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# Drinfeld $A$ -quasi-modular forms.

Vincent Bosser, Federico Pellarin

May 1, 2010

## 1 Introduction

The aim of this article is twofold: first, improve the multiplicity estimate obtained by the second author in [11] for Drinfeld quasi-modular forms; and then, study the structure of certain algebras of *almost- $A$ -quasi-modular forms*, which already appeared in [11].

In order to motivate and describe more precisely our results, let us introduce some notation. Let  $q = p^e$  be a power of a prime number  $p$  with  $e > 0$  an integer, let  $\mathbb{F}_q$  be the finite field with  $q$  elements. Let  $\theta$  be an indeterminate over  $\mathbb{F}_q$ , and write  $A = \mathbb{F}_q[\theta]$ ,  $K = \mathbb{F}_q(\theta)$ . Let  $|\cdot|$  be the absolute value on  $K$  defined by  $|x| = q^{\deg_\theta x}$ , and denote by  $K_\infty = \mathbb{F}_q((1/\theta))$  the completion of  $K$  with respect to  $|\cdot|$ , by  $K_\infty^{\text{alg}}$  an algebraic closure of  $K_\infty$ , and by  $C$  the completion of  $K_\infty^{\text{alg}}$  for the unique extension of  $|\cdot|$  to  $K_\infty^{\text{alg}}$ .

Let us denote by  $\Omega$  the rigid analytic space  $C \setminus K_\infty$  and by  $\Gamma := \mathbf{GL}_2(A)$  the group of  $2 \times 2$ -matrices with determinant in  $\mathbb{F}_q^*$ , having coefficients in  $A$ . The group  $\Gamma$  acts on  $\Omega$  by homographies. In this setting, we can define *Drinfeld modular forms* and *Drinfeld quasi-modular forms* for  $\Gamma$  in the usual way (see [3] or Section 2 below for a definition). One of the problems considered in this paper is to prove a *multiplicity estimate* for Drinfeld quasi-modular forms, that is, an upper bound for the vanishing order at infinity of such forms, as a function of the *weight* and the *depth*. In [4] and [11], the following conjecture is suggested ( $\nu_\infty(f)$  denotes the vanishing order of  $f$  at infinity, see Section 3 for the definition):

**Conjecture 1.1** *There exists a real number  $c(q) > 0$  such that, for all non-zero quasi-modular form  $f$  of weight  $w$  and depth  $l \geq 1$ , one has*

$$\nu_\infty(f) \leq c(q) l(w - l). \tag{1}$$

In fact, it is plausible that we can choose  $c(q) = 1$  in the bound (1). We refer to [4, § 1] or [11, § 1] for further discussion about this question.

In the classical (complex) case, the analogue of Conjecture 1.1 is actually an easy exercise using the resultant  $\mathbf{Res}_{E_2}(f, df/dz)$  in the polynomial ring  $\mathbb{C}[E_2, E_4, E_6]$  (here  $E_{2i}$  denotes the classical Eisenstein series of weight  $2i$ ).

Thus, a natural idea to attack Conjecture 1.1 is to try to mimic this easy proof. However, as explained in [4, § 1.2] and [11, § 1.1], if we do this we are led to use not only the first derivative of  $f$  but also its *higher divided derivatives*, or more precisely the sequence of its *hyperderivatives*  $D_n f$ ,  $n \geq 0$ , as defined in [3]. But then, due to the erratic behaviour of the operators  $D_n$ , obstacles arise which are not easy to overcome, and this approach appears as unfruitful to solve conjecture 1 (see [4, § 1.2] and [11, § 1] for more details).

Another approach to prove Conjecture 1.1 was carried out in [4]. The idea was here to use a *constructive* method: namely, we have constructed explicit families of *extremal* Drinfeld quasi-modular forms, and, by using a resultant argument as above (the function  $df/dz$  being replaced now by a suitable extremal form), we were able to get partial multiplicity estimates in the direction of Conjecture 1.1. Unfortunately, we could not construct enough families of extremal forms to prove a general estimate. Thus, also this approach to conjecture 1 seemed unfruitful.

Recently, a new approach was introduced, this time successfully, in [11] to get a general multiplicity estimate (although not optimal) toward Conjecture 1.1. The result obtained is a bound of the form

$$\nu_\infty(f) \leq c(q) l^2 w \max\{1, \log_q w\}, \quad (2)$$

where  $c(q)$  is explicit and  $\log_q$  is the logarithm in base  $q$ . Moreover, it is also proved in [11] that a bound like (1) holds *if* an extra condition of the form  $w > c_0(q)l^{5/2}$  is fulfilled ( $c_0(q)$  being explicit).

One of the main results of this paper is an improvement of the bound (2), yielding Conjecture 1.1 "up to a logarithm", namely:

**Theorem 1.2** *There exists a real number  $c(q) > 0$  such that the following holds. Let  $f$  be a non zero quasi-modular form of weight  $w$  and depth  $l \geq 1$ . Then*

$$\nu_\infty(f) \leq c(q) l(w - l) \max\{1, \log_q(w - l)\}.$$

*Moreover, one can take  $c(q) = 252q(q^2 - 1)$ .*

The proof of this result will be given in Section 3. It consists in a refinement of the method used in [11]. Recall that the main idea is to introduce a new indeterminate  $t$  as in Anderson's theory of  $t$ -motives, and to work with certain *deformations* of Drinfeld quasi-modular forms (called *almost  $A$ -quasi-modular forms* in [11]), on which the Frobenius  $\tau : x \mapsto x^q$  acts. Roughly speaking, these forms are functions  $\Omega \rightarrow C[[t]]$  satisfying certain regularity properties, as well as transformation formulas under the action of  $\Gamma$  involving *two factors of automorphy*. The precise definitions require quite long preliminaries: they are collected for convenience in Section 2, which is mostly a review of facts taken from [11].

Section 4 is devoted to the problem of clarifying the structure of almost  $A$ -quasi-modular forms. More precisely, let  $\mathbb{T}_{>0}$  denote the sub- $C$ -algebra of  $C[[t]]$  consisting of series having positive convergence radius, and let  $\widetilde{\mathcal{M}}$  denote the  $\mathbb{T}_{>0}$ -algebra of almost  $A$ -quasi-modular forms. As for standard Drinfeld quasi-modular forms, almost  $A$ -quasi-modular forms have a depth. Denote by  $\mathcal{M}$  the sub-algebra of  $\widetilde{\mathcal{M}}$  generated by forms of zero depth. Let  $E$  denote the "false" Eisenstein series of weight 2 and type 1 defined in [7]. One can define a particular almost  $A$ -quasi-modular form denoted by  $\mathbf{E}$  (see Section 2.3.2), which is a *deformation* of  $E$ . In Section 4, we obtain the following partial description of the structure of the algebra  $\widetilde{\mathcal{M}}$  (see Theorem 4.1 for the complete statement):

**Theorem 1.3** *The  $\mathbb{T}_{>0}$ -algebra  $\mathcal{M}$  has dimension 3 and the algebra  $\widetilde{\mathcal{M}}$  has dimension 5. Moreover, we have  $\widetilde{\mathcal{M}} = \mathcal{M}[E, \mathbf{E}]$ .*

We conjecture that  $\mathcal{M}$  is generated by three elements that can be explicitly given (see Conjecture 4.9). However, we don't know how to prove this yet.

In the very last part of Section 4, we define the notion of  *$A$ -modular forms*: they generate a sub- $\mathbb{T}_{>0}$ -algebra of  $\widetilde{\mathcal{M}}$  denoted by  $\mathbb{M}$ . We show (Theorem 4.11) that the algebra  $\mathbb{M}$  is of finite type and dimension three over  $\mathbb{T}_{>0}$  and we determine explicit generators.

## 2 Preliminaries

This section collects the preliminaries which will be needed in the next two sections. This is essentially a review of the paper [11].

### 2.1 Drinfeld modular forms and quasi-modular forms.

The now classical theory of *Drinfeld modular forms* started with the work of Goss (see [6]) and was improved by Gekeler (cf. [7]). We recall here briefly the basic definitions and properties of Drinfeld quasi-modular forms. The reader is referred to [3] for more details and proofs.

We will use the notations of the preceding section. For  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$  and  $z \in \Omega$ , we will denote by  $\gamma(z) = \frac{az+b}{cz+d}$  the image of the homographic action of the matrix  $\gamma$  on  $z$ . We will further denote by  $\tau : c \mapsto c^q$  the Frobenius endomorphism, generator of the skew polynomial ring  $C[\tau] = \mathbf{End}_{\mathbb{F}_q\text{-lin.}}(\mathbb{G}_a(C))$ .

Let  $\Phi_{\text{Car}} : A \rightarrow C[\tau]$  be the Carlitz module, defined by

$$\Phi_{\text{Car}}(\theta) = \theta\tau^0 + \tau.$$

Let  $\tilde{\pi}$  be one of its fundamental periods (fixed once for all), and let  $e_{\text{Car}} : C \rightarrow C$  be the associated exponential function. We have  $\ker e_{\text{Car}} = \tilde{\pi}A$  and the function  $e_{\text{Car}}$  has the following entire power series expansion, for all  $z \in C$ :

$$e_{\text{Car}}(z) = \sum_{i \geq 0} \frac{z^{q^i}}{d_i}, \quad (3)$$

where, borrowing classical notations,

$$d_0 = 1, \quad d_i = [i][i-1]^q \dots [1]^{q^{i-1}} \text{ for } i \geq 1 \quad (4)$$

and  $[i] = \theta^{q^i} - \theta$ . We define the parameter at infinity <sup>(1)</sup>  $u$  by setting, for  $z \in \Omega$ ,

$$u = u(z) := \frac{1}{e_{\text{Car}}(\tilde{\pi}z)}.$$

We will say that a function  $f : \Omega \rightarrow C$  is holomorphic on  $\Omega$  if it is analytic in the rigid analytic sense, and will say that it is *holomorphic at infinity* if

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<sup>1</sup>Note that this parameter is sometimes denoted by  $t(z)$  in the literature, e.g. in [7] and [3].

it is  $A$ -periodic (that is,  $f(z + a) = f(z)$  for all  $z \in \Omega$  and all  $a \in A$ ) and if there is a real number  $\epsilon > 0$  such that, for all  $z \in \Omega$  satisfying  $|u(z)| < \epsilon$ ,  $f(z)$  is equal to the sum of a convergent series

$$f(z) = \sum_{n \geq 0} f_n u(z)^n,$$

where  $f_n \in C$ . In the sequel, we will often identify such a function with a formal series in  $C[[u]]$ , thus simply writing

$$f = \sum_{n \geq 0} f_n u^n.$$

We can now recall the definition of a quasi-modular form for the group  $\Gamma$  [3].

**Definition 2.1** Let  $w \geq 0$  be an integer and  $m \in \mathbb{Z}/(q-1)\mathbb{Z}$ . A holomorphic function  $f : \Omega \rightarrow C$  is a *quasi-modular form of weight  $w$  and type  $m$*  if there exist  $A$ -periodic functions  $f_0, \dots, f_l$ , holomorphic in  $\Omega$  and at infinity, such that, for all  $z \in \Omega$  and all  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ :

$$f(\gamma(z)) = (cz + d)^w (\det \gamma)^{-m} \sum_{i=0}^l f_i(z) \left( \frac{c}{cz + d} \right)^i.$$

In the definition above, the functions  $f_0, \dots, f_l$  are uniquely determined by  $f$ , and moreover we have  $f_0 = f$ . When  $f \neq 0$ , the weight  $w$  and the type  $m$  are also uniquely determined by  $f$ . If  $f \neq 0$ , we can take  $f_l \neq 0$  in the definition above and  $l$  is then called the *depth* of  $f$ . The zero function is by convention of weight  $w$ , type  $m$  and depth  $l$  for all  $w, m$  and  $l$ . A quasi-modular form of depth 0 is by definition a *Drinfeld modular form* as in *e.g.* [7].

We will denote by  $M_{w,m}$  the  $C$ -vector space of Drinfeld modular forms of weight  $w$  and type  $m$ , and by  $\widetilde{M}_{w,m}^{\leq l}$  the  $C$ -vector space of Drinfeld quasi-modular forms of weight  $w$ , type  $m$ , and depth  $\leq l$ . We further denote by  $M$  (resp.  $\widetilde{M}$ ) the  $C$ -algebra of functions  $\Omega \rightarrow C$  generated by modular forms (resp. by quasi-modular forms). By [7, Theorem (5.13)] and [3, Theorem 1], we have

$$M = C[g, h] \quad \text{and} \quad \widetilde{M} = C[E, g, h], \tag{5}$$

where  $E$ ,  $g$ ,  $h$  are three algebraically independent functions defined in [7]. The function  $g$  is modular of weight  $q-1$ , type 0 (this is a kind of normalized Eisenstein series),  $h$  is modular of weight  $q+1$  and type 1 (it is very similar to a normalized Poincaré series), and  $E$  is quasi-modular of weight 2, type 1 and depth 1.

Drinfeld quasi-modular forms of weight  $w$ , type  $m$  and depth  $l$  coincide with polynomials  $f \in C[E, g, h]$  which are homogeneous of weight  $w$ , type  $m$ , and such that  $\deg_E f = l$ . Moreover, by [7, (8.4)] we have the following transformation formula for  $E$ :

$$E(\gamma(z)) = \frac{(cz+d)^2}{\det \gamma} \left( E(z) - \frac{1}{\tilde{\pi}} \frac{c}{cz+d} \right) \quad (z \in \Omega, \gamma \in \Gamma). \quad (6)$$

Finally, we have ([7, § 10])

$$\begin{aligned} g &= 1 + \cdots \in A[[u^{q-1}]] \\ h &= -u + \cdots \in uA[[u^{q-1}]] \\ E &= u + \cdots \in uA[[u^{q-1}]] \end{aligned}$$

where the dots stand for terms of higher order.

## 2.2 The functions $\mathbf{s}_{\text{Car}}$ , $\mathbf{s}_1$ and $\mathbf{s}_2$ .

The notion of  $A$ -quasi modular forms and  $A$ -modular forms involves in a crucial way three particular *Anderson generating functions*, which will be denoted by  $\mathbf{s}_{\text{Car}}$ ,  $\mathbf{s}_1$  and  $\mathbf{s}_2$  as in [11]. We recall here the definitions and properties that will be needed later. The results quoted here are taken from [11] but some of them are already implicit in [1].

### 2.2.1 Notations and definitions concerning formal series.

Let  $t$  be an indeterminate. As in [11], for any positive real number  $r > 0$  we will denote by  $\mathbb{T}_{<r}$  the sub- $C$ -algebra of  $C[[t]]$  whose elements are formal series converging for  $t \in C$  with  $|t| < r$ , and similarly we will write  $\mathbb{T}_{\leq r}$  for the sub-algebra of elements of  $C[[t]]$  that converge for  $t \in C$  with  $|t| \leq r$ . We will denote by  $\mathbb{T}_{>0}$  the sub- $C$ -algebra of  $C[[t]]$  of elements  $f \in C[[t]]$  that have a convergence radius  $> 0$ .

If a function  $\mathbf{f} : \Omega \rightarrow C[[t]]$  is given, we will denote the image of  $z$  indifferently by  $\mathbf{f}(z)$  or  $\mathbf{f}(z, t)$ , the latter notation being used when we want

to stress the dependence in  $t$ . If the series  $\mathbf{f}(z, t)$  converges at  $t = t_0$  for some  $z \in \Omega$ , we will use both notations  $\mathbf{f}(z, t_0)$  or  $\mathbf{f}(z)|_{t=t_0}$ .

Let  $\mathbf{f} : \Omega \rightarrow C[[t]]$  be a function. We will call *convergence radius* (or simply *radius*) of  $\mathbf{f}$  the supremum of the real numbers  $r > 0$  such that for all  $z \in \Omega$  and all  $t_0 \in C$  with  $|t_0| < r$ , the formal series  $\mathbf{f}(z, t) \in C[[t]]$  converges at  $t_0$ . If we denote by  $r_z$  the usual convergence radius of the series  $\mathbf{f}(z, t)$  (for  $z \in \Omega$  fixed), the convergence radius of  $\mathbf{f}$  is nothing else than  $\inf\{r_z \mid z \in \Omega\}$ .

If  $k \in \mathbb{Z}$  is an integer and  $f = \sum_{i \geq 0} f_i t^i$  is an element of  $C[[t]]$ , we define the  $k$ -th *Anderson's twist* of  $f$  by

$$f^{(k)} := \sum_{i \geq 0} f_i^{q^k} t^i.$$

It is straightforward to check that if  $f \in C[[t]]$  has a convergence radius equal to  $r$ , then  $f^{(k)}$  has a convergence radius equal to  $r^{q^k}$ . Similarly, if  $\mathbf{f} : \Omega \rightarrow C[[t]]$  has a convergence radius equal to  $r$ , then the convergence radius of  $\mathbf{f}^{(k)}$  is  $r^{q^k}$ .

Most of the functions  $\mathbf{f} : \Omega \rightarrow C[[t]]$  that will be considered in this paper share regularity properties that will play an important role later. We have gathered these properties in the following definition:

**Definition 2.2** Let  $\mathbf{f}$  be a function  $\Omega \rightarrow C[[t]]$ . We say that  $\mathbf{f}$  is *regular* if the following properties hold.

1. The function  $\mathbf{f}$  has a convergence radius  $> 0$ .
2. There exists  $\varepsilon > 0$  such that, for all  $t_0 \in C$ ,  $|t_0| < \varepsilon$ , the map  $z \mapsto \mathbf{f}(z, t_0)$  is holomorphic on  $\Omega$ .
3. For all  $a \in A$ ,  $\mathbf{f}(z + a) = \mathbf{f}(z)$ . Moreover, there exists  $c > 0$  such that for all  $z \in \Omega$  with  $|u(z)| < c$  and  $t$  with  $|t| < c$ , there is a convergent expansion

$$\mathbf{f}(z, t) = \sum_{n, m \geq 0} c_{n, m} t^n u^m,$$

where  $c_{n, m} \in C$ .

We denote by  $\mathcal{O}$  the set of regular functions; it is a  $\mathbb{T}_{>0}$ -algebra. It is plain that  $\mathcal{O}$  contains at least all the Drinfeld quasi-modular forms. We also notice that if  $\mathbf{f}$  belongs to  $\mathcal{O}$ , then  $\mathbf{f}^{(k)}$  belongs to  $\mathcal{O}$  for all  $k \geq 0$ .



### 2.2.2 Anderson generating functions.

Let  $\Lambda \subset C$  be an  $A$ -lattice of rank  $r \geq 1$ . We will denote by  $\Phi_\Lambda$  the Drinfeld module associated with  $\Lambda$ , and by  $e_\Lambda : C \rightarrow C$  its exponential map. The map  $e_\Lambda$  has a power series expansion of the form

$$e_\Lambda(\zeta) = \sum_{i \geq 0} \alpha_i(\Lambda) \zeta^{q^i}$$

for lattice functions  $\alpha_i$ , with  $\alpha_0(\Lambda) = 1$ . For any  $\omega \in \Lambda$  we introduce, following Anderson [1], the formal series  $\mathbf{s}_{\Lambda, \omega} \in C[[t]]$  defined by

$$\mathbf{s}_{\Lambda, \omega}(t) = \sum_{i \geq 0} e_\Lambda\left(\frac{\omega}{\theta^{i+1}}\right) t^i. \quad (7)$$

In [11] the following result is proved.

**Proposition 2.3** *Let  $\Lambda$  be an  $A$ -lattice, and let  $\omega \in \Lambda \setminus \{0\}$ .*

1. *The series  $\mathbf{s}_{\Lambda, \omega}(t)$  lies in  $\mathbb{T}_{< q}$  and its convergence radius is  $q$ .*
2. *For all  $t \in C$  with  $|t| < q$ , we have*

$$\mathbf{s}_{\Lambda, \omega}(t) = \sum_{i \geq 0} \frac{\alpha_i(\Lambda) \omega^{q^i}}{\theta^{q^i} - t}. \quad (8)$$

3. *The function  $t \mapsto \mathbf{s}_{\Lambda, \omega}(t)$  extends to a meromorphic function on  $C$  by means of the r.h.s. of (8). It has a simple pole at  $t = \theta^{q^i}$  for all  $i \geq 0$ , with residue  $-\alpha_i(\Lambda) \omega^{q^i}$ .*
4. *Let us write  $\Phi_\Lambda(\theta) = \theta \tau^0 + l_1 \tau + \cdots + l_r \tau^r$ . Then the following relation holds:*

$$\sum_{k=1}^r l_k \mathbf{s}_{\Lambda, \omega}^{(k)}(t) = (t - \theta) \mathbf{s}_{\Lambda, \omega}(t).$$

*Proof.* It is easy to show that, for  $\omega \neq 0$ , the power series (7) is convergent if and only if  $|t| < q$ . The first assertion follows from this. For the others, see [10, § 4.2.2].  $\square$

### 2.2.3 The function $\mathbf{s}_{\text{Car}}$ .

We take here  $\Lambda = \tilde{\pi}A$  and  $\omega = \tilde{\pi}$  (rank 1-case). We set:

$$\mathbf{s}_{\text{Car}}(t) := \mathbf{s}_{\tilde{\pi}A, \tilde{\pi}}.$$

In this case,  $\Phi_\Lambda$  is the Carlitz module  $\Phi_{\text{Car}}$  and  $e_\Lambda$  is the Carlitz exponential  $e_{\text{Car}}$ , so  $\alpha_i(\Lambda) = 1/d_i$  by (3).

The main properties of the series  $\mathbf{s}_{\text{Car}}$  are summarized in the following proposition.

**Proposition 2.4** 1. *The following expansion holds, for all  $|t| < q$ :*

$$\mathbf{s}_{\text{Car}}(t) = \sum_{i \geq 0} \frac{\tilde{\pi}^{q^i}}{d_i(\theta^{q^i} - t)}.$$

2. *We have:*

$$(t - \theta)\mathbf{s}_{\text{Car}}(t)|_{t=\theta} = -\tilde{\pi}. \quad (9)$$

3. *The series  $\mathbf{s}_{\text{Car}}$  satisfies the following  $\tau$ -difference equation:*

$$\mathbf{s}_{\text{Car}}^{(1)}(t) = (t - \theta)\mathbf{s}_{\text{Car}}(t). \quad (10)$$

4. *The following product expansion holds, for  $t \in C$ :*

$$\mathbf{s}_{\text{Car}}(t) = \frac{1}{\Omega^{(-1)}(t)} = \frac{(-\theta)^{1/(q-1)}}{\prod_{i \geq 0} (1 - t/\theta^{q^i})}, \quad (11)$$

where  $(-\theta)^{1/(q-1)}$  is an appropriate  $(q-1)$ -th root of  $-\theta$  and  $\Omega(t)$  is the function defined in [2, 3.1.2].

*Proof.* The points 1, 2 and 3 immediately follow from Proposition 2.3. Point 4 follows from [11, Formula (31)].  $\square$

### 2.2.4 The functions $\mathbf{s}_1$ and $\mathbf{s}_2$ .

Let us now choose  $z \in \Omega$ , and consider the  $A$ -lattice  $\Lambda_z := A + zA$ , which is of rank 2, with associated exponential function  $e_z := e_{\Lambda_z}$ . By [7], the corresponding Drinfeld module  $\Phi_z$  satisfies

$$\Phi_z(\theta) = \theta\tau^0 + \tilde{g}(z)\tau + \tilde{\Delta}(z)\tau^2,$$

where

$$\tilde{g}(z) = \tilde{\pi}^{q-1}g(z) \quad \text{and} \quad \tilde{\Delta}(z) = \tilde{\pi}^{q^2-1}\Delta(z) \quad \text{with} \quad \Delta = -h^{q-1}. \quad (12)$$

Here the functions  $g$  and  $h$  are those already introduced in Section 2.1. We define:

$$\mathbf{s}_1(z, t) := \mathbf{s}_{\Lambda_z, z}(t) \quad \text{and} \quad \mathbf{s}_2(z, t) := \mathbf{s}_{\Lambda_z, 1}(t).$$

We have:

**Proposition 2.5** *1. For all  $z \in \Omega$ , the series  $\mathbf{s}_1(z)$  and  $\mathbf{s}_2(z)$  are units in  $\mathbb{T}_{>0}$ .*

*2. We have:*

$$(t - \theta)\mathbf{s}_1(z, t)|_{t=\theta} = -z \quad \text{and} \quad (t - \theta)\mathbf{s}_2(z, t)|_{t=\theta} = -1. \quad (13)$$

*3. The series  $\mathbf{s}_1$  and  $\mathbf{s}_2$  satisfy the following linear  $\tau$ -difference equations of order 2:*

$$\mathbf{s}_1^{(2)} = -\frac{\tilde{g}}{\tilde{\Delta}}\mathbf{s}_1^{(1)} + \frac{t - \theta}{\tilde{\Delta}}\mathbf{s}_1 \quad \text{and} \quad \mathbf{s}_2^{(2)} = -\frac{\tilde{g}}{\tilde{\Delta}}\mathbf{s}_2^{(1)} + \frac{t - \theta}{\tilde{\Delta}}\mathbf{s}_2. \quad (14)$$

*4. The function  $\mathbf{s}_2$  is a regular function of convergence radius  $q$ . Moreover, there exists a real number  $c > 0$  such that the following expansion holds, for  $|t| < q$  and  $|u(z)| < c$ :*

$$\mathbf{s}_2(z, t) = \frac{1}{\tilde{\pi}}\mathbf{s}_{Car}(t) + \sum_{i \geq 1} \kappa_i(t)u^i \in C[[u]], \quad (15)$$

where  $\kappa_i \in \mathbb{T}_{<q}$  for all  $i \geq 1$ .

*Proof.* As already remarked in [11], the first statement follows from the fact that for all  $z \in \Omega$ , the constant term of the series  $\mathbf{s}_2(z)$  and  $\mathbf{s}_1(z)$  are  $e_{\Lambda_z}(1/\theta)$  and  $e_{\Lambda_z}(z/\theta)$  respectively, which never vanish for  $z \in \Omega$ . The statements 2 and 3 follow from Proposition 2.3. The last assertion is proved in [11, proof of Proposition 5].  $\square$

We notice that the function  $\mathbf{s}_1$  is not regular, since it is not  $A$ -periodic by (19).

A remarkable feature concerning the series  $\mathbf{s}_1$  and  $\mathbf{s}_2$  is that their first twists are related to the periods of second kind of the Drinfeld module  $\Phi_z$ . Let  $F_z : C \rightarrow C$  be the unique  $\mathbb{F}_q$ -linear function satisfying

$$F_z(0) = 0 \quad \text{and} \quad e_z^q(\zeta) = F_z(\theta\zeta) - \theta F_z(\zeta).$$

Following [8, § 7], we define  $\eta_1 := F_z(z)$  and  $\eta_2 := F_z(1)$  (periods of second kind of  $\Phi_z$ ). We further define:

$$\widehat{\Psi} := \begin{pmatrix} \mathbf{s}_1 & \mathbf{s}_2 \\ \mathbf{s}_1^{(1)} & \mathbf{s}_2^{(1)} \end{pmatrix}.$$

Then we have:

**Proposition 2.6** 1. *One has*

$$\mathbf{s}_1^{(1)}(z, \theta) = \eta_1, \quad \mathbf{s}_2^{(1)}(z, \theta) = \eta_2, \quad (16)$$

and

$$(t - \theta) \det \widehat{\Psi}(z, t) \Big|_{t=\theta} = \det \begin{pmatrix} -z & -1 \\ \eta_1 & \eta_2 \end{pmatrix} = -\frac{1}{\tilde{\pi}^q h(z)}. \quad (17)$$

2. *For all  $z \in \Omega$ , we have the following equality in  $\mathbb{T}_{<q}$ :*

$$\det \widehat{\Psi}(z, t) = \frac{\mathbf{s}_{Car}(t)}{\tilde{\pi}^{q+1} h(z)}. \quad (18)$$

*Proof.* Formulas (16) follow from [8, formula (5.3)], and Formula (17) follows from (13), (16) and [8, Theorem 6.2]. The relation (18) is proved in [11], during the proof of Proposition 4.  $\square$

The last important property of the series  $\mathbf{s}_1$  and  $\mathbf{s}_2$  is their behaviour under the action of  $\Gamma$ . We need to introduce new notation. Let us denote by  $\overline{A}$  the polynomial ring  $\mathbb{F}_q[t]$ . If  $a = a(\theta) \in A$ , we define  $\overline{a} := a(t) \in \overline{A}$ . Similarly, we write

$$\overline{\gamma} := \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix} \in \mathbf{GL}_2(\overline{A}) \quad \text{if} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

We further define the vectorial map:

$$\Sigma(z, t) := \begin{pmatrix} \mathbf{s}_1(z, t) \\ \mathbf{s}_2(z, t) \end{pmatrix}.$$

**Proposition 2.7** *For all  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$  and all  $z \in \Omega$ , we have the following identity of series in  $\mathbb{T}_{<q}$ :*

$$\Sigma(\gamma(z), t) = (cz + d)^{-1} \overline{\gamma} \cdot \Sigma(z, t). \quad (19)$$

*Proof.* See [11, Lemma 2]. □

Let us now define, for  $\gamma \in \Gamma$  and  $z \in \Omega$ ,

$$\boldsymbol{\xi} := \frac{\mathbf{s}_1}{\mathbf{s}_2}, \quad J_\gamma(z) = J_\gamma = cz + d \quad \text{and} \quad \mathbf{J}_\gamma(z) = \mathbf{J}_\gamma = \overline{c}\boldsymbol{\xi} + \overline{d}. \quad (20)$$

By Proposition 2.5,  $\mathbf{s}_2(z)$  is a unit in  $\mathbb{T}_{>0}$  for all  $z \in \Omega$  and  $\boldsymbol{\xi}(z)$  and  $\mathbf{J}_\gamma(z)$  are well defined elements of  $\mathbb{T}_{>0}^\times$  for all  $z$ . We also notice that the function  $\mathbf{J}_\gamma : \Gamma \times \Omega \rightarrow \mathbb{T}_{>0}^\times$  is a factor of automorphy for the group  $\Gamma$  since (see [11, § 3.2] for a proof)

$$\mathbf{J}_{\gamma\delta}(z) = \mathbf{J}_\gamma(\delta(z))\mathbf{J}_\delta(z). \quad (21)$$

We further notice that

$$\boldsymbol{\xi}(z)|_{t=\theta} = z, \quad \mathbf{J}_\gamma(z)|_{t=\theta} = J_\gamma(z).$$

Now, Formula (19) implies

$$\mathbf{s}_2(\gamma(z), t) = J_\gamma^{-1} \mathbf{J}_\gamma \mathbf{s}_2(z, t). \quad (22)$$

It turns out that this formula has a generalisation for all the twists  $\mathbf{s}_2^{(k)}$ ,  $k \geq 1$ . Let us write, for  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$  and  $z \in \Omega$ ,

$$L_\gamma(z) := \frac{c}{cz + d} \quad \text{and} \quad \mathbf{L}_\gamma(z) := \frac{\bar{c}}{(\theta - t)(\bar{c}\mathbf{s}_1 + \bar{d}\mathbf{s}_2)}. \quad (23)$$

Then

$$\mathbf{L}_\gamma|_{t=\theta} = L_\gamma$$

by (13). We note here that this definition of  $\mathbf{L}_\gamma$  differs from the one in [11, § 3.2].

Define further the sequence of series  $(g_k^*)_{k \geq -1}$  by:

$$g_{-1}^* = 0, \quad g_0^* = 1$$

and

$$g_k^* = g^{q^{k-1}} g_{k-1}^* + (t - \theta^{q^{k-1}}) \Delta^{q^{k-2}} g_{k-2}^*, \quad k \geq 1.$$

We have  $g_1^* = g$ . Moreover, the identity  $g_k^*(z, \theta) = g_k(z)$  holds for all  $k \geq 0$ , where  $g_k$  is the Eisenstein series defined in [7, Formula (6.8)].

**Proposition 2.8** *For all  $k \geq 0$ , all  $\gamma \in \Gamma$  and all  $z \in \Omega$ , we have:*

$$\mathbf{s}_2^{(k)}(\gamma(z)) = J_\gamma^{-q^k} \mathbf{J}_\gamma \left( \mathbf{s}_2^{(k)}(z) + (t - \theta) \frac{g_{k-1}^*(z) \mathbf{s}_{\text{Car}}}{\tilde{\pi}^{q^k+1} h(z)^{q^{k-1}}} \mathbf{L}_\gamma \right). \quad (24)$$

*Sketch of proof.* We give only the main steps of the proof and refer to [11, Proof of Proposition 4] for the details. We first note that for  $k = 0$  Formula (24) is Formula (22). Thanks to (18) one then shows the identity:

$$\mathbf{J}_\gamma^{(1)} = \mathbf{J}_\gamma \left( 1 + \frac{(t - \theta) \tilde{\mathbf{s}}_{\text{Car}}}{\tilde{\pi}^{q+1} h \mathbf{s}_2^{(1)}} \mathbf{L}_\gamma \right). \quad (25)$$

Applying Anderson's twist to the formula (22) and using the identity (25) above, we get the formula (24) for  $k = 1$ . The proof then goes by induction on  $k$ , using the formula

$$\mathbf{s}_2^{(k)} = -\frac{\tilde{g}^{q^{k-2}}}{\tilde{\Delta}^{q^{k-2}}} \mathbf{s}_2^{(k-1)} + \frac{t - \theta^{q^{k-2}}}{\tilde{\Delta}^{q^{k-2}}} \mathbf{s}_2^{(k-2)}$$

for  $k \geq 2$ , which follows from (14).  $\square$

## 2.3 Almost- $A$ -quasi-modular forms and the functions $\mathbf{h}$ and $\mathbf{E}$ .

In this section, we first define the notion of *almost- $A$ -quasi-modular forms*. Then we introduce and study two particular such forms, denoted by  $\mathbf{h}$  and  $\mathbf{E}$  in [11], which are deformations of the Drinfeld modular and quasi-modular forms  $h$  and  $E$ , respectively. As we will see in Section 4, the functions  $\mathbf{h}$  and  $\mathbf{E}$  will be the basic examples of  *$A$ -quasi-modular forms*.

### 2.3.1 Almost- $A$ -quasi-modular forms

Recall (Section 2.2.1) that we denote by  $\mathcal{O}$  the  $\mathbb{T}_{>0}$ -algebra of regular functions. Following [11, § 4.2], we define:

**Definition 2.9** Let  $\mathbf{f}$  be a regular function  $\Omega \rightarrow C[[t]]$ . We say that  $\mathbf{f}$  is an *almost- $A$ -quasi-modular form of weight  $(\mu, \nu)$ , type  $m$  and depth  $\leq l$*  if there exist regular functions  $\mathbf{f}_{i,j} \in \mathcal{O}$ ,  $0 \leq i + j \leq l$ , such that for all  $\gamma \in \Gamma$  and all  $z \in \Omega$ ,

$$\mathbf{f}(\gamma(z), t) = \det(\gamma)^{-m} J_\gamma^\mu J_\gamma^\nu \left( \sum_{i+j \leq l} \mathbf{f}_{i,j} L_\gamma^i L_\gamma^j \right). \quad (26)$$

For  $\mu, \nu \in \mathbb{Z}$ ,  $m \in \mathbb{Z}/(q-1)\mathbb{Z}$ ,  $l \in \mathbb{Z}_{\geq 0}$ , we denote by  $\widetilde{\mathcal{M}}_{\mu, \nu, m}^{\leq l}$  the  $\mathbb{T}_{>0}$ -module of almost  $A$ -quasi-modular forms of weight  $(\mu, \nu)$ , type  $m$  and depth  $\leq l$ . We have

$$\widetilde{\mathcal{M}}_{\mu, \nu, m}^{\leq l} \widetilde{\mathcal{M}}_{\mu', \nu', m'}^{\leq l'} \subset \widetilde{\mathcal{M}}_{\mu+\mu', \nu+\nu', m+m'}^{\leq l+l'}.$$

We also denote by  $\widetilde{\mathcal{M}}$  the  $\mathbb{T}_{>0}$ -algebra generated by all the almost- $A$ -quasi-modular forms. It was proved in [11] that this algebra is graded by the group  $\mathbb{Z}^2 \times \mathbb{Z}/(q-1)\mathbb{Z}$  and filtered by the depths. The problem of determining the structure of  $\widetilde{\mathcal{M}}$  will be considered in Section 4.

It is clear that  $\widetilde{\mathcal{M}}_{w, m}^{\leq l} \subset \widetilde{\mathcal{M}}_{w, 0, m}^{\leq l}$  for all  $w, m$ . A non-trivial example of an almost- $A$ -quasi-modular form is given by the function  $\mathbf{s}_2$ . By (22) we have indeed  $\mathbf{s}_2 \in \widetilde{\mathcal{M}}_{-1, 1, 0}^{\leq 0}$ . In the following section we introduce two other examples of almost- $A$ -quasi-modular forms.

### 2.3.2 The functions $\mathbf{h}$ and $\mathbf{E}$ .

**Definition 2.10** We define the two functions  $\mathbf{h} : \Omega \rightarrow \mathbb{T}_{>0}$  and  $\mathbf{E} : \Omega \rightarrow \mathbb{T}_{>0}$  by

$$\mathbf{h} = \frac{\tilde{\pi}h\mathbf{s}_2}{\mathbf{s}_{\text{Car}}} \quad \text{and} \quad \mathbf{E} = \frac{\mathbf{h}^{(1)}}{\Delta}.$$

We remark that these definitions make sense since  $\mathbf{s}_{\text{Car}}$  is a unit in  $\mathbb{T}_{>0}$  and since  $\Delta(z)$  does not vanish on  $\Omega$ . We also note that by (10), this definition of  $\mathbf{E}$  coincides with the definition of [11, § 3].

The main properties of  $\mathbf{h}$  and  $\mathbf{E}$  are summarized in the following two propositions:

**Proposition 2.11** *The following properties hold.*

1. *The function  $\mathbf{h}$  is regular with an infinite convergence radius, and  $\mathbf{h}(z, \theta) = h(z)$  for all  $z \in \Omega$ .*
2. *The function  $\mathbf{h}$  satisfies the following  $\tau$ -difference equation:*

$$\mathbf{h}^{(2)} = \frac{\Delta^{q-1}}{t - \theta^q} (g\mathbf{h}^{(1)} + \Delta\mathbf{h}). \quad (27)$$

3. *We have the following  $u$ -expansion, for  $|t| < q$  and  $|u(z)|$  sufficiently small:*

$$\mathbf{h}(z, t) = -u + \cdots \in u\mathbb{F}_q[t, \theta][[u^{q-1}]].$$

4. *The function  $\mathbf{h}$  satisfies the following functional equations, for all  $z \in \Omega$  and all  $\gamma \in \Gamma$ :*

$$\mathbf{h}(\gamma(z)) = \frac{J_\gamma^q \mathbf{J}_\gamma}{\det \gamma} \mathbf{h}(z).$$

*Proof.* It is easily seen that the product occurring in (11) expands as a series with infinite convergence radius. Since  $\mathbf{s}_2$  is regular of radius  $q$  by Proposition 2.5, it follows that the function  $\mathbf{s}_2/\mathbf{s}_{\text{Car}}$ , hence  $\mathbf{h}$ , is regular of radius  $\geq q$ . Moreover, we have  $\mathbf{h}(z, \theta) = h(z)$  by (9) and (13). The equation (27) follows easily from the definition of  $\mathbf{h}$  and from the  $\tau$ -difference equations (10) and (14) satisfied by  $\mathbf{s}_{\text{Car}}$  and  $\mathbf{s}_2$  (taking the relations (12) into account). The point 3 of the proposition follows from [11, Proposition 5] and [11, Lemma 16]. The point 4 is an immediate consequence of the formula (22) and of the



fact that  $h \in M_{q+1,1}$ . It remains to prove that the convergence radius of  $\mathbf{h}$  is infinite. Rewrite the formula (27) as follows:

$$\mathbf{h} = -\frac{g}{\Delta}\mathbf{h}^{(1)} + \frac{t - \theta^q}{\Delta^q}\mathbf{h}^{(2)}. \quad (28)$$

Let  $z$  be fixed, and let  $r \geq q$  be the radius of  $\mathbf{h}(z, t)$ . Then the left hand side of (28) has radius  $r$ , but the right hand side has obviously a radius  $r^q$  because of the twists. Hence  $r = \infty$ .  $\square$

**Proposition 2.12** *The following properties hold.*

1. *The function  $\mathbf{E}$  is regular with an infinite convergence radius, and  $\mathbf{E}(z, \theta) = E(z)$  for all  $z \in \Omega$ .*
2. *The function  $\mathbf{E}$  satisfies the following  $\tau$ -difference equation:*

$$\mathbf{E}^{(2)} = \frac{1}{t - \theta^{q^2}}(g^q \mathbf{E}^{(1)} + \Delta \mathbf{E}). \quad (29)$$

3. *We have the following  $u$ -expansion, for  $|t| < q^q$  and  $|u(z)|$  sufficiently small:*

$$\mathbf{E}(z) = u + \cdots \in u\mathbb{F}_q[t, \theta][[u^{q-1}]].$$

4. *The function  $\mathbf{E}$  satisfies the following functional equations, for all  $z \in \Omega$  and all  $\gamma \in \Gamma$ :*

$$\mathbf{E}(\gamma(z)) = \frac{J_\gamma \mathbf{J}_\gamma}{\det \gamma}(\mathbf{E}(z) - \frac{1}{\tilde{\pi}} \mathbf{L}_\gamma). \quad (30)$$

*Proof.* Follows easily from Proposition 2.11 (for the point 4 use (25)).  $\square$

It follows at once from Propositions 2.11 and 2.12 that  $\mathbf{h}$  and  $\mathbf{E}$  are almost- $A$ -quasi-modular forms.

**Corollary 2.13** *we have*

$$\mathbf{h} \in \widetilde{\mathcal{M}}_{q,1,1}^{\leq 0} \quad \text{and} \quad \mathbf{E} \in \widetilde{\mathcal{M}}_{1,1,1}^{\leq 1}.$$

### 3 Proof of Theorem 1.2

We prove here Theorem 1.2, following the same method of [11]. We need a few more notations. If  $f : \Omega \rightarrow C$  is a non-zero Drinfeld quasi-modular form having a  $u$ -expansion (for  $|u(z)|$  small)

$$f = \sum_{m \geq 0} f_m u^m,$$

we denote by  $\nu_\infty(f)$  its *vanishing order at infinity*, that is

$$\nu_\infty(f) := \min\{m \mid f_m \neq 0\}.$$

We extend the notation  $\nu_\infty(\mathbf{f})$  to regular functions  $\mathbf{f} \in \mathcal{O}$  as follows. If  $\mathbf{f} \in \mathcal{O}$ , non-zero, then by definition there exists  $c > 0$  such that, for  $(t_0, z) \in C \times \Omega$  with  $|t_0| < c$  and  $|u(z)| < c$ , one has

$$f(z, t_0) = \sum_{m \geq 0} f_m(t_0) u^m,$$

where  $f_m = \sum_{n \geq 0} f_{nm} t^n \in C[[t]]$  is a formal series. We define

$$\nu_\infty(\mathbf{f}) := \min\{m \mid f_m \neq 0\}.$$

Following [11], we introduce the  $\mathbb{T}_{>0}$ -algebra

$$\mathbb{M}^\dagger = \mathbb{T}_{>0}[g, h, \mathbf{E}, \mathbf{E}^{(1)}] = \mathbb{T}_{>0}[g, h, \mathbf{E}, \mathbf{h}],$$

the equality being an easy consequence of the definition of  $\mathbf{E}$  and of (27) (see also [11, Lemma 16]). The functions  $g, h, \mathbf{E}, \mathbf{h}$  are algebraically independent over  $\mathbb{T}_{>0}$  by [11, Proposition 14]. For  $\mu, \nu \in \mathbb{Z}$ ,  $m \in \mathbb{Z}/(q-1)\mathbb{Z}$ , we denote by  $\mathbb{M}_{\mu, \nu, m}^\dagger$  the sub- $\mathbb{T}_{>0}$ -module of  $\mathbb{M}^\dagger$  consisting of forms of weight  $(\mu, \nu)$  and type  $m$ .

By (29), it is clear that  $\mathbb{M}^\dagger$  is stable under twisting. More precisely, we have the following result, already remarked in [11]:

**Lemma 3.1** *Let  $\mu, \nu \in \mathbb{Z}$ ,  $m \in \mathbb{Z}/(q-1)\mathbb{Z}$  and  $k \in \mathbb{Z}_{\geq 0}$ . If  $\mathbf{f} \in \mathbb{M}_{\mu, \nu, m}^\dagger$ , then  $\mathbf{f}^{(k)} \in \mathbb{M}_{q^k \mu, \nu, m}^\dagger$*

*Proof.* It suffices to prove the assertion for  $k = 1$  and for  $\mathbf{f} \in \{g, h, \mathbf{h}, \mathbf{E}\}$ . If  $\mathbf{f} = g$  or  $\mathbf{f} = h$  the result is clear. If  $\mathbf{f} = \mathbf{h}$  this is clear too since  $\mathbf{h}^{(1)} = \Delta \mathbf{E} \in \mathbb{M}_{q^2, 1, 1}^\dagger$ . Suppose now that  $\mathbf{f} = \mathbf{E}$ . We have  $\mathbf{E}^{(1)} = \frac{\mathbf{h}^{(2)}}{\Delta^q}$  by definition of  $\mathbf{E}$ . But then the  $\tau$ -difference equation (27) implies immediately  $\mathbf{E}^{(1)} \in \mathbb{M}_{q, 1, 1}^\dagger$ .  $\square$

We recall the following multiplicity estimate for elements of  $\mathbb{M}_{\mu, \nu, m}^\dagger$ :

**Lemma 3.2** *Let  $\mathbf{f}$  be a non-zero element of  $\mathbb{M}_{\mu, \nu, m}^\dagger$  with  $\nu \neq 0$ . Then*

$$\nu_\infty(\mathbf{f}) \leq \mu\nu.$$

*Proof.* [11, Proposition 19].  $\square$

The following lemma is a refinement of [11, Lemma 24].

**Lemma 3.3** *Let  $\mu, \nu$  be two integers such that  $\nu \geq 0$  and  $\mu - \nu \geq 6(q^2 - 1)$ . Define  $V := \text{rank}_{\mathbb{T}_{>0}}(\mathbb{M}_{\mu, \nu, m}^\dagger)$ . Then*

$$\frac{(\mu - \nu)^2}{3(q^2 - 1)(q - 1)} \leq V \leq \frac{3}{(q^2 - 1)(q - 1)}(\mu - \nu)^2.$$

*Proof.* A basis of  $\mathbb{M}_{\mu, \nu, m}^\dagger$  is given by

$$(\phi_{is} \mathbf{h}^s \mathbf{E}^{\nu-s})_{0 \leq s \leq \nu, 1 \leq i \leq \sigma(s)},$$

where, for all  $s$ ,  $(\phi_{is})_{1 \leq i \leq \sigma(s)}$  is a basis of  $M_{\mu-s(q-1)-\nu, m-\nu}$ . Thus

$$V = \sum_{0 \leq s \leq \lfloor \frac{\mu-\nu}{q-1} \rfloor} \dim_C(M_{\mu-s(q-1)-\nu, m-\nu}).$$

Now, we have (e.g. [5, Proposition 4.3]),

$$\frac{\mu - \nu - s(q - 1)}{q^2 - 1} - 1 \leq \dim_C(M_{\mu-s(q-1)-\nu, m-\nu}) \leq \frac{\mu - \nu - s(q - 1)}{q^2 - 1} + 1.$$

A simple computation gives

$$(\lfloor x \rfloor + 1) \left( \frac{1}{q+1} (x - \lfloor x \rfloor / 2) - 1 \right) \leq V \leq (\lfloor x \rfloor + 1) \left( \frac{1}{q+1} (x - \lfloor x \rfloor / 2) + 1 \right)$$

with  $x = \frac{\mu-\nu}{q-1}$ . Using the inequalities  $x/2 \leq x - \lfloor x \rfloor / 2 \leq x/2 + 1/2$  and  $x \leq \lfloor x \rfloor + 1 \leq x + 1$ , we get

$$x\left(\frac{x}{2(q+1)} - 1\right) \leq V \leq (x+1)\left(\frac{x}{2(q+1)} + \frac{1}{2(q+1)} + 1\right).$$

Since  $\frac{x}{2(q+1)} - 1 \geq \frac{x}{3(q+1)}$ ,  $x+1 \leq 2x$ ,  $\frac{1}{2(q+1)} \leq \frac{x}{2(q+1)}$  and  $1 \leq \frac{x}{2(q+1)}$ , we obtain the result.  $\square$

**Lemma 3.4** *Let  $\mu, \nu$  be two integers such that  $\nu \geq 1$  and  $\mu - \nu \geq 6(q^2 - 1)$ . Let  $m$  be an element of  $\{0, 1, \dots, q-2\}$ , and suppose that  $\mathbb{M}_{\mu, \nu, m}^\dagger \neq \{0\}$ . There exists a form  $\mathbf{f}_{\mu, \nu, m} \in \mathbb{M}_{\mu, \nu, m}^\dagger$ , non-zero, satisfying the following properties.*

1. We have

$$\nu_\infty(\mathbf{f}_{\mu, \nu, m}) \geq \frac{(\mu - \nu)^2}{9(q^2 - 1)}. \quad (31)$$

2. The function  $\mathbf{f}_{\mu, \nu, m}$  has a  $u$ -expansion of the form

$$\mathbf{f}_{\mu, \nu, m} = u^m \sum_{n \geq n_0} b_n u^{n(q-1)} \quad (32)$$

with  $b_{n_0}(t) \neq 0$ ,  $b_n(t) \in \mathbb{F}_q[t, \theta]$  for all  $n$ , and

$$\deg_t b_{n_0} \leq \nu \log_q \left( \frac{3(\mu - \nu)^2}{2(q^2 - 1)(q - 1)} \right) + \nu \log_q \left( \frac{\mu\nu}{q - 1} \right). \quad (33)$$

*Proof.* Set  $V := \text{rank}_{\mathbb{T}_{>0}}(\mathbb{M}_{\mu, \nu, m}^\dagger)$  as above, and put  $U = \lfloor V/2 \rfloor$ . It follows from the proof of [11, Proposition 5] that there exists a non-zero form  $\mathbf{f}_{\mu, \nu, m} \in \mathbb{M}_{\mu, \nu, m}^\dagger$  having an expansion of the form (32) with  $b_{n_0}(t) \neq 0$ ,  $b_n(t) \in \mathbb{F}_q[t, \theta]$  for all  $n$ , and satisfying

$$U \leq n_0 \leq \frac{\mu\nu}{q-1} \quad \text{and} \quad \deg_t b_n \leq \nu \log_q U + \nu \log_q \max\{1, n\}. \quad (34)$$

We have obviously

$$\nu_\infty(\mathbf{f}_{\mu, \nu, m}) = m + n_0(q-1) \geq n_0(q-1). \quad (35)$$

We deduce from this

$$\nu_\infty(\mathbf{f}_{\mu,\nu,m}) \geq (q-1)U \geq \frac{(q-1)V}{3},$$

hence the bound (31) by Lemma 3.3. The inequality (33) follows from (34), from the estimate  $U \leq V/2$  and from Lemma 3.3.  $\square$

*Proof of Theorem 1.2.*

Let  $f \in C[E, g, h]$  be a non-zero quasi-modular form of weight  $w$ , type  $m$  and depth  $l \geq 1$ . Without loss of generality we may suppose that  $f$  is irreducible. We choose

$$\nu = 1$$

and

$$\mu = 12(q^2 - 1)(w - l).$$

Since

$$\mu - \nu \geq 12(q^2 - 1) - 1 \geq 6(q^2 - 1),$$

we may apply Lemma 3.4 and we thus get the existence of a form  $\mathbf{f}_{\mu,\nu,m}$ . Let  $k$  be the smallest integer  $\geq 0$  such that

$$q^k > 3 \log_q \mu. \tag{36}$$

Using Lemma 3.4 and its notation, we have

$$\begin{aligned} \deg_t b_{n_0} &\leq \nu \log_q \left( \frac{3(\mu - \nu)^2}{2(q^2 - 1)(q - 1)} \right) + \nu \log_q \left( \frac{\mu\nu}{q - 1} \right) \\ &\leq \log_q(\mu^2) + \log_q \mu = 3 \log_q \mu, \end{aligned}$$

hence  $\deg_t b_{n_0} < q^k$ . We then define

$$f_k := \mathbf{f}_{\mu,\nu,m}^{(k)} \Big|_{t=\theta}.$$

By [11, Lemma 22], we have  $\nu_\infty(f_k) = \nu_\infty(\mathbf{f}_{\mu,\nu,m}^{(k)})$ , hence

$$\nu_\infty(f_k) = q^k \nu_\infty(\mathbf{f}_{\mu,\nu,m}). \tag{37}$$

On the other hand, it follows from Lemma 3.1 that

$$w(f_k) = q^k \mu + \nu = q^k \mu + 1.$$

Suppose first that  $f$  does not divide  $f_k$ . Consider the resultant

$$\rho := \mathbf{Res}_E(f, f_k) \in C[g, h].$$

Since  $\rho$  is a non-zero modular form, we have  $\nu_\infty(\rho) \leq w(\rho)/(q+1)$ . Thus, since there exist polynomials  $A, B \in C[E, g, h]$  with  $Af + Bf_k = \rho$ , we find

$$\min\{\nu_\infty(f); \nu_\infty(f_k)\} \leq \nu_\infty(\rho) \leq \frac{w(\rho)}{q+1} = \frac{\nu(w-l) + q^k \mu l}{q+1}. \quad (38)$$

If we had  $\min\{\nu_\infty(f); \nu_\infty(f_k)\} = \nu_\infty(f_k)$ , then (38), (37) and (31) would imply

$$q^k \frac{(\mu - \nu)^2}{9(q^2 - 1)} \leq \frac{\nu(w-l) + q^k \mu l}{q+1},$$

or

$$q^k \frac{(\mu - 1)^2}{9(q-1)} \leq w - l + q^k \mu l. \quad (39)$$

But on one hand

$$q^k \frac{(\mu - 1)^2}{9(q-1)} > q^k \frac{\mu^2}{18(q-1)} \geq \frac{3\mu \log_q \mu}{18(q-1)} \geq \frac{\mu}{6(q-1)} \geq 2(w-l),$$

and on the other hand

$$q^k \frac{(\mu - 1)^2}{9(q-1)} \geq q^k \frac{\mu^2}{18(q-1)} \geq q^k \frac{12(q^2 - 1)\mu l}{18(q-1)} \geq 2q^k \mu l.$$

This contradicts (39), hence  $\min\{\nu_\infty(f); \nu_\infty(f_k)\} = \nu_\infty(f)$ , and (38) gives the following upper bound:

$$\nu_\infty(f) \leq \frac{(w-l) + q^k \mu l}{q+1}. \quad (40)$$

Now, we have  $\mu \geq 12(q^2 - 1) \geq q$ , so  $q^k > 3$  by (36) and thus  $k \geq 1$ . It then follows from the definition of  $k$  that

$$q^{k-1} \leq 3 \log_q \mu. \quad (41)$$

But the definition of  $\mu$  gives

$$\mu \leq 12(q^2 - 1)(w-l), \quad (42)$$

hence

$$\log_q \mu \leq \log_q(12) + 2 + \log_q(w - l) \leq 7 \max\{1, \log_q(w - l)\}. \quad (43)$$

By (41), (43) and (42) we get

$$q^k \mu l \leq 252 q(q^2 - 1)l(w - l) \max\{1, \log_q(w - l)\}.$$

The estimate (40) now gives:

$$\nu_\infty(f) \leq 252 q^2 l(w - l) \max\{1, \log_q(w - l)\}$$

which implies the bound of the theorem.

Suppose now that  $f$  divides  $f_k$ . Then

$$\nu_\infty(f) \leq \nu_\infty(f_k) = q^k \nu_\infty(\mathbf{f}_{\mu, \nu, m}) \leq q^k \mu \nu = q^k \mu$$

by Lemma 3.2. Hence, again by (41), (43) and (42):

$$\nu_\infty(f) \leq 252 q(q^2 - 1)(w - l) \max\{1, \log_q(w - l)\}.$$

□

## 4 $A$ -modular and $A$ -quasi-modular forms

Let us define, for  $i = 0, \dots, q$ :

$$\mathbf{h}_i := \tilde{\pi}^i \mathbf{s}_{\text{Car}}^{-i} h \mathbf{s}_2^i,$$

so that  $\mathbf{h}_0 = h$  and  $\mathbf{h}_1 = \mathbf{h}$ , the function introduced in [11] and in Section 2. We recall that in Section 2, we have seen that the functions  $E, g, h, \mathbf{s}_2, \mathbf{E}, \mathbf{h}$  are regular (and the radii of  $E, g, h, \mathbf{E}, \mathbf{h}$  are infinite). It is a simple exercise to show that also the  $\mathbf{h}_i$ 's (for  $i = 0, \dots, q$ ) are regular, of infinite radius.

We recall that  $\mathbf{s}_2$  is regular, with radius  $q$ . We will use again that  $\mathbf{s}_2(z_0, t) \in C[[t]]^\times$  is a unit in the formal series, for all  $z_0 \in \Omega$  fixed. At once, for all  $t_0$  with  $|t_0|$  small,  $\mathbf{s}_2(z, t_0)$  identifies with a unit of  $C[[u]]$  by (15).

## 4.1 Almost $A$ -quasi-modular forms

Recall that we have defined the  $\mathbb{T}_{>0}$ -algebra  $\widetilde{\mathcal{M}}$  of almost- $A$ -quasi-modular forms in Section 2.3.1. It contains the five algebraically independent functions  $E, g, h, \mathbf{E}, \mathbf{E}^{(1)}$ . However, in [11] there is no information about the structure of this algebra (the multiplicity estimate that was the main objective of that paper only required the use of the fourth-dimensional algebra  $\widetilde{\mathcal{M}}^\dagger = \mathbb{T}_{>0}[g, h, \mathbf{E}, \mathbf{E}^{(1)}]$ ). In this paper we enlighten part of the structure of  $\widetilde{\mathcal{M}}$ .

We denote by  $\mathcal{M}$  the  $\mathbb{T}_{>0}$ -sub-algebra of  $\widetilde{\mathcal{M}}$  generated by almost  $A$ -quasi-modular forms of depth 0. Obviously,  $\mathcal{M}$  inherits the graduation from  $\widetilde{\mathcal{M}}$  and we can write:

$$\mathcal{M} = \bigoplus_{\mu, \nu, m} \mathcal{M}_{\mu, \nu, m}$$

with  $\mathcal{M}_{\mu, \nu, m} = \widetilde{\mathcal{M}}_{\mu, \nu, m}^{\leq 0}$ .

We will prove the following theorem which supplies partial information on the structure of  $\widetilde{\mathcal{M}}$ :

**Theorem 4.1** *The  $\mathbb{Z}^2 \times \mathbb{Z}/(q-1)\mathbb{Z}$ -algebra  $\widetilde{\mathcal{M}}$  has dimension five over  $\mathbb{T}_{>0}$  and we have*

$$\widetilde{\mathcal{M}} = \mathcal{M}[E, \mathbf{E}]. \quad (44)$$

Moreover, the following inclusions hold, implying that  $\mathcal{M}$  has dimension three over  $\mathbb{T}_{>0}$ :

$$\mathbb{T}_{>0}[g, h, \mathbf{s}_2] \subset \mathcal{M} \subset \mathbb{T}_{>0}[g, h, \mathbf{s}_2, \mathbf{s}_2^{-1}]. \quad (45)$$

As exercises to familiarise with Theorem 4.1, the reader can verify the following properties (hint: do not forget to use, for example, the functional equations (24)):

1.  $M \subset \mathcal{M}, \widetilde{M} \subset \widetilde{\mathcal{M}}$ .
2.  $g \in \mathcal{M}_{q-1,0,0}, h \in \mathcal{M}_{q+1,0,1}$  and  $E \in \widetilde{\mathcal{M}}_{2,0,1}^{\leq 1}$ .
3.  $\mathbf{s}_2 \in \mathcal{M}_{-1,1,0}$  but  $\mathbf{s}_2^{(1)} \notin \widetilde{\mathcal{M}}$ .
4. For all  $i = 0, \dots, q, \mathbf{h}_i \in \mathcal{M}_{q+1-i,i,1}$ .
5.  $\mathbf{E} \in \widetilde{\mathcal{M}}_{1,1,1}^{\leq 1}$  and  $\mathbf{E}^{(1)} \in \widetilde{\mathcal{M}}_{q,1,1}^{\leq 1}$ .



6. There exists an element  $\lambda \in \mathbb{F}_q(t, \theta)^\times$  (compute it!) such that  $\mathbf{E}^{(1)} = \lambda(\mathbf{h}_1 + g\mathbf{E})$ .
7.  $\mathbb{M}^\dagger = \mathbb{T}_{>0}[g, h, \mathbf{E}, \mathbf{E}^{(1)}] = \mathbb{T}_{>0}[g, h, \mathbf{E}, \mathbf{h}_1]$ .
8.  $\mathbf{E} = -\sqrt[q]{\mathbf{h}_q^{(1)}}$ .

## 4.2 Structure of $\widetilde{\mathcal{M}}$

In this subsection, we show that (44) holds; this will be a consequence of Proposition 4.3 that will need the proposition below.

**Proposition 4.2** *For  $l > 0$ , let  $\mathbf{f}$  be an element of  $\widetilde{\mathcal{M}}_{\mu, \nu, m}^{\leq l} \setminus \widetilde{\mathcal{M}}_{\mu, \nu, m}^{\leq l-1}$  satisfying (26). Then, if  $i + j = l$ , we have, in (26),  $\mathbf{f}_{i,j} \in \mathcal{M}_{\mu-2i-j, \nu-j, m-i-j}$ .*

*Proof.* Let us consider three matrices  $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \Gamma$  as follows:

$$\mathcal{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \mathcal{C} = \mathcal{A} \cdot \mathcal{B} = \begin{pmatrix} * & * \\ x & y \end{pmatrix}. \quad (46)$$

We recall from [11] that we have the functional equations:

$$\begin{aligned} L_{\mathcal{A}}(\mathcal{B}(z)) &= \det(\mathcal{B})^{-1} J_{\mathcal{B}}(z)^2 (L_{\mathcal{C}}(z) - L_{\mathcal{B}}(z)), \\ L_{\mathcal{A}}(\mathcal{B}(z)) &= \det(\mathcal{B})^{-1} J_{\mathcal{B}}(z) \mathbf{J}_{\mathcal{B}}(z) (\mathbf{L}_{\mathcal{C}}(z) - \mathbf{L}_{\mathcal{B}}(z)). \end{aligned} \quad (47)$$

We now compute, by using (47) and the fact that  $(\mathbf{J}_\gamma)_{\gamma \in \Gamma}$  is a factor of automorphy (cf. (21)):

$$\begin{aligned}
\mathbf{f}(\mathcal{C}(z)) &= \mathbf{f}(\mathcal{A}(\mathcal{B}(z))) = \\
&= \det(\mathcal{A})^{-m} J_{\mathcal{A}}(\mathcal{B}(z))^\mu \mathbf{J}_{\mathcal{A}}(\mathcal{B}(z))^\nu \sum_{i+j \leq l} \mathbf{f}_{i,j}(\mathcal{B}(z)) L_{\mathcal{A}}(\mathcal{B}(z))^i \mathbf{L}_{\mathcal{A}}(\mathcal{B}(z))^j \\
&= \det(\mathcal{A})^{-m} J_{\mathcal{A}}(\mathcal{B}(z))^\mu \mathbf{J}_{\mathcal{A}}(\mathcal{B}(z))^\nu \times \\
&\quad \sum_{i+j \leq l} \mathbf{f}_{i,j}(\mathcal{B}(z)) (\det(\mathcal{B})^{-1} J_{\mathcal{B}}(z)^2 (L_{\mathcal{C}}(z) - L_{\mathcal{B}}(z)))^i (\det(\mathcal{B})^{-1} J_{\mathcal{B}}(z) \times \\
&\quad \mathbf{J}_{\mathcal{B}}(z) (\mathbf{L}_{\mathcal{C}}(z) - \mathbf{L}_{\mathcal{B}}(z)))^j \\
&= \det(\mathcal{A})^{-m} J_{\mathcal{A}}(\mathcal{B}(z))^\mu \mathbf{J}_{\mathcal{A}}(\mathcal{B}(z))^\nu \times \\
&\quad \sum_{i+j \leq l} \mathbf{f}_{i,j}(\mathcal{B}(z)) \det(\mathcal{B})^{-i-j} J_{\mathcal{B}}(z)^{2i+j} \mathbf{J}_{\mathcal{B}}(z)^j \times \\
&\quad \sum_{s=0}^i (-1)^{i-s} \binom{i}{s} L_{\mathcal{C}}^s L_{\mathcal{B}}^{i-s} \sum_{s'=0}^j (-1)^{j-s'} \binom{j}{s'} \mathbf{L}_{\mathcal{C}}^{s'} \mathbf{L}_{\mathcal{B}}^{j-s'} \\
&= \det(\mathcal{A})^{-m} J_{\mathcal{A}}(\mathcal{B}(z))^\mu \mathbf{J}_{\mathcal{A}}(\mathcal{B}(z))^\nu \times \\
&\quad \sum_{s+s' \leq l} L_{\mathcal{C}}^s \mathbf{L}_{\mathcal{C}}^{s'} \sum_{i=s}^l \times \\
&\quad \sum_{j=s'}^{l-i} \mathbf{f}_{i,j}(\mathcal{B}(z)) \det(\mathcal{B})^{-i-j} J_{\mathcal{B}}^{2i+j} \mathbf{J}_{\mathcal{B}}^j (-1)^{i+j-s-s'} \binom{i}{s} \binom{j}{s'} L_{\mathcal{B}}^{i-s} \mathbf{L}_{\mathcal{B}}^{j-s'}.
\end{aligned}$$

On the other side, we see that

$$\begin{aligned}
\mathbf{f}(\mathcal{C}(z)) &= \det(\mathcal{A}\mathcal{B})^{-m} J_{\mathcal{C}}^\mu \mathbf{J}_{\mathcal{C}}^\nu \sum_{i+j \leq l} \mathbf{f}_{i,j} L_{\mathcal{C}}^i \mathbf{L}_{\mathcal{C}}^j \\
&= \det(\mathcal{A}\mathcal{B})^{-m} J_{\mathcal{A}}(\mathcal{B}(z))^\mu J_{\mathcal{B}}^\mu \mathbf{J}_{\mathcal{A}}(\mathcal{B}(z))^\nu \mathbf{J}_{\mathcal{B}}^\nu \sum_{i+j \leq l} \mathbf{f}_{i,j} L_{\mathcal{C}}^i \mathbf{L}_{\mathcal{C}}^j.
\end{aligned}$$

We let  $\mathcal{A}$  span  $\mathbf{GL}_2(A)$  leaving  $\mathcal{B}$  fixed at the same time. Since in this way the matrix  $\mathcal{C}$  covers the whole group  $\Gamma$ , if  $s, s'$  are such that  $s + s' \leq l$ , then

$$\begin{aligned}
\det(\mathcal{B})^{-m} J_{\mathcal{B}}^\mu \mathbf{J}_{\mathcal{B}}^\nu \mathbf{f}_{s,s'}(z) &= \\
&= \sum_{i=s}^l \sum_{j=s'}^{l-i} \mathbf{f}_{i,j}(\mathcal{B}(z)) \det(\mathcal{B})^{-i-j} J_{\mathcal{B}}^{2i+j} \mathbf{J}_{\mathcal{B}}^j (-1)^{i+j-s-s'} \binom{i}{s} \binom{j}{s'} L_{\mathcal{B}}^{i-s} \mathbf{L}_{\mathcal{B}}^{j-s'}.
\end{aligned}$$

If  $s + s' = l$ , there is only one term in the sum above, corresponding to  $i = s, j = s'$ . Thus we find

$$\mathbf{f}_{s,s'}(\mathcal{B}(z)) = \det(\mathcal{B})^{s+s'-m} J_{\mathcal{B}}^{\mu-2s-s'} \mathbf{J}_{\mathcal{B}}^{\nu-s'} \mathbf{f}_{s,s'}(z).$$

This identity of formal series of  $C[[t]]$  holds for every  $\mathcal{B} \in \Gamma$  and every  $z \in \Omega$ . Since on the other side we know already that  $\mathbf{f}_{s,s'} \in \mathcal{O}$ , the proposition follows.  $\square$

**Proposition 4.3** *Let  $\mathbf{f}$  be in  $\widetilde{\mathcal{M}}_{\mu,\nu,m}^{\leq l}$ . Then*

$$\mathbf{f} \in \bigoplus_{i+j \leq l} E^i \mathbf{E}^j \mathcal{M}_{\mu-2i-j, \nu-j, m-i-j}.$$

*Proof.* Let us assume that  $\mathbf{f}$  is an almost  $A$ -quasi-modular form, such that for  $\gamma \in \Gamma$ , (26) holds. By Proposition 4.2, if  $i + j = l$  in (26), then  $\mathbf{f}_{i,j} \in \mathcal{M}_{\mu-2i-j, \nu-j, m-i-j}$ . By the functional equations of  $E, \mathbf{E}$  (6) and (30), the almost  $A$ -quasi-modular form  $\rho_{i,j} := E^i \mathbf{E}^j$ , for  $\gamma \in \Gamma$ , transforms like:

$$\rho_{i,j}(\gamma(z)) = \det(\gamma)^{-i-j} J_{\gamma}^{2i+j} \mathbf{J}_{\gamma}^j \left( \rho_{i,j}(z) + \cdots + (-1)^{i+j} \frac{1}{\widetilde{\pi}^{i+j}} L_{\gamma}^i L_{\gamma}^j \right).$$

Hence, the function:

$$\mathbf{f}' := \mathbf{f} - (-1)^l \widetilde{\pi}^l \sum_{i+j=l} \rho_{i,j} \mathbf{f}_{i,j}$$

is an almost  $A$ -quasi-modular form with same weight and type as  $\mathbf{f}$ , and depth strictly less than  $l$ . We can apply Proposition 4.2 again on this form, and construct in this way another almost  $A$ -quasi-modular form of depth strictly less than  $l - 1$  and so on. Since the depth is positive, we will end the inductive process with an almost  $A$ -quasi-modular form of depth  $\leq 0$ . Summing up all the terms, we obtain what we wanted, noticing that the proposition immediately implies that (44) holds.  $\square$

### 4.3 The algebra $\mathcal{M}$

We prove (45) in this subsection. We first need some preliminaries.

**Lemma 4.4** *Let  $(z_0, t_0)$  be in  $\Omega \times (C \setminus \{\theta, \theta^q, \theta^{q^2}, \dots\})$ . There exists  $\gamma \in \Gamma$  such that  $\mathbf{s}_2(\gamma(z_0), t_0) \neq 0$ .*

*Proof.* We can suppose that  $(z_0, t_0)$  are such that  $\mathbf{s}_2(z_0, t_0) = 0$ , otherwise the result is trivial.

If  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ , by (19),

$$\mathbf{s}_2(\gamma(z_0), t_0) = (cz_0 + d)^{-1}(\bar{c}(t_0)\mathbf{s}_1(z_0, t_0) + \bar{d}(t_0)\mathbf{s}_2(z_0, t_0)).$$

Let us choose  $\gamma$  so that  $c \in A$  satisfies  $\bar{c}(t_0) \neq 0$ . If also  $\mathbf{s}_2(\gamma(z_0), t_0)$  vanishes, then,  $\mathbf{s}_1(z_0, t_0) = 0$ . By (18),  $\mathbf{s}_{\text{car}}(t_0) = 0$  (the form  $h$  is holomorphic with no zeros on  $\Omega$  so its inverse is holomorphic and does not vanish). But the formula (11) is contradictory with the vanishing of  $\mathbf{s}_{\text{car}}(t_0)$  just obtained.

□

**Remark 4.5** Thanks to Lemma 4.4, we will use  $\mathbf{s}_2$  to detect the structure of our automorphic functions following closely the usual procedure for Drinfeld modular forms, where one uses the fact that  $h$  does not vanish in  $\Omega$  to deduce the structure of  $S_{w,m}$  (space of cusp forms) from the structure of  $M_{w-q-1, m-1}$ .

We will also need the following proposition:

**Proposition 4.6** *For given  $\mu, \nu, m$ , let  $\mathbf{f}$  be a non-zero element of  $\mathcal{M}_{\mu, \nu, m}$ . Then, we have  $\nu \geq -\mu$  and the function:*

$$F := \mathbf{f} \mathbf{s}_2^{-\nu}$$

belongs to  $M_{\mu+\nu, m} \otimes_C \mathbb{T}_{>0}$ .

*Proof.* By (22) the function  $F$  satisfies, for all  $z \in \Omega$ , the following identities in  $C[[t]]$ :

$$F(\gamma(z)) = \det(\gamma)^{-m} J_\gamma^{\mu+\nu} F(z).$$

Let  $t_0$  be such that  $|t_0| < r$ , with  $r$  the minimum of the radius of  $\mathbf{f}$  and  $q$ , the radius of  $\mathbf{s}_2$ . We know that the functions  $z \mapsto \mathbf{f}(z, t_0)$  and  $z \mapsto \mathbf{s}_2(z, t_0)$  are holomorphic functions  $\Omega \rightarrow C$ . Therefore,  $\varphi_{t_0}(z) := F(z, t_0)$  is a meromorphic function over  $\Omega$  (more precisely, it is holomorphic if  $\nu \leq 0$ ). Moreover, it is plain that for all  $\gamma \in \Gamma$  and  $z \in \Omega$  at which  $\varphi_{t_0}$  is defined,

$$\varphi_{t_0}(\gamma(z)) = \det(\gamma)^{-m} J_\gamma^{\mu+\nu} \varphi_{t_0}(z).$$

We also see that  $\varphi_{t_0}$  is holomorphic at infinity. Indeed, for all  $t_0$  with  $|t_0|$  small,  $\mathbf{s}_2(z, t_0)$  is a unit of  $C[[u]]$ , by (15). Moreover, there exists  $c > 0$

depending on  $t_0$  such that  $\mathbf{s}_2(z, t_0)$  has no zeroes for  $z$  such that  $|u| = |u(z)| < c$  and this property is obviously shared with  $\mathbf{s}_2(z, t_0)^{-1}$ . Hence,  $\varphi_{t_0}$  is a “meromorphic Drinfeld modular form” of weight  $\mu + \nu$  and type  $m$  which is holomorphic at infinity.

We claim that in fact,  $\varphi_{t_0}$  is holomorphic on  $\Omega$  (hence regular) regardless of the sign of  $\nu$  (this implies that it is a “genuine” Drinfeld modular form). Indeed, if the claim is false,  $\nu > 0$  and  $\varphi_{t_0}$  has a pole at  $z_0 \in \Omega$ , as well as at any point  $\gamma(z_0)$  with  $\gamma \in \Gamma$ . Since  $z \mapsto \mathbf{f}(z, t_0)$  is holomorphic in  $\Omega$ ,  $\gamma(z_0)$  is a zero of  $z \mapsto \mathbf{s}_2(z, t_0)$  for all  $\gamma \in \Gamma$  as above. But this is contradictory with Lemma 4.4.

Hence, for all  $t_0$  as above,  $\varphi_{t_0} = F(\cdot, t_0) \in M_{\mu+\nu, m}$ . Let  $(b_1, \dots, b_\ell)$  be a basis of  $M_{\mu+\nu, m}$  over  $C$ . It is easy to show, looking at the  $(u_1, \dots, u_\ell)$ -expansion in  $C[[u_1, \dots, u_\ell]]$  of the function  $\det(b_i(z_j))_{1 \leq i, j \leq \ell}$  and the existence of the “triangular” basis  $(g^i h^j)$  in  $M_{\mu+\nu, m}$ , that there exist (obviously distinct)  $z_1, \dots, z_\ell \in \Omega$  such that the matrix  $(b_i(z_j))_{1 \leq i, j \leq \ell} \in \mathbf{Mat}_{\ell \times \ell}(C)$  is non-singular.

For all  $t$  with  $|t| < r$ , there exist  $d_1(t), \dots, d_\ell(t) \in C$  such that

$$F(z, t) = d_1(t)b_1(z) + \dots + d_\ell(t)b_\ell(z).$$

Since for all  $i = 1, \dots, \ell$ ,  $F(z_i, t)$  belongs to  $\mathbb{T}_{>0}$ , we find:

$$d_1(t)b_1(z_j) + \dots + d_\ell(t)b_\ell(z_j) \in \mathbb{T}_{>0}, \quad j = 1, \dots, \ell.$$

Solving a linear system in the indeterminates  $d_1, \dots, d_\ell$  we find  $d_1, \dots, d_\ell \in \mathbb{T}_{>0}$ . Hence,  $F \in M_{\mu+\nu, m} \otimes_C \mathbb{T}_{>0}$  and  $\mathbf{f}$  being non-zero, we have  $\mu + \nu \geq 0$ , concluding the proof of the proposition.  $\square$

We deduce from Proposition 4.6 and from the discussion above, the inclusions (45).

**Remark 4.7** If  $f$  is a Drinfeld modular form in  $M_{w, m}$ ,  $\mathbf{f}_i := \tilde{\pi}^i \mathbf{s}_{\text{Car}}^{-i} f \mathbf{s}_2^i$  belongs to  $\mathcal{M}_{w-i, i, m}$  for all  $i \geq 0$ , has infinite radius, and is such that  $\mathbf{f}_i|_{t=\theta} = f$  (this is how we have defined the forms  $\mathbf{h}_i$ ). For example, if  $f = 1$ , we have  $\mathbf{1}_i = \tilde{\pi}^i \mathbf{s}_{\text{Car}}^{-i} \mathbf{s}_2^i$ . Proposition 4.6 gives a converse of this fact: every element of  $\mathcal{M}_{\mu, \nu, m}$  comes from a Drinfeld modular form in this way. However, it is an open question to prove or disprove that  $\mathbf{s}_2^{-1}$  is regular. In fact, we expect that  $\mathbf{s}_2^{-1} \notin \mathcal{O}$ . Several arguments led us to believe that there exists a sequence  $(z_n, t_n)_{n \geq 0}$  in  $\Omega \times C$  with  $\lim_{n \rightarrow \infty} t_n = 0$  and  $t_n \neq 0$  for all  $n$ , such

that  $\mathbf{s}_2(z_n, t_n) = 0$  for all  $n$  but, at the time being, we do not know how to precisely locate the zeroes of the function  $(z, t) \mapsto \mathbf{s}_2(z, t)$ .

**Question 4.8** Find the exact image of  $\mathcal{M}$  in  $\mathbb{T}_{>0}[g, h, \mathbf{s}_2, \mathbf{s}_2^{-1}]$ .

The following conjecture agrees with our guess that  $\mathbf{s}_2^{-1}$  is not regular.

**Conjecture 4.9** We have  $\mathcal{M} = \mathbb{T}_{>0}[g, h, \mathbf{s}_2]$ .

In the next subsection we will construct a three-dimensional sub-algebra  $\mathbb{M}$  of  $\mathcal{M}$  of  $A$ -modular forms with a very precise structure, which also is finitely generated over  $\mathbb{T}_{>0}$ . To define it, we will use the Frobenius structure over  $\mathcal{O}$ . With this task in view, we will first need to introduce the modules  $\widehat{\mathcal{M}}_{\mu, \nu, m}^{\leq l}$  at the beginning of the next subsection.

#### 4.4 $A$ -modular forms and their structure

For  $(\mu, \nu, m)$  an element of  $\mathbb{Z}^2 \times \mathbb{Z}/(q-1)\mathbb{Z}$  and  $l \in \mathbb{N}$ , we will use, in all the following, the sub- $\mathbb{T}_{>0}$ -module  $\widehat{\mathcal{M}}_{\mu, \nu, m}^{\leq l} \subset \widetilde{\mathcal{M}}_{\mu, \nu, m}^{\leq l}$  whose elements  $\mathbf{f}$  satisfy functional equations of the more particular form:

$$\mathbf{f}(\gamma(z), t) = \det(\gamma)^{-m} J_\gamma^\mu J_\gamma^\nu \left( \sum_{i \leq l} \mathbf{f}_i L_\gamma^i \right),$$

with  $\mathbf{f}_i \in \mathcal{O}$  for all  $i$ . In such a kind of functional equation, there is no dependence on  $L_\gamma$ . Hence,  $E \notin \widehat{\mathcal{M}}$  where  $\widehat{\mathcal{M}}$  is the  $\mathbb{T}_{>0}$ -algebra, graded by  $\mathbb{Z}^2 \times \mathbb{Z}/(q-1)\mathbb{Z}$ , generated by the modules  $\widehat{\mathcal{M}}_{\mu, \nu, m}^{\leq l}$ . We deduce from the last subsection that  $\widehat{\mathcal{M}} = \widetilde{\mathcal{M}}[\mathbf{E}]$  is a  $\mathbb{T}_{>0}$ -algebra of dimension 4.

**Definition 4.10** Let  $\mathbf{f}$  be an element of  $\widehat{\mathcal{M}}_{\mu, \nu, m}^{\leq l}$ . We say that  $\mathbf{f}$  is an  $A$ -quasi-modular form of weight  $(\mu, \nu)$ , type  $m$  and depth  $\leq l$ , if, for all  $k \geq 0$ ,

$$\mathbf{f}^{(k)} \in \widehat{\mathcal{M}}. \quad (48)$$

If  $l = 0$ , we will say that  $\mathbf{f}$  is an  $A$ -modular form of weight  $(\mu, \nu)$  and type  $m$ .

For  $\mu, \nu \in \mathbb{Z}, m \in \mathbb{Z}/(q-1)\mathbb{Z}, l \in \mathbb{Z}_{\geq 0}$ , we denote by  $\widetilde{\mathbb{M}}_{\mu, \nu, m}^{\leq l}$  the  $\mathbb{T}_{>0}$ -module of  $A$ -quasi-modular forms of weight  $(\mu, \nu)$ , type  $m$  and depth  $\leq l$ . Again, we have

$$\widetilde{\mathbb{M}}_{\mu, \nu, m}^{\leq l} \widetilde{\mathbb{M}}_{\mu', \nu', m'}^{\leq l'} \subset \widetilde{\mathbb{M}}_{\mu+\mu', \nu+\nu', m+m'}^{\leq l+l'}$$

and the  $\mathbb{T}_{>0}$ -algebra  $\widetilde{\mathbb{M}}$  generated by all these modules is again graded by  $\mathbb{Z}^2 \times \mathbb{Z}/(q-1)\mathbb{Z}$  (and filtered by the depths). The graded sub-algebra of  $\widetilde{\mathbb{M}}$  generated by  $A$ -modular forms is denoted by  $\mathbb{M}$ . We have thus  $\mathbb{M} = \bigoplus_{\mu, \nu, m} \mathbb{M}_{\mu, \nu, m}$ , where  $\mathbb{M}_{\mu, \nu, m} = \widetilde{\mathbb{M}}_{\mu, \nu, m}^{\leq 0}$ .

Earlier in this paper and in [11], we have introduced the  $\mathbb{T}_{>0}$ -algebra  $\mathbb{M}^\dagger = \mathbb{T}_{>0}[g, h, \mathbf{E}, \mathbf{E}^{(1)}]$ , which turned out to be equal to  $\mathbb{T}_{>0}[g, h, \mathbf{E}, \mathbf{h}_1]$ . We have  $\mathbb{M}^\dagger \subset \widetilde{\mathbb{M}}$  but these algebras are not equal. Indeed, with the help of (24) <sup>(2)</sup>, one sees that  $\mathbb{T}_{>0}[g, h, \mathbf{h}_1, \dots, \mathbf{h}_q] \subset \mathbb{M}$ . The next theorem provides the equality of the latter two algebras. We notice that at the time being, we do not have a complete structure theorem for the algebra  $\widetilde{\mathbb{M}}$ .

**Theorem 4.11** *The algebra  $\mathbb{M}$  of  $A$ -modular forms is finitely generated of dimension three. Moreover, we have  $\mathbb{M} = \mathbb{T}_{>0}[g, \mathbf{h}_0, \mathbf{h}_1, \dots, \mathbf{h}_q]$ .*

*Proof.* Since it is clear that  $\mathbb{T}_{>0}[g, \mathbf{h}_0, \mathbf{h}_1, \dots, \mathbf{h}_q]$  has dimension three, the first part of the statement is a consequence of the second that we prove now. Let  $\mathbf{f}$  be a non-constant element of  $\mathbb{M}_{\mu, \nu, m}$ . Since  $\mathbb{M}_{\mu, \nu, m} \subset \mathcal{M}_{\mu, \nu, m}$ , Proposition 4.6 implies that  $\mathbf{f} \mathbf{s}_2^{-\nu} \in \mathbb{T}_{>0}[g, h]$ . Hence, there exists a non-zero element  $\varphi \in \mathcal{M}_{\mu+\nu, m} \otimes_{\mathbb{C}} \mathbb{T}_{>0}$  such that  $\mathbf{f} = \varphi \mathbf{s}_2^\nu$ .

Let  $s$  be the integer  $\nu_\infty(\mathbf{f})$ ; we have that  $s = \nu_\infty(\varphi)$  because  $\nu_\infty(\mathbf{s}_2) = 0$  as we remarked earlier. We claim that  $0 \leq \nu \leq sq$ . We first proceed to prove that  $\nu \geq 0$ .

Assume conversely that  $\nu < 0$ . The hypotheses and (24) imply, for  $k \geq 0$ , for all  $z \in \Omega$  and for all  $\gamma \in \Gamma$ :

$$\begin{aligned} (\varphi \mathbf{s}_2^\nu)^{(k)}(\gamma(z), t) &= J_\gamma^{\mu q^k} \mathbf{J}_\gamma^\nu \det(\gamma)^{-m} \varphi(z)^{(k)} (\mathbf{s}_2^{(k)} + \alpha_k \mathbf{L}_\gamma)^\nu \\ &= \sum_{a, b, c} J_\gamma^a \mathbf{J}_\gamma^b \det(\gamma)^{-c} \sum_j \mathbf{f}_{a, b, c, j, k} \mathbf{L}_\gamma^j, \end{aligned}$$

where

$$\alpha_k = (t - \theta) \frac{g_{k-1}^*(z, t) \mathbf{s}_{\text{Car}}(t)}{\widetilde{\pi}^{q^k+1} h(z)^{q^{k-1}}}$$

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<sup>2</sup>Identity (24) also certifies that  $\mathbf{s}_2$  does not belong to  $\widetilde{\mathbb{M}}$ . Indeed,  $\mathbf{s}_2^{(1)} \notin \widetilde{\mathcal{M}}$ .

is the (non-zero) coefficient of  $\mathbf{L}_\gamma$  in the inner bracket of the right-hand side of (24), where all the sums have finitely many terms, and where the  $\mathbf{f}_{a,b,c,j,k}$ 's are regular functions. Comparing the two right-hand sides we get:

$$1 = \det(\gamma)^m J_\gamma^{-\mu q^k} \mathbf{J}_\gamma^{-\nu} (\varphi^{-1})^{(k)} \times \\ (\mathbf{s}_2^{(k)} + \alpha_k \mathbf{L}_\gamma)^{-\nu} \sum_{a,b,c} J_\gamma^a \mathbf{J}_\gamma^b \det(\gamma)^{-c} \sum_j \mathbf{f}_{a,b,c,j,k} \mathbf{L}_\gamma^j$$

which provides a contradiction with the non-vanishing of  $\alpha_k$  because the constant function  $z \mapsto 1$  is  $A$ -quasi-modular of weight  $(0,0)$ , type 0 and depth  $\leq 0$ .

We have proved that  $\nu \geq 0$ . We now prove the upper bound  $\nu \leq qs$ . We can write  $\varphi = h^s \psi$  with  $\psi \in \mathbb{T}_{>0}[g, h]$  such that  $\nu_\infty(\psi) = 0$ . Since  $\mathbf{f} = h^s \psi \mathbf{s}_2^\nu$ , we have, for all  $k$ ,  $\mathbf{f}^{(k)} = h^{sq^k} \psi^{(k)} (\mathbf{s}_2^\nu)^{(k)}$  and

$$\mathbf{f}^{(k)}(\gamma(z)) = \det(\gamma)^{-m} J_\gamma^{\mu q^k} \mathbf{J}_\gamma^\nu h(z)^{sq^k} \psi^{(k)}(z) (\mathbf{s}_2^{(k)}(z) + \alpha_k \mathbf{L}_\gamma)^\nu.$$

A necessary condition for  $\mathbf{f}^{(k)}$  to be in  $\widehat{\mathcal{M}}$  is that  $h^{sq^k} \alpha_k^\nu$  belongs to  $\mathcal{O}$  for all  $k \geq 1$ . Since  $\nu_\infty(\alpha_k) = -q^{k-1}$  (when  $k \geq 1$ ), this condition is equivalent to  $sq - \nu \geq 0$ . This gives  $\nu \leq qs$  as claimed.

Write now  $\nu = qa + r$  with  $0 \leq r < q$  (Euclidean division), from which we deduce  $a \leq s$  and  $a < s$  if  $r \neq 0$ .

If  $r = 0$  then we have  $\nu = qa$  and  $h^s \mathbf{s}_2^\nu = h^s \mathbf{s}_2^{qa} = h^{s-a} h^a \mathbf{s}_2^{qa}$  which belongs to  $\mathbb{T}_{>0} \mathbf{h}_0^{s-a} \mathbf{h}_q^a$ . If  $r \neq 0$ , then  $s - a - 1 \geq 0$  and

$$h^s \mathbf{s}_2^\nu = h^{s-a-1} \cdot h \mathbf{s}_2^r \cdot h^a \mathbf{s}_2^{qa} \in \mathbb{T}_{>0} \mathbf{h}_0^{s-a-1} \mathbf{h}_r \mathbf{h}_q^a.$$

In all cases, we thus have  $h^s \mathbf{s}_2^\nu \in \mathbb{T}_{>0}[\mathbf{h}_0, \mathbf{h}_1, \dots, \mathbf{h}_q]$ . Since  $\mathbf{f} = h^s \psi \mathbf{s}_2^\nu$  with  $\psi \in \mathbb{T}_{>0}[g, h]$ , we finally obtain  $\mathbf{f} \in \mathbb{T}_{>0}[g, \mathbf{h}_0, \mathbf{h}_1, \dots, \mathbf{h}_q]$  as required.  $\square$

## 5 Final remarks

**Remark 5.1** The image of the map (Anderson's twist)  $\tau : \widetilde{\mathcal{M}} \rightarrow \mathcal{O}$  is not contained in  $\widetilde{\mathcal{M}}$ . At least, it can be proved that given  $\mathbf{f} \in \widetilde{\mathcal{M}}_{\mu,\nu,m}^{\leq l}$ , we have  $\tau^k \mathbf{f} \in \widetilde{\mathcal{M}}[1/h]_{q^k \mu, \nu, m}^{\leq l}$  for all  $k \geq 0$ .

In Lemma 3.1 (cf. [11]), we have showed that  $\mathbb{F}_q[[t]]$ -linear Anderson's operator  $\tau : \mathcal{O} \rightarrow \mathcal{O}$ , which operates on double formal series of  $C[[t, u]]$  as:

$$\tau : \sum_{i,j \geq 0} c_{i,j} t^i u^j \mapsto \sum_{i,j \geq 0} c_{i,j}^q t^i u^{qj}$$



(the  $c_{i,j}$ 's being elements of  $C$ ) defines an operator  $\mathbb{M}^\dagger \rightarrow \mathbb{M}^\dagger$  and is  $\mathbb{T}_{>0}$ -linear on the modules  $\mathbb{M}_{\mu,\nu,m}^\dagger \rightarrow \mathbb{M}_{q\mu,\nu,m}^\dagger$ . Thanks to this result, it is possible to prove that every form of  $\mathbb{M}_{\mu,\nu,m}^\dagger$  satisfies a non-trivial linear  $\tau$ -difference equation.

Since we still do not know the exact structure of the algebra  $\widetilde{\mathbb{M}}$ , we presently cannot extend this observation to the settings of this paper.

**Remark 5.2** It is natural to ask whether any of the finitely generated  $\mathbb{T}_{>0}$ -modules we have introduced so far, like  $\mathcal{M}_{\mu,\nu,m}, \mathbb{M}_{\mu,\nu,m}$ , can be endowed with a natural extension of Hecke operators as defined, on the vector spaces  $M_{w,m}$ , in [7]. Let us fix  $(\mu, \nu, m)$  in  $\mathbb{Z}^2 \times \mathbb{Z}/(q-1)\mathbb{Z}$ . One is then tempted to choose, for  $\mathfrak{P}$  a prime ideal of  $A$  generated by the monic polynomial  $\mathfrak{p} \in A \setminus \{0\}$  of degree  $d > 0$ , a  $\mathbb{T}_{>0}$ -linear map

$$T_{\mathfrak{P}} : \mathcal{M}_{\mu,\nu,m} \rightarrow \mathcal{O}$$

by:

$$(T_{\mathfrak{P}}\mathbf{f})(z) := \mathfrak{p}^\mu \bar{\mathfrak{p}}^\nu \mathbf{f}(\mathfrak{p}z) + \sum_{b \in A, \deg_\theta b < d} \mathbf{f}\left(\frac{z+b}{\mathfrak{p}}\right),$$

for  $\mathbf{f} \in \mathcal{M}_{\mu,\nu,m}$ .

This map reduces to the Hecke operator  $T_{\mathfrak{P}} : M_{w,m} \rightarrow M_{w,m}$  of [7] if we set  $t = \theta$  (when this operation makes sense). However, it is unclear when the image  $T_{\mathfrak{P}}(\mathcal{M}_{\mu,\nu,m})$  is contained in  $\mathcal{M}_{\mu,\nu,m}$ . For instance, it can be proved that  $\Delta^2$  is not eigenform for all the operators  $T_{\mathfrak{P}}$ . Since it is easy, applying the second inclusion in (45), to prove that if  $q \neq 2, 3$  then  $\mathcal{M}_{2(q-1), 2q(q-1), 0} = \mathbb{T}_{>0} \mathbf{h}_q^{2(q-1)}$ , it follows that  $T_{\mathfrak{P}}(\mathcal{M}_{2(q-1), 2q(q-1), 0}) \subsetneq \mathcal{M}_{2(q-1), q(q-1), 0}$ .

The problem arises with our second factor of automorphy  $(\mathbf{J}_\gamma)_{\gamma \in \Gamma}$ : it does not extend to a factor of automorphy  $\mathbf{GL}_2(K) \times \Omega \rightarrow C[[t]]$ . In other words, almost  $A$ -quasi-modular forms do not always come from lattice functions. In this sense, our algebra  $\mathcal{M}$  might still be “too small”, needing to be embedded in a larger algebra which is presently unknown.

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