_+$ (see [3]). Denote by $\Pi_q(\theta) : L^2(\Lambda_{2L}) \rightarrow L^2(\Lambda_{2L})$ the orthogonal projection onto $\text{Ker}(h(\theta) - 2bq)$, and by $r_{q,L,\omega}(\theta)$ the operator $\Pi_q(\theta) V_{L,\omega}^{\text{per}} \Pi_q(\theta)$ defined and self-adjoint on the finite-dimensional Hilbert space $\Pi_q(\theta) L^2(\Lambda_{2L})$. Then, one has

$$\rho_{q,L,\omega}(E) = (2\pi)^{-2} \int_{\Lambda_{2L}^*} N(E; r_{q,L,\omega}(\theta)) d\theta, \quad E \in \mathbb{R}.$$

By analogy with (5), let the function $\rho_{q,L,\omega}$ denote the IDS for the operator

$$\mathcal{R}_{q,L,\omega} := \int_{\Lambda_{2L}^*} \oplus r_{q,n,\omega} d\theta$$

defined and self-adjoint on $\mathcal{P}_q L^2(\Lambda_{2L} \times \Lambda_{2L}^*)$ where $\mathcal{P}_q := \int_{\Lambda_{2L}^*} \oplus \Pi_q(\theta) d\theta$. Note that $\mathcal{R}_q = \mathcal{P}_q V_{L,\omega}^{\text{per}} \mathcal{P}_q$ and $\mathcal{P}_q = U \Pi_q U^*$. So $\rho_{q,L,\omega}(E)$ is the integrated density of states of the periodic operator $\Pi_q V_{L,\omega}^{\text{per}} \Pi_q$. One has

Theorem 3 ([5]). *Assume that the hypotheses of Theorem 2 hold. Let $q \in \mathbb{Z}_+$, $\eta > 0$. If $q \geq 1$, assume $M < 2b$. Then, there exist $\nu = \nu(\eta) > 0$, $C > 1$ and $\tilde{E}_0 > 0$, such that for each $E \in (0, \tilde{E}_0)$ and $L \geq E^{-\nu}$, we have*

$$\begin{aligned} \mathbb{E}(\rho_{q,L,\omega}(E/C)) &- e^{-E^{-\eta}} \\ &\leq \mathcal{N}_b(2bq + E) - \mathcal{N}_b(2bq) \leq \mathbb{E}(\rho_{q,L,\omega}(CE)) + e^{-E^{-\eta}}. \end{aligned}$$

The estimate (3) and thus Theorem 1 is then a consequence of

Theorem 4. For $\eta \in (0, 1)$, there exists $C_\eta > 0$ such that, for E sufficiently small and $L \geq 1$, one has, for almost all ω ,

$$(6) \quad e^{|\log E|^{1-\eta} \log |\log E|} \Pi_q V_{L,\omega}^{\text{per}} \Pi_q \geq \left[\inf_{\gamma \in \Lambda_{2L}} \left(\sum_{|\beta-\gamma| \leq |\log E|^{(1-\eta)/2}} \omega_\beta \right) - e^{-|\log E|^{1-\eta}/C_\eta} \right] \Pi_q.$$

The proof of Theorem 4 relies on Lemma 3 that shows that we can “enlarge” the support of the potentials at the expense of a small price. Lemma 3 is stated and proved in section 2.

Let us now use Theorem 4 to complete the proof of (3) and thus of Theorem 1. Pick $L \asymp E^{-\nu}$, ν given by Theorem 3 and fix $\eta \in (0, 1)$ arbitrary. Thus, by the definition of $\rho_{q,L,\omega}$, using (6), we obtain, for $E > 0$ small,

$$\begin{aligned} \mathcal{N}_b(2bq + E) - \mathcal{N}_b(2bq) &\leq CL^d \mathbb{P} \left(\sigma(\Pi_q V_{L,\omega}^{\text{per}} \Pi_q) \cap (-\infty, CE] \neq \emptyset \right) + e^{-E^{-\eta}} \\ &\leq CL^d \mathbb{P} \left(\inf_{\gamma \in \Lambda_{2L}} \sum_{|\beta-\gamma| \leq |\log E|^{(1-\eta)/2}} \omega_\beta - e^{-|\log E|^{1-\eta}/C_\eta} \leq e^{-|\log E|/2} \right) + e^{-E^{-\eta}} \\ &\leq CL^{2d} \mathbb{P} \left(\sum_{|\beta| \leq |\log E|^{(1-\eta)/2}} \omega_\beta \leq 2e^{-|\log E|^{1-\eta}/C_\eta} \right) + e^{-E^{-\eta}}. \end{aligned}$$

as the random variables are i.i.d. Hence, by a classical standard large deviation result (see e.g. [2]), using the assumption (1), we obtain that

$$\mathcal{N}_b(2bq + E) - \mathcal{N}_b(2bq) \leq C_\eta e^{-|\log E|^{2-2\eta}/C_\eta}.$$

As this holds for any $\eta > 0$, we obtain (3) and thus complete the proof of Theorem 1. \square

2. THE PROOF OF THEOREM 4

Recall that $\Pi_q = \Pi_q(b)$, $q \in \mathbb{Z}_+$ is the orthogonal projection on eigenspace corresponding to the q -th Landau level for the Landau Hamiltonian H_0 with constant magnetic field $b > 0$. We recall

Lemma 1 ([7]). *Pick $p > 1$ and let $V \in L^p(\mathbb{R}^2)$ be radially symmetric. Let $(\mu_{q,k}(V))_{k \in \mathbb{Z}_+}$ be the eigenvalues of the compact operator $\Pi_q V \Pi_q$ repeated according to multiplicity. Then, for $k \in \mathbb{Z}_+$, one has*

$$\mu_{q,k}(V) = \langle V \varphi_{q,k}, \varphi_{q,k} \rangle$$

where

- the functions $\varphi_{q,k}$ are given by

$$\varphi_{q,k}(x) := \sqrt{\frac{q!}{\pi k!}} \left(\frac{b}{2} \right)^{(k-q+1)/2} (x_1 + ix_2)^{k-q} L_q^{(k-q)} \left(\frac{b|x|^2}{2} \right) e^{-b|x|^2/4},$$

for $x \in \mathbb{R}^2$,

- $L_q^{(k-q)}$ are the generalized Laguerre polynomials given by

$$L_q^{(k-q)}(\xi) := \sum_{l=\max\{0, q-k\}}^q \binom{k}{q-l} \frac{(-\xi)^l}{l!}, \quad \xi \geq 0, \quad q \in \mathbb{Z}_+, \quad k \in \mathbb{Z}_+,$$

- $\langle \cdot, \cdot \rangle$ denotes the scalar product in $L^2(\mathbb{R}^2)$.

Finally, for $k \in \mathbb{Z}_+$, a normalized eigenfunctions of $\Pi_q V \Pi_q$ corresponding to the eigenvalue $\mu_{q,k}(V)$ is equal to $\varphi_{q,k}$. In particular, the eigenfunctions are independent of V .

We denote by $D(x, R)$ the disk of radius $R > 0$, centered at $x \in \mathbb{R}^2$. We set $\nu_{q,k}(R) := \mu_{q,k}(\chi_{D(0,R)})$ where χ_A is the characteristic function of the set A .

Lemma 2. Fix $q \in \mathbb{Z}_+$. Define $\varrho = \varrho(R) := bR^2/2$ and

$$(7) \quad \nu_{q,k}^a(R) = \frac{e^{-\varrho} \varrho^{-q+1+k} (k-\varrho)^{2q-1}}{q! k!}.$$

Pick $\beta \in (0, 2)$. Let $f : [1, +\infty) \rightarrow [1, +\infty)$ be such that

$$(8) \quad k^{2q-1} f^{-2q}(k) \xrightarrow[k \rightarrow +\infty]{} 0 \quad \text{and} \quad k f^{-\beta}(k) \xrightarrow[k \rightarrow +\infty]{} 0.$$

Then, there exists $k_0 \geq 1$ and $C > 0$ such that, for $k \geq k_0$,

$$(9) \quad \sup_{\substack{R > 0 \\ \varrho(R) \leq k-f(k)}} \left| \frac{\nu_{q,k}(R)}{\nu_{q,k}^a(R)} - 1 \right| \leq C \left(\frac{k^{2q-1}}{f^{2q}(k)} + \frac{k}{f^{\beta+1}(k)} \right).$$

This lemma is an extension of Corollary 2 in [1] to a larger range of radii R .

Proof of Lemma 2. By Lemma 1, passing to polar coordinates (r, θ) in the integral $\langle \chi_{D(0,R)} \varphi_{q,k}, \varphi_{q,k} \rangle$, and changing the variable $br^2/2 = \xi$, we immediately that the eigenvalues $\nu_{q,k}(R)$ of the operator $\Pi_q \chi_{D(0,R)} \Pi_q$ can be written as

$$(10) \quad \nu_{q,k}(R) = \frac{q!}{k!} \int_0^\varrho \xi^k \left[L_q^{(k-q)}(\xi) \right]^2 e^{-\xi} d\xi.$$

For $q = 0$, we have

$$(11) \quad \nu_{0,k}(R) = \frac{1}{k!} \int_0^\varrho \xi^k e^{-\xi} d\xi = \frac{e^{-\varrho} \varrho^{k+1}}{k!} \int_0^1 e^{\rho t + k \log(1-t)} dt.$$

Now, using a Taylor expansion at 0 and the concavity of $t \mapsto \rho t + k \log(1-t)$, write

$$\begin{aligned} \int_0^1 e^{\rho t + k \log(1-t)} dt &= \int_0^{(k-\rho)^{-\beta/2}} e^{\rho t + k \log(1-t)} dt + \int_{(k-\rho)^{-\beta/2}}^1 e^{\rho t + k \log(1-t)} dt \\ &= \int_0^{(k-\rho)^{-\beta/2}} e^{-(k-\rho)t} \left(1 + O(k(k-\rho)^{-\beta}) \right) dt \\ &\quad + O\left(e^{-(k-\rho)^{1-\beta/2}} \right) \\ &= \frac{1}{k-\rho} + O\left(\frac{k}{(k-\rho)^{\beta+1}} \right) \end{aligned}$$

This and (11) yields (9) when $q = 0$.

Consider now the case $q \geq 1$. For some $C_q > 0$, one has

$$(12) \quad \forall k \geq 1, \quad \sup_{s \in \{0, \dots, q\}} \left| k^{s-q} \binom{k}{q-s} (q-s)! - 1 \right| \leq \frac{C_q}{k}.$$

In order to check (9), we assume first of all that $k \geq q$. In this case, using (12), we have

$$(13) \quad \begin{aligned} \nu_{q,k}(R) &= \frac{q!}{k!} \sum_{l,m=0}^q (-1)^{l+m} \frac{1}{m!l!} \binom{k}{q-l} \binom{k}{q-m} \int_0^\varrho e^{-\xi} \xi^{k-q+m+l} d\xi \\ &= V(k, q) + R(k, q) \end{aligned}$$

where

$$(14) \quad \begin{aligned} V(k, q) &= \frac{1}{k!q!} \sum_{l,m=0}^q (-1)^{l+m} \binom{q}{l} \binom{q}{m} k^{2q-l-m} \int_0^\varrho e^{-\xi} \xi^{k-q+m+l} d\xi \\ &= \frac{1}{k!q!} \int_0^\varrho e^{-\xi} \xi^{k-q} (k-\xi)^{2q} d\xi, \end{aligned}$$

and

$$(15) \quad \begin{aligned} |R(k, q)| &\leq \frac{C_q}{k} \frac{1}{k!q!} \sum_{l,m=0}^q \binom{q}{l} \binom{q}{m} k^{2q-l-m} \int_0^\varrho e^{-\xi} \xi^{k-q+m+l} d\xi \\ &\leq \frac{C_q}{k} \frac{1}{k!q!} \int_0^\varrho e^{-\xi} \xi^{k-q} (k+\xi)^{2q} d\xi. \end{aligned}$$

For $\rho \leq k - f(k)$, one computes

$$(16) \quad \left| \frac{|R(k, q)|}{V(k, q)} \right| \leq C \frac{k^{2q-1}}{f^{2q}(k)} \xrightarrow{k \rightarrow +\infty} 0$$

by our assumption on f .

On the other hand, as in the case $q = 0$, we have

$$\int_0^\varrho e^{-\xi} \xi^{k-q} (k-\xi)^{2q} d\xi = e^{-\rho} \rho^{k-q+1} (k-\rho)^{2q} I(k, \rho)$$

where

$$I(k, \rho) = \int_0^1 e^{\rho\xi} (1-\xi)^{k-q} \left(1 + \frac{\rho}{k-\rho} \xi \right)^{2q} d\xi.$$

The function $t \mapsto \rho t + (k-q) \log(1-t) + 2q \log \left(1 + \frac{\rho}{k-\rho} t \right)$ is concave on $[0, 1]$ and its derivative at 0 is

$$\rho - k + q + 2q\rho/(k-\rho) = (\rho - k) (1 + O(k(k-\rho)^{-2})).$$

Hence, as in the case $q = 0$, we obtain that

$$I(k, \rho) = \frac{1}{k-\rho} + O\left(\frac{k}{(k-\rho)^{\beta+1}} \right).$$

Plugging this into (14), using (16) and (15), and replacing in (13), we obtain (9) for $q \geq 1$.

This completes the proof of Lemma 2. \square

We will now use Lemma 2 to obtain “enlarge” our potentials and prove

Lemma 3. *Let $q \in \mathbb{Z}_+$ and fix $b > 0$. Fix $\varepsilon > 0$. There exists a constant $C_0 > 0$ and $R_0 > 1$ such that, for each $R \geq 1$,*

$$(17) \quad \Pi_q \chi_{D(0,\varepsilon)} \Pi_q \geq e^{-C_0 R^2 \log R} \left(\Pi_q \chi_{D(0,R)} \Pi_q - e^{-R^2/C_0} \Pi_q \chi_{D(0,2R)} \Pi_q \right).$$

This lemma is basically Lemma 2 in [1] except that we want to control the behavior of the constants coming up in the inequality in terms of R .

Proof of Lemma 3. We fix $\delta \in (0, 1)$, and, bearing in mind Lemma 3, namely (9) and (7), we know that for $k \geq k_0 = CR^2$ with $C > b/2$ (hence, for some $R_0 > 0$, $k - f(k) \geq \rho(R)$ for $R \geq R_0$ large where f is defined in Lemma 3), implies

$$(1 - \delta) \frac{e^{-\varrho(\tilde{R})} \varrho(\tilde{R})^{k-q+1} (k - \rho(\tilde{R}))^{2q-1}}{q! k!} \leq \nu_{n,k}(\tilde{R}) \\ \leq (1 + \delta) \frac{e^{-\varrho(\tilde{R})} \varrho(\tilde{R})^{k-q+1} (k - \rho(\tilde{R}))^{2q-1}}{q! k!},$$

for $\tilde{R} \in [R/2, 2R]$. We will show that, if $R \geq R_0$, then, the operator inequality

$$(18) \quad \Pi_q \chi_{D(0,\varepsilon)} \Pi_q \geq C_1 \left(\Pi_q \chi_{D(0,R)} \Pi_q - C_2 \Pi_q \chi_{D(0,2R)} \Pi_q \right)$$

holds with the following constants:

-

$$(19) \quad C_1 := \min_{k \in \{0, \dots, k_0\}} \frac{\nu_{n,k}(\varepsilon)}{\nu_{n,k}(R)} \geq \frac{1}{C_0} e^{-2CR^2 \log R};$$

as $k_0 = CR^2$, the lower bound is a consequence of (9) and (7) written for $\nu_{n,k}(\varepsilon)$;

-

$$(20) \quad C_2 := \frac{1 + \delta}{1 - \delta} 2^{-2(k_0 - n + 1)} e^{-\varrho(R) + \varrho(2R)} \leq e^{-R^2/C_0}.$$

By Lemma 1, the operators $\Pi_q \chi_{D(0,\varepsilon)} \Pi_q$, $\Pi_q \chi_{D(0,R)} \Pi_q$, and $\Pi_q \chi_{D(0,2R)} \Pi_q$, are reducible in the same basis $\{\varphi_{n,k}\}_{k \in \mathbb{Z}_+}$. Hence, in order to prove (18), it suffices to check that the numerical inequalities

$$(21) \quad \nu_{n,k}(\varepsilon) \geq C_1 (\nu_{n,k}(R) - C_2 \nu_{n,k}(2R))$$

hold for each $k \in \mathbb{Z}_+$. If $k \leq k_0$, then (21) is valid because in this case (19) implies $\nu_{n,k}(\varepsilon) \geq C_1 \nu_{n,k}(R)$ which obviously entails (21). If $k > k_0$, by (19)

and (20) we have

$$\begin{aligned}
\nu_{n,k}(R) - C_2\nu_{n,k}(2R) &\leq (1 + \delta) \frac{e^{-\varrho(R)} \varrho(R)^{-q+1} (k - \rho(R))^{2q-1} \varrho(R)^k}{q! k!} \\
&\quad - \left(\frac{1 + \delta}{1 - \delta} 2^{-2(k_0 - q + 1)} e^{-\varrho(R) + \varrho(2R)} \right. \\
&\quad \quad \left. \times (1 - \delta) \frac{e^{-\varrho(2R)} \varrho(2R)^{-q+1} (k - \rho(R))^{2q-1} \varrho(2R)^k}{q! k!} \right) \\
&= (1 + \delta) \frac{e^{-\varrho(R)} (k - \rho(R))^{2q-1} \varrho(2R)^{k-q+1}}{q! k!} 2^{2(q-1)} \left(2^{-2k} - 2^{-2k_0} \right).
\end{aligned}$$

Hence, we find that $\nu_{n,k}(R) - C_2\nu_{n,k}(2R) \leq 0$ if $k \geq k_0$, which again implies (21). This completes the proof of Lemma 3. \square

We now prove Theorem 4.

The magnetic translations for the constant magnetic field problem in two-dimensions are defined as follows (see e.g. [8]). For any field strength $b \in \mathbb{R}$, any vector $\alpha \in \mathbb{R}^2$, and $f \in C_0^\infty(\mathbb{R}^2)$, we define the magnetic translation by α as

$$U_\alpha^b f(x) := e^{\frac{ib}{2}(x_1\alpha_2 - x_2\alpha_1)} f(x + \alpha).$$

In a standard way, the family $\{U_\alpha^b \mid \alpha \in \mathbb{R}^2\}$ extends to a projective unitary representation of \mathbb{R}^2 on $L^2(\mathbb{R}^2)$. We note that for any magnetic field strength b , For $\gamma \in \mathbb{Z}^2$, the invariance of $H_L(b)$ with respect to group of magnetic translations gives

$$U_\gamma^b \Pi_q \chi_{D(0,\varepsilon)} \Pi_q U_{-\gamma}^b = \Pi_q \chi_{D(\gamma,\varepsilon)} \Pi_q.$$

Hypothesis (H1) on the single-site potential u guarantees that there exists $\varepsilon \in (0, 1/2)$ so that $V_\omega \geq \sum_{\gamma \in \mathbb{Z}^2} \omega_\gamma \chi_{D(\gamma,\varepsilon)}$. Plugging this into (4), we get

$$V_{L,\omega}^{\text{per}} \geq \sum_{\gamma \in 2L\mathbb{Z}^2} \sum_{\beta \in \Lambda_{2L}} \omega_\beta \chi_{D(\gamma+\beta,\varepsilon)}.$$

Fix $\eta \in (0, 1)$ and pick $R \asymp |\log E|^{(1-\eta)/2}$. Using Lemma 3, we get that

$$\Pi_q \chi_{D(\gamma,\varepsilon)} \Pi_q \geq e^{-C_0 R^2 \log R} \left(\Pi_q \chi_{D(\gamma,R)} \Pi_q - e^{-R^2/C_0} \Pi_q \chi_{D(\gamma,2R)} \Pi_q \right).$$

Hence, as the random variables are bounded

$$\begin{aligned}
e^{C_0 R^2 \log R} \Pi_q V_{L,\omega}^{\text{per}} \Pi_q &\geq e^{C_0 R^2 \log R} \sum_{\gamma \in 2L\mathbb{Z}^2} \sum_{\beta \in \Lambda_{2L}} \omega_\beta \Pi_q \chi_{D(\gamma+\beta,\varepsilon)} \Pi_q \\
&\geq \sum_{\gamma \in 2L\mathbb{Z}^2} \sum_{\beta \in \Lambda_{2L}} \omega_\beta \Pi_q \chi_{D(\gamma+\beta,R)} \Pi_q \\
&\quad - C e^{-R^2/C_0} \sum_{\gamma \in 2L\mathbb{Z}^2} \sum_{\beta \in \Lambda_{2L}} \Pi_q \chi_{D(\gamma+\beta,2R)} \Pi_q \\
&\geq \Pi_q \sum_{\gamma \in 2L\mathbb{Z}^2} \sum_{\beta \in \Lambda_{2L}} \omega_\beta \sum_{|\nu-\gamma-\beta| \leq R/2} \chi_{|x-\nu| \leq 1/2} \Pi_q - C e^{-R^2/C_0} R^2 \Pi_q \\
&\geq \left(\inf_{\gamma \in \Lambda_{2L}} \left(\sum_{|\beta-\gamma| \leq R/2} \omega_\beta \right) - C R^2 e^{-R^2/C_0} \right) \Pi_q
\end{aligned}$$

Taking into account $R \asymp |\log E|^{(1-\eta)/2}$, this completes the proof of Theorem 4.

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