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► **To cite this version:**

Marc Bellon. An Efficient Method for the Solution of Schwinger–Dyson equations for propagators. *Letters in Mathematical Physics*, 2010, 94 (1), pp.77-86. 10.1007/s11005-010-0415-3 . hal-00479697v2

HAL Id: hal-00479697

<https://hal.science/hal-00479697v2>

Submitted on 28 Sep 2010

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AN EFFICIENT METHOD FOR THE SOLUTION OF SCHWINGER–DYSON EQUATIONS FOR PROPAGATORS.

MARC P. BELLON

ABSTRACT. Efficient computation methods are devised for the perturbative solution of Schwinger–Dyson equations for propagators. I show how a simple computation allows to obtain the dominant contribution in the sum of many parts of previous computations. This allows for an easy study of the asymptotic behavior of the perturbative series. In the cases of the four-dimensional supersymmetric Wess–Zumino model and the ϕ_3^3 complex scalar field, the singularities of the Borel transform for both positive and negative values of the parameter are obtained and compared.

1. INTRODUCTION

In preceding works [1, 2], we have shown how to solve Schwinger–Dyson equations for propagators. This allows to compute the β -function in models without vertex divergencies, e.g., the supersymmetric Wess–Zumino model and some version of a six-dimensional theory of a scalar field with cubic interactions. The resolution method in the first of these works [1] is exact, but does not allow for an easy understanding of the properties of the computed perturbative series. This method is also computationally heavy and would scale badly for Schwinger–Dyson equations with a greater number of propagators.

The approximate differential equations introduced in [2] have the double advantage of allowing an easy derivations of the asymptotic behavior of the perturbative series and of a complexity which is only quadratic in the perturbative order. However, as can be seen in the second equation proposed for the Wess–Zumino model, this simplicity is lost if we have to get more precise results.

In this work, I propose a procedure which allows to systematically improve the second method, while remaining computationally simple and making the asymptotic behavior of the perturbative series transparent. In particular, I shall derive the dominant singularity of the Borel transform of the perturbative series on the positive axis, through the added precision on the asymptotic behavior of the perturbative series. In this work, I limit myself to the contributions of the dominant poles, which are sufficient for the asymptotic behavior, but the inclusion of additional terms would allow to reach any desired precision, for a fraction of the computational cost of the method used in [1].

All these computations are based on two techniques, which have been described extensively in [1, 2], elaborating on the work of Kreimer and Yeats [5]:

2010 *Mathematics Subject Classification.* 81T15;81T17.

Key words and phrases. Renormalization; Schwinger-Dyson equation; Borel summation.

a renormalization group equation allows to deduce the full propagator from the renormalization group functions and a Mellin transform of the one loop diagram gives the renormalization group functions from the propagators. These results will be freely used here.

In the present paper, we will see in a first step how the contribution of a simple pole in the Mellin transform of the diagram can be computed recursively, allowing the computation of an infinite sum of derivatives of the propagator while sidestepping the computation of the individual derivatives. This is immediately applied in the following section to the solution of the linear Schwinger–Dyson equation for a scalar model in dimension 6 which was first solved in [4]. The new method surely recovers the same numerical values and is computationally comparable, but it is far superior for the asymptotic analysis of the perturbative series and it prepares for the more complex examples.

In the cases of non-linear Schwinger–Dyson equations, some terms cannot be computed with this first method and I extend it to the case where the denominator depends on the sum of some variables of the Mellin transform. This method is then applied to the supersymmetric Wess–Zumino model. The main result thus obtained is a precise asymptotic study of the perturbative series: beyond the dominant contribution which gives a pole for the Borel transform of the series, the exact nature of the singularity on the positive axis, which was discovered in the numerical computation of [1], is uncovered. Finally, the scalar model in dimension 6 is studied in the case of a non-linear Schwinger–Dyson equation. Again, a singularity on the positive real axis is predicted, but it would have been difficult to spot numerically, since the non alternating part is smaller by a factor bigger than n^4 . Finally, I conclude on the possible extensions of this work, in particular with respect to the inclusion of higher loop corrections in the Schwinger–Dyson equations. The singularities of the Mellin transforms for higher loop diagrams are poles with denominators of the type envisioned here: the dominant contributions of the high order derivatives of the Mellin transform can be computed by the methods presented here.

2. CONTRIBUTION FROM A SIMPLE POLE.

We keep the fundamental insight in [2] that the Taylor series is dominated by the contribution of the singularities near the origin. In the one-loop case, the Mellin transform is expressible through Γ functions and these singularities are single poles. The first case to consider is therefore the one of a simple pole $1/(k+x)$.

The corresponding contribution to the γ -function is:

$$(1) \quad \frac{1}{k+x} = \frac{1}{k} \sum_{p=0}^{\infty} \left(\frac{-x}{k}\right)^p \longrightarrow \mathcal{I} = \frac{1}{k} \sum_{p=0}^{\infty} \left(\frac{-1}{k}\right)^p \gamma_p$$

In this expression, γ_p is the p^{th} derivative with respect to the logarithm of the impulsion L of the correction to the propagator $G(L)$. However, the seemingly

local sum of derivatives in equation (1) can be given an equivalent integral form:

$$(2) \quad \mathcal{I}_k = \int_{-\infty}^0 G(L)e^{kL}dL$$

If k is negative, the integral has to be taken from 0 to $+\infty$. For $k = 1$, we recover the integral

$$\int_0^{\mu^2} G(p^2)d(p^2),$$

which appeared in the work of Broadhurst and Kreimer [4].

The point is that \mathcal{I} can be efficiently evaluated through the use of the renormalization group. The action of $\gamma + \beta a \partial_a$ on $G(L)$ gives its derivative with respect to L ;

$$(3) \quad (\gamma + \beta a \partial_a)\mathcal{I}_k = \int_{-\infty}^0 \partial_L G(L)e^{kL}dL = G(0) - k\mathcal{I}_k$$

The last equality comes from an integration by part. Equation (3) can be used to obtain an efficient recursive computation of the perturbative expansion of \mathcal{I} .

3. THE LINEAR CASE.

As a first application, let us go back to the linear Schwinger–Dyson equation for a 6-dimensional scalar field first studied in [4]. The one-variable Mellin transform is a rational function and a simple solution method was presented in [4]. This amounts to acting on the Schwinger–Dyson equation with an adequate differential operator in the logarithm of the momentum L to reduce the right-hand side to a constant. In [2], I proposed to act with a simpler differential operator. This allowed in particular to more easily account for the asymptotic behavior of the perturbative solution, but the truncated right-hand side precluded an exact solution.

With the insight of the preceding section, it is possible to obtain the exact solution in a form which allows for an easy asymptotic analysis. Indeed, through the use of \mathcal{I}_2 and \mathcal{I}_3 which represent the effect of the propagator on the single poles $1/(2+x)$ and $1/(3+x)$, equation (10) of [2] can be written:

$$(4) \quad \gamma + \gamma(2a\partial_a - 1)\gamma = a(\mathcal{I}_2 - \mathcal{I}_3).$$

The recursive evaluation of \mathcal{I}_2 and \mathcal{I}_3 through the use of equation (3), completed by the identity $\beta = 2\gamma$, allows for a rapid evaluation of the perturbative series for γ . The result coincides with the one stemming from the partial differential equation of [4], providing a check of the computation.

It is then easy to obtain from equation (4) the asymptotic behavior of the perturbative series. Let us fix the notation for the perturbative coefficients of γ , \mathcal{I}_2 and \mathcal{I}_3 :

$$(5) \quad \gamma = \sum_{n=1}^{\infty} c_n a^n, \quad \mathcal{I}_2 = \sum_{n=0}^{\infty} d_n a^n, \quad \mathcal{I}_3 = \sum_{n=0}^{\infty} f_n a^n.$$

It is convenient to convert the term bilinear in γ to the form $(a\partial_a - 1)\gamma^2$, so that with the first values $c_1 = 1/6$ and $c_2 = -11/6^3$, equation (4) gives the

following relation:

$$(6) \quad c_{n+1} \simeq -\frac{2n}{6}c_n + \frac{22(n-1)}{6^3}c_{n-1} + d_n - f_n.$$

In the equation (3), the dominant components come from the cases where either γ or \mathcal{I}_k is of maximal order, so that one obtains:

$$(7) \quad d_n \simeq -\frac{1}{4}c_n - \frac{2n-1}{12}d_{n-1}$$

$$(8) \quad f_n \simeq -\frac{1}{9}c_n - \frac{2n-1}{18}f_{n-1}$$

Now, it is consistent to suppose that d_n and f_n are proportional to c_n and replace c_{n-1} in the right hand side of the previous equations by the its approximate value $-\frac{3}{n}c_n$ to solve for the unknown factor. One obtains:

$$(9) \quad d_n \simeq -\frac{1}{2}c_n, \quad f_n \simeq -\frac{1}{6}c_n.$$

Putting all together, one obtains the following asymptotic relation:

$$(10) \quad c_{n+1} \simeq -\left(\frac{n}{3} + \frac{23}{36}\right)c_n,$$

which is well verified on the exact solution. By considering additional terms in the expansion of γ^2 and the recursive definition of d_n and f_n , one could obtain additional terms in the expansion in $\frac{1}{n}$ of the ratio of c_n over c_{n-1} .

4. GENERAL POLES.

When studying non-linear Schwinger–Dyson equations, Mellin transforms with many variables must be used. However, the obtained singularities are really simple, since they correspond to the divergences appearing when some subgraph becomes scale invariant: the singularities are simple poles whose denominators only depend on the sum of the Mellin variables associated to the subgraph.

The formulation of the pairing between propagator and Mellin transform appears quite dissymmetric, but it is in fact totally symmetric. The evaluation of a function of the propagator $G(L)$ can be reduced to the following pairing of $G(L)$ with the Mellin transform $H(x)$:

$$(11) \quad \sum_{n=0}^{\infty} \frac{1}{n!} \partial_L^n G(L)|_{L=0} \partial_x^n H(x)|_{x=0}$$

For simple enough Mellin transforms H , it will be more convenient therefore to write this duality in reverse order, i.e., as some differential operator acting on the propagator $G(L)$. The structure will simply be:

$$(12) \quad H(\partial_L)G(L),$$

since for an analytic H , the Taylor series gives the value of the function. The situation becomes more interesting when there are many variables, since a sum of derivatives with respect to L_i can be converted to a simple derivative if we identify the different variables. The methods of section 2 can then be applied to evaluate the effect of a simple pole.

However, one must pay special attention to the numerator: if in one variable, any numerator can be reduced to a scalar, it is not the case in a multidimensional setting. It is even possible, if we start from a multiloop diagram, that the numerator itself be a meromorphic function with poles. The identification of the different variables L_i has therefore to be done after the application of the differential operator corresponding to the part of the numerator which depends on the variables in the denominator. The final result is that we can evaluate easily the following type of pairings:

$$(13) \quad \mathcal{I} = \frac{1}{k + \sum_{j=1}^n \partial_{L_j}} N(\partial_{L_1}, \dots, \partial_{L_n}) \prod_{j=1}^n G(L_j) \Big|_{L_1=\dots=L_n=0}$$

We obtain an equation similar to eq. (3), apart that the anomalous dimension γ is multiplied by the number of fields n and that $G(0)$ is replaced by an expression involving N .

$$(14) \quad (k + n\gamma + \beta a \partial_a) \mathcal{I} = N(\partial_{L_1}, \dots, \partial_{L_n}) \prod_{j=1}^n G(L_j) \Big|_{L_1=\dots=L_n=0}$$

5. WESS–ZUMINO MODEL.

Equipped with these new tools, let us consider the supersymmetric Wess–Zumino model. It is now possible to compute additional terms of the asymptotic behavior of the perturbative series. In particular, I shall obtain the exact nature of the singularity on the positive real axis. The aim is to include all the poles at unit distance from the origin, that is to use the approximation of the Mellin transform introduced in equation (21) of [1], corrected to have the exact xy term:

$$(15) \quad h(x, y) = (1 + xy) \left(\frac{1}{1+x} + \frac{1}{1+y} - 1 \right) + \frac{1}{2} \frac{xy}{1-x-y} + \frac{1}{2} xy.$$

To solve the Schwinger–Dyson equation with this approximate Mellin transform, we need two functions in addition to γ itself: F , the sum of the γ_n associated to $1/(1+x)$ and H , the function associated to the term $xy/(1-x-y)$. For the same approximation, the differential equation obtained in section 3.4 of [2] is very complex.

The renormalization group yields to the following functional equations for F and H :

$$(16) \quad F = 1 - \gamma(3a\partial_a + 1)F,$$

$$(17) \quad H = \gamma^2 + \gamma(3a\partial_a + 2)H.$$

Using the approximate Mellin transform (15), the Schwinger–Dyson equation then takes the form:

$$(18) \quad \gamma = 2aF - a - 2a\gamma(F - 1) + \frac{1}{2}a(H - \gamma^2)$$

The three equations (16,17,18) can be converted in recursion equations for the perturbative expansions of the functions γ , F and G .

The numerical solution is easy. Compared with the complete calculations of [1], the obtained precision is good, with errors which are always less than 1 percent and are asymptotically of the order of half a percent. This divides the error by a factor 10 with respect to the cruder approximation of [2] and moreover captures the non-alternating part in the development of γ : an asymptotic analysis of this system of equations will allow to obtain the exact nature of the singularity on the positive axis in the Borel transform of the series for γ .

Let us write $\gamma = \sum c_n a^n$, $F = \sum f_n a^n$ and $H = \sum h_n a^n$. It is easy to show that $c_1 = 1$, $c_2 = -2$, $f_2 = 1$. With the rapid growth of all the coefficients, one obtains the following asymptotic forms of the equations (16,17):

$$(19) \quad f_{n+1} \simeq -(3n+1)f_n + 2(3n-2)f_{n-1} - c_{n+1},$$

$$(20) \quad h_{n+1} \simeq 2c_n - 4c_{n-1} + (3n+2)h_n - 2(3n-1)h_{n-1} + 8c_{n-1}.$$

In the first of these equations, $(3n-2)f_{n-1}$ and c_{n+1} are respectively proportional to $-f_n$ and $2f_n$, yielding the asymptotic relation:

$$(21) \quad f_{n+1} \simeq -(3n+5)f_n.$$

Remarking that the dominant term in c_n is $2f_{n-1}$, one recovers the dominant asymptotic relation for c_n , $c_{n+1} \simeq -(3n+2)c_n$, observed in [1] and proved in [2]. For h_n , one obtains similarly the asymptotic relation

$$(22) \quad h_{n+1} \simeq 3nh_n.$$

This term is non-alternating and is the dominant contribution to the singularity of the Borel transform on the positive real axis. The ratio of the growth of f_n and g_n is in absolute value $1-5/3n$. This translates in a ratio $n^{-5/3}$ between the absolute values of these two contributions: this ratio coincides nicely with the numeric results and confirms the subleading character of the c_n term in eq. (20).

6. THE ϕ_6^3 MODEL.

In this case, one considers the action of $\partial_L + \partial_L^2$ on the Schwinger–Dyson equation. The left-hand side is then

$$(23) \quad \gamma + \gamma(3a\partial_a - 1)\gamma,$$

and the right-hand side is based on the following Mellin transform:

$$(24) \quad H(x, y) = \frac{\Gamma(2+x)\Gamma(2+y)\Gamma(1-x-y)}{\Gamma(1-x)\Gamma(1-y)\Gamma(4+x+y)}$$

I approximate H with the contributions from the poles for x or y equal to -2 or -3 and the one for $1-x-y=0$. For $y=-2$, the residue is $(1-x)(1-x/2)$ and is converted to $(1+xy/2)(1+xy/4)$ not to mar the low order terms. For $y=-3$, the residue has two more factors and is $-(1+x)(1-x)(1-x/2)(1-x/3)$. For $1-x-y=0$, the residue is $xy/12(1+xy/2)$. We therefore will use the following

approximation of H :

$$\begin{aligned}
 h(x, y) &= \left(1 + \frac{xy}{2}\right)\left(1 + \frac{xy}{4}\right)\left(\frac{1}{2+x} + \frac{1}{2+y} - \frac{1}{2}\right) \\
 &\quad - \left(1 - \frac{xy}{3}\right)\left(1 + \frac{xy}{3}\right)\left(1 + \frac{xy}{6}\right)\left(1 + \frac{xy}{9}\right)\left(\frac{1}{3+x} + \frac{1}{3+y} - \frac{1}{3}\right) \\
 (25) \quad &\quad + \frac{1}{12}xy\left(1 + \frac{xy}{2}\right)\frac{1}{1-x-y} - \frac{5}{216}xy.
 \end{aligned}$$

The last term in this equation adjust the coefficient of xy to the exact value, so that the cubic term in γ is exact. Most remarkably, this coefficient appears as the sum of the contributions from poles more distant from the origin, with poles in $k+x$ giving negative contributions and the poles in $k-x-y$ giving positive contributions. For the terms in x^2y or xy^2 where both contributions add up, the remainder is larger.

For the asymptotic analysis, these additional terms do not matter. We need three functions in addition to the anomalous dimension γ , \mathcal{I}_2 and \mathcal{I}_3 as in the case of the linear Schwinger–Dyson equation and a function H associated to the term $xy(1+xy/2)/(1-x-y)$. We have similar equations for \mathcal{I}_2 and \mathcal{I}_3 than in the linear case:

$$(26) \quad 2\mathcal{I}_2 = 1 - \gamma(1 + 3a\partial_a)\mathcal{I}_2,$$

$$(27) \quad 3\mathcal{I}_3 = 1 - \gamma(1 + 3a\partial_a)\mathcal{I}_3.$$

The recursion equation for H just has additional source terms with respect to the Wess–Zumino case, due to the more complex numerator:

$$(28) \quad H = \gamma^2 + \frac{1}{2}\gamma_2^2 + \gamma(2 + 3a\partial_a)H.$$

Finally, using the approximate form of the Mellin transform (25) and neglecting terms divisible by $(xy)^3$ which are subdominant, one obtains the following equation:

$$\begin{aligned}
 \gamma + \gamma(3a\partial_a - 1)\gamma &= a(2\mathcal{I}_2 - \frac{1}{2}) + a\frac{3}{4}\gamma(2\mathcal{I}_2 - 1 - \frac{1}{4}\gamma) + a\frac{1}{8}\gamma_2(2\mathcal{I}_2 - 1 - \frac{1}{2}\gamma) \\
 &\quad - a(2\mathcal{I}_3 - \frac{1}{3}) - \frac{5}{18}a\gamma(2\mathcal{I}_3 - \frac{2}{3} - \frac{1}{9}\gamma) \\
 (29) \quad &\quad + \frac{5}{34}a\gamma_2(2\mathcal{I}_3 - \frac{2}{3} - \frac{2}{9}\gamma) + \frac{1}{12}aH - \frac{5}{216}a\gamma^2.
 \end{aligned}$$

This gives the first two non-zero coefficients of γ , $c_1 = \frac{1}{6}$, $c_2 = -\frac{11}{108}$. The dominant contribution for c_{n+1} comes from the second term of the left-hand side and is proportional to $n/2$. The dominant terms for \mathcal{I}_2 and \mathcal{I}_3 are in fact exactly the same than in the case of the linear Schwinger–Dyson equation, since the changes from $2n$ to $3n$ in the recursions for c_n and in equations (26,27) have compensating effects. Eq. (9) remains valid at the dominant level. At the next to leading order, we have therefore:

$$(30) \quad c_{n+1} \simeq -(3n+1)(c_1c_n + c_2c_{n-1}) + 2d_n - 2f_n \simeq -\left(\frac{n}{2} + \frac{13}{9}\right)c_n.$$

The recursion operation for H (28) will differ from the Wess–Zumino case through the values of the first coefficients of γ . The source terms are not

important for the non-alternating component of H . We obtain

$$(31) \quad h_{n+1} \simeq \frac{1}{6}(3n+2)h_n - \frac{11}{108}(3n-1)h_{n-1} \simeq \left(\frac{n}{2} - \frac{5}{18}\right)h_n.$$

The absolute values of the factors in the recursions for h_n and c_n are in the ratio $1 - 31/(9n)$, meaning that the absolute value of h_n is smaller by a power $31/9 = 3,444\dots$ of n than c_n . Compounded with the fact that h_n intervenes only in c_{n+1} , this non-alternating component in c_n could easily have been missed in a numerical study. In a sense, the situation is even worse, since the γ^2 term in equation (28) will give a larger contribution, so that h_n itself is dominated by this alternating component. However, this is not really a problem since the asymptotic recurrence relations are linear and the full solution will be a superposition of the particular solution proportional to c_{n-1} and a solution of the equation (31).

7. CONCLUSION.

In this work, the solution of Schwinger–Dyson equations for propagators has made a new step forward: I reach not only the leading asymptotic behavior of the perturbative terms, but the subleading, “wrong sign” contribution, as well as systematic corrections in powers of $1/n$. These computations can be made easily more precise through the inclusion of additional poles of the Mellin transform or the adjunction of a few monomials: a suitable combination of the two methods should allow to obtain the solution of the Schwinger–Dyson equations with very high precision for a fraction of the computational cost of the methods of [1]. However, such a full precision computation is not really useful since the Schwinger–Dyson equations I consider here are but the first approximation to the full system of Schwinger–Dyson equations.

In the computation of higher order corrections to the Schwinger–Dyson equations, the power of the present methods should be precious. Indeed, in a higher loop primitively divergent diagram, the number of individual propagators is higher and the full evaluation through the straightforward methods of [1], much more complex. For a Mellin transform in k variables, the number of different derivatives up to total order n scales as n^{k-1} , with the evaluation of each of these terms at perturbative order $2n$ again implying approximately n^{k-1} operations. Some time for space bartering could reduce somewhat this growth by reusing partial products, but in any case, this complexity would make any computation, whether explicit or asymptotic, very complex, even if the evaluation of the derivatives were not difficult per se.

In the approach presented here, the situation is much more manageable: the pole structure of the Mellin transform, which is linked to the divergences of sub-diagrams, is of the form analyzed in section 4 and their contributions can be computed in linear time. Furthermore, one can identify the residues which are important for the asymptotic behavior of the perturbative series and focus the analytical evaluations on them. This will be the subject of a forthcoming publication [3]. If, up to now, these works have dealt only with Schwinger–Dyson equations for propagators, the efficiency of the methods introduced here should

be important in the more challenging cases of systems including Schwinger–Dyson equations for vertices.

In the spirit of [2], we could also deduce systems of differential equations for the anomalous dimension and some auxiliary functions, with the hope of obtaining information on the asymptotic behavior of the anomalous dimension at large coupling. However, the singularity of the Borel transform on the positive axis indicates that the Borel resummation is not uniquely defined. It is therefore a challenge to determine whether such a system of differential equations, which would be determined from purely perturbative considerations, can be given a non-perturbative meaning.

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