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CLASSIFYING TAME BLOCKS AND RELATED ALGEBRAS UP TO STABLE EQUIVALENCES OF MORITA TYPE

GUODONG ZHOU AND ALEXANDER ZIMMERMANN

ABSTRACT. We contribute to the classification of finite dimensional algebras under stable equivalence of Morita type. More precisely we give a classification of the class of Erdmann's algebras of dihedral, semi-dihedral and quaternion type and obtain as byproduct the validity of the Auslander-Reiten conjecture for these classes of algebras.

INTRODUCTION

Stable categories were introduced very early in the representation theory of algebras and played a major rôle in the development of Auslander-Reiten theory for example. Nevertheless, already in the 1970's Auslander and Reiten knew that equivalences of stable categories can behave very badly. For example there are indecomposable finite dimensional algebras which are stably equivalent to a direct product of two algebras none of which is separable [1, Example 3.5].

Around 1990 the concept of derived categories became popular in the representation theory of groups and algebras by mainly two developments. First Happel interpreted successfully tilting theory in the framework of derived categories and secondly Broué formulated his famous abelian defect group conjecture in this framework. Many homological constructions are more natural in the language of derived categories. Work of Rickard [16] and Keller-Vossieck [8] show that an equivalence between derived categories of self-injective algebras imply an equivalence between the stable categories of these algebras of a very particular shape. They are induced by bimodules which are invertible almost as for Morita equivalences. This discovery in mind Broué defined two algebras A and B to be stably equivalent of Morita type if there is an $A - B$ -bimodule M and a $B - A$ -bimodule N , which are projective considered as module on either side only and so that there are isomorphisms of bimodules $M \otimes_B N \simeq A \oplus P$ for a projective $A - A$ -bimodule P and $N \otimes_A M \simeq B \oplus Q$ for a projective $B - B$ -bimodule Q .

It soon became clear that stable equivalences of Morita type are much better behaved than abstract stable equivalences. Nevertheless, classes of algebras which are classified up to stable equivalence of Morita type are rare. In recent joint work with Yuming Liu [13] we gave several invariants which proved to be very sophisticated and powerful so that a classification of big classes of symmetric algebras up to stable equivalence of Morita type becomes feasible. The additional problem mainly is that the number of simple modules is not proven to be an invariant under stable equivalence of Morita type. This fact is the long-standing open Auslander-Reiten conjecture.

Erdmann gave an (up to parameters) finite list [3] of algebras which are defined by properties on their Auslander Reiten quiver and which include all blocks of finite groups of tame representation type. Her classification is up to Morita equivalence. Holm pursued further this approach and classified the algebras in Erdmann's list up to derived equivalence [5]. We shall give an account of his results in Section 2.

In the present work we classify the algebras of dihedral, semi-dihedral and quaternion type up to stable equivalence of Morita type. Our classification is almost as complete as for derived equivalences and the classification coincides in some sense with the derived equivalence classification. In particular we show the Auslander-Reiten conjecture for these classes of algebras and note that the classes are closed under stable equivalences of Morita type.

The paper is organised as follows. In Section 1 we recall some of the invariants under stable equivalence of Morita type we use in the sequel. Section 2 explains Holm's derived equivalence

classification of algebras of dihedral, semi-dihedral and quaternion type. In Section 3 we present an independent classification for the case of tame blocks of group rings. The proof is much simpler than the general case, and hence we decided to present the arguments separately, though, of course, the general theorem includes this case as well. Moreover, a short summary of Holm's result on Hochschild cohomology of tame blocks is given there. Section 4 shows that the derived equivalence classification of dihedral type algebras coincides with the classification up to stable equivalence of Morita type. The main tool is a result of Pogorzały [15, Theorem 7.3]. This section is the first technical core of the paper. Section 5 computes the centres of the algebras of semi-dihedral and of quaternion type. This section prepares the classification result for these classes of algebras. Section 6 distinguishes then stable equivalence classes of Morita type using basically invariants derived from the centre. This part is the second technical core of the paper. Section 7 finally summarises large parts of what was proved before and contains the main result Theorem 7.1 of the paper as well as some results derived from Külshammer like invariants computed initially to distinguish derived equivalence classes.

1. STABLE INVARIANTS

The stable category $A\text{-}\underline{\text{mod}}$ of a finite dimensional K -algebra A has the same objects as the category of A -modules and morphisms, denoted by $\underline{\text{Hom}}_A(M, N)$ from M to N , are equivalence classes of morphisms of A -modules modulo those factoring through a projective A -module.

In this section we shall explain and state most of the various properties of algebras invariant under stable equivalences of Morita type used in the sequel.

The first reduction is a result of Keller-Vossieck and Rickard.

Theorem 1.1. (*Keller-Vossieck [8] and Rickard [16]*) *Let K be a field and let A and B be two self-injective K algebras. If the bounded derived categories of A and B are equivalent, $D^b(A) \simeq D^b(B)$, then the algebras A and B are stably equivalent of Morita type.*

Hence in order to give a classification of a class of algebras up to stable equivalence of Morita type we can start from a classification up to derived equivalence and decide for two representatives of the derived equivalence classes whether they are stably equivalent of Morita type.

In order to do so we use several criteria, some linked to questions around the centre of the algebras.

We first recall a construction due to Broué. Let A and B be K -algebras. If A is stably equivalent of Morita type to B , then the subcategory of the stable category generated by left and right projective $A \otimes_K A^{op}$ -modules is equivalent to the analogous category of $B \otimes_K B^{op}$ -modules. The $A \otimes_K A^{op}$ -module A is mapped to the $B \otimes_K B^{op}$ -module B under this equivalence. Therefore

$$\underline{\text{End}}_{A \otimes_K A^{op}}(A) \simeq \underline{\text{End}}_{B \otimes_K B^{op}}(B).$$

Broué denotes by

$$Z^{st}(A) = \underline{\text{End}}_{A \otimes_K A^{op}}(A)$$

the stable centre and by

$$Z^{pr}(A) := \ker \left(\text{End}_{A \otimes_K A^{op}}(A) \longrightarrow \underline{\text{End}}_{A \otimes_K A^{op}}(A) \right)$$

the projective centre of A .

Theorem 1.2. (*Broué [2, Proposition 5.4]*) *Let A and B be two algebras which are stably equivalent of Morita type, then $Z^{st}(A) \simeq Z^{st}(B)$.*

The centre is usually not an invariant under stable equivalences of Morita type. However one of the main results of [13] gives a partial answer.

Theorem 1.3. (*Liu, Zhou, Zimmermann [13, Theorem 1.1]*) *Let K be an algebraically closed field and let A and B be two indecomposable finite dimensional K -algebras which are stably equivalent of Morita type. Then $\dim_K(HH_0(A)) = \dim_K(HH_0(B))$ if and only if the number of simple A -modules up to isomorphism equals the number of simple B -modules up to isomorphism.*

Since for symmetric algebras $\text{Hom}_K(HH_0(A), K) \simeq Z(A)$, we get that for symmetric indecomposable finite dimensional algebras over algebraically closed fields the dimension of the centres coincide if and only if the number of simple modules coincide.

Moreover, a very useful criterion was given in [13] as well in order to estimate the dimension of the projective centre.

Proposition 1.4. (*Liu, Zhou, Zimmermann [13, Proposition 2.4 and Corollary 2.9]*) *Let K be an algebraically closed field and let A be an indecomposable symmetric K -algebra with n simple modules up to isomorphism. Then the dimension of the projective centre equals the rank of the Cartan matrix, seen as linear mapping $K^n \rightarrow K^n$.*

A classical invariant, popularised recently by Külshammer [11], is the Reynolds ideal defined for any K -algebra as $R(A) := Z(A) \cap \text{soc}(A)$. For symmetric algebras A and a perfect field K of strictly positive characteristic Külshammer constructed a descending sequence of ideals $T_n(A)^\perp$ of the centre of A , for $n \in \mathbb{N}$ with $R(A) = \bigcap_{n \in \mathbb{N}} T_n^\perp(A)$.

Proposition 1.5. [13, Proposition 2.4 and proof of Proposition 2.5] *The projective centre of an algebra equals the Higman ideal of A and the Higman ideal of an algebra is in the socle of the algebra.*

We shall use the following fact.

Theorem 1.6. *Let K be an algebraically closed field and let A and B be two finite dimensional symmetric indecomposable K -algebras which are stably equivalent of Morita type. Then*

$$\dim_K(Z(A)/R(A)) = \dim_K(Z(B)/R(B)).$$

Furthermore if K is of positive characteristic or the Cartan matrix of A is non singular, then we have an isomorphism of algebras $Z(A)/R(A) \simeq Z(B)/R(B)$.

Proof The first statement is the dual of [13, Corollary 5.4], as all algebras in question are symmetric. For the second statement, the case of positive characteristic is contained in [9, Proposition 5.8]. In case of non singular Cartan matrix, by [17, Proposition 5.1] (or see the discussions at the end of this section), B has also non singular Cartan matrix. So the rank is the Cartan matrix is equal to the number of simple modules, that is, the dimension of the Reynolds ideal. So we have $Z^{pr}(A) = R(A)$ and $Z^{st}(A) = Z(A)/R(A)$. Now use Theorem 1.2. □

Let A be an indecomposable finite dimensional algebra and let C_A be its Cartan matrix. The Cartan matrix induces in a natural way a mapping of the Grothendieck group $G_0(A)$ of abelian groups (the Grothendieck group taken in the sense of A -modules modulo exact sequences). The stable Grothendieck group $G_0^{st}(A)$ is defined as the cokernel of

$$G_0(A) \xrightarrow{C_A} G_0(A) \rightarrow G_0^{st}(A) \rightarrow 0$$

Proposition 1.7. (*Xi [17]*) *Let A and B be finite dimensional indecomposable K -algebras and suppose that A and B are stably equivalent of Morita type. Then $G_0^{st}(A) \simeq G_0^{st}(B)$.*

It is clear by this statement that a stable equivalence of Morita type preserves those elementary divisors of the Cartan matrix which are different from 1, including their multiplicity. (Note that as usual the elementary divisors are supposed to be non negative). In particular the absolute value of the Cartan determinant is preserved.

2. ALGEBRAS OF DIHEDRAL, SEMI-DIHEDRAL AND QUATERNION TYPE

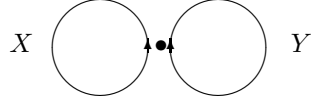
Let K be an algebraically closed field. In this section we shall give Karin Erdmann's list of algebras of dihedral, semi-dihedral and quaternion type.

By Theorem 1.1 of Keller-Vossieck and Rickard, for two self-injective algebras A and B , an equivalence $D^b(A) \simeq D^b(B)$ of the bounded derived categories implies that A and B are stably equivalent of Morita type. Hence, as basis of our discussion we shall use the list of Thorsten Holm [5] of algebras of dihedral, semi-dihedral and quaternion type up to derived equivalences. There are three families: the algebras of dihedral type, the algebras of semi-dihedral type, the algebras of quaternion type. Each family is subdivided into three subclasses: algebras with one simple module, algebras with

two simple modules and algebras with three simple modules. Each subfamily contains algebras with quivers and relations, depending on parameters.

	dihedral	semidihedral	quaternion
1 simple	$K[X, Y]/(XY, X^m - Y^n),$ $m \geq n \geq 2, m + n > 4;$ $D(1\mathcal{A})_1^1 = K[X, Y]/(X^2, Y^2);$ $(\text{char} K = 2)$ $K[X, Y]/(X^2, YX - Y^2);$ $D(1\mathcal{A})_1^k, k \geq 2;$ $(\text{char} K = 2) D(1\mathcal{A})_2^k(d),$ $k \geq 2, d = 0 \text{ or } 1;$	$SD(1\mathcal{A})_1^k, k \geq 2;$ $(\text{char}(K) = 2) SD(1\mathcal{A})_2^k(c, d)$ $k \geq 2, (c, d) \neq (0, 0);$	$Q(1\mathcal{A})_1^k, k \geq 2;$ $(\text{char} K = 2) Q(1\mathcal{A})_2^k(c, d),$ $k \geq 2, (c, d) \neq (0, 0);$
2 simples	$D(2\mathcal{B})^{k,s}(c),$ $k \geq s \geq 1, c \in \{0, 1\}$	$SD(2\mathcal{B})_1^{k,t}(c)$ $k \geq 1, t \geq 2, c \in \{0, 1\};$ $SD(2\mathcal{B})_2^{k,t}(c)$ $k \geq 1, t \geq 2,$ $k + t \geq 4, c \in \{0, 1\};$	$Q(2\mathcal{B})_1^{k,s}(a, c)$ $k \geq 1, s \geq 3, a \neq 0;$
3 simples	$D(3\mathcal{K})^{a,b,c},$ $a \geq b \geq c \geq 1;$ $D(3\mathcal{R})^{k,s,t,u},$ $s \geq t \geq u \geq k \geq 1, t \geq 2$	$SD(3\mathcal{K})^{a,b,c}$ $a \geq b \geq c \geq 1, a \geq 2;$	$Q(3\mathcal{K})^{a,b,c}$ $a \geq b \geq c \geq 1, b \geq 2,$ $(a, b, c) \neq (2, 2, 1);$ $Q(3\mathcal{A})_1^{2,2}(d)$ $d \notin \{0, 1\}$

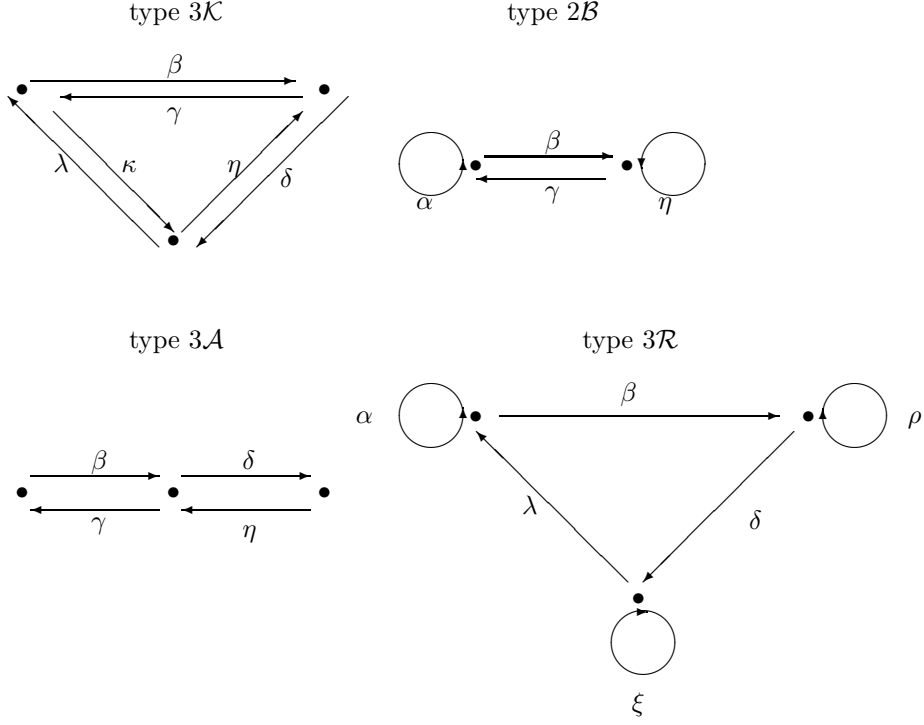
All algebras with one simple module in the above list has the quiver of type $1\mathcal{A}$



and with relations

$$\begin{aligned}
D(1\mathcal{A})_1^k &: X^2, Y^2, (XY)^k - (YX)^k; \\
D(1\mathcal{A})_2^k(d) &: X^2 - (XY)^k, Y^2 - d \cdot (XY)^k, (XY)^k - (YX)^k, (XY)^k X, (YX)^k Y; \\
SD(1\mathcal{A})_1^k &: (XY)^k - (YX)^k, (XY)^k X, Y^2, X^2 - (YX)^{k-1} Y; \\
SD(1\mathcal{A})_2^k(c, d) &: (XY)^k - (YX)^k, (XY)^k X, Y^2 - d(XY)^k, \\
&\quad X^2 - (YX)^{k-1} Y + c(XY)^k; \\
Q(1\mathcal{A})_1^k &: (XY)^k - (YX)^k, (XY)^k X, Y^2 - (XY)^{k-1} X, X^2 - (YX)^{k-1} Y; \\
Q(1\mathcal{A})_2^k(c, d) &: X^2 - (YX)^{k-1} Y - c(XY)^k, Y^2 - (XY)^{k-1} X - d(XY)^k, \\
&\quad (XY)^k - (YX)^k, (XY)^k X, (YX)^k Y.
\end{aligned}$$

The quivers of the algebras of type $2\mathcal{B}$, $3\mathcal{K}$, $3\mathcal{A}$ and $3\mathcal{R}$ are respectively:



The relations are respectively

$$\begin{aligned}
D(2\mathcal{B})^{k,s}(c) &: \beta\eta, \eta\gamma, \gamma\beta, \alpha^2 - c(\alpha\beta\gamma)^k, (\alpha\beta\gamma)^k - (\beta\gamma\alpha)^k, \eta^s - (\gamma\alpha\beta)^k; \\
SD(2\mathcal{B})_1^{k,t}(c) &: \gamma\beta, \eta\gamma, \beta\eta, \alpha^2 - (\beta\gamma\alpha)^{k-1}\beta\gamma - c(\alpha\beta\gamma)^k, \eta^t - (\gamma\alpha\beta)^k, (\alpha\beta\gamma)^k - (\beta\gamma\alpha)^k; \\
SD(2\mathcal{B})_2^{k,t}(c) &: \beta\eta - (\alpha\beta\gamma)^{k-1}\alpha\beta, \eta\gamma - (\gamma\alpha\beta)^{k-1}\gamma\alpha, \gamma\beta - \eta^{t-1}, \alpha^2 - c(\alpha\beta\gamma)^k, \beta\eta^2, \eta^2\gamma; \\
Q(2\mathcal{B})_1^{k,s}(a, c) &: \gamma\beta - \eta^{s-1}, \beta\eta - (\alpha\beta\gamma)^{k-1}\alpha\beta, \eta\gamma - (\gamma\alpha\beta)^{k-1}\gamma\alpha, \\
&\quad \alpha^2 - a(\beta\gamma\alpha)^{k-1}\beta\gamma - c(\beta\gamma\alpha)^k, \alpha^2\beta, \gamma\alpha^2; \\
D(3\mathcal{K})^{a,b,c} &: \beta\delta, \delta\lambda, \lambda\beta, \gamma\kappa, \kappa\eta, \eta\gamma, (\beta\gamma)^a - (\kappa\lambda)^b, (\lambda\kappa)^b - (\eta\delta)^c, (\delta\eta)^c - (\gamma\beta)^a; \\
D(3\mathcal{R})^{k,s,t,u} &: \alpha\beta, \beta\rho, \rho\delta, \delta\xi, \xi\lambda, \lambda\alpha, \alpha^s - (\beta\delta\lambda)^k, \rho^t - (\delta\gamma\beta)^k, \xi^u - (\lambda\beta\delta)^k; \\
SD(3\mathcal{K})^{a,b,c} &: \kappa\eta, \eta\gamma, \gamma\kappa, \delta\gamma - (\gamma\alpha)^{a-1}\gamma, \beta\delta - (\kappa\lambda)^{b-1}\kappa, \lambda\beta - (\eta\delta)^{c-1}\eta; \\
Q(3\mathcal{K})^{a,b,c} &: \beta\delta - (\kappa\lambda)^{a-1}\kappa, \eta\gamma - (\lambda\kappa)^{a-1}\lambda, \delta\lambda - (\gamma\beta)^{b-1}\gamma, \kappa\eta - (\beta\gamma)^{b-1}\beta, \lambda\beta - (\eta\delta)^{c-1}\eta, \\
&\quad \gamma\kappa - (\delta\eta)^{c-1}\delta, \gamma\beta\delta, \delta\eta\gamma, \lambda\kappa\eta; \\
Q(3\mathcal{A})_1^{2,2}(d) &: \beta\delta\eta - \beta\gamma\beta, \delta\eta\gamma - \gamma\beta\gamma, \eta\gamma\beta - d\eta\delta\eta, \gamma\beta\delta - d\delta\eta\delta, \beta\delta\eta\delta, \eta\gamma\beta\gamma.
\end{aligned}$$

The following result suggests that we only need to consider internally these three classes of algebras in order to classify them up to stable equivalences of Morita type.

Proposition 2.1. *If two indecomposable algebras A and B are stably equivalent of Morita type and A is of dihedral (resp. semi-dihedral, quaternion) type, then so is B .*

Proof These classes of algebras are defined in terms of the nature of their Auslander-Reiten quiver. Roughly, an algebra A is of one of these types if

- A is symmetric, indecomposable and tame;
- the Cartan matrix of A is non-singular.
- the stable Auslander Reiten quiver of A has the following components

	dihedral type	semidihedral type	quaternion type
tubes	rank 1 and 3 at most two 3-tubes	rank at most 3 at most one 3-tube	rank at most 2
others	$\mathbb{Z}A_\infty/\Pi$	$\mathbb{Z}A_\infty$ and $\mathbb{Z}D_\infty$	

for a certain group Π .

For more details we refer to [3].

By a result of Yuming Liu ([12, Corollary 2.4])(resp. Henning Krause ([10, last corollary of the article])), if two algebras are stably equivalent of Morita type and one of them is symmetric (resp. tame), so is the other. If two algebras are stably equivalent of Morita type, they are stably equivalent and thus their stable Auslander-Reiten quiver are isomorphic. By a result of Chang-Chang Xi ([17, Proposition 5.1]), if two algebras are stably equivalent of Morita type, the absolute values of the determinant of their Cartan matrices are the same and thus if the Cartan matrix of one algebra is non-singular, so is that of the other. Therefore, the defining properties are preserved by a stable equivalence of Morita type between two indecomposable algebras. \square

3. TAME BLOCKS

3.1. Derived classification. The following is a classification of algebras up to derived equivalence, as given by Holm [5], which could occur as blocks of group algebras. For some cases the question if there is a block of a group with this derived equivalence type is not clear yet. We include in this case the algebra as well. Now let K be an algebraically closed field of characteristic two. Let A be a tame block of defect $n \geq 2$. Then A is derived equivalent to one of the following algebras.

	dihedral	semidihedral	quaternion
1 simple	$D(1\mathcal{A})_1^{2^{n-2}}, n \geq 2;$	$SD(1\mathcal{A})_1^{2^{n-2}}, n \geq 4;$	$Q(1\mathcal{A})_1^{2^{n-2}}, n \geq 3;$
2 simples	$D(2\mathcal{B})_1^{1,2^{n-2}}(c),$ $c \in \{0, 1\}, n \geq 3;$	$SD(2\mathcal{B})_1^{1,2^{n-2}}(c),$ $c \in \{0, 1\}, n \geq 4;$ $SD(2\mathcal{B})_2^{2,2^{n-2}}(c),$ $c \in \{0, 1\}, n \geq 4;$	$Q(2\mathcal{B})_1^{2,2^{n-2}}(a, c),$ $n \geq 3, a \in K^*, c \in K;$
3 simples	$D(3\mathcal{K})^{2^{n-2},1,1}, n \geq 2;$	$SD(3\mathcal{K})^{2^{n-2},2,1}, n \geq 4;$	$Q(3\mathcal{K})^{2^{n-2},2,2}, n \geq 3.$

3.2. Hochschild cohomology of tame blocks. If A and B are two algebras which are stably equivalent of Morita type, then Xi shows [17, Theorem 4.2] that the Hochschild cohomology groups $HH^m(A)$ and $HH^m(B)$ are isomorphic for any $m \geq 1$. Furthermore, in a recent paper of the first author with Shengyong Pan ([14]), we proved that a stable equivalence of Morita type preserve the algebra structure of the stable Hochschild cohomology, that is, the Hochschild cohomology modulo the projective center.

For the sake of completeness we resume results of Holm [5] which allow to distinguish a certain number of pairs of algebras up to stable equivalence of Morita type, although we could avoid using these results in the sequel, mainly because they only deal with blocks of group rings with one or three simple modules.

3.2.1. Dihedral type. By [5, Theorem 3.2.2] the Hochschild cohomology ring of a block with dihedral defect group of order 2^n with $n \geq 2$ and one simple module has dimension $\dim(HH^i(B)) = 2^{n-2} + 3 + 4i$.

By [5, Theorem 3.2.8] the Hochschild cohomology ring of a block with dihedral defect group of order 2^n with $n \geq 2$ and three simple modules has dimension $2^{n-2} + 3$ in degree 0, and dimension

$2^{n-2} + 1$ in degree 1. Further, for all $i \geq 1$, $\dim(HH^{3i-1}(B)) = 2^{n-2} - 1 + 4i$ and $\dim(HH^{3i}(B)) = \dim(HH^{3i+1}(B)) = 2^{n-2} + 1 + 4i$.

3.2.2. Semi-dihedral type. By [5, Theorem 3.3.2] the Hochschild cohomology ring of a block with semi-dihedral defect group of order 2^n with $n \geq 4$ and one simple module has dimension $2^{n-2} + 3$ in degree 0, dimension $2^{n-2} + 6$ in degree 1, dimension $2^{n-2} + 7$ in degree 2, and dimension $2^{n-2} + 8$ in degree 3. Further, $\dim(HH^{i+4}(B)) = \dim(HH^i(B)) + 8$.

By [5, Theorem 3.3.3] the Hochschild cohomology ring of a block with semi-dihedral defect group of order 2^n with $n \geq 4$ and three simple modules has dimension $2^{n-2} + 4$ in degrees 0 and 3, dimension $2^{n-2} + 2$ in degrees 1 and 2, and dimension $2^{n-2} + 5$ in degree 4. Further, $\dim(HH^{i+4}(B)) = \dim(HH^i(B)) + 2 + x(i)$, where $x(i)$ is 0 if 3 divides i , and $x(i) = 1$ else.

3.2.3. Quaternion type. By [5, Theorem 3.4.2] a block with one simple modules and quaternion defect group of order 2^n with $n \geq 3$ has periodic Hochschild cohomology ring with period 4 and dimension $2^{n-2} + 3$ in degrees congruent 0 or 3 mod 4 and of dimension $2^{n-2} + 5$ in degrees congruent 1 or 2 mod 4.

By [5, Theorem 3.4.6] a block with three simple modules and quaternion defect group of order 2^n with $n \geq 3$ has periodic Hochschild cohomology ring with period 4 and dimension $2^{n-2} + 5$ in degrees congruent 0 or 3 mod 4 and of dimension $2^{n-2} + 3$ in degrees congruent 1 or 2 mod 4.

3.3. Blocks of dihedral defect groups.

Proposition 3.1. *Let K be an algebraically closed field of characteristic 2 and let A be a dihedral block of defect $n \geq 2$. Then A is stably equivalent of Morita type to one and exactly one of the following algebras: $D(1A)_1^{2^{n-2}}$; $D(2B)^{1,2^{n-2}}(c)$ (for $n \geq 3$) with $c = 0$ or $c = 1$; $D(3K)^{2^{n-2},1,1}$.*

As a consequence, the derived classification coincides with the classification up to stable equivalences of Morita type.

Remark 3.2. Before giving the proof, we remark that for a dihedral block with two simple modules, we don't know whether the case $c = 1$ really occurs. All known examples have zero as the value of this scalar. But this doesn't influence our result, since $D(2B)^{k,s}(0)$ is NOT derived equivalent to $D(2B)^{k,s}(1)$. There are several proofs of this fact (cf [7, Corollary 5.3] [6, Theorem 1.1]). One can also use a result of Pogorzały ([15, Theorem 7.3]), which says that an algebra stably equivalent to a self-injective special biserial algebra which is not a Nakayama algebra is itself a selfinjective special biserial algebra. Notice that $D(2B)^{k,s}(0)$ is a symmetric special biserial algebra, but $D(2B)^{k,s}(1)$ is not. As a consequence the algebras $D(2B)^{k,s}(1)$ cannot be stably equivalent to any algebra of the other classes.

Proof Since Holm's result [5] implies that any algebra of dihedral type is derived equivalent to one in the list we gave, we just need to show that any two algebras in the list are not stably equivalent of Morita type.

We prove that for different parameter $s \neq t$, $D(2B)^{1,s}(1)$ is NOT stably equivalent of Morita type to $D(2B)^{1,t}(1)$. To this end, one computes the dimension of the stable centre, that is, the quotient of the centre by the projective centre. By Proposition 1.4, for a symmetric algebra, the dimension of the projective centre is the p -rank of the Cartan matrix, where p is the characteristic of the ground field, which is two for tame blocks. We have thus that for $A = D(2B)^{1,2^{n-2}}(1)$, $\dim Z^{st}(A) = 2^{n-2} + 2$ for $n \geq 3$. Since $n \geq 3$ this dimension distinguishes two algebras with different parameters in this class. Another way to see this is to use the absolute value of the determinant of the Cartan matrix, which is invariant under stable equivalences of Morita type, by a result of Chang-Chang Xi ([17, Proposition 5.1]). In fact, the absolute value of the determinant of the Cartan matrix of $D(2B)^{1,s}(1)$ is $4s$.

Now consider other classes of algebras. Pogorzały proved the Auslander-Reiten conjecture for self-injective special biserial algebras ([15, Theorem 0.1]), that is, if two self-injective special biserial algebras are stably equivalent, they have the same number of non projective simple modules. Thus two indecomposable non-simple self-injective special biserial algebras with different numbers of simple modules cannot be stably equivalent. By [13, Corollary 1.2], we know that for symmetric algebras, this is equivalent to say that their centre have the same dimension. Now by computing the dimension of the centre, we obtain easily that the number of simple modules and the defect n characterise

equivalence classes under stable equivalences of Morita type of dihedral blocks which are special biserial. One can also use the computations of Holm about Hochschild cohomology of dihedral blocks resumed in Section 3.2 to distinguish dihedral blocks with one simple module from those with three simple modules. \square

3.4. Blocks with semi-dihedral defect groups.

Proposition 3.3. *Let K be an algebraically closed field of characteristic 2 and let A be a semi-dihedral block of defect $n \geq 4$. Then A is stably equivalent of Morita type to one of the following algebras: $SD(1A)^{2^{n-2}}$ with $n \geq 4$; $SD(2B)_1^{1,2^{n-2}}(c)$ with $n \geq 4$, $c \in \{0, 1\}$; $SD(2B)_2^{2,2^{n-2}}(c)$ with $n \geq 4$, $c \in \{0, 1\}$; $SD(3K)^{2^{n-2},2,1}$, $n \geq 4$.*

Remark 3.4. (1) The list of algebras occurring as blocks of group algebras is taken from [5].
 (2) In the above classification we have two problems still. There is a scalar problem, that is, as in the case of derived equivalence classification, we cannot determine whether for different values of c , $SD(2B)_1^{1,2^{n-2}}(0)$ (resp. $SD(2B)_2^{2,2^{n-2}}(0)$) is not stably equivalent of Morita type to $SD(2B)_1^{1,2^{n-2}}(1)$ (resp. $SD(2B)_2^{2,2^{n-2}}(1)$).
 Moreover, we do not know whether the two algebras $SD(2B)_1^{1,2^{n-2}}(c_1)$ and $SD(2B)_2^{1,2^{n-2}}(c_2)$ are stably equivalent of Morita type. Therefore, up to these problems, the derived classification coincides with the classification up to stable equivalences of Morita type.

Proof Since a derived equivalence between self-injective algebras induces a stable equivalence of Morita type, the statement of the proposition is true simply by the derived equivalence classification of Thorsten Holm. We now prove that the classification is complete up to the problems cited above.

By the result of Thorsten Holm on Hochschild cohomology of semi-dihedral blocks, a semi-dihedral block with one simple module can not be stably equivalent of Morita type to a semi-dihedral block with three simple modules. The dimension of the stable centre of $SD(1A)^{2^{n-2}}$ with $n \geq 4$ is $2^{n-2} + 3$, it is $2^{n-2} + 2$ for $SD(2B)_1^{1,2^{n-2}}(c)$ and is $2^{n-2} + 4$ for $SD(2B)_2^{2,2^{n-2}}(c)$, while for $SD(3K)^{2^{n-2},2,1}$, it is $2^{n-2} + 3$. This invariant distinguishes semi-dihedral blocks with two simple modules from those with one or three simple modules and it also distinguishes $SD(2B)_1^{1,2^{n-2}}(c)$ from $SD(2B)_2^{2,2^{n-2}}(c)$. \square

3.5. Blocks with quaternion defect groups.

Proposition 3.5. *Let K be an algebraically closed field of characteristic 2 and let A be a block with generalised quaternion defect groups of defect $n \geq 3$. Then A is stably equivalent of Morita type to one of the following algebras: $Q(1A)^{2^{n-2}}$ with $n \geq 3$; $Q(2B)_1^{2,2^{n-2}}(a, c)$ with $n \geq 3$, $a \in K^*$, $c \in K$; $Q(3K)^{2^{n-2},2,2}$ with $n \geq 3$.*

Remark 3.6. The above classification is complete up to some scalar problem, that is, as in the case of derived equivalence classification, we cannot determine whether for different values of a and c , $Q(2B)_1^{2,2^{n-2}}(a, c)$ is not stably equivalent of Morita type to $Q(2B)_1^{2,2^{n-2}}(a', c')$. Therefore, up to these scalar problems, the derived classification coincides with the classification up to stable equivalences of Morita type.

Proof Since a derived equivalence between self-injective algebras induces a stable equivalence of Morita type, the statement of the proposition is true simply by the derived equivalence classification of Thorsten Holm. We now prove that the classification is complete up to the scalar problem.

The dimension of the stable centre is $2^{n-2} + 3$ for $Q(1A)^{2^{n-2}}$, is $2^{n-2} + 4$ for $Q(2B)_1^{2,2^{n-2}}(a, c)$ and is $2^{n-2} + 5$ for $Q(3K)^{2^{n-2},2,2}$. This invariant thus distinguishes these algebras up to stable equivalences of Morita type up to the scalar problem. One can also use the result of Thorsten Holm on Hochschild cohomology of blocks with generalised quaternion defect groups to distinguish blocks with generalised quaternion defect groups having one simple module from those having three simple modules. \square

4. ALGEBRAS OF DIHEDRAL TYPE

We classify algebras of dihedral type up to stable equivalences of Morita type in this section. Notice that all algebras except $B_1 = K[X, Y]/(X^2, Y^2 - XY)$ and $D(1\mathcal{A})_2^k(d)$ are special biserial. By the result of Pogorzały [15, Theorem 0.1], one only needs to consider separately dihedral algebras with one, two or three simple modules.

4.1. One simple module. Let K be an algebraically closed field of characteristic $p \geq 0$. By the classification of Erdmann ([3]), an algebra of dihedral type with one simple module is Morita equivalent to one of the following algebras.

$$A_1(m, n) = K[X, Y]/(XY, X^m - Y^n) \text{ with } m \geq n \geq 2 \text{ and } m + n > 4;$$

$$C_1 = K[X, Y]/(X^2, Y^2);$$

$$D(1\mathcal{A})_1^k = K\langle X, Y \rangle / (X^2, Y^2, (XY)^k - (YX)^k) \text{ with } k \geq 2;$$

and if $p = 2$,

$$B_1 = K[X, Y]/(X^2, Y^2 - XY);$$

and

$$D(1\mathcal{A})_2^k(d) = K\langle X, Y \rangle / (X^2 - (XY)^k, Y^2 - d(XY)^k, (XY)^k - (YX)^k, (XY)^k X, (YX)^k Y).$$

Proposition 4.1. *Let K be an algebraically closed field of characteristic $p \geq 0$ and let A be an algebra of dihedral type with one simple module. Then A is stably equivalent of Morita type to one and exactly one algebra in the following list:*

- $A_1(n, m)$ with $m \geq n \geq 2$ and $m + n > 4$;
- C_1 ;
- $D(1\mathcal{A})_1^k$ with $k \geq 2$;
- if $p = 2$, B_1 and $D(1\mathcal{A})_2^k(d)$ with $k \geq 2$ and $d \in \{0, 1\}$, except that we don't know whether $D(1\mathcal{A})_2^k(0)$ and $D(1\mathcal{A})_2^k(1)$ are stably equivalent of Morita type or not.

The proof combines the following five claims below using some invariants of these algebras shown in the following table.

Characteristic zero case

algebra A	$A_1(m, n)$	C_1	$D(1\mathcal{A})_1^k$
$\dim Z(A)$	$n + m$	4	$k + 3$
$\dim Z^{pr}(A)$	1	1	1
$\dim Z^{st}(A)$	$n + m - 1$	3	$k + 2$
C_A	$[n + m]$	[4]	$[4k]$
G_0^{st}	$\mathbb{Z}/(n + m)$	$\mathbb{Z}/4$	$\mathbb{Z}/4k$

Characteristic two case

algebra A	$A_1(m, n)$	C_1	$D(1\mathcal{A})_1^k$	B_1	$D(1\mathcal{A})_2^k(d)$
$\dim Z(A)$	$n + m$	4	$k + 3$	4	$k + 3$
$\dim Z^{pr}(A)$	0 or 1	0	0	0	0
$\dim Z^{st}(A)$	$n + m$ or $n + m - 1$	4	$k + 3$	4	$k + 3$
C_A	$[n + m]$	[4]	$[4k]$	[4]	$[4k]$
G_0^{st}	$\mathbb{Z}/(n + m)$	$\mathbb{Z}/4$	$\mathbb{Z}/4k$	$\mathbb{Z}/4$	$\mathbb{Z}/4k$

Characteristic $p > 2$ case

algebra A	$A_1(m, n)$	C_1	$D(1\mathcal{A})_1^k$
$\dim Z(A)$	$n + m$	4	$k + 3$
$\dim Z^{pr}(A)$	0 or 1	1	0 or 1
$\dim Z^{st}(A)$	$n + m$ or $n + m - 1$	3	$k + 3$ or $k + 2$
C_A	$[n + m]$	[4]	$[4k]$
G_0^{st}	$\mathbb{Z}/(n + m)$	$\mathbb{Z}/4$	$\mathbb{Z}/4k$

By the result of Pogorzały ([15, Theorem 7.3]), we only need to compare $A_1(m, n)$, C_1 with $D_1(k)$, since they are special biserial, and compare B_1 with $D_1(1\mathcal{A})^k(d)$, since they are not special biserial.

Claim 1. C_1 cannot be stably equivalent of Morita type to $A_1(m, n)$ or $D(1\mathcal{A})_1^k$.

Comparing the stable Grothendieck groups gives the result. Since $m + n > 4$ and $k \geq 2$.

Similarly one proves

Claim 1'. B_1 cannot be stably equivalent of Morita type to $D(1\mathcal{A})_2^k(d)$.

Claim 2. $A_1(m, n)$ cannot be stably equivalent of Morita type to $D(1\mathcal{A})_1^k$.

Compare their stable centres and their stable Grothendieck groups.

Claim 3. $A = A_1(m, n)$ is not stably equivalent of Morita type to $A' = A_1(m', n')$ for $(m, n) \neq (m', n')$

Now suppose that $A = A_1(m, n)$ is stably equivalent of Morita type to $A' = A_1(m', n')$, then by comparing their stable Grothendieck groups, $n + m = n' + m'$. The Loewy length of the stable centre of $A(m, n)$ is $\max(m, n)$ if the characteristic p divides $m + n$ and is $\max(m, n) + 1$, otherwise. Thus the stable centres are not isomorphic.

Claim 4. $D(1\mathcal{A})_1^k$ cannot be stably equivalent of Morita type to $D(1\mathcal{A})_1^l$ for $k \neq l$

Comparing the orders of the stable Grothendieck groups gives the result.

Claim 5. $D(1\mathcal{A})_2^k(d)$ cannot be stably equivalent of Morita type to $D(1\mathcal{A})_2^l(d)$ for $k \neq l$.

Consider the stable Grothendieck groups or the stable centres.

4.2. Two simple modules. For algebras of dihedral type with two simple modules, we have the following result of Holm.

Proposition 4.2. ([5, Proposition 2.3.1]) *Let K be an algebraically closed field of characteristic $p \geq 0$ and let A be an algebra of dihedral type with two simple module. Then A is derived equivalent to $D(2\mathcal{B})^{k,s}(0)$ with $k \geq s \geq 1$ or ($p = 2$ and $D(2\mathcal{B})^{k,s}(1)$ with $k \geq s \geq 1$).*

Proposition 4.3. *Let K be an algebraically closed field of characteristic $p \geq 0$ and let A be an algebra of dihedral type with two simple module. Then A is stably equivalent of Morita type to one and exactly one of the following algebras: $D(2\mathcal{B})^{k,s}(0)$ with $k \geq s \geq 1$ or if $p = 2$, $D(2\mathcal{B})^{k,s}(1)$ with $k \geq s \geq 1$.*

Proof By the result of Pogorzały ([15, Theorem 7.3]), in case of characteristic two, the algebras $D(2\mathcal{B})^{k,s}(0)$ and $D(2\mathcal{B})^{k,s}(1)$ are not stably equivalent of Morita type.

Now for any characteristic p and for different parameters $(k, s) \neq (k', s')$ such $k \geq s \geq 1$ and $k' \geq s' \geq 1$, if $D(2\mathcal{B})^{k,s}(c)$ is stably equivalent to $D(2\mathcal{B})^{k',s'}(c)$, then comparing the dimension of the centre modulo the Reynolds ideal gives $k + s = k' + s'$. Since the absolute values of the determinants of the Cartan matrices are the same, we get $ks = k's'$. This implies that $k = k'$ and $s = s'$. \square

4.3. Three simple modules.

Proposition 4.4. *Let K be an algebraically closed field of characteristic $p \geq 0$ and let A be an algebra of dihedral type with two simple modules. Then A is stably equivalent of Morita type to one and exactly one of the following algebras: $D(3\mathcal{K})^{a,b,c}$ with $a \geq b \geq c \geq 1$ or $D(3\mathcal{R})^{k,s,t,u}$ with $s \geq t \geq u \geq k \geq 1$ and $t \geq 2$.*

Proof Holm shows [5, page 58] that the stable Auslander-Reiten quivers of algebras of type $D(3\mathcal{K})^{a,b,c}$ and of algebras of type $D(3\mathcal{R})^{k,s,t,u}$ is different. Hence algebras of these two types cannot be stably equivalent of Morita type.

Again, we consider different parameters of type $D(3\mathcal{K})^{a,b,c}$ or of type $D(3\mathcal{R})^{k,s,t,u}$. By Theorem 1.6, one can use the algebra structure of the centre modulo the Reynolds ideal to distinguish stable equivalences classes of Morita type.

Using the the explicit basis of the centres (Holm[5, Lemma 2.3.16]) allows to determine the the quotient

$$Z(D(3\mathcal{K})^{a,b,c})/R(D(3\mathcal{K})^{a,b,c}) \simeq K[A, B, C]/(A^a, B^b, C^c, AB, AC, BC)$$

and hence two algebras of type $D(3\mathcal{K}^{a,b,c})$ can only be stably equivalent of Morita type if the parameters a, b, c are equal (cf Theorem 1.6).

Using (Holm[5, Lemma 2.3.17]), we also get that

$$Z(D(3\mathcal{R})^{k,s,t,u})/R(D(3\mathcal{R})^{k,s,t,u}) \simeq K[U, V, W, T]/(U^s, V^t, W^u, T^k, UV, UW, UT, VW, VT, WT)$$

and again two algebras of type $D(3\mathcal{R})^{k,s,t,u}$ can only be stably equivalent of Morita type if the parameters coincide. \square

Although our above result is only a complete classification up to a scalar problem in one simple module case, we can prove nevertheless the following special case of the Auslander-Reiten conjecture.

Corollary 4.5. *Let A be an indecomposable algebra which is stably equivalent of Morita type to an algebra of dihedral type. Then this algebra has the same number of simple modules as the algebra of dihedral type.*

Proof By Proposition 2.1, A is necessarily of dihedral type. Then apply our classification results above. Notice that although we cannot determine whether $D(1\mathcal{A})^k(0)$ and $D(1\mathcal{A})^k(1)$ are stably equivalent of Morita type or not, they have the same number of simple modules. \square

5. CENTRES OF SEMI-DIHEDRAL AND QUATERNION TYPE ALGEBRAS

We shall study the centres and the stable centres of the involved algebras.

5.1. Semi-dihedral type. An algebra of semi-dihedral type with one simple module is Morita equivalent to $SD(1\mathcal{A})_1^k$ with $k \geq 2$ or to (in case of characteristic 2) $SD(1\mathcal{A})_2^k(c, d)$ with $k \geq 2$ and $(c, d) \neq (0, 0)$. Recall from [3, Corollary III.1.3] that for each of these algebras, the centre has dimension $4k$ and the dimension of the centre is $k + 3$. Denote by A one of the above algebras. The centre $Z(A)$ has a K -basis given by

$$\{1; (XY)^i + (YX)^i; (XY)^k; X(YX)^{k-1}; (YX)^{k-1}Y \mid 1 \leq i \leq k-1\}$$

Lemma 5.1. *Let K be an algebraically closed field and let A be one of the algebras $SD(1\mathcal{A})_1^k$ with $k \geq 2$ or (in case of characteristic 2) $SD(1\mathcal{A})_2^k(c, d)$ with $k \geq 2$ and $(c, d) \neq (0, 0)$.*

If K is of characteristic 2, then

$$Z(A) \simeq K[U, T, V, W]/(U^k, T^2, V^2, W^2, UT, UV, UW, TV, TW, VW)$$

and $R(A) = Z(A) \cap \text{Soc}(A) = K \cdot T$.

If K is of characteristic different from 2, then

$$Z(A) \simeq K[U, V, W]/(U^{k+1}, V^2, W^2, UV, UW, VW)$$

and $R(A) = Z(A) \cap \text{Soc}(A) = K \cdot U^k$.

Proof We need to identify U with $XY + YX$, observe that $((XY) + (YX))^i = (XY)^i + (YX)^i$, and identify T with $(XY)^k$ and V and W with the other two remaining elements. If K is of characteristic 2, then $U^k = (XY)^k + (YX)^k = 0$, and if K is of characteristic different from 2, then $U^k = (XY)^k + (YX)^k = 2(XY)^k \neq 0$. \square

The Cartan matrix of the algebra A is the matrix $(4k)$ of size 1×1 . Recall that the dimension of the projective centre is the p -rank of the Cartan matrix where p is the characteristic of the base field. If the characteristic of K divides $4k$, then the p -rank of the Cartan matrix is 0, and is 1 otherwise.

Remark 5.2. Using the dimension of the center modulo the Reynolds ideal, we see that different values of k give different stable equivalent classes of Morita type for the above algebras.

Now we turn to the cases of two simple modules. An algebra of semi-dihedral type with two simple modules is derived equivalent to $SD(2\mathcal{B})_1^{k,s}(c)$ with $k \geq 1$, $s \geq 2$ and $c \in \{0, 1\}$ or to $SD(2\mathcal{B})_2^{k,s}(c)$ with $k \geq 1$, $s \geq 2$, $k + s \geq 4$ and $c \in \{0, 1\}$.

Lemma 5.3. *Let A be the algebra $SD(2\mathcal{B})_1^{k,s}(c)$ or the algebra $SD(2\mathcal{B})_2^{k,s}(c)$.*

(1) If K is of characteristic 2, then

$$Z(A) \simeq K[u, v, w, t]/(u^k - v^s, w^2, t^2, uv, uw, vw, tw, ut, vt)$$

and $R(A) = K \cdot u^k \oplus K \cdot w$.

(2) If K is of characteristic different from 2, then

$$Z(A) \simeq K[u, v, t]/(u^{k+1}, v^{s+1}, t^2, uv, ut, vt)$$

and $R(A) = K \cdot u^k \oplus K \cdot v^s$.

Proof By [3, IX 1.2 LEMMA], a basis of the centre of $SD(2\mathcal{B})_1^{k,s}(c)$ is given by

$$\{1; (\alpha\beta\gamma)^i + (\beta\gamma\alpha)^i + (\gamma\alpha\beta)^i; (\beta\gamma\alpha)^{k-1}\beta\gamma; (\alpha\beta\gamma)^k; \eta^j \mid 1 \leq i \leq k-1; 1 \leq j \leq s\}$$

Now let

$$u = \alpha\beta\gamma + \beta\gamma\alpha + \gamma\alpha\beta, v = \eta, t = (\beta\gamma\alpha)^{k-1}\beta\gamma, w = (\alpha\beta\gamma)^k.$$

If $\text{char}(K) = 2$, then $u^t = v^s$; otherwise, $u^t = v^s + 2w$. Hence, w may be eliminated from the relations by the equation $u^k = v^s + 2w$ in case $\text{char}(K) \neq 2$. It is easy to verify all other relations. An argument of comparing dimensions gives the result.

As for $SD(2\mathcal{B})_2^{k,s}(c)$, by [3, IX 1.2 LEMMA], a basis of the centre of $SD(2\mathcal{B})_1^{k,s}(c)$ is given by

$$\{1; (\alpha\beta\gamma)^i + (\beta\gamma\alpha)^i + (\gamma\alpha\beta)^i; (\beta\gamma\alpha)^{k-1}\beta\gamma; (\alpha\beta\gamma)^k; \eta + (\alpha\beta\gamma)^{k-1}\alpha; \eta^j \mid 1 \leq i \leq k-1; 2 \leq j \leq s\}$$

Now let

$$u = \alpha\beta\gamma + \beta\gamma\alpha + \gamma\alpha\beta, v = \eta + (\alpha\beta\gamma)^{k-1}\alpha, t = (\beta\gamma\alpha)^{k-1}\beta\gamma, w = (\alpha\beta\gamma)^k.$$

Similar argument as above gives the result. \square

It is important to know that in this presentation the element t is not in the socle of $SD(2\mathcal{B})_1^{k,s}(c)$ and can therefore not be in the projective centre (cf Proposition 1.5).

The Cartan matrix of $SD(2\mathcal{B})_1^{k,s}(c)$ with $k \geq 1, s \geq 2$ and $c \in \{0, 1\}$ and of $SD(2\mathcal{B})_2^{k,s}(c)$ with $k \geq 1, s \geq 2, k+s \geq 4$ and $c \in \{0, 1\}$ is

$$\begin{pmatrix} 4k & 2k \\ 2k & s+k \end{pmatrix}$$

The determinant is $4ks$. If the base field is of characteristic 2, then the 2-rank is 1 if and only if $k+s$ is odd, 0 else. If the base field is of characteristic $p > 2$, then the p -rank of the Cartan matrix is 1 if and only if p divides k or p divides s but not both; the p -rank is 0 if and only if p divides k and s ; and the p -rank is 2 if and only if p divides neither k nor s . If the characteristic of the field is 0, then the rank of the Cartan matrix is 2.

Recall that Holm proved in [5, Lemma 2.4.16] that the centre of $SD(3\mathcal{K})^{a,b,c}$ with $a \geq b \geq c \geq 1$ and $a \geq 2$ has a basis given by

$$\{1, (\beta\gamma + \gamma\beta)^{i_1}; (\kappa\lambda + \lambda\kappa)^{i_2}; (\delta\eta + \eta\delta)^{i_3}; (\lambda\kappa)^b; (\beta\gamma)^b; (\delta\eta)^c \mid 1 \leq i_1 < a, 1 \leq i_2 < b, 1 \leq i_3 \leq c\}$$

and so

Lemma 5.4.

$$\begin{aligned} Z(SD(3\mathcal{K})^{a,b,c}) &\simeq K[A, B, C, S_1, S_2, S_3]/(A^{a+1}, B^{b+1}, C^{c+1}, A^a - S_2 - S_3, B^b - S_3 - S_1, \\ &\quad C^c - S_1 - S_2, AS_i, BS_i, CS_i, S_i S_j, AB, AC, BC; i, j \in \{1, 2, 3\}) \end{aligned}$$

and $R(SD(3\mathcal{K})^{a,b,c}) = K \cdot A^a \oplus K \cdot B^b \oplus K \cdot C^c$.

Proof Let

$$A = \beta\gamma + \gamma\beta, B = \kappa\lambda + \lambda\kappa, C = \delta\eta + \eta\delta, S_1 = \lambda\beta\delta, S_2 = \delta\lambda\beta, S_3 = \beta\delta\lambda.$$

Then it is a straight forward verification that A, B, C, S_1, S_2, S_3 satisfy the relations on the right-handed side. Now the isomorphism follows from a dimension argument. \square

Corollary 5.5. *Let $a \geq b \geq c \geq 1$ with $a \geq 2$ and let $a' \geq b' \geq c' \geq 1$ with $a' \geq 2$. Then $SD(3\mathcal{K})^{a,b,c}$ is stably equivalent of Morita type to $SD(3\mathcal{K})^{a',b',c'}$ if and only if $a = a'$, $b = b'$ and $c = c'$.*

Proof As in the proof of Proposition 4.4, one can consider the centre modulo the Reynolds ideal. The Reynolds ideal of the centre is of dimension three and is spanned by the elements S_1, S_2, S_3 . Hence

$$Z(SD(3\mathcal{K})^{a,b,c})/R(SD(3\mathcal{K})^{a,b,c}) \simeq K[A, B, C]/(A^a, B^b, C^c, AB, AC, BC)$$

so that an isomorphism of the centres modulo the Reynolds ideal implies that the parameters are identical. The statement then follows from Theorem 1.6. \square

The Cartan matrix of $SD(3\mathcal{K})^{a,b,c}$ equals (cf [3])

$$\begin{pmatrix} a+b & a & b \\ a & a+c & c \\ b & c & b+c \end{pmatrix}$$

which has determinant $4abc$. Since all coefficients of the Cartan matrix are positive integers, the rank of the Cartan matrix for fields of characteristic 0 is always 3.

Suppose that K is a base field of characteristic $p > 2$.

The p -rank 0 occurs if and only if all parameters a, b, c are divisible by p ;

the p -rank 1 occurs if and only if exactly one parameter a, b, c is not divisible by p ;

the p -rank is 2 if and only if exactly one of the parameters is divisible by p ;

the p -rank is 3 if and only if p doesn't divide abc .

Suppose that K is a base field of characteristic 2.

The 2-rank 0 occurs if and only if all parameters a, b, c are all even;

the 2-rank 1 occurs if and only if exactly one parameter a, b, c is odd;

the 2-rank is 2 if and only if at least two of a, b, c are odd.

5.2. Quaternion type. An algebra of quaternion type with one simple module is Morita equivalent to $Q(1\mathcal{A})_1^\ell$ with $\ell \geq 2$ or to $Q(1\mathcal{B})_2^\ell(c, d)$ with $\ell \geq 2$ and $(c, d) \neq (0, 0)$. Again, by [3] the centre is of dimension $\ell + 3$ and the algebra is of dimension 4ℓ . Let A be one of the above algebras. In the above presentation, the centre has a K -basis given by

$$\{1; (XY)^i + (YX)^i; (XY)^\ell; X(YX)^{\ell-1}; (YX)^{\ell-1}Y \mid 1 \leq i \leq \ell - 1\}.$$

Lemma 5.6. (1) *If K is of characteristic 2, then*

$$Z(A) \simeq K[U, T, V, W]/(U^k, T^2, V^2, W^2, UT, UV, UW, TV, TW, VW)$$

and $R(A) = Z(A) \cap \text{soc}(A) = K \cdot T$.

(2) *If K is of characteristic different from 2, then*

$$Z(A) \simeq K[U, V, W]/(U^{k+1}, V^2, W^2, UV, UW, VW)$$

and $R(A) = Z(A) \cap \text{soc}(A) = K \cdot U^k$.

Proof The proof is a straight forward verification. \square

An algebra of quaternion type with two simple modules is derived equivalent to $Q(2\mathcal{B})_1^{k,s}(a, c)$ with $k \geq 1$, $s \geq 3$ and $a \neq 0$. By [3, IX 1.2 LEMMA], the centre of $Q(2\mathcal{B})_1^{k,s}(a, c)$ has a basis

$$\{1; (\alpha\beta\gamma)^i + (\beta\gamma\alpha)^i + (\gamma\alpha\beta)^i, (\beta\gamma\alpha)^{k-1}\beta\gamma, (\alpha\beta\gamma)^k, \eta + (\alpha\beta\gamma)^{k-1}\alpha, \eta^j \mid 1 \leq i \leq k-1; 2 \leq j \leq s\}.$$

By a similar proof as that of Proposition 5.3, we have

Lemma 5.7. (1) *If $\text{char}(K) = 2$, then*

$$Z(Q(2\mathcal{B})_1^{k,s}(a, c)) \simeq K[u, v, w, t]/(u^k - v^s, w^2, t^2, uv, uw, vw, tw, ut, vt)$$

and $R(A) = Z(A) \cap \text{soc}(A) = K \cdot u^k \oplus K \cdot w$.

(2) *If $\text{char}(K) \neq 2$, then*

$$Z(Q(2\mathcal{B})_1^{k,s}(a, c)) \simeq K[u, v, t]/(u^{k+1}, v^{s+1}, t^2, uv, ut, vt)$$

and $R(A) = Z(A) \cap \text{soc}(A) = K \cdot u^k \oplus K \cdot v^s$.

The Cartan matrix of $Q(2\mathcal{B})_1^{k,s}(a, c)$ is

$$\begin{pmatrix} 4k & 2k \\ 2k & k+s \end{pmatrix}.$$

An algebra of quaternion type with three simple modules is derived equivalent to $Q(3\mathcal{K})^{a,b,c}$ with $a \geq b \geq c \geq 1$, $b \geq 2$ and $(a, b, c) \neq (2, 2, 1)$ or to $Q(3\mathcal{A})_1^{2,2}(d)$ with $d \notin \{0, 1\}$.

The dimension of the centre of $Q(3\mathcal{K})^{a,b,c}$ is $a + b + c + 1$ and has a basis

$$\{1, (\beta\gamma + \gamma\beta)^{i_1}, (\kappa\lambda + \lambda\kappa)^{i_2}, (\delta\eta + \eta\delta)^{i_3}; (\lambda\kappa)^b; (\beta\gamma)^b; (\delta\eta)^c \mid 1 \leq i_1 < a; 1 \leq i_2 < b; 1 \leq i_3 \leq c\}.$$

The Cartan matrix of the algebra $Q(3\mathcal{K})^{a,b,c}$ is

$$\begin{pmatrix} a+b & a & b \\ a & a+c & c \\ b & c & b+c \end{pmatrix}$$

The dimension of the centre of $Q(3\mathcal{A})_1^{2,2}(d)$ is 6 and has a basis

$$\{1, \beta\gamma + \gamma\beta + d\eta\delta, \beta\gamma + \eta\delta + \delta\eta, (\beta\gamma)^2, (\gamma\beta)^2 = d(\delta\eta)^2, (\eta\delta)^2\}.$$

The Cartan matrix of $Q(3\mathcal{A})_1^{2,2}(d)$ is

$$\begin{pmatrix} 4 & 2 & 2 \\ 2 & 3 & 1 \\ 2 & 1 & 3 \end{pmatrix}$$

Indeed, the fact that the above elements are central as is readily verified and the dimensions are as they should be. The statement on the Cartan matrix is taken from [3].

Lemma 5.8. *We have*

$$\begin{aligned} Z(Q(3\mathcal{K})^{a,b,c}) &\simeq K[A, B, C, S_1, S_2, S_3]/(A^{a+1}, B^{b+1}, C^{c+1}, A^a - S_2 - S_3, B^b - S_3 - S_1, \\ &\quad C^c - S_1 - S_2, AS_i, BS_i, CS_i, S_i S_j, AB, AC, BC; i, j \in \{1, 2, 3\}) \\ Z(Q(3\mathcal{A})_1^{2,2}(d)) &\simeq K[A, B, C, S_1, S_2, S_3]/(A^3, B^3, C^2, A^2 - S_2 - S_3, B^2 - S_3 - S_1, \\ &\quad C - S_1 - S_2, AS_i, BS_i, CS_i, S_i S_j, AB, AC, BC; i, j \in \{1, 2, 3\}) \end{aligned}$$

Proof The proof for $Q(3\mathcal{K})^{a,b,c}$ is identical to the one of Lemma 5.4. For $Q(3\mathcal{A})_1^{2,2}(d)$, let

$$\begin{aligned} A &:= \beta\gamma + \gamma\beta + d\eta\delta, \\ B &:= \beta\gamma + \eta\delta + \delta\eta, \\ C &:= (1-d)(\delta\eta)^2 + d^2(\eta\delta)^2, \\ S_1 &:= (1-d)(\delta\eta)^2, \\ S_2 &:= d^2(\eta\delta)^2, \\ S_3 &:= (\beta\gamma)^2 + d(\delta\eta)^2. \end{aligned}$$

The rest is a straight forward verification. □

Note that, in order to simplify the notation we may put $Q(3\mathcal{K})^{2,2,1} = Q(3\mathcal{A})_1^{2,2}(d)$.

Corollary 5.9. *Let $a \geq b \geq c \geq 1$ with $b \geq 2$ and let $a' \geq b' \geq c' \geq 1$ with $b' \geq 2$. Then $Q(3\mathcal{K})^{a,b,c}$ is stably equivalent of Morita type to $Q(3\mathcal{K})^{a',b',c'}$ if and only if $a = a'$, $b = b'$ and $c = c'$.*

Proof As in the proof of Proposition 4.4, one can consider the centre modulo the Reynolds ideal. The Reynolds ideal of the centre is of dimension three and is spanned by the elements S_1, S_2, S_3 and as in the proof of the semi-dihedral type

$$Z(Q(3\mathcal{K})^{a,b,c})/R(Q(3\mathcal{K})^{a,b,c}) \simeq K[A, B, C]/(A^a, B^b, C^c, AB, AC, BC)$$

so that as in the semi-dihedral case an isomorphism of the centres modulo the Reynolds ideal implies that the parameters are identical. Finally apply Theorem 1.6. □

6. ALGEBRAS WITH STABLE CENTRES AND CARTAN DATA AS FOR SEMI-DIHEDRAL AND QUATERNION TYPE; STABLE EQUIVALENCES

For $a, b, c \geq 1$, **let** $A_3^{a,b,c}$ be a basic indecomposable symmetric K -algebra with centre isomorphic to

$$Z(A_3^{a,b,c}) \simeq K[A, B, C, S_1, S_2, S_3] / (A^{a+1}, B^{b+1}, C^{c+1}, A^a - S_2 - S_3, B^b - S_3 - S_1, C^c - S_1 - S_2, AS_i, BS_i, CS_i, S_i S_j, AB, AC, BC; i, j \in \{1, 2, 3\})$$

and Cartan matrix

$$\begin{pmatrix} a+b & a & b \\ a & a+c & c \\ b & c & b+c \end{pmatrix}$$

and the Reynolds ideal $R(A_3^{a,b,c}) = KA^a \oplus KB^b \oplus KC^c$.

For $k, s \geq 1$, **let** $A_2^{k,s}$ be a basic indecomposable symmetric algebra with Cartan matrix

$$\begin{pmatrix} 4k & 2k \\ 2k & s+k \end{pmatrix}$$

so that in case K is of characteristic 2, the centre

$$Z(A_2^{k,s}) \simeq K[u, v, w, t] / (u^k - v^s, w^2, t^2, uv, uw, vw, tw, ut, vt)$$

and $R(A_2^{k,s}) = Ku^k \oplus Kw$ and if K is of characteristic different from 2, then

$$Z(A_2^{k,s}) \simeq K[u, v, t] / (u^{k+1}, v^{s+1}, t^2, uv, ut, vt)$$

and $R(A_2^{k,s}) = Ku^k \oplus Kv^s$

For $\ell \geq 2$, **let** A_1^ℓ be a basic indecomposable symmetric algebra of dimension 4ℓ so that in case K is of characteristic 2,

$$Z(A_1^\ell) \simeq K[U, T, V, W] / (U^\ell, T^2, V^2, W^2, UT, UV, UW, TV, TW, VW)$$

and the Reynolds ideal $R(A_1^\ell) = K \cdot T$ and if K is of characteristic different from 2, then

$$Z(A_1^\ell) \simeq K[U, V, W] / (U^{\ell+1}, V^2, W^2, UV, UW, VW)$$

and $R(A_1^\ell) = K \cdot U^\ell$.

6.1. Two simples versus three simples; characteristic different from 2. Concerning the relations of $Z(A_3^{a,b,c})$ we see that the elements $S_1 + S_2, S_2 + S_3, S_3 + S_1$ generate the same space as S_1, S_2, S_3 , which is the whole socle of the algebra, if and only if

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

is a regular matrix. This is the case if and only if K is a field of characteristic different from 2. Therefore if the characteristic of the base field is different from 2, we get

$$Z(A_3^{a,b,c}) \simeq K[A, B, C] / (A^{a+1}, B^{b+1}, C^{c+1}, AB, AC, BC).$$

Lemma 6.1. *Let K be an algebraically closed field of characteristic $p > 2$ or of characteristic 0. Suppose $p \nmid (ks)$. Then $A_2^{k,s}$ and $A_3^{a,b,c}$ cannot be stably equivalent of Morita type.*

Proof If K is of characteristic 0, then the rank of the Cartan matrix of $A_3^{a,b,c}$ is 3 and the rank of the Cartan matrix of $A_2^{k,s}$ is 2. Hence

$$\begin{aligned} Z^{st}(A_2^{k,s}) &= K[u, v, t] / (u^k, v^s, t^2, uv, ut, vt) \\ Z^{st}(A_3^{a,b,c}) &= K[A, B, C] / (A^a, B^b, C^c, AB, AC, BC) \end{aligned}$$

The same holds if $p \nmid (ks)$ because then $abc = ks$ since the Cartan determinants coincide, and since therefore the Cartan matrices are regular. Hence in order to get the stable centres isomorphic we may assume $c = 2, a = k, b = s$, else a permutation of the letters a, b, c will do. Hence $ks = abc$ becomes $ks = 2ks$, a contradiction.

□

Lemma 6.2. *Let K be an algebraically closed field of characteristic $p > 2$. Suppose $p|k$ and $p|s$. Then $A_2^{k,s}$ and $A_3^{a,b,c}$ cannot be stably equivalent of Morita type.*

Proof Suppose that the algebras are stably equivalent of Morita type. We know that the Cartan determinants coincide and hence $p^2|abc$. If p divides a and b and c , then the Cartan matrix of $A_3^{a,b,c}$ has three elementary divisors divisible by p , which implies that the stable Grothendieck group of $A_3^{a,b,c}$ tensored by K has rank 3. This gives a contradiction since the stable Grothendieck group of $A_2^{k,s}$ tensored by K can only be of rank 2 at most.

Since the p -rank of the Cartan matrix of $A_2^{k,s}$ is 0, we get that

$$Z^{st}(A_2^{k,s}) = K[u, v, t]/(u^{k+1}, v^{s+1}, t^2, uv, ut, vt)$$

If p divides two of the parameters a, b, c , then

$$Z^{st}(A_3^{a,b,c}) = K[A, B, C]/(A^{a+1}, B^{b+1}, C^{c+1}, AB, AC, BC, \lambda_A A^a + \lambda_B B^b + \lambda_C C^c)$$

for some parameters $\lambda_A, \lambda_B, \lambda_C$ not all 0. If p divides only one of the parameters a, b, c , then

$$Z^{st}(A_3^{a,b,c}) = K[A, B, C]/(A^{a+1}, B^{b+1}, C^{c+1}, AB, AC, BC, \lambda_A A^a + \lambda_B B^b + \lambda_C C^c, \mu_A A^a + \mu_B B^b + \mu_C C^c)$$

for a matrix

$$\begin{pmatrix} \lambda_A & \lambda_B & \lambda_C \\ \mu_A & \mu_B & \mu_C \end{pmatrix}$$

of rank 2.

The socle of the stable centre of $A_2^{k,s}$ is three-dimensional, and so we need to assure that this is the case of the stable centre of $A_3^{a,b,c}$ as well. But this implies that the projective centre of $A_3^{a,b,c}$ is generated by B^b and C^c (say) if p divides only one of a, b, c and by C^c (say) if p divides two of the parameters a, b and c .

In the first case, p divides only one of the parameters a, b, c , we get $\{a+1, b, c\} = \{k+1, s+1, 2\}$, taken with multiplicities. If $c = 2$ (or $b = 2$, case which is studied analogously), then $abc = ks$ becomes $2ab = ks$. Moreover, $a \in \{k, s\}$ and $b \in \{k+1, s+1\}$ gives $2k(s+1) = ks$ or $2s(k+1) = ks$. Hence $ks+k = 0$ or $ks+s = 0$, a contradiction. Hence $a = 1$. But then $b = k+1$ and $c = s+1$. Now, p was assumed to divide k and s , and so p divides none of the parameters a, b, c , a contradiction to the hypothesis.

If p divides two of the parameters a, b, c , the projective centre is one-dimensional and we got that C^c , say, generates the projective centre. Then $\{a+1, b+1, c\} = \{k+1, s+1, 2\}$ again taken with multiplicities. If $c = 2$, then $k = a$ and $b = s$, say, and $abc = 2ks = ks$, a contradiction. Hence by symmetry we may assume $b = 1$. If $c = s+1$ and $k = a$, then $abc = k(s+1) = ks$ gives a contradiction; if $c = k+1$ and $a = s$ the same contradiction holds.

□

Lemma 6.3. *Let K be an algebraically closed field of characteristic $p > 2$. Suppose $p|k$ and $p \nmid s$ or $p|s$ and $p \nmid k$. Then $A_2^{k,s}$ and $A_3^{a,b,c}$ cannot be stably equivalent of Morita type.*

Proof The hypothesis implies that the projective centre of $A_2^{k,s}$ is one-dimensional, and hence there are parameters ν_u, ν_v not both 0 with

$$Z^{st}(A_2^{k,s}) = K[u, v, t]/(u^{k+1}, v^{s+1}, t^2, \nu_u u^k + \nu_v v^s, uv, ut, vt)$$

Again, as before, the stable Grothendieck groups need to be isomorphic and so not all parameters a, b, c can be divisible by p . Actually, since one of the parameters k and s is not divisible by p , one of the elementary divisors of the Cartan matrix of $A_2^{k,s}$ is not divisible by p and the other is divisible by p . Hence one of the elementary divisors of the Cartan matrix of $A_3^{a,b,c}$ is 1, one is not divisible by p and the third is divisible by p . If p divides two of the parameters a, b and c , then two elementary

divisors of the Cartan matrix of $A_3^{a,b,c}$ are divisible by p , whence a contradiction. Hence p divides exactly one of the parameters a, b, c , and the projective centre of $A_3^{a,b,c}$ is two-dimensional. We get

$$\begin{aligned} Z^{st}(A_3^{a,b,c}) &= K[A, B, C] / (A^{a+1}, B^{b+1}, C^{c+1}, AB, AC, BC, \\ &\quad \lambda_A A^a + \lambda_B B^b + \lambda_C C^c, \mu_A A^a + \mu_B B^b + \mu_C C^c) \end{aligned}$$

for a matrix

$$\begin{pmatrix} \lambda_A & \lambda_B & \lambda_C \\ \mu_A & \mu_B & \mu_C \end{pmatrix}$$

of rank 2.

Suppose $\nu_u = 0$ or $\nu_v = 0$. By symmetry we may suppose $\nu_u = 0$. Then the socle of $Z^{st}(A_2^{k,s})$ is three-dimensional. Hence in order to get this we need to have that B^b and C^c , say, generate the projective centre of $A_3^{a,b,c}$. But then $\{a+1, b, c\} = \{k, s+1, 2\}$, taken with multiplicities. The case $a = 1$ gives a contradiction to $abc = ks$ as well as the case $c = 2$ (or likewise $b = 2$).

Hence $\nu_u \neq 0 \neq \nu_v$. The socle of $A_2^{k,s}$ is two-dimensional and therefore the socle of $Z^{st}(A_3^{a,b,c})$ has to be two-dimensional as well. This implies that one of the elements A^a, B^b, C^c has to be in the projective centre, say C^c . Therefore

$$Z^{st}(A_3^{a,b,c}) = K[A, B, C] / (A^{a+1}, B^{b+1}, C^c, AB, AC, BC, \lambda_A A^a + \lambda_B B^b)$$

This give $c = 2$ and $\{a, b\} = \{k, s\}$. Now, the equality of Cartan determinants $abc = ks$ is not satisfied, a contradiction. \square

6.2. Two simples versus three simples; characteristic 2. We are now dealing with the case $p = 2$. Recall that

$$Z(A_2^{k,s}) \simeq K[u, v, w, t] / (u^k - v^s, w^2, t^2, uw, vw, tw, ut, vt)$$

and

$$\begin{aligned} Z(A_3^{a,b,c}) &\simeq K[A, B, C, S_1, S_2, S_3] / (A^{a+1}, B^{b+1}, C^{c+1}, A^a - S_1 - S_2, B^b - S_2 - S_3, \\ &\quad C^c - S_1 - S_3, AS_i, BS_i, CS_i, S_i S_j, AB, AC, BC; i, j \in \{1, 2, 3\}) \end{aligned}$$

In case $p = 2$ the subspace of the socle of the algebra generated by $S_1 + S_2, S_2 + S_3, S_1 + S_3$ is of codimension 1, namely given by the condition

$$(S_1 + S_2) + (S_2 + S_3) + (S_1 + S_3) = 0$$

and so

$$A^a + B^b + C^c = 0.$$

Hence, in characteristic 2 we get

$$Z(A_3^{a,b,c}) \simeq K[A, B, C, S] / (A^{a+1}, B^{b+1}, C^{c+1}, S^2, A^a + B^b + C^c, AS, BS, CS, AB, AC, BC)$$

We shall show

Proposition 6.4. *Let K be an algebraically closed field of characteristic 2. $A_3^{a,b,c}$ cannot be stably equivalent of Morita type to $A_2^{k,s}$.*

The proof will consist of two technical lemmata 6.5 and 6.6 contradicting each other.

Lemma 6.5. *Let K be an algebraically closed field of characteristic 2. Suppose that $A_3^{a,b,c}$ is stably equivalent of Morita type to $A_2^{k,s}$. Then k and s are both even, two of a, b, c are odd and the third is even.*

Proof Supposing that $A_3^{a,b,c}$ is stably equivalent of Morita type to $A_2^{k,s}$, then $abc = ks$. The dimensions of the centre modulo the Reynolds ideal gives $k + s = a + b + c - 2$.

If a, b and c are all even, then all elementary divisors of the Cartan matrix of the algebra A_3^{abc} are divisible by 2 and hence the stable Grothendieck group tensored by K of A_3^{abc} is of dimension 3. But the stable Grothendieck group (tensored by K) of $A_2^{k,s}$ is 2 at most. Therefore this cannot happen. We get that at least one of a, b or c is odd.

If k and s are both odd, then a, b, c are all odd, but then $k + s = a + b + c - 2$ cannot hold.

Suppose now that $ks = abc$ is even.

The stable centre of $A_2^{k,s}$ is of dimension $k + s + 2$ if $k + s$ is even and of dimension $k + s + 1$ if $k + s$ is odd. Hence the stable centre of $A_2^{k,s}$ is of even dimension in any case. If two parameters, say a and b are even, then the 2-rank of the Cartan matrix of $A_3^{a,b,c}$ is one, and hence the stable centre of $A_3^{a,b,c}$ is of dimension $a + b + c$. Since c is odd $a + b + c$ is odd and we get a contradiction.

We have proved that two among a, b or c are odd and the third is odd now and the equality of the dimensions of the quotient of the centres by the Reynolds ideals $k + s = a + b + c - 2$ shows that k and s are both even. □

Lemma 6.6. *Let K be an algebraically closed field of characteristic 2. Suppose that $A_3^{a,b,c}$ is stably equivalent of Morita type to $A_2^{k,s}$. Then $k + s$ is odd.*

Proof If $k + s$ is even then

$$Z(A_2^{k,s}) = Z^{st}(A_2^{k,s}) \simeq K[u, v, w, t]/(u^k - v^s, w^2, t^2, uv, uw, vw, tw, ut, vt)$$

is isomorphic to the quotient of

$$Z(A_3^{a,b,c}) \simeq K[A, B, C, S]/(A^{a+1}, B^{b+1}, C^{c+1}, S^2, A^a + B^b + C^c, AS, BS, CS, AB, AC, BC)$$

by a two-dimensional subspace of $\langle A^a, B^b, S \rangle_K$ since by Lemma 6.5 at most one of the parameters a, b, c is even. The socle of $A_3^{a,b,c}$ is generated by A^a, B^b, S and the projective centre is generated by two elements $\mu_A A^a + \mu_B B^b + \mu_S S$ and $\nu_A A^a + \nu_B B^b + \nu_S S$.

If $\mu_S \neq 0$ or $\nu_S \neq 0$ then the stable centre of $SD(3K)$ is isomorphic to

$$Z^{st}(A_3^{a,b,c}) \simeq K[A, B, C]/(A^{a+1}, B^{b+1}, C^{c+1}, A^a + B^b + C^c, \mu'_A A^a + \mu'_B B^b, AB, AC, BC)$$

for some parameters μ'_A and μ'_B not both 0. In case $\min(k, s) \geq 2$ we consider the quotient modulo the radical squared of both stable centres. The one of $A_2^{k,s}$ has a basis given by $1, u, v, w, t$ and the one of $A_3^{a,b,c}$ is K -linearly generated by $1, A, B, C$ (knowing that in case some of the parameters a, b, c are 1, then these 4 elements are not linearly independent). In any case this gives a contradiction and so $s = 1$ or $k = 1$ in this case. So k and s are both odd, which is impossible by Lemma 6.5.

So we get $\mu_S = 0 = \nu_S$. But then the projective centre of $A_3^{a,b,c}$ is generated by A^a and B^b and we get

$$Z^{st}(A_3^{a,b,c}) \simeq K[A, B, C, S]/(A^a, B^b, C^c, S^2, AS, BS, CS, AB, AC, BC)$$

which needs to be isomorphic to

$$Z^{st}(A_2^{k,s}) = K[u, v, w, t]/(u^k - v^s, w^2, t^2, uv, uw, vw, tw, ut, vt)$$

By symmetry we may assume again $a \geq b \geq c$ and $k \geq s$. If $c \geq 2$, the socle of $Z^{st}(A_3^{a,b,c})$ is four-dimensional, whereas the socle of $Z^{st}(A_2^{k,s})$ is three-dimensional. Hence $c = 1$. But then, comparing the quotient modulo the radical squared gives $s = 1$ and therefore $a = k + 1$ and $b = 2$. The equation $abs = ks$ becomes $2(k + 1) = k$, a contradiction. □

6.3. One simple versus two simples. We shall deal with the possibility that an algebra of type A_1^ℓ with one simple module is stably equivalent of Morita type to an algebra of type $A_2^{k,s}$ with 2 simple modules.

Lemma 6.7. *An algebra of type A_1^ℓ with one simple module cannot be stably equivalent of Morita type to an algebra of type $A_2^{k,s}$.*

Proof Suppose that A_1^ℓ and $A_2^{k,s}$ are stably equivalent of Morita type. Since the Cartan determinants are equal, we have $4\ell = 4ks$. Since the centre modulo the Reynolds ideal is invariant under a stable equivalence of Morita type in our case by Theorem 1.6, we have $(\ell + 3) - 1 = (k + s + 2) - 2$. This means that k and s are integer solutions of the equation of second order $X^2 - (\ell + 2)X + \ell = 0$. But the discriminant of this equation is equal to $(\ell + 2)^2 - 4\ell = \ell^2 + 4$ which should be a square of an integer m . We look for pythagorean triples $(2, \ell, m)$. It is well known that ℓ and m have to be odd,

whence writing down the pythagorean equation one gets $m = 1$ or $l = 1$. This contradiction proves that such a triple does not exist. \square

6.4. One simple versus three simples.

Lemma 6.8. *An algebra of type A_1^ℓ is not stably equivalent of Morita type to an algebra of type $A_3^{a,b,c}$.*

Proof The proof follows the lines of the proof of Lemma 6.7. The equality of Cartan determinants give $4\ell = 4abc$ and the centre modulo the Reynolds ideal give $(\ell + 3) - 1 = (a + b + c + 1) - 3$. This means that we have two equalities. $\ell = abc$ and $\ell = a + b + c - 4$. By symmetry we may suppose $a \geq b \geq c \geq 1$. Then $c = 1$, otherwise

$$\ell = abc \geq 4a > 3a \geq a + b + c - 4.$$

But now $l = ab$ and $l = a + b - 3$. The same argument gives $b = 1$ and now we have $a = \ell = a - 2$, which is a contradiction. \square

We resume the situation.

Proposition 6.9. *Let K be a field of characteristic 2.*

- $A_3^{a,b,c}$ cannot be stable equivalent of Morita type to $A_2^{k,s}$ (cf Proposition 6.4).
- $A_2^{k,s}$ cannot be stable equivalent of Morita type to A_1^ℓ (cf Lemma 6.7).
- $A_3^{a,b,c}$ cannot be stably equivalent of Morita type to A_1^ℓ (cf Lemma 6.8).

Let K be a field of characteristic different from 2.

- There is no stable equivalence of Morita type between $A_3^{a,b,c}$ and $A_2^{k,s}$ (cf Lemma 6.1, Lemma 6.2 and Lemma 6.3).
- There is no stable equivalence of Morita type between A_1^ℓ and $A_2^{k,s}$ (cf Lemma 6.7).
- There is no stable equivalence of Morita type between A_1^ℓ and $A_3^{a,b,c}$ (cf Lemma 6.8).

Although we cannot classify completely algebras of semi-dihedral and quaternion type up to stable equivalences of Morita type, we can nevertheless prove the following

Corollary 6.10. *Let A be an indecomposable algebra which is stably equivalent to an algebra B of semi-dihedral type (resp. quaternion type). Then A has the same number of simple modules as B .*

Proof This is an immediate consequence of the above proposition. \square

7. THE MAIN THEOREM AND CONCLUDING REMARKS

We resume the results of this paper in a single theorem. We use the notations introduced above, which coincides with the notations in [3] or [5].

Theorem 7.1. *Let K be an algebraically closed field.*

Suppose A and B are indecomposable algebras which are stably equivalent of Morita type.

- If A is an algebra of dihedral type, then B is of dihedral type. If A is of semi-dihedral type, then B is of semi-dihedral type. If A is of quaternion type then B is of quaternion type.
- If A and B are of dihedral, semidihedral or quaternion type, then A and B have the same number of simple modules.
- Let A be an algebra of dihedral type.
 - (1) If A is local, then A is stably equivalent of Morita type to one and exactly one algebra in the following list:
 - $A_1(n, m)$ with $m \geq n \geq 2$ and $m + n > 4$;
 - C_1 ;
 - $D(1A)_1^k$ with $k \geq 2$;
 - if $p = 2$, B_1 and $D(1A)_2^k(d)$ with $k \geq 2$ and $d \in \{0, 1\}$, except that we don't know whether $D(1A)^k(0)$ and $D(1A)^k(1)$ are stably equivalent of Morita type or not.

- (2) If A has two simple modules, then A is stably equivalent of Morita type to one and exactly one of the following algebras: $D(2\mathcal{B})^{k,s}(0)$ with $k \geq s \geq 1$ or if $p = 2$, $D(2\mathcal{B})^{k,s}(1)$ with $k \geq s \geq 1$.
- (3) If A has three simple modules then A is stably equivalent of Morita type to one and exactly one of the following algebras: $D(3\mathcal{K})^{a,b,c}$ with $a \geq b \geq c \geq 1$ or $D(3\mathcal{R})^{k,s,t,u}$ with $s \geq t \geq u \geq k \geq 1$ and $t \geq 2$.
- Let A be an algebra of semi-dihedral type.
 - (1) If A has one simple module then A is stably equivalent of Morita type to one of the following algebras: $SD(1\mathcal{A})_1^k$ for $k \geq 2$ or $SD(1\mathcal{A})_2^k(c, d)$ for $k \geq 2$ and $(c, d) \neq (0, 0)$ if the characteristic of K is 2. Different parameters k yield algebras in different equivalence classes of Morita type.
 - (2) If A has two simple modules then A is stably equivalent of Morita type to $SD(2\mathcal{B})_1^{k,s}(c)$ for $k \geq 1, s \geq 2, c \in \{0, 1\}$ or to $SD(2\mathcal{B})_2^{k,s}(c)$ for $k \geq 1, s \geq 2, c \in \{0, 1\}, k + s \geq 4$.
 - (3) If A has three simple modules, then A is stably equivalent of Morita type to one and only one algebra of the type $SD(3\mathcal{K})^{a,b,c}$ for $a \geq b \geq c \geq 1$.
- Let A be an algebra of quaternion type.
 - (1) If A has one simple modules, then A is stably equivalent of Morita type to one of the algebras $Q(1\mathcal{A})_1^k$ for $k \geq 2$ or $Q(1\mathcal{A})_2^k(c, d)$ for $k \geq 2, (c, d) \neq (0, 0)$ if characteristic if the K is 2. Different parameters k yield algebras in different equivalence classes of Morita type.
 - (2) If A has two simple modules then A is stably equivalent of Morita type to one of the algebras $Q(2\mathcal{B})_1^{k,s}(a, c)$ for $k \geq 1, s \geq 3, a \neq 0$.
 - (3) If A has three simple modules, then A is stably equivalent of Morita type to one of the algebras $Q(3\mathcal{K})^{a,b,c}$ for $a \geq b \geq c \geq 1, b \geq 2, (a, b, c) \neq (2, 2, 1)$ or $Q(3\mathcal{A})_1^{2,2}(d)$ for $d \in K \setminus \{0, 1\}$. Different parameters a, b, c yield algebras in different stable equivalence classes of Morita type.

Proof The first point is Proposition 2.1 and the second point is Corollary 4.5 and Corollary 6.10. The third point is Proposition 4.1, Proposition 4.3 and Proposition 4.4. The fourth point is Proposition 6.9 together with Section 5.1 and the fifth point is Proposition 6.9 together with Section 5.2. □

Remark 7.2. For algebras of dihedral type, we proved in Section 4 that the classification up to stable equivalences of Morita type coincide with derived equivalence classification, up to a scalar problem in $D(1\mathcal{A})_2^k(d)$. The only piece that is missing for a complete classification is the question if $D(1\mathcal{A})_2^k(0)$ is stably equivalent of Morita type to $D(1\mathcal{A})_2^k(1)$.

Derived equivalent local algebras are Morita equivalent as is shown by Roggenkamp and the second author (cf [18]). Observe that tame local symmetric algebras are classified in [3, Chapter III]. Actually, the classification coincides with the algebras with one simple module we already dealt with in the text. So, a complete classification of the algebras of dihedral type with one simple module would give a classification of tame local symmetric algebras.

Corollary 7.3. *The Auslander Reiten conjecture holds for tame local symmetric algebras, i.e. if A is a tame local symmetric algebra and if B is an algebra without simple direct factor which is stably equivalent of Morita type to A , then B is local tame symmetric as well.*

Proof By Liu [12] B is indecomposable since A is indecomposable. Erdmann classified tame local symmetric algebras [3, III.1 Theorem]. The classification coincides with the list of local algebras of dihedral, semi-dihedral or quaternion type. □

We cannot give any answer to the classification of algebras of dihedral, semi-dihedral or quaternion type up to derived equivalence beyond the information that is already known. Nevertheless, one more statement for algebras of semi-dihedral type was obtained by Holm and the second author.

Theorem 7.4. (Holm and Zimmermann [6])

- (1) Let F be an algebraically closed field of characteristic 2. For any given integers $k, s \geq 1$, consider the algebras of semi-dihedral type $SD(2\mathcal{B})_1^{k,s}(c)$ for the scalars $c = 0$ and $c = 1$. Put $B_c^{k,s} := SD(2\mathcal{B})_1^{k,s}(c)$. Suppose that if $k = 2$ then $s \geq 3$ is odd, and if $s = 2$ then $k \geq 3$ is odd. Then the factor rings $Z(B_0^{k,s})/T_1(B_0^{k,s})^\perp$ and $Z(B_1^{k,s})/T_1(B_1^{k,s})^\perp$ are not isomorphic.

In particular, the algebras $SD(2\mathcal{B})_1^{k,s}(0)$ and $SD(2\mathcal{B})_1^{k,s}(1)$ are not derived equivalent.

- (2) Let F be an algebraically closed field of characteristic 2. For any given integers $k, s \geq 1$, consider the algebras of semi-dihedral type $SD(2\mathcal{B})_2^{k,s}(c)$ for the scalars $c = 0$ and $c = 1$. Put $C_c^{k,s} := SD(2\mathcal{B})_2^{k,s}(c)$. If the parameters k and s are both odd, then the factor rings $Z(C_0^{k,s})/T_1(C_0^{k,s})^\perp$ and $Z(C_1^{k,s})/T_1(C_1^{k,s})^\perp$ are not isomorphic. Hence the algebras $SD(2\mathcal{B})_2^{k,s}(0)$ and $SD(2\mathcal{B})_2^{k,s}(1)$ have different sequences of generalised Reynolds ideals.

In particular, for k and s odd, the algebras $SD(2\mathcal{B})_2^{k,s}(0)$ and $SD(2\mathcal{B})_2^{k,s}(1)$ are not derived equivalent.

We get the following positive result.

- Corollary 7.5.** (1) Let F be an algebraically closed field of characteristic 2. For any given integers $k, s \geq 1$, consider the algebras of semi-dihedral type $SD(2\mathcal{B})_1^{k,s}(c)$ for the scalars $c = 0$ and $c = 1$. Suppose that if $k = 2$ then $s \geq 3$ is odd, and if $s = 2$ then $k \geq 3$ is odd. Then the algebras $SD(2\mathcal{B})_1^{k,s}(0)$ and $SD(2\mathcal{B})_1^{k,s}(1)$ are not stably equivalent of Morita type.
- (2) Let F be an algebraically closed field of characteristic 2. For any given integers $k, s \geq 1$, consider the algebras of semi-dihedral type $SD(2\mathcal{B})_2^{k,s}(c)$ for the scalars $c = 0$ and $c = 1$. If the parameters k and s are both odd, then the algebras $SD(2\mathcal{B})_2^{k,s}(0)$ and $SD(2\mathcal{B})_2^{k,s}(1)$ are not stably equivalent of Morita type.

Proof Since the quotients $Z^{st}(A) := Z(A)/Z^{pr}(A)$ and $T_n^\perp(A)^{st} := T_n(A)^\perp/Z^{pr}(A)$ are invariants under stable equivalences of Morita type, so are the quotients $Z^{st}(A)/T_n^\perp(A)^{st} = Z(A)/T_n^\perp(A)$.

Hence the parameters in the theorem yield not only algebras in different derived equivalence classes, but also algebras in different equivalence classes up to stable equivalences of Morita type. \square

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